RELEASE FOR

DOCUME

GPO PRICE $

CFSTI PRICE(S) $

Hard copy (HC) 3.00

Microfiche (MF) .50

ff 653 July 65

N65- 10 5

(ACCESSION NUMBER)

(THRU)

53

(PAGES)

1

(CODE)

CR 64123

(NASA CR OR TMX OR AD NUMBER)

07

(CATEGORY)

FACILITY FORM 602

REPORT

by

THE OHIO STATE UNIVERSITY RESEARCH FOUNDATION
COLUMBUS 12, OHIO

Cooperator	National Aeronautics and Space Administration 1520 H Street, Northwest Washington 25, D.C.
Grant No.	NsG-74-60
Investigation of	Receiver Techniques and Detectors for Use at Millimeter and Submillimeter Wave Lengths
Subject of Report	On the Correlation Radiometer Technique
Submitted by	K. Fujimoto Antenna Laboratory Department of Electrical Engineering
Date	11 January 1962

TABLE OF CONTENTS

ON THE CORRELATION RADIOMETER TECHNIQUE

I. INTRODUCTION

The most important problem in radiometer technique is to obtain the lowest detectable temperature. Such a minimum detectable temperature is usually determined by the noise fluctuations in the receiver output level, when the real signal is absent. Since the temperature is observed in the form of thermal radiation special techniques must be employed to reduce spurious fluctuations and internal noises produced by the receiver circuits, and to differentiate them from the real signal. The conventional method for reducing the spurious fluctuation and noise effects of the receiver is to employ an optimum modulation of signal, so that the spurious spectrum and noise would be cancelled out, as in the well known Dicke-type system.[1]

Many types of radiometers have been investigated previously,[2,3,4,5,6,7] but the most commonly used one is still the Dicke-type radiometer and its various modified versions. In this paper, we wish to study the possibility of the remarkable usefulness of the correlation radiometer, and to compare it with the Dicke-type radiometer with special emphasis on millimeter applications.

The correlation radiometer consists of two receiver systems with separate antennas, as shown in Fig. 1. Both antennas are looking at the same signal source, thus the two signals S_1 and S_2 will be correlated in time, and upon multiplication they will provide an output proportional to S. Noise, N_1 and N_2, introduced by each receiver will necessarily have a low degree of correlation because of the random nature of N_1 and N_2; thus the correlated output will represent the signal S plus some low level correlation between N_1 and N_2. In other words, the sensitivity of the radiometer could be greatly increased due to the low degree of correlation of N_1 and N_2.

There are many advantages and disadvantages of the correlation radiometer. For example, this system of two identical receivers would be difficult to design in practice, and spurious fluctuations of individual receivers may have more serious effects at the output of the receiver as compared to the Dicke-type. Furthermore, the practical correlator may have a bandwidth much narrower than the receiver bandwidth; in

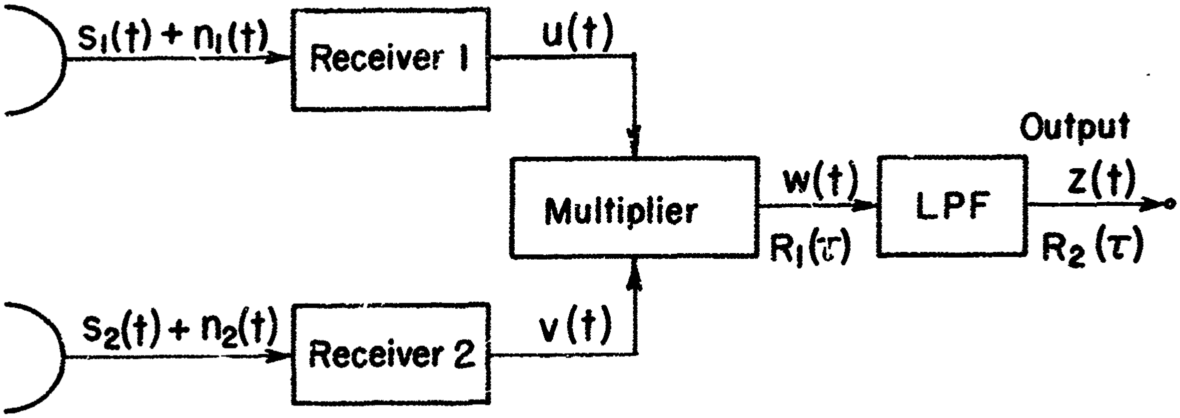

Fig. 1. Simple block diagram for the correlation radiometer.

this case the minimum detectable temperature would be increased due to the loss of bandwidth. However, in the millimeter and submillimeter wavelength region, and for radiometers used in space probes, these disadvantages may be offset by some of the advantages of such a system.

In the first place, the correlation radiometer is a natural system to use for the interferometer. It means that tracking or beam steering can be performed on space probes by introducing a phase shift in one of the antennas without any mechanical movements.

Secondly, the microwave switches in Dicke radiometers become very lossy and hard to build in the millimeter wavelength region. Although the optical chopper may be used to replace it, it is hard to achieve noise compensation with the optical type of chopper. Since the correlation radiometer does not employ a switching mechanism, it may have a better sensitivity in this respect.

Thirdly, in the correlation radiometer we can observe simultaneously two signals, which are coherent but have different signal strengths at the receiver inputs. For example, a radio source and its wave reflected by some body like the moon can be captured by each antenna separately. The correlation radiometer can then be used to analyze the scattering and reflection characteristics of the body.

Finally, it may happen that the radio wave from a source is interfered with by some natural phenomena for one antenna while the signal in the other antenna is unaffected. Then the correlated signal can be utilized for the measurement of the characteristics of the obstacles, similar to the double beam optical spectrometer.

Notice that all these advantages (except the elimination of a chopper) can be achieved to a limited degree by the interferometer type of radiometer using a Dicke-type of detection system. However, its sensitivity may be lower than the correlation type of radiometer. Consider again the example of the reflection from the moon, as shown in Fig. 2. The reflected signal of the sun by the moon, S_s' would be quite weak compared to the background radiation of the moon. If the Dicke-type of detection is employed both S_s' and the background radiation are chopped. Hence it may be quite difficult to distinguish the contribution of S_s' from the contribution of the "noise" (i.e., the background radiation). But in the correlation radiometer, S_{sun} and S_s' would have a high degree of correlation, which may yield a much better sensitivity for measuring S_s'. The beam steering properties would be the same for both types of radiometers.

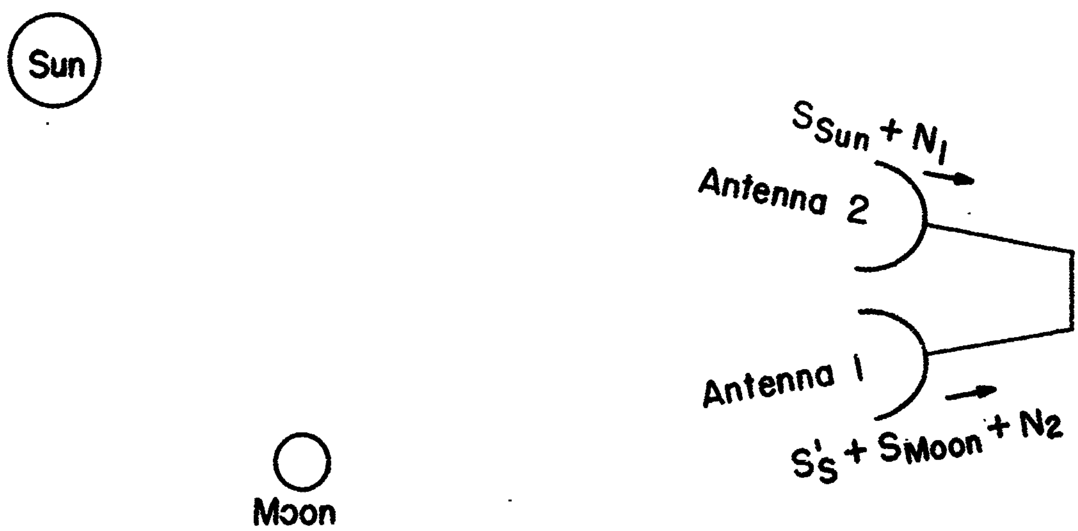

Fig. 2. A proposed experiment to study the bi-static reflection from moon using a correlation radiometer.

In this report we have made an analysis of the gain fluctuation effects and other receiver characteristics upon the sensitivity of a correlation radiometer. An experimental model of the correlator, the Halltron, has also been tested to examine the practical capability of such a correlator. And finally, some comparisons between the two radiometer techniques are discussed to evaluate the usefulness of correlation radiometers.

II. MINIMUM DETECTABLE SENSITIVITY OF A CORRELATION RADIOMETER

Consider the simplified block diagram of the correlation radiometer shown in Fig. 1. Its final output power is produced by correlating the two incident signals, S_1 and S_2, and the two noises, N_1 and N_2, with the multiplier. Therefore, in order to determine the minimum detectable signal in terms of an equivalent temperature, ΔT, one must evaluate the signal-to-noise ratio at the output of the radiometer system. This may be found by calculating the autocorrelation function of the output of the low pass filter. Let us first consider the input terminal voltage of the receivers,

$$S_j(t) + n_j(t) \quad (j = 1, 2) \tag{1}$$

where $S_j(t)$ is the signal voltage and $n_j(t)$ is the noise voltage. Both S_j and n_j are assumed to be random functions with zero mean having Gaussian distributions (j denotes the receiver number). The output voltages of the receivers, before the multiplier, are expressed by

$$u(t) = A_1(t)\,(S_1(t) + n_1(t)) + n_{i1}(t) \tag{2}$$

$$v(t) = A_2(t)\,(S_2(t) + n_2(t)) + n_{i2}(t) \tag{3}$$

where $A_j(t)$ $(j = 1, 2)$ is the voltage gain of the receiver amplifier and $N_{ij}(t)$ $(j = 1, 2)$ is the internal noise of the receiver. In order to make calculation easier, we replace the noise terms $\{n_j(t) + n_{ij}(t)/A_j(t)\}$ by the equivalent noise voltages at the receiver input, $N_j(t)$ $(j = 1, 2)$.

(4) $$u(t) = A_1(t)\{S_1(t) + N_1(t)\}$$

(5) $$v(t) = A_2(t)\{S_2(t) + N_2(t)\} .$$

The gain of the receiver $A_j(t)$ $(j = 1, 2)$ can be expressed by

(6) $$A_j(t) = A_j + \Delta A_j(t) \quad (j = 1, 2)$$

where A_j is the averaged gain and $\Delta A_j(t)$ is the gain variation. ΔA_j can be assumed to be a random function with zero mean and Gaussian distribution.* Since signals coming into the input terminals of the two receivers are coherent, the relation between $S_1(t)$ and $S_2(t)$ may be given by

(7) $$S_2(t) = k\, S_1(t + \theta)$$

where k is the amplitude factor,** and θ is the phase factor, including phase differences resulting from both the wave path length and the internal delay in the receiver. Then the output voltage of the multiplier is given by

(8) $$\begin{aligned} w(t) &= u(t) \cdot v(t) \\ &= A_1 A_2 \left(1 + \frac{\Delta A_1(t)}{A_1}\right)\left(1 + \frac{\Delta A_2(t)}{A_2}\right)\{S_1(t) + N_1(t)\}\{k S_1(t+\theta) + N_2(t)\}. \end{aligned}$$

---

* The gain variation of the system is represented by the gain variation of the amplifier here. There may be many effects contributing to the output amplitude fluctuation in the system; such as incident signal strength fluctuations, mixer gain, IF amplifier gain, correlator non-linearity, and so on. Thus the gain variation function is quite complicated. It can be assumed as a random function as indicated by Strum.[8]

** For the case where two identical antennas are looking directly at the same source, K = 1.

The autocorrelation function of w(t) is expressed by

$$(9) \quad R_1(\tau) = \overline{w(t)\, w(t+\tau)}$$

$$= A_1^2 A_2^2 \left(1 + \frac{\phi_{A_1}}{A_1^2}\right)\left(1 + \frac{\phi_{A_2}}{A_2^2}\right) \{k^2 (\phi_s^2(\tau) + \phi_s^2(\theta) + \phi_s(\tau+\theta)\,\phi_s(\tau-\theta))$$

$$+ \phi_s \phi_{N_2} + k^2 \phi_s \phi_{N_1} + \phi_{N_1}\phi_{N_2}\}$$

where ϕ_{A_j}, ϕ_s and ϕ_{N_j} corresponds to the autocorrelation function of $\Delta A_j(t)$, $S(t)$ and $N_j(t)$, respectively. Let us now assume that these signal and noise functions are obtained by passing the white noise through a high Q band pass amplifier (i.e., pre-detector RF or IF circuit) and that the fluctuation spectrum can be treated as RC-noise, such that they are expressed[9] by

$$(10) \qquad \phi_{A_j} = \psi_{A_j}\, e^{-\omega_{A_j}|\tau|}$$

$$(11) \qquad \phi_s = \psi_s\, e^{-\omega_s|\tau|} \cos \omega_o\tau$$

$$(12) \qquad \phi_{N_j} = \psi_{N_j}\, e^{-\omega_{N_j}|\tau|} \cos \omega_o\tau$$

where ψ_A, ψ_s and ψ_N are the mean square value of ΔA, S and N_j, respectively; ω_{A_j} is the cutoff frequency of the gain fluctuation spectrum; ω_s and ω_{N_j} are the halfbandwidths of the bandpass amplifier; and ω_o is the center frequency of the amplifier. By substituting Eqs. (10), (11) and (12) into Eq. (9), we get

$$(13) \quad R_1(\tau) = \left(1 + \frac{\psi_{A_1}}{A_1^2} e^{-\omega_{A_1}|\tau|}\right)\left(1 + \frac{\psi_{A_2}}{A_2^2} e^{-\omega_{A_2}|\tau|}\right) \{k^2 [\psi_s^2\, e^{-2\omega_s|\tau|} \cos^2 \omega_o\tau$$

$$+ \psi_s^2\, e^{-2\omega_s\theta} \cos^2 \omega_o\theta + \psi_s^2\, e^{-\omega_s(|\tau+\theta| + |\tau-\theta|)} \cos\omega_o(\tau+\theta)\cos\omega_o(\tau-\theta)]$$

$$+ \psi_s \psi_{N_2}\, e^{-(\omega_s+\omega_{N_2})|\tau|} \cos^2 \omega_o\tau + k^2 \psi_s\psi_{N_1}\, e^{-(\omega_s+\omega_{N_1})|\tau|} \cos^2\omega_o\tau$$

$$+ \psi_{N_1}\psi_{N_2}\, e^{-(\omega_{N_1}+\omega_{N_2})|\tau|} \cos^2 \omega_o\tau\}$$

where we have neglected the constant product $A_1^2 A_2^2$. The dc component of $R_1(\tau)$ is

$$(14) \qquad R_1(\tau)\Big]_{dc} = k^2 \psi_s^2 \, e^{-2\omega_s\theta} \cos^2 \omega_o\theta$$

which represents the signal component. Thus $R_1(\tau)$ is the correlated output after the multiplier but before the integrator, which is essentially a low pass filter. Thus the output of the low pass filter is

$$(15) \qquad R_2(\tau') = \int\int R_1(\tau) \, |w(f)|^2 \, e^{-j2\pi f(\tau - \tau')} \, d\tau \, df$$

where w (f) is the transfer function of the low pass filter. If we assume that the integrator (i.e., the low pass filter) has the same transfer function as an RC network, then

$$(16) \qquad |w(f)| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_L}\right)^2}} \qquad (\omega = 2\pi f)$$

where ω_L is the cut off angular frequency. The corresponding impulse response is

$$(17) \qquad G(\tau) = \int_{-\infty}^{\infty} |w(f)|^2 \, e^{j2\pi f\tau} \, df$$

$$= \int_{-\infty}^{\infty} \frac{1}{1 + \left(\frac{\omega}{\omega_L}\right)^2} \, e^{j2\pi f\tau} \, df \qquad (\omega = 2\pi f)$$

$$= \frac{\omega_L}{2} \, e^{-\omega_L|\tau|} \quad .$$

The output noise power is the mean square value of the ac terms of the low pass filter output voltages; in this case it can be derived from Eq. (15) with $\tau' = 0$, that is,

$$(18) \qquad R_2(0) = \int_{-\infty}^{\infty} \{ R_1(\tau) - R_1(\tau)\big]_{dc}\} \; \frac{\omega_L}{2} \; e^{-\omega_L |\tau|} \, d\tau \quad .$$

Substitution of Eq. (13) in Eq. (18) yields the following $R_2(0)$ after making some algebraic manipulations and neglecting the terms containing $2\omega_o$, which are filtered out by the pass characteristics of the network.

$$(19) \quad R_2(0) \cong \frac{\omega_L}{2} \Big[k^2 \psi_s^2 \Big\{ \frac{1}{2\omega_s + \omega_L} + \Big(e^{-2\omega_s \theta} \frac{1-e^{-\omega_L \theta}}{\omega_L} + \frac{e^{-(2\omega_s + \omega_L)\theta}}{2\omega_s + \omega_L} \Big) \cos 2\omega_o \theta \Big\}$$

$$+ \psi_s \psi_{N_2} \frac{1}{\omega_s + \omega_{N_2} + \omega_L} + k^2 \psi_s \psi_{N_1} \frac{1}{\omega_s + \omega_{N_1} + \omega_L}$$

$$+ \psi_{N_1} \psi_{N_2} \frac{1}{\omega_{N_1} + \omega_{N_2} + \omega_L}$$

$$+ \frac{\psi_{A_1}}{A_1^2} \Big\{ k^2 \psi_s^2 \Big(\frac{1}{2\omega_s + \omega_L + \omega_{A_1}} + \Big(e^{-2\omega_s \theta} \frac{1-e^{-(\omega_L + \omega_{A_1})\theta}}{\omega_L + \omega_{A_1}}$$

$$+ \frac{e^{-(2\omega_s + \omega_L + \omega_{A_1})\theta}}{2\omega_s + \omega_L + \omega_{A_1}} \Big) \cos 2\omega_o \theta \Big)$$

$$+ \psi_s \psi_{N_2} \frac{1}{\omega_s + \omega_{N_2} + \omega_L + \omega_{A_1}} + k^2 \psi_s \psi_{N_1} \frac{1}{\omega_s + \omega_{N_1} + \omega_L + \omega_{A_1}}$$

$$+ \psi_{N_1} \psi_{N_2} \frac{1}{\omega_{N_1} + \omega_{N_2} + \omega_L + \omega_{A_1}} \Big\}$$

$$+ \frac{\psi_{A_2}}{A_2^2} \Big\{ k^2 \psi_s^2 \Big(\frac{1}{2\omega_s + \omega_L + \omega_{A_2}} + \Big(e^{-2\omega_s \theta} \frac{1-e^{-(\omega_L + \omega_{A_2})\theta}}{\omega_L + \omega_{A_2}}$$

$$+ \frac{e^{-(2\omega_s + \omega_L + \omega_{A_2})\theta}}{2\omega_s + \omega_L + \omega_{A_2}} \Big) \cos 2\omega_o \theta$$

$$+ \psi_s \psi_{N_2} \frac{1}{\omega_s + \omega_{N_2} + \omega_L + \omega_{A_2}} + k^2 \psi_s \psi_{N_1} \frac{1}{\omega_s + \omega_{N_1} + \omega_L + \omega_{A_2}}$$

$$+ \psi_{N_1} \psi_{N_2} \frac{1}{\omega_{N_1} + \omega_{N_2} + \omega_L + \omega_{A_2}} \Big\} \Big]$$

In Eq. (19), higher order terms of the gain fluctuation ratio, such as ψ_{A_1}/A_1^2 times ψ_{A_2}/A_2^2, are neglected, since they are very small quantities. The bandwidth of the signal spectrum and noise spectrum are usually much larger than the low pass filter bandwidth, that is, $\omega_s >> \omega_L$, $\omega_{N_j} >> \omega_L$, and $\omega_o >> \omega_L$, $\omega_o >> \omega_{A_i}$, $\omega_o >> \omega_s$ and $\omega_o >> \omega_{N_j}$. Thus $R_2(0)$ may be expressed approximately as

$$(20) \quad R_2(0) \cong \frac{\omega_L}{2} \left[k^2 \psi_s^2 \left\{ \frac{1}{2\omega_s} + \left(e^{-2\omega_s\theta} \cdot \frac{1-e^{-2\omega_L\theta}}{\omega_L} + \frac{e^{-2\omega_s\theta}}{2\,\omega_s} \right) \cos 2\omega_o\theta \right\} + \psi_s \psi_{N_2} \frac{1}{\omega_s + \omega_{N_2}} + k^2 \psi_s \psi_{N_1} \cdot \frac{1}{\omega_s + \omega_{N_1}} + \psi_{N_1} \psi_{N_2} \cdot \frac{1}{\omega_{N_1} + \omega_{N_2}} \right] \left(1 + \frac{\psi_{A_1}}{A_1^2} + \frac{\psi_{A_2}}{A_2^2} \right) .$$

Since the noise spectrum and the signal spectrum may be assumed to have nearly equal bandwidths, then $\omega_s = \omega_{N_1} = \omega_{N_2} = \omega_i$. Thus we get

$$(21) \quad R_2(0) = \frac{1}{4} \frac{\omega_L}{\omega_i} \left[k^2 \psi_s^2 \left\{ 1 + \left(e^{-2\omega_i\theta} \cdot \frac{1-e^{-2\omega_L\theta}}{\omega_L} 2\omega_i + e^{-2\omega_i\theta} \right) \cos 2\omega_o\theta \right\} + \psi_s \psi_{N_2} + k^2 \psi_s \psi_{N_1} + \psi_{N_1} \psi_{N_2} \right] \left(1 + \frac{\psi_{A_1}}{A_1^2} + \frac{\psi_{A_2}}{A_2^2} \right) .$$

The output signal-to-noise ratio is given by

$$(22) \quad \frac{S}{N} = \frac{R_1(\tau)\big]_{dc}}{R_2(0)} = \frac{k^2 \psi_s^2 e^{-2\omega_i\theta} \cos^2 \omega_o\theta}{\frac{1}{4}\frac{\omega_L}{\omega_i}\left[k^2 \psi_s^2\left\{1+\left(e^{-2\omega_i\theta}\frac{2\omega_i}{\omega_L} \cdot (1-e^{-2\omega_L\theta})+e^{-2\omega_i\theta}\right)\cos 2\omega_o\theta\right\}+\psi_s\psi_{N_2} + k^2 \psi_s\psi_{N_1} + \psi_{N_1}\psi_{N_2}\right]\left(1 + \frac{\psi_{A_1}}{A_1^2} + \frac{\psi_{A_2}}{A_2^2}\right)} .$$

We may put θ equal to zero in Eq. (22) without losing any generality, since it is always possible to cancel out any phase difference by adjusting the phase of the receiver. Then Eq. (22) becomes, for $\theta = 0$ and $2\omega_i = \omega_I$(ω_I is the bandwidth of the pre-detector amplifier).

$$\text{(23)} \qquad \frac{S}{N} = \frac{2\,k^2\,\psi_s^{\,2}}{(2k^2\,\psi_s^{\,2} + \psi_s\psi_{N_2} + k^2\,\psi_s\psi_{N_1} + \psi_{N_1}\psi_{N_2})(1 + \frac{\psi_{A_1}}{A_1^{\,2}} + \frac{\psi_{A_2}}{A_2^{\,2}})}\,.$$

This S/N ratio result can be simplified further if one knows the approximate magnitudes of the signal and the noise.

2.1. The Effect of Input Signal Strength

a. Small input signal case

If ψ_N^2 is assumed much greater than $\psi_s^{\,2}$, we get

$$\text{(24)} \qquad \frac{S}{N} = \frac{2\,k^2\,\psi_s^{\,2}\,\omega_I}{\psi_{N_1}\,\psi_{N_2}\,\omega_L} \cdot \frac{1}{(1 + \frac{\psi_{A_1}}{A_1^{\,2}} + \frac{\psi_{A_2}}{A_2^{\,2}})}$$

where ψ_s, ψ_N and ψ_A correspond to the mean square value of the voltage functions, σ_s^2, σ_N^2 and $\sigma_A^{\,2}$, respectively, that is, they are proportional to the power in the signal, noise and mean square value of the amplitude fluctuations, respectively.

The minimum detectable signal temperature ΔT is usually defined as the ΔT that would produce enough signal power to give unity output signal to noise ratio.

Putting S/N equal to unity, and substituting σ^2 for ψ, one obtains

$$\text{(25)} \qquad \sigma_s^{\,4} = \sigma_{N_1}^{\,2}\,\sigma_{N_2}^{\,2} \cdot \frac{1}{2\,k^2} \cdot \frac{\omega_L}{\omega_I}\,(1 + \frac{\sigma_{A_1}^{\,2}}{A_1^{\,2}} + \frac{\sigma_{A_2}^{\,2}}{A_2^{\,2}})\,.$$

From the equation for Johnson noise power, we know

$$\text{(26)} \qquad \sigma_s^{\,2} = k \cdot \Delta T$$

(27) $$\sigma_{N_j}^{\ 2} = k\, T_o F_j \quad (j = 1, 2)$$

where k is the Boltzman constant, T_o is the input noise temperature of the receiver, and F_j is the operating noise figure of the amplifiers.

Therefore the minimum detectable temperature difference under the condition where there exist gain fluctuation in the receiver is:

(28) $$\Delta T = \frac{T_o \sqrt{F_1\ F_2}}{\sqrt{2}\ k} \sqrt{\frac{\omega_L}{\omega_I}} \sqrt{1 + \frac{\sigma_{A_1}^{\ 2}}{A_1^{\ 2}} + \frac{\sigma_{A_2}^{\ 2}}{A_2^{\ 2}}} \quad .$$

$$(\Delta T \ll T_e)$$

The fluctuation factor in Eq. (28) is approximately expressed by

(29) $$\Gamma = \frac{1}{2} \left(\frac{\sigma_{A_1}^{\ 2}}{A_1^{\ 2}} + \frac{\sigma_{A_2}^{\ 2}}{A_2^{\ 2}} \right) \quad .$$

When $F_1 = F_2 = F$, Eq. (28) could be combined with Eq. (29) to give

(30) $$\Delta T = \frac{T_o F}{\sqrt{2}\ k} \sqrt{\frac{\omega_L}{\omega_I}} \cdot (1 + \Gamma).$$

Usually the gain of both amplifiers would be the same. However, if one input signal is much smaller than the other one, the gain of one of the receivers may need to be increased to become larger than that of the other receiver.

Considering such a case, we put $A_2^{\ 2} = \alpha A_1^{\ 2}$, $\sigma_{A_2}^{\ 2} = \beta \sigma_{A_1}^{\ 2}$ and $\sigma_{A_1}^{\ 2}/A_1^{\ 2} = k_1$ and substitute these quantities into Eq. (29). Then

(31) $$\Gamma = \frac{1}{2} k_1 (1 + \frac{\beta}{\alpha}) $$
$$= k_1\ x$$

where

$$(32) \qquad x = \frac{1}{2}\left(1 + \frac{\beta}{\alpha}\right) .$$

The relation expressed by Eqs. (31) and (32) is plotted in Figs. 3 and 4. One could easily obtain the gain fluctuation factor for any combination of α and β from these figures. For example, if the gain of one receiver is 6 db larger than that of the other receiver, say $\alpha = 4$, and the fluctuation power of one receiver is five times larger than that of the other receiver, say $\beta = 5$ we get $x = 1.125$, or $T = 1.125(\%)$ when the fluctuation rate of one receiver is 1%, say $k_1 = 1(\%)$.

From Fig. 4, we could determine directly the maximum β that could be allowed for a particular set of values of gain fluctuation factor, x, and the gain ratio of the receivers, α.

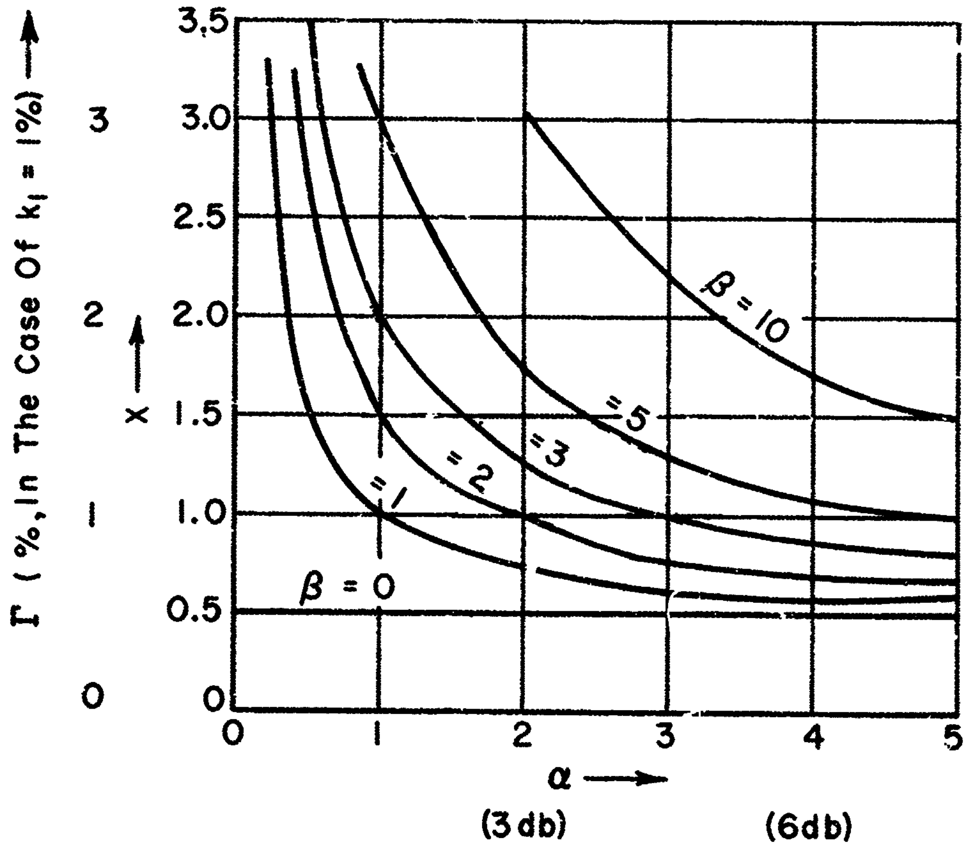

Fig. 3. The relation between gain ratio in each receiver and fluctuation factor — $x = 1/2\ (1 + \beta/\alpha)$ (fluctuation factor) $\Gamma = x \cdot k_1$.

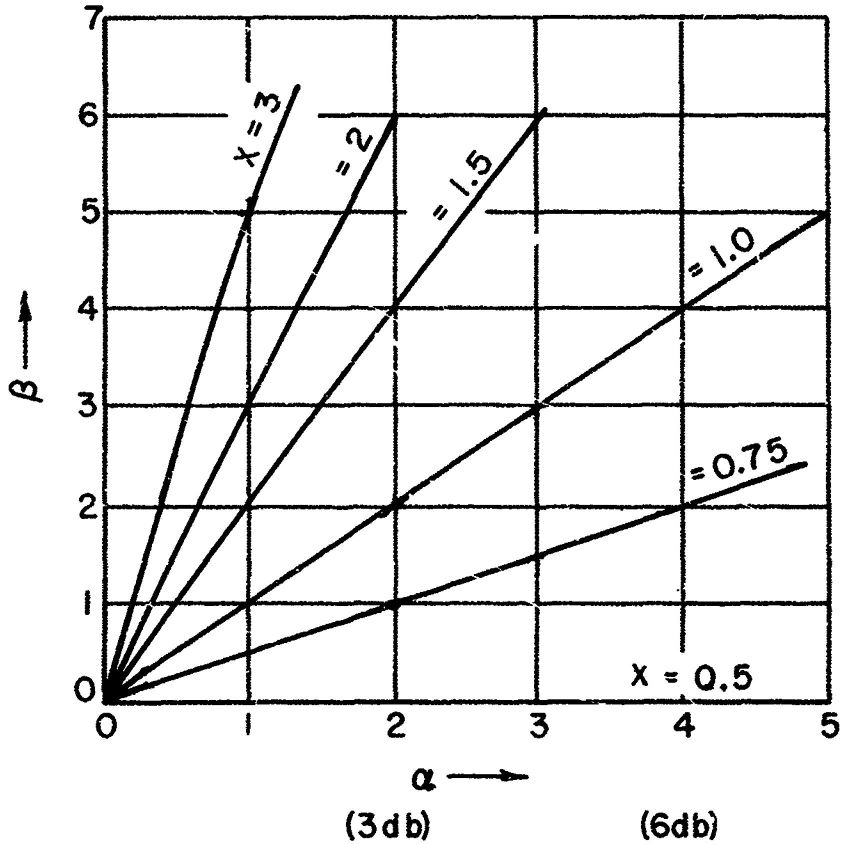

Fig. 4. The relation between gain ratio and fluctuation ratio of each receiver.

b. Different signal input case

In this case we assume that there exists no gain fluctuation in the system; i.e., $\sigma_{A_1}{}^2 = \sigma_{A_2}{}^2 = 0$.

From Eq. (23), we obtain

$$\frac{S}{N} = \frac{2}{2 + \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_1 R_2}} \qquad \frac{\omega_I}{\omega_L} \tag{33}$$

where

$$R_1 = \frac{\psi_s}{\psi_{N_1}} \text{ and } R_2 = \frac{k^2 \psi_s}{\psi_{N_2}} \tag{34}$$

If the one input signal strength is much larger than the other and if we assume that $\psi_{N_1} = \psi_{N_2}$ we obtain the following relations;

(1) when $R_1 \gg 1 \gg R_2$

(35) $$\frac{S}{N} = 2R_2 \cdot \frac{\omega_I}{\omega_L}$$

(2) when $R_1 \gg R_2 \gg 1$

(36) $$\frac{S}{N} = \frac{\omega_I}{\omega_L} \quad .$$

This result means that in the case of large input signal-to-noise ratio the output signal-to-noise ratio is only determined by the ratio of the amplifier bandwidth to low pass filter bandwidth, similar to the case of detection of a continuous modulated signal.

c. Equal input signal strength case

In this case $k = 1$; i.e., $R_1 = R_2 = R$ (when $\psi_{N_1} = \psi_{N_2}$), thus

(37a) $$\frac{S}{N} = \frac{2}{2 + \frac{2}{R} + \frac{1}{R^2}} \cdot \frac{\omega_I}{\omega_L}$$

(37b) $$= \frac{2R^2}{2R^2 + 2R + 1} \cdot \frac{\omega_I}{\omega_L} \quad .$$

When the input signal-to-noise ratio is very large

(38) $$\frac{S}{N} = \frac{\omega_I}{\omega_L} \cdots (R \gg 1) \quad .$$

When the input signal-to-noise ratio is very small

(39) $$\frac{S}{N} = 2R^2 \frac{\omega_I}{\omega_L} \cdots (R \ll 1) \quad .$$

Equation (38) shows the same result as Eq. (36) with unity k and no fluctuation. Equation (39) shows the same result as Eq. (24), (putting $\psi_{N_1} = \psi_{N_2} = \psi_N$ and $\psi_s/\psi_N = R$ in Eq. (24)).

Comparison of these results with Goldstein's results[3] shows a slight difference in the factor $\sqrt{2}$ of the coefficient of minimum detectable ΔT. The reason for the difference is that Goldstein assumed a different band pass characteristic for the receivers and filters.

This difference of $\sqrt{2}$ is just the ratio of the rectangular characteristics (assumed by Goldstein for both the receiver and low pass filter) to the band pass and RC characteristics (assumed by us in this report for the receiver and the low pass filter, respectively). With this difference in mind, our results should be considered to be equivalent to Goldstein's results.

2.2. The Effect of Gain Fluctuation to Noise Figure

The noise figure of the system may be described as the root mean product of the noise figure of each receiver, neglecting the noise in the multiplier and low frequency amplifiers, as shown in Eq. (28).

In the preceding analysis the noise figure was assumed to be constant. However, the noise figure would be varied by the gain fluctuation in the system, depending upon the type of receivers used for the radiometer. For example, if one used a conventional superheterodyne receiver, where the noise figure of the receiver system is predominantly the noise figure of the mixer, then the effect of gain fluctuations on the total noise figure would be small. On the other hand if some low noise mixer is used such that the noise of the receiver is predominantly due to the noise of the post mixer circuit, the noise figure would be affected significantly by the amplifier gain. To estimate the seriousness of gain fluctuation upon the noise figure of the amplifier we shall consider a special case where N_i is considered to be independent of the gain. The noise figure F is expressed by

$$(40) \qquad F = 1 + \frac{N_i}{A\,N_1} \quad .$$

Thus if the gain, A, is changed, F should be varied by the influence of the gain variation. That is

(41) $$\frac{\partial F}{\partial A} = -\frac{1}{A^2} \cdot \frac{N_i}{N_1} .$$

From Eqs. (40) and (41), we get the relation

(42) $$\frac{\partial F}{F - 1} = -\frac{\partial A}{A} .$$

If we could assume that ∂F equals $(F(t) - F)$ and ∂A to $(A(t)-A)$ approximately, the following relation would be obtained,

(43) $$F(t) = (1 + \Gamma)(F - 1) + 1$$

where $$\Gamma = \frac{|A(t) - A|}{A} .$$

The noise figure for effective temperature is expressed by

(44) $$F_e = F_o + \frac{T_a}{T_o} - 1$$

$$= (1 + \Gamma)(F_o - 1) + \frac{T_a}{T_o}$$

where T_a is the effective input temperature of the receiver. F_o is the standard noise figure and T_o is the standard temperature, say 290°K. Thus the variation of the noise figure due to gain variation is given by

(45) $$\sqrt{F_1 F_2} = \left[\left\{(1 + \Gamma_1)(F_{o1} - 1) + \frac{T_{a1}}{T_o}\right\}\left\{(1 + \Gamma_2)(F_{o2} - 1) + \frac{T_{a2}}{T_o}\right\}\right]^{\frac{1}{2}} .$$

2.3. The Effect of Unequal Bandwidth

The receivers in the correlation radiometer may have different bandwidths. In order to calculate this effect on the minimum detectable temperature, let us assume that the bandwidths (i.e., one half of total width) of the receivers are $\omega_{s_1} = \omega_{n_1} = \omega_{I_1}$ and $\omega_{s_2} = \omega_{n_2} = \omega_{I_2}$. From Eq. (19), we obtain the following relation, for $\theta = 0$,

$$(46) \quad R_2(0) = \frac{\omega_L}{2} \left[k^2 \psi_s^2 \cdot \frac{2}{\omega_{I_1} + \omega_{I_2}} + \psi_s \psi_{N_2} \cdot \frac{1}{\omega_{I_1} + \omega_{I_2}} \right.$$

$$\left. + k^2 \psi_s \psi_{N_1} \cdot \frac{1}{\omega_{I_2} + \omega_{I_1}} + \frac{\psi_{N_1} \psi_{N_2}}{\omega_{I_1} + \omega_{I_2}} \right]$$

$$= \frac{1}{2} \cdot \frac{\omega_L}{\omega_{I_1} + \omega_{I_2}} \left[2k^2 \psi_s^2 + \psi_s \psi_{N_2} + k^2 \psi_s \psi_{N_1} + \psi_{N_1} \psi_{N_2} \right]$$

where the gain fluctuation terms are neglected. Assuming $\omega_{I_2} = \alpha \omega_{I_1}$, then

$$(47) \quad R_2(0) = \frac{1}{2} \frac{\omega_L}{\omega_{I_1}} \cdot \frac{1}{1+\alpha} \left[2k^2 \psi_s^2 + \psi_s \psi_{N_2} + k^2 \psi_s \psi_{N_1} + \psi_{N_1} \psi_{N_2} \right].$$

From this result, we conclude that the effect of the different bandwidths of the receivers would increase the minimum detectable temperature by a factor of $(1 + \alpha)$. For carefully constructed receiver, α should be small. Thus it would not affect the minimum detectable temperature significantly, even if α varies with time, as we sometimes see in practice.

2.4. The Effect of Receiver Phase Errors in the Dicke and the Correlation Radiometers

In the Dicke system, phase errors between the chopper and the coherent detector in the receiver may be observed at the receiver output as a correlation error, and the signal-to-noise ratio would be changed to some extent.

For convenience's sake let us consider Goldstein's treatment,[3] in which the multiplier output $\omega(t)$ is expressed by $\omega(t) = v(t) \sin 2\pi qt$, where $v(t)$ is the output of the band-pass filter (Fig. 5) and q is the

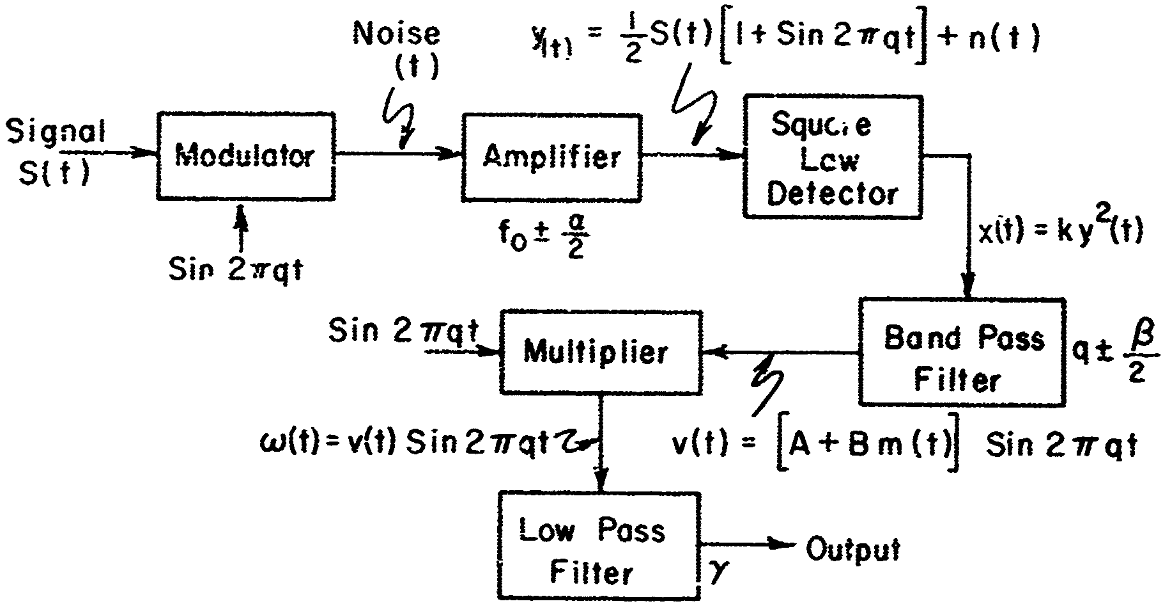

Fig. 5. Model of the Dicke-radiometer.

frequency of the multiplier input, which is also equal to the modulation frequency. v(t) can be written, as Goldstein showed,

(48) $$v(t) = [A + B\, m(t)] \sin 2\pi\, qt$$

as the output of the band pass filter, where A and B are constants and m(t) is a random voltage, which represents noise components. If there would be some phase error of amount ϕ, $\omega(t)$ can be written

(49) $$\omega(t) = v(t) \sin (2\pi\, qt + \phi) ,$$

By substituting Eq. (48) into Eq. (49), $\omega(t)$ is given by

$$\omega(t) = [A + B\, m(t)] \sin 2\pi\, qt \sin (2\pi\, qt + \phi) \tag{50}$$

while $\omega(t) = [A + B\, m(t)]\sin^2 2\pi\, qt$ in the case for no phase error. The autocorrelation function of $\omega(t)$, neglecting terms that are removed by the low pass filter, is then

$$\overline{\omega(t)\, \omega(t + \tau)} = R_3(\tau) \cdot \cos^2 \phi \tag{51}$$

where $R_3(\tau)$ is the autocorrelation function of $\omega(t)$ in the case of no phase variation. Thus we obtain

$$\frac{S}{N} = \frac{S_o}{N_o} \cdot \frac{1}{\cos^2 \phi} \tag{52}$$

where S_o/N_o is the signal-to-noise ratio in the case of no phase variation.

In practice, ϕ may possibly be changing with time, and should not be considered a constant. However, if the amount of phase variation in the receiver were very small, $\cos^2 \phi$ would be near unity, so the signal-to-noise ratio would be practically unchanged.

For a correlation radiometer, phase errors would cause much smaller effects at the output, because the multiplier output of the two signals in the correlation radiometer is determined by the autocorrelation function of these two variables, and the autocorrelation function is independent of the phase factor, θ shown in Eq. (17).

2.5. The Effects of Correlator Characteristics

In practice, one would expect that the correlator (i.e., the multiplier) may have certain characteristics such as nonlinear characteristics, limitation in bandwidth, and correlation error due to finite integration time, that would tend to increase the minimum detectable temperature of the correlation radiometer.

(a) Nonlinearity

Let us consider first the nonlinearity of the multiplier. Since the multiplier characteristic is not perfectly linear, an error due to nonlinearit of the multiplier would be anticipated. Assuming that the correlator has a nonlinearity factor δ, the output of the multiplier can be expressed by the following formula:

$$\omega(t) = \{u(t) \cdot v(t)\}^{1+\delta} \tag{53}$$

where all the notations are the same as the ones used in Fig. 1.

Hence

$$\begin{aligned} \omega(t) &= \{(S_1 + N_1)(S_2 + N_2)\}^{1+\delta} \\ &= (N_1 N_2)^{1+\delta} \{(1 + \frac{S_1}{N_1})(1 + \frac{S_2}{N_2})\}^{1+\delta} \\ &\cong N_1 N_2 \{(1 + \frac{S_1}{N_1})(1 + \frac{S_2}{N_2})\}^{1+\delta} \quad . \end{aligned} \tag{54}$$

In the case of small input signal-to-noise ratio, $\omega(t)$ is expressed by

$$\begin{aligned} \omega(t) &= N_1 N_2 \{1 + (1+\delta)\frac{S_1}{N_1}\}\{1 + (1+\delta)\frac{S_2}{N_2}\} \\ &= N_1 N_2 \{1 + (1+\delta)(\frac{S_1}{N_1} + \frac{S_2}{N_2}) + (1+\delta)^2 \frac{S_1 S_2}{N_1 N_2}\} \\ &= N_1 N_2 + (1+\delta)(S_1 N_2 + S_2 N_1) + (1+\delta)^2 S_1 S_2 \, . \end{aligned} \tag{55}$$

$(1 + x)^{1+\delta}$ has been approximated by $1 + (1+\delta)x$ for $x \ll 1$.
The autocorrelation function of $\omega(t)$ is given by

$$(56) \qquad R_1(\tau) = \overline{N_1 N_2} + (1+\delta)^2 (\overline{S_1 N_2} + \overline{S_2 N_1}) + (1+\delta)^4 \overline{S_1 S_2} \; .$$

Here, as in Section 2.1 we assumed that

$$S_1 = S(\tau)$$

$$S_2 = k\, S(\tau + \theta)$$

and the autocorrelation function of N and S are taken to be ϕ_n and ϕ_s, respectively. Thus Eq. (56) can be written as

$$(57) \qquad R_1(\tau) = \phi_{n_1}(\tau)\, \phi_{n_2}(\tau) + (1+\delta)^2 (\phi_s(\tau)\, \phi_{n_2}(\tau) + k^2 \phi_s(\tau)\, \phi_{n_1}(\tau))$$

$$+ k^2 (1+\delta)^4 \{\phi_s^2(\tau) + \phi_s^2(\theta) + \phi_s(\tau+\theta)\, \phi_s(\tau-\theta)\} \; .$$

Again signal and noise passed through a band limited circuit are assumed to have the following properties:

$$(58) \qquad \phi_s(\tau) = \psi_s\, e^{-\omega_s |\tau|} \cos \omega_o \tau$$

$$\phi_n(\tau) = \psi_n\, e^{-\omega_n |\tau|} \cos \omega_o \tau$$

Hence,

$$(59) \qquad R_1(\tau) = \psi_{n_1} \psi_{n_2}\, e^{-(\omega_{n_1} + \omega_{n_2}) |\tau|} \cos^2 \omega_o \tau$$

$$+ (1+\delta)^2 (\psi_s\, \psi_{n_2}\, e^{-(\omega_s + \omega_{n_2})|\tau|} + k^2\, \psi_s \psi_{n_1}\, e^{-(\omega_s + \omega_{n_1})|\tau|}) \cos^2 \omega_o \tau$$

$$+ k^2 (1+\delta)^4 \psi_s^2 \{ e^{-2\omega_s |\tau|} \cdot \cos^2 \omega_o \tau$$

$$+ e^{-\omega_s (|\tau+\theta| + |\tau-\theta|)} \cdot \cos \omega_o (\tau+\theta) \cos \omega_o (\tau-\theta)$$

$$+ e^{-2\omega_s \theta} \cdot \cos^2 \omega_o \theta \} \; .$$

The signal component is the dc component of $R_1(\tau)$, that is

$$R_1(\tau)\,\Big]_{dc} = \psi_s^2\, k^2\, (1+\delta)^4\, e^{-2\omega_s\theta} \cdot \cos^2 \omega_o \theta \quad .$$

The noise output power of the low pass filter is given by Eq. (59). Therefore we obtain the following equation:

$$(60)\qquad R_2(0) = \frac{\omega_L}{2}\Big[\psi_{n_1}\psi_{n_2} \cdot \frac{1}{\omega_{n_1}+\omega_{n_2}} + (1+\delta)^2 \Big\{\psi_s\psi_{n_2} \cdot \frac{1}{\omega_s+\omega_{n_2}} + k^2\psi_s\psi_{n_1} \cdot \frac{1}{\omega_s+\omega_{n_1}}\Big\} + k^2\psi_s^2 (1+\delta)^4 \Big\{\frac{1}{2\omega_s} + \frac{e^{-2\omega_s\theta}}{2\,\omega_s} + \frac{1-e^{-\omega_L\theta}}{\omega_L}\, e^{-2\omega_s\theta}\Big\}\Big] \quad .$$

For $\theta = 0$ and $\omega_s = \omega_{n_1} = \omega_{n_2} = \frac{\omega_I}{2}$,

$$(61)\qquad R_2(0) = \frac{1}{2}\,\frac{\omega_L}{\omega_I}\Big[\psi_{n_1}\psi_{n_2} + (1+\delta)^2 (\psi_s\psi_{n_2} + k^2\psi_s\psi_{n_1}) + 2k^2\psi_s^2 (1+\delta)^4\Big] \quad .$$

Hence the signal-to-noise ratio is given by

$$(62)\qquad \frac{S}{N} = \frac{2\,\psi_s^2 k^2 (1+\delta)^4}{\left[\psi_{n_1}\psi_{n_2} + (1+\delta)^2 (\psi_s\psi_{n_2} + k^2\psi_s\psi_{n_1}) + 2k^2\psi_s^2 (1+\delta)^4\right]} \cdot \frac{\omega_I}{\omega_L}$$

which can be rewritten as follows, similar to Eq. (33),

$$(63)\qquad \frac{S}{N} = \frac{2(1+\delta)^4}{2(1+\delta)^4 + (1+\delta)^2\left(\frac{1}{R_1}+\frac{1}{R_2}\right) + \frac{1}{R_1 R_2}} \cdot \frac{\omega_I}{\omega_L}$$

where we assumed $\psi_{N_1} = \psi_{N_2}$. Error due to nonlinearity may be defined by α, that is

(64) $$\alpha = \frac{\left|\frac{S}{N} - \frac{S_o}{N_o}\right|}{\frac{S_o}{N_o}}$$

where S_o/N_o is the signal-to-noise ratio without nonlinearity. From Eqs. (33), (63) and (64), α is given approximately by

(65) $$\alpha = \frac{2\delta(R_1 + R_2 + 2)}{2(1 + 4\delta) R_1 R_2 + (1 + 2\delta)(R_1 + R_2) + 1} .$$

$$(\delta << 1)$$

Let us consider some special cases.

(1) for the condition $R_1 << 1$ and $R_2 << 1$,

(66) $$\alpha = 4\delta$$

(2) for the condition $R_1 \gg 1 \gg R_2$,

(67) $$\alpha = 2\delta$$

(3) for the condition $R_1 \gg 1$ and $R_2 \gg 1$,

$$\alpha = 0 .$$

In all cases, as the nonlinearity factor δ is always much less than unity, we are able to say that small nonlinearity of the multiplier does not significantly affect the output signal-to-noise ratio.

One thing for which we must take care is the input signal level, which may be amplified up to or near to the saturation level at the input of the multiplier. This occurs when the input signal strength is large and the amplifier has sufficient gain to cause saturation of the signal level. At or near the saturation region, δ would no longer be considered much less than unity, and the error due to the nonlinearity factor would tend to increase as δ becomes large.

(b) Bandwidth limitation

The limitation of the multiplier bandwidth and its operating frequency is a serious problem in a correlation radiometer. Our derivation is valid when the multiplier is able to operate at the frequency of the preceding amplifier and has a bandwidth as wide as that of the amplifier. Consider the following example. Let the receivers be two superheterodyne receivers having two if amplifiers operating at 30 mc with 4 mc bandwidth. If the multiplier could operate only between 0 and 1 mc, then the output voltages of the if amplifiers will not be correlated by the multiplier at all. If a square law detector is inserted between the amplifier and the multiplier, then the correlator would correlate effectively the outputs of the detectors within its bandwidth. In this case the minimum detectable signal (M. D. S.) is proportional to the ratio of ω_L to ω_F, (c. f., Appendix I) where ω_L is the cut off frequency of the post multiplier integrator and ω_F the cut off frequency of the low pass filter following the detector.

The bandwidth of the multiplier ω_M should be the same or larger than that of the low pass filter. It follows that with the small bandwidth of the multiplier the M. D. S. would be increased by the factor of ω_F/ω_M. The same result would be obtained if the square law detector is replaced by an heterodyne receiver which transfers the band pass spectrum of the output of the if amplifier to a low pass spectrum. Usually ω_M is much smaller than ω_I, thus the signal-to-noise ratio is limited by ω_M/ω_L. For the case where ω_M is larger than ω_I the maximum signal-to-noise ratio would then be limited by the ratio of ω_I/ω_L.

If we could expect to utilize a correlator that has high operating frequency and wide bandwidth by means of, for example, electronic devices such as a heterodyne detector, the discussion made above might no longer be necessary.

(c) Error due to finite integration time

Another important problem in the correlator is the error due to the finite integration time. Output signal-to-noise ratio is affected by the integration time of the correlator. In our derivation we have assumed that the integration time is infinite, while in practice it should be finite, as the averaging over an infinite time interval would be experimentally impossible. When the integrating circuit has the

the form of RC low pass filter and a finite duration of observation interval T_o is used, the S/N measured is expressed by

(68) $$\frac{S}{N}\Big]_{meas.} = \frac{R_{12}^{2}(\tau_o)\,(1 - e^{-\omega_L T_o})^2}{\omega_L \int_0^{T_o} e^{-\omega_L \tau}\,[1 - e^{-2\omega_L (T_o - \tau)}]\,R_\xi(\tau)\,d\tau}$$

as shown by Davenport,[13] where $R_{12}(\tau_o)$ is the mean value of the product function and $R_\xi(\tau)$ is the autocorrelation function of the product function minus the mean value $R_{12}(\tau_o)$, that is to say,

(69) $$R_\xi(\tau) = R_x(\tau) - R_{12}(\tau_o)$$

where $R_x(\tau)$ is the autocorrelation function of the product function.

The specification of "small" T_o or "large" T_o corresponds to the requirement that T_o be small or large compared to the filter time constant $1/\omega_L$. Let us consider the $S/N]_{meas.}$ for these two limiting case for T_o. For very large values of T_o, the $S/N]_{meas.}$ becomes independent of the duration of the observation interval as follows:

(70) $$\frac{S}{N}\Big]_{meas.} = \frac{R_{12}^{2}(\tau_o)}{\omega_L \int_0^{\infty} e^{-\omega_L \tau}\,R_\xi(\tau)\,d\tau} \quad \text{as } T_o \to \infty \,.$$

For very small values of T_o,

(71) $$\frac{S}{N}\Big]_{meas.} = \frac{R_{12}^{2}(\tau_o)T_o}{2\int_0^{T_o} (1 - \frac{\tau}{T_o})\,R_\xi(\tau)\,d\tau} \quad \text{as } T_o \to 0 \,.$$

Equation (65) is derived for the case where the upper limit of integration is assumed to be infinite. From Eq. (63) signal-to-noise ratio can be maximized[5] for a given observation time with

(72) $$\omega_L = \frac{1.3}{T} \quad .$$

Substituting Eq. (72) into Eq. (24), we obtain

(73) $$\frac{S}{N} = 0.27\, \omega_I T \, \frac{k^2 \, \psi_s^{\,2}}{\psi_{N_1} \psi_{N_2}} \quad .$$

The minimum detectable sensitivity is given by

(74) $$\sigma_s^{\,2} = 0.7\, \sigma_n^{\,2} \; \frac{1}{k} \; (\omega_I T)^{-\frac{1}{2}} \quad .$$

The relation is plotted in Fig. 6 for the case of k = 1 and 4 mc bandwidth; say $f_I = 4$(mc) where

(75) $$f_I = \frac{\omega_I}{2\pi} \quad .$$

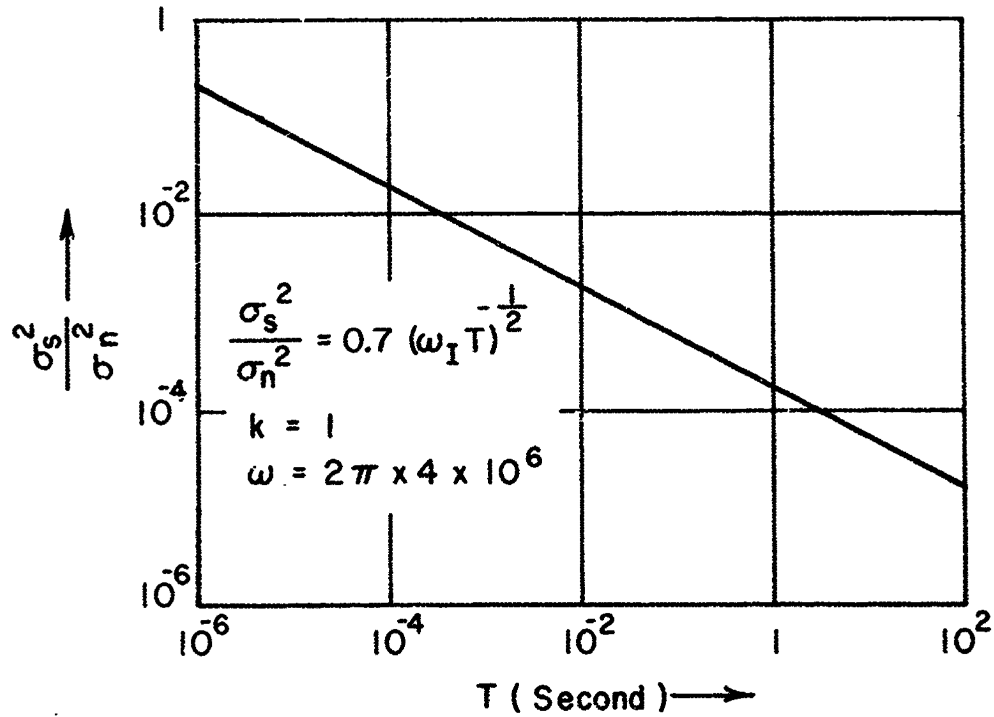

Fig. 6. The minimum detectable signal power and permissible output fluctuation rate versus observation time.

III. COMPARISON OF THE DICKE AND CORRELATION RADIOMETER

(a) General Conception

The Dicke type radiometer has been most commonly used at microwave frequencies because of its ability to overcome noise and gain fluctuation effects, to give low minimum detectable signals.

However, in the millimeter wave region, low noise electronic switching schemes are extremely difficult to employ. Although mechanical choppers could be used in front of the receiving horn, noise compensation would be difficult to achieve with mechanical choppers. Additional noise from the millimeter wave components plus the noise contributed by the choppers would tend to increase substantially the minimum detectable temperature, especially at the shorter m.m. wavelength.

On the other hand, the correlation radiometer eliminates the chopper system. We have shown that the effects of gain fluctuation on the minimum detectable sensitivity (M.D.S.) are nearly the same as in the Dicke system. Furthermore, there are many merits of the correlation radiometer for applications in space exploratory observations as discussed in the introduction. Therefore if an ideal correlation radiometer could be built, it would be superior to the Dicke system, especially for m.m. wave applications. In practice, the most serious drawback of the correlation radiometer is the limitation of the M.D.S. due to the bandwidth limitation of the multiplier. Because the multiplier bandwidth is usually much smaller than the amplifier bandwidth, the M.D.S. is limited by the ratio of the multiplier bandwidth to the integrator bandwidth rather than the ratio of the amplifier bandwidth to the integrator bandwidth. Therefore, the practical construction of the multiplier seems to be the most serious problem in the correlation radiometer.

(b) Sensitivity

To make clear the comparison of a correlation radiometer with a Dicke radiometer, let us consider the temperature sensitivity of the receiver as an example, taking some special cases.

The output signal-to-noise ratio is expressed by

$$(33) \qquad \frac{S}{N} = \frac{2}{2 + \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_1 R_2}} \quad \frac{\omega I}{\omega L}$$

as we have already shown in the preceding section.

For the case of small input signal strengths, with both signals and amplifiers identical,

$$(39) \qquad \frac{S}{N} = 2R^2 \cdot \frac{\omega_I}{\omega_L} \quad (R_1 = R_2 = R).$$

In this case the sensitivity is defined as the signal power to give unity output signal-to-noise ratio. Input signal-to-noise ratio R can be directly expressed in terms of temperature as

$$(76) \qquad R = \frac{\psi_S}{\psi_N} = \frac{\Delta T}{T_n}$$

where T_n is the internal noise temperature, $F_e T_o$ (Fig. 7). Thus by

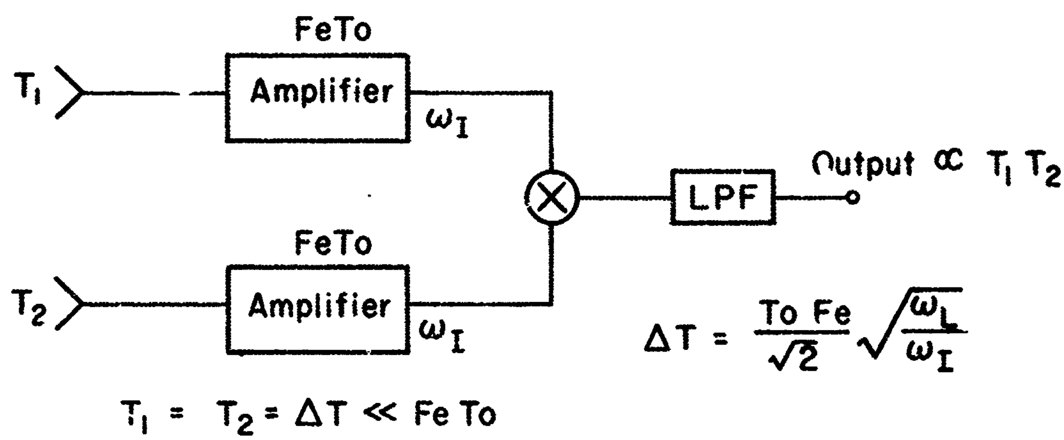

Fig. 7. Model diagram of the correlation radiometer in the case of small input signal strengths.

substituting Eq. (76) into Eq. (39), we obtain

$$\Delta T = \frac{F_e T_o}{\sqrt{2}} \sqrt{\frac{\omega L}{\omega_I}} \tag{77}$$

In the Dicke-type radiometer, (Fig. 8), we can expect the same sensitivity as the correlation radiometer shown in Eq. (78).

$$\Delta T = K\, F_e T_o \sqrt{\frac{\omega L}{\omega_I}} \tag{78}$$

where K is a constant.

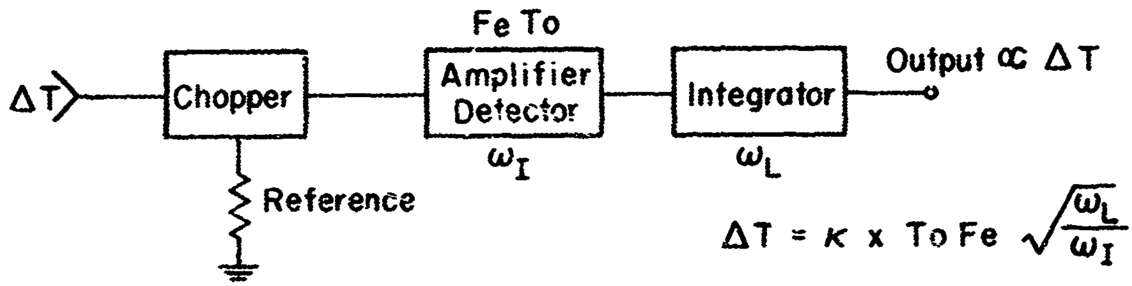

Fig. 8. Model diagram of the Dicke type radiometer.

We may say that in the simplest case, where both amplifiers and both signals are identical and the input signals are very small compared to the internal noise of the receiver, the correlation and Dicke-type radiometer are comparable, except for a factor of order unity from the coefficient K.

For the case of different input signal strengths, in which we are most interested because of the possibility of making reflection measurements, the output signal-to-noise ratio, for a special case, is expressed by

$$\frac{S}{N} = 2\, R_2 \,\frac{\omega L}{\omega_I} \qquad (R_1 >> 1 >> R_2) \tag{35}$$

where the assumption was made that $R_1 \gg 1 >> R_2$, and $\psi_{N_1} = \psi_{N_2}$, that is, one input signal strength is much larger than the internal noise of the receiver $(T_1 \gg F_e T_o)$, and the other input signal strength is much smaller than the internal noise $(T_2 << F_e T_o)$.

For this case, if the output signal power is much larger than the noise power, the concept of the minimum detectable sensitivity would no longer be adequate for characterizing the receiver (c.f., Appendix II). It would be better to define a differential temperature ΔT, which corresponds to an increment in the output signal power, which also has the meaning of an accuracy of measurement. It is expressed by

$$\frac{\Delta S}{S} = k \,\frac{\Delta T}{T}\ , \tag{79}$$

where T is the temperature corresponding to the output signal power, and k is a constant. Since the output signal power is proportional to the product of the two input signal temperatures (Fig. 9)

$$\begin{aligned} S &= C\, T_1 T_2 \\ &= C\, K^2\, T_1{}^2 \end{aligned} \tag{80}$$

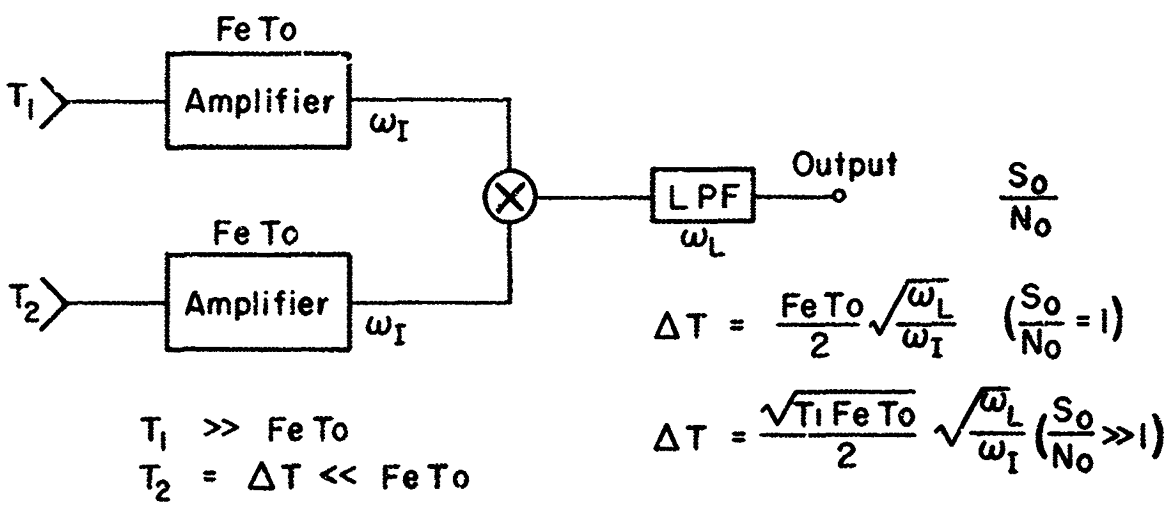

Fig. 9. Model diagram of the correlation radiometer in the case of different input signal strengths.

where C is a constant and $K^2 = T_2 / T_1$. Thus

(81) $$\frac{\Delta S}{S} = \frac{2\Delta T_1}{T_1} .$$

Since in this case ΔS corresponds to a fluctuation component, it corresponds to the noise term, so that

(82) $$\frac{S}{N} = \frac{S}{\Delta S}$$

and thus

(83) $$\frac{T_1}{2} \cdot \frac{1}{\Delta T_1} = 2R_2 \frac{\omega_I}{\omega_L} = 2 \frac{\Delta T_2}{T_n} \frac{\omega_I}{\omega_L} .$$

The fluctuation of the output is predominantly given by ΔT_1, since T_1 is assumed to be much longer than T_2. Thus we may set $\Delta T_2 = \Delta T_1 = \Delta T$ in Eq. (83).

Thus ΔT is expressed by

(84) $$\Delta T = \frac{\sqrt{T_1 F_e T_o}}{2} \sqrt{\frac{\omega_L}{\omega_I}} .$$

That means that the differential temperature depends on one input signal temperature T_1 and would be much larger than that of the Dicke-type radiometer, if we could define such an accuracy for the Dicke radiometer. Then we cannot say that the correlation radiometer would be more effective than Dicke radiometer for the case of high output signal-to-noise ratio. However for the Dicke radiometer, we cannot exactly introduce such an "accuracy" in the case of receiving two signals, because by means of Dicke radiometer, we cannot simultaneously receive two signals. Even if we could use two antennas for the Dicke system, we would not be able to obtain a better signal-to-noise ratio than in the correlation radiometer, as each channel acts as an independent Dicke-receiver.

If the output signal level is comparable to the noise level, we should take the signal power to give unity output signal-to-noise ratio as the M.D.S., that is

(85) $$2R_2 \cdot \frac{\omega_I}{\omega_L} = 1 \, .$$

Thus

(86) $$\Delta T = \frac{F_e T_o}{2} \cdot \frac{\omega_L}{\omega_I} \quad (R_1 \gg 1 \gg R_2)$$

for the case of $R_1 \gg 1 \gg R_2$. This result means that the sensitivity is greatly affected by the internal noise and is directly proportional to the noise temperature T_n and ω_L/ω_I. In this case ($R_1 \gg 1 \gg R_2$) the M.D.S. would seem to be better than in the Dicke radiometer by a factor of $\sqrt{\omega_L/\omega_I}$, except for factor of order unity.

We may conclude that for the case of a low output signal-to-noise ratio, the correlation radiometer would be superior because of the suppression of the background noise by correlation, even when signal-to-internal-noise ratio would be the same for the Dicke and correlation radiometers. However, we must be careful for the correlation radiometer that is receiving two signals, one of which has a much larger strength than the other, we cannot expect sufficient signal-to-noise ratio, unless the two signals could effectively be correlated with each other.

IV. CONCLUSIONS

Based upon the preceding discussions, we can conclude that at the present time the usefulness of correlation radiometer probably would be limited to some special applications only. For example, if one is interested in measuring the emission from a discrete emission line spectrum, then one would normally use a narrow bandwidth radiometer and the limitations on the multiplier would not be serious. Instead, one would gain from the flexibility of the correlation radiometer for such an application. On the other hand, if one is interested in measuring the reflected radiation of the sun from the moon, the correlation radiometer is also superior to the Dicke radiometer. Here, the correlation radiometer would correlate only the radiation of the sun while the Dicke radiometer would chop the background radiation of the moon as well as the radiation of the sun reflected from the moon. This may increase the sensitivity of measuring the reflected signal despite the loss of bandwidth due to the multiplier. Of course, with the advancement of the state of the art of electronic devices, multipliers with a large bandwidth and high operating frequency may be found in the future such that the limitation on bandwidths is no longer a problem. In that case, the flexibility of the correlation radiometer would make it a more attractive radiometer than the Dicke system.

ACKNOWLEDGEMENTS

The author wishes to express his thanks to Drs. W.S.C. Chang and W.H. Peake for their frequent guidance and many valuable suggestions.

REFERENCES

1. Dicke, R.H., "The Measurement of Thermal Radiation at Microwave Frequencies," Radiation Laboratory Report 787, August 22, 1945.

2. Selove, W., "A DC Comparison Radiometer," Rev. Sci. Instrument, v. 25, 1954, p. 120.

3. Goldstein, S.J., "A Comparison of Two Radiometer Circuits," Proceedings of the I.R.E., v. 43, 1955, p. 1663. This comparison was shown to be in error by D.G. Tucker and M.H.Graham in their correspondence in Proceedings of the I.R.E., v. 45, 1957, p. 365.

4. Strom, L.D., "The Theoretical Sensitivity of the Dicke Radiometer," Proceedings of the I.R.E., v. 45, 1957, p. 1291. Correction of this report can be seen in Proc. of the I.R.E., v. 45, 1957, p. 1631.

5. Galejs, "Comparison of Subtraction-type and Multiple-type Radiometers," Proc. of the I.R.E., v. 45, 1957, p. 1420.

6. Porter,R.A., et. al., "A Survey of Microwave Radiometer with Terrain Mapping Applications," Tech. Rep. RAD-TR 9-6020, Res. and Advanced Dev. Div. AVCO Corp., 23 September 1960.

7. Graham, M., "Radiometer Circuit," Proc. of the I.R.E., v. 46, 1958, p, 1966.

8. Strum, P.D., "Consideration in High-Sensitivity Microwave Radiometry," Proc. of the I.R.E., vol. 46, 1958, p. 43.

9. Middleton, D., <u>An Introduction to Statistical Communication Theory</u>, McGraw Hill Book Co., 1960.

10. Long, M.W., "Submillimeter Wave Radiometry," Proc. of the I.R.E., v. 49, 1961, p. 1024.

11. Brown, H., "A New Type of Interferometer for Use in Radioastronomy," The Philosophical Mag., v. 45, 1954, p. 663.

12. Drake, F.D. and Ewen, H.I., "A Broad-Band Microwave Source Comparison Radiometer for Advanced Research in Radio Astronomy," Proc. of the I.R.E., v. 46, 1958, p. 53.

13. Davenport, W.B., "Correlator Errors due to Finite Observation Intervals," M.I.T. Tech. Report No. 191, March 8, 1951.

APPENDIX I: SIGNAL-TO-NOISE RATIO OF THE CORRELATION RADIOMETER WITH A SQUARE-LAW DETECTOR BETWEEN THE AMPLIFIER AND THE MULTIPLIER

Let us consider Fig. A-1 and Fig. A-2. The autocorrelation function of the output of the square-law detector in Fig. A-1 is expressed by

(A-1)
$$
\begin{aligned}
R_1(\tau) &= \overline{yy'} \\
&= \overline{(S_1+n_1)^2\,(S_1'+n_1')^2} \\
&= \overline{S_1^2}\cdot\overline{S_1'^2} + 2\overline{S_1S_1'}^2 + \overline{S_1^2}\cdot\overline{n_1'^2} + \overline{S_1'^2}\cdot\overline{n_1^2} \\
&\quad + 4\,\overline{S_1S_1'}\cdot\overline{n_1n_1'} + \overline{n_1^2}\cdot\overline{n_1'^2} + 2\,\overline{n_1n_1'}^2
\end{aligned}
$$

where the primed quantities are functions of $(t+\tau)$ and the unprimed quantities are functions of t; the detector's specific constant has been neglected; and the gain of the amplifier has been assumed to be unity.

If we assume that the amplifier has a rectangular band pass characteristic, the quantities in Eq. (A-1) are given by

(A-2)
$$
\begin{aligned}
\overline{S^2} &= \overline{S'^2} = \sigma_s^2 \\
\overline{n^2} &= \overline{n'^2} = \sigma_n^2 \\
\overline{SS'} &= \sigma_s^2 \cdot \frac{\sin \pi\alpha\tau}{\pi\alpha\tau}\,,\ \cos \omega_0\tau \\
\overline{nn'} &= \sigma_n^2 \cdot \frac{\sin \pi\alpha\tau}{\pi\alpha\tau}\,,\ \cos \omega_0\tau \quad .
\end{aligned}
$$

Then the spectral density of the output of the detector is obtained as follows,

(A-3)
$$
\begin{aligned}
G_1(f) &= 4\int_0^\infty R_1(\tau)\cos 2\pi f\tau\, df \\
&= (\sigma_{s_1}^2 + \sigma_{n_1}^2)^2 \left[\delta(f) + 2\cdot\frac{\alpha_1 - f}{\alpha_1^2}\right] \qquad (0 < f < \alpha)
\end{aligned}
$$

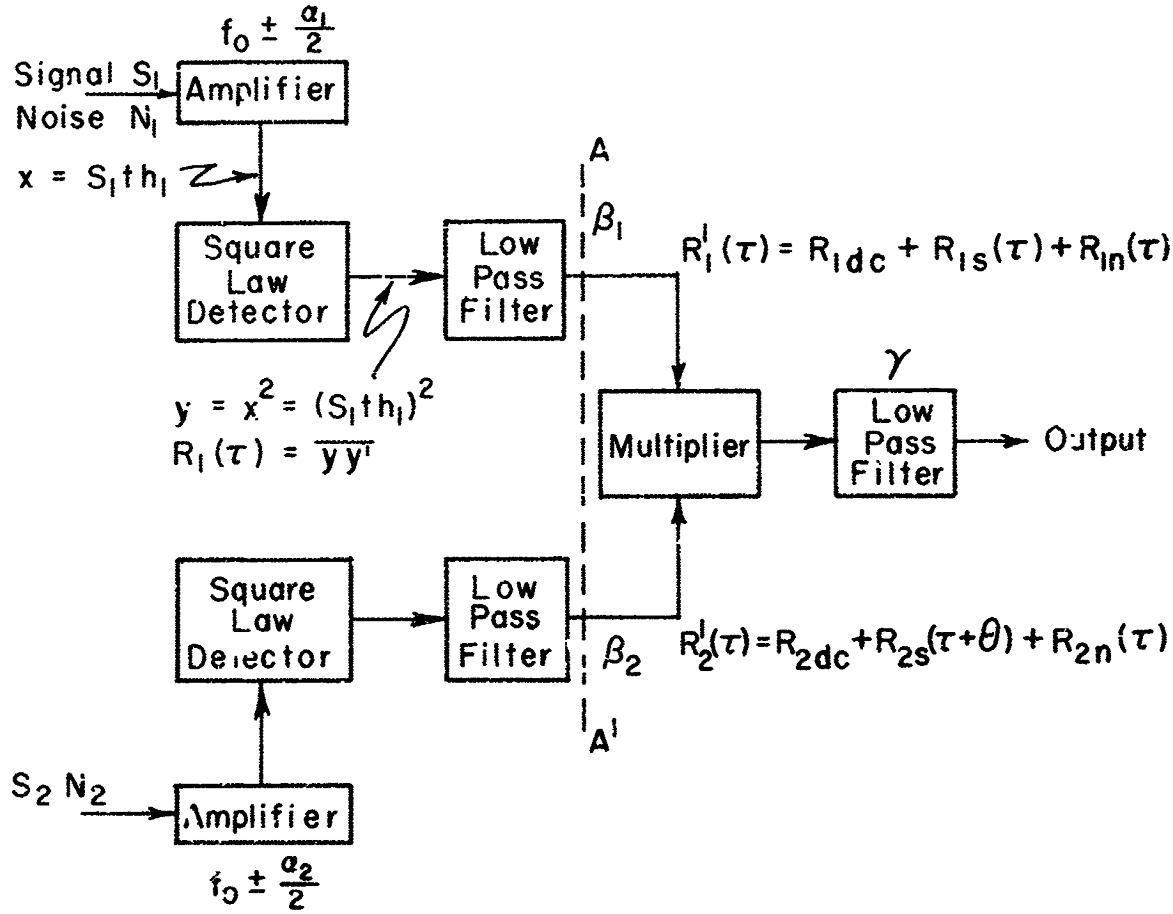

Fig. A-1. Model diagram of the correlation radiometer with square-law-detector between amplifier and multiplier.

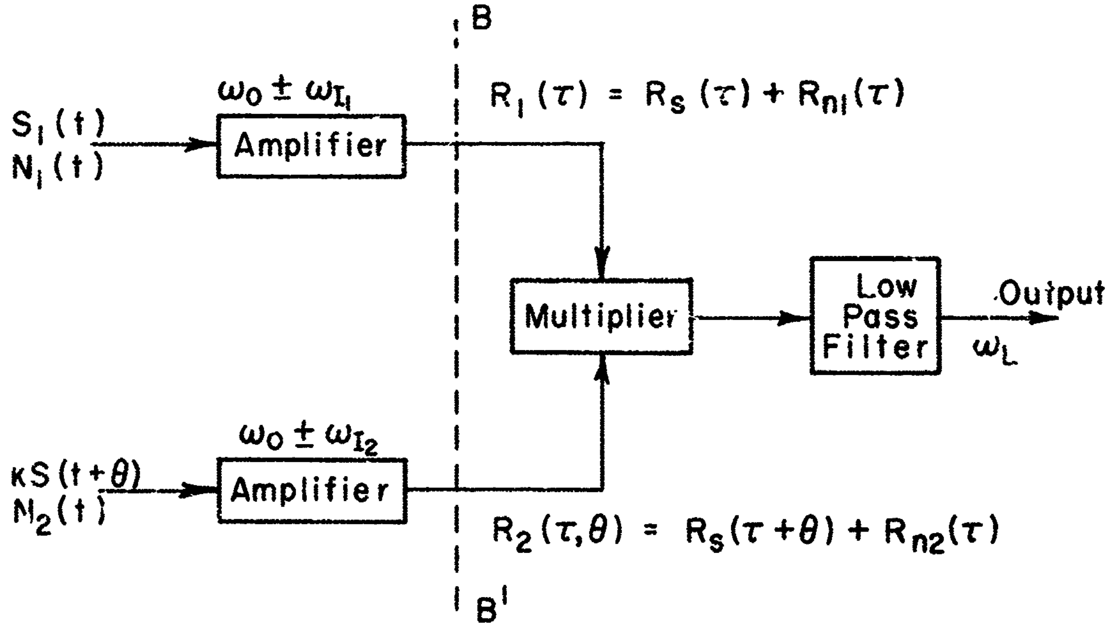

Fig. A-2. Model diagram of the correlation radiometer.

where $\delta(f)$ is a delta function. The first term of Eq. (A-3) shows the dc component of the output and the second term shows the ac part of the output. The output power of the low pass filter, which has a rectangular pass band from 0 to β, is expressed by

$$P_1 = \int_0^{\beta} G_1(f)\,df \tag{A-4}$$

$$= P_1 dc + P_1 ac$$

where

$$P_1 dc = (\sigma_{S_1}^2 + \sigma_{n_1}^2)^2 \tag{A-5}$$

$$P_1 ac = 2(\sigma_{S_1}^2 + \sigma_{n_1}^2)^2 \frac{\beta_1}{\alpha_1} \quad (\alpha_1 >> \beta_1) \ . \tag{A-6}$$

Thus the autocorrelation function of the input voltage to the multiplier is given by

$$R_1(\tau) = (\sigma_{S_1}^2 + \sigma_{n_1}^2)^2 + 2(\sigma_{S_1}^2 + \sigma_{n_1}^2)^2 \frac{\beta_1}{\alpha_1} \phi_1(\tau) \tag{A-7}$$

where $\phi_1(\tau)$ is the Fourier transform of the spectral density of the low pass filter output.

Let

$$R_1'(\tau) = R_{dc_1} + R_{acS_1} + R_{acn_1} \tag{A-8}$$

where

$$R_{dc_1} = (\sigma_{S_1}^2 + \sigma_{n_1}^2)^2 \tag{A-9}$$

$$R_{ac\,S_1} = 2\sigma_{S_1}^4 \frac{\beta_1}{\alpha_1} \phi_1(\tau)$$

$$R_{acn_1} = 2\sigma_{n_1}^2 (2\sigma_{S_1}^2 + \sigma_{n_1}^2) \frac{\beta_1}{\alpha_1} \phi_1(\tau)$$

In the equation (A-9) R_{dc_1} is the d-c term of the autocorrelation function of both the signal and the noise components; R_{acS_1} is the a-c term of the autocorrelation function of the signal; and R_{acn_1} is that of the noise. By a similar manipulation we can obtain $R_2'(\tau, \tau+\theta)$, where θ is the time delay, with which S_2 should be correlated to S_1, that is

$$\text{(A-10)} \qquad R_2'(\tau, \tau+\theta) = R_{dc_2} + R_{ac\,S_2}(\tau+\theta) + R_{ac\,n_2}(\tau) \quad .$$

The output of the multiplier is obtained by means of the correlation between those two components. In practice we can balance the components R_{dc_1} and R_{dc_2} out before the input stage of the multiplier, so it is sufficient for us to consider only the ac-terms in Eq. (A-8) and (A-10). They are expressed by

$$\text{(A-11)} \qquad R_{ac\,S_1} \propto \frac{\beta_1}{\alpha_1}\,\phi_1(\tau)$$

$$R_{ac\,n_1} \propto \frac{\beta_1}{\alpha_1}\,\phi_1(\tau)$$

$$R_{ac\,S_2} \propto \frac{\beta_2}{\alpha_2}\,\phi_1(\tau+\theta)$$

$$R_{ac\,n_2} \propto \frac{\beta_2}{\alpha_2}\,\phi_2(\tau) \quad .$$

In order to obtain the signal-to-noise ratio of the output, let us consider that the all components in the right side of the dotted line AA$'$ in Fig. A-1 correspond to all the components in the right side of the dotted line BB$'$ in Fig. A-2.

The autocorrelation functions of the signal and the noise components, $R_{ac}\,S(\tau)$ and $R_{ac}\,n(\tau)$ in Fig. A-1 then correspond to those of the signal and noise components $R_S(\tau)$ and $R_n(\tau)$ in Fig. A-2. These relations are

$$R_{acS_1} = R_S(\tau) = \psi_S\, x(\tau)$$

(A-12)
$$R_{acn_1} = R_{n_1}(\tau) = \psi_{n_1}\, y_1(\tau)$$
$$R_{acS_2} = R_S(\tau+\theta) = \psi_S\, x(\tau+\theta)$$
$$R_{acn_2} = R_{n_2}(\tau) = \psi_{n_2}\, y_2(\tau)$$

where $x(\tau)$ and $y(\tau)$ are the Fourier transforms of the spectral densities of the amplifier output, related to the signal and noise, respectively, and ψ_S and ψ_n are their constant terms as used in the preceding sections.

From Eqs. (A-9) and (A-12), we obtain the following relations by equating the constant terms of (A-11) to those of (A-12), and $\phi_1(\tau)$ and $\phi_2(\tau)$ to $x(\tau)$ and $y(\tau)$;

$$\psi_{S_1} \propto \frac{\alpha_1}{\beta_1}$$

(A-13)
$$\psi_{n_1} \propto \frac{\alpha_1}{\beta_1}$$
$$\psi_{S_2} \propto \frac{\alpha_2}{\beta_2}$$
$$\psi_{n_2} \propto \frac{\alpha_2}{\beta_2} \quad .$$

Thus the output signal-to-noise ratio can be obtained by substituting the relations (A-13) into Eq. (24), that is,

(24)
$$\frac{S}{N} = \frac{k^2\, \psi_S^{\,2}}{\psi_{N_1}\psi_{N_2}} \cdot \frac{\omega_I}{\omega_L}$$

hence

(A-14)
$$\frac{S}{N} \propto k^2 \cdot \frac{\beta}{\gamma}$$

where we have assumed $\alpha = \alpha_1 = \alpha_2$, $\beta = \beta_1 = \beta_2$; and ω_L and ω_I in Fig. A-2 are replaced by γ and β in Fig. A-1. Equation (A-14) means that the signal-to-noise ratio is proportional to β/γ.

From this result we are able to show that the M.D.S. is proportional to the ratio of the bandwidth of the integrator to that of the preceding multiplier, that is to say, the ratio of ω_I to ω_F. (Notations, ω_I and ω_L, used in the preceding section then correspond to β and γ directly.)

APPENDIX II. SENSITIVITY CONCEPT

When the two signal strengths at the input terminals of the two receivers are very different, we must treat carefully the concept of the receiver sensitivity. For example, if the output signal level is much larger than the noise level, minimum detectable sensitivity would no longer be a suitable parameter to characterize the receiver. It would be better to define the differential temperature ΔT, which corresponds to the smallest detectable increment ΔS in the output signal power S. In other words ΔT signifies some accuracy of measurement ΔA. The relation between ΔS and ΔT is expressed by

$$\text{(AII-1)} \qquad \frac{\Delta S}{S} = k \frac{\Delta T}{T}$$

where T is the input signal temperature and k is a constant.

The output signal power is proportional to the product of the two input signal temperatures,

$$\text{(AII-2)} \qquad \begin{aligned} S &= C\, T_1 T_2 \\ &= C\, K^2\, T_1^2 \end{aligned}$$

where C is a constant and $K^2 = T_2 / T_1$. Thus

$$\text{(AII-3)} \qquad \frac{\Delta S}{S} = \frac{2\Delta T_1}{T_1} .$$

Since in this case ΔS represents a signal fluctuation, it corresponds to a noise component (for example N in Eq. (35)),

$$\text{(AII-4)} \qquad \frac{S}{N} = \frac{S}{\Delta S} .$$

Therefore

$$\text{(AII-5)} \qquad \frac{S}{N} = \frac{T_1}{2} \cdot \frac{1}{\Delta T_1} .$$

We can introduce the concept of accuracy ΔA from this result, by setting ΔT_1 in Eq. (AII-5) equal to ΔA, since T_1 is much larger than T_2 and ΔT_1 is dominant in the fluctuation term.

Concerning the input signal levels, there are four typical cases, according to the variation of the two input signal-to-noise ratio, say R_1 and R_2 in Eq. (35). These are:

(1) $R_1 \gg R_2 \gg 1$; both signals are much larger than the noise level and one signal is much larger than the other.

((1') $R_1 \approx R_2 \gg 1$; both signals are nearly the same and much larger than the noise level.)

(2) $R_1 \gg R_2 \approx 1$; one signal is much larger than the other which is nearly equal to the noise level.

((2') $R_1 \approx R_2 \approx 1$; signals and noise level are comparable.)

(3) $R_1 \gg 1 \gg R_2$; one signal is much larger than the noise level, while the other is much smaller.

((3') $R_1 \approx 1 \gg R_2$, one signal is nearly equal level to the noise level, but much larger than the other.)

(4) $1 \gg R_1 \gg R_2$, both signals are much smaller than the noise level and one signal is much larger than the other.

((4') $1 \gg R_1 \approx R_2$, both signals are nearly same level and much smaller than the noise level.) For all cases we assume that two receivers are identical.

Let us consider the signal-to-noise ratio first and the sensitivity next in each case. In the case of (2')

$$\frac{S}{N} = \frac{2}{2 + \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_1 R_2}} \cdot \alpha \qquad \text{(AII-6)}$$

where $\alpha = \omega_I/\omega_L$.

This is the most general expression for the signal-to-noise ratio and cannot be further simplified.

In the case of (1) (and (1'))

$$\text{(AII-7)} \qquad \frac{S}{N} \approx \alpha = \frac{\omega_I}{\omega_L} \quad \begin{pmatrix} R_1 \gg 1 \\ R_2 \gg 1 \end{pmatrix} \quad .$$

In this case the signal output level would greatly exceed the noise level; thus we must use the concept of accuracy here by means of Eq. (AII-5),

$$\text{(AII-8)} \qquad \frac{S}{N} = \frac{S}{\Delta S} = \frac{1}{2}\,\frac{T_1}{\Delta T_1} = \alpha$$

where

$$S = C\, T_1\, T_2$$

$$= C \cdot K^2\, T_1^2$$

$$\Delta S = C \cdot K^2 \cdot 2 T_1 \Delta T_1 \quad .$$

Then

$$\text{(AII-9)} \qquad \Delta T_1 = \frac{T_1}{2}\,\frac{1}{\alpha} \quad .$$

As the definition of accuracy, put $\Delta T_1 = \Delta A$

$$\text{(AII-10)} \qquad \Delta A_{(1)} = \frac{T_1}{2} \cdot \frac{\omega_L}{\omega_I} \quad .$$

In the case of (2)

$$\text{(AII-11)} \qquad \frac{S}{N} \approx \frac{2R_2}{2R_2 + 1} \cdot \alpha \quad \begin{pmatrix} R_1 \gg 1 \\ R_2 \approx 1 \end{pmatrix} \quad .$$

Since usually $\alpha \gg 1$, the signal-to-noise ratio in this case is much larger than unity, hence we must consider again the accuracy as follows; from the relation (AII-8) and (AII-11)

$$\text{(AII-12)} \qquad \frac{2R_2}{2R_2+1}\,\alpha = \frac{T_1}{2}\cdot\frac{1}{\Delta T_1}$$

putting $\Delta T_1 = \Delta T_2 = \Delta A$, and $R = \frac{\Psi_S}{\Psi_N} = \frac{T}{T_n} = \frac{T}{F_e T_o}$, where T_n is the receiver noise and T in R is substituted by ΔT or ΔA, thus

$$\text{(AII-13)} \qquad \frac{2\frac{\Delta A}{T_n}}{\frac{2\Delta A}{T_n}+1}\,\alpha = \frac{T_1}{2}\,\frac{1}{\Delta A} \quad .$$

Then

$$\text{(AII-14)} \qquad \Delta A = \frac{T_1 + \sqrt{T_1^2 + 8\,T_1\,T_n\alpha}}{4\alpha}$$

$$= \frac{T_1}{4\alpha}\left(1 + \sqrt{1 + 8\,\frac{T_n}{T_1}\,\alpha}\,\right)$$

where the minus sign is neglected. If $\frac{T_1}{T_n} \ll 8\alpha$

$$\text{(AII-15)} \qquad \Delta A_{(2)} = \frac{T_1}{4\alpha}\cdot\sqrt{\frac{T_n}{T_1}\,8\alpha}$$

$$= \frac{\sqrt{T_1 T_n}}{\sqrt{2\alpha}}$$

$$= \frac{\sqrt{T_1 F_e T_o}}{\sqrt{2}}\cdot\sqrt{\frac{\omega_L}{\omega_I}} \quad .$$

If $\frac{T_1}{T_n} \gg 8\alpha$

(AII-16) $$\Delta A'_{(2)} = \frac{T_1}{4\alpha} = \frac{T_1}{4} \cdot \frac{\omega_L}{\omega_I} .$$

In the case of (3),

(AII-17) $$\frac{S}{N} \approx 2R_2 \cdot \alpha \quad \begin{pmatrix} R_1 \gg 1 \\ R_2 \ll 1 \end{pmatrix}.$$

In this case there exist two possible conditions, one is $2R_2 \cdot \alpha \gg 1$ and another $2R_2 \cdot \alpha \lesssim 1$.

Under the condition $2R_2 \cdot \alpha \gg 1$,

(AII-18) $$2R_2 \cdot \alpha = \frac{T_1}{2} \cdot \frac{1}{\Delta T}$$

hence, putting $\Delta T_1 = \Delta T_2 = \Delta A_{(3)}$

(AII-19) $$\Delta A_{(3)} = \frac{\sqrt{T_1 F_e T_e}}{2} \sqrt{\frac{\omega_L}{\omega_I}} .$$

Under the condition $2R_2 \alpha \lesssim 1$,

(AII-20) $$2R_2 \alpha = 1$$

hence

(AII-21) $$\Delta T_{(3)} = \frac{1}{2} F_e T_e \cdot \frac{\omega_L}{\omega_I} .$$

In the case of (3'),

(AII-22) $$\frac{S}{N} \approx \frac{2R_1 R_2}{R_1 + 1} \cdot \alpha \begin{pmatrix} R_1 \approx 1 \\ R_2 \ll 1 \end{pmatrix} .$$

Under the condition $R_2 \cdot \alpha \lesssim 1$

(AII-23) $$\frac{2R_1 R_2}{R_1 + 1} \alpha = 1 .$$

Thus

(AII-24) $$\Delta T = \frac{T_n}{4\alpha} (1 + \sqrt{1 + 8K^2 \alpha}) .$$

If $8K^2 << \frac{1}{\alpha}$

(AII-25) $$\Delta T_{(3)} = \frac{T_n}{4\alpha} \cdot 2$$

$$= \frac{F_e T_o}{2} \cdot \frac{\omega_L}{\omega_I} .$$

When $8K^2 >> \frac{1}{\alpha}$

(AII-26) $$\Delta T'_{(3')} = \frac{K F_e T_o}{\sqrt{2}} \sqrt{\frac{\omega_L}{\omega_I}} .$$

Under the condition $R_2 \alpha >> 1$,

(AII-27) $$\frac{2R_1 R_2}{R_1 + 1} \cdot \alpha = \frac{T_1}{2} \cdot \frac{1}{\Delta T_1} .$$

By putting $\Delta T_1 = \Delta T_2 = \Delta A'_{(3')}$

(AII-28) $$\Delta A'_{(3')} = \frac{1}{2} \sqrt{T_1 + F_e T_o} \cdot \sqrt{\frac{\omega_L}{\omega_I}}$$

$$\approx \frac{1}{\sqrt{2}} \sqrt{F_e T_o} \sqrt{\frac{\omega_L}{\omega_I}} .$$

In the case of (4)

(AII-29) $$\frac{S}{N} \approx 2R_1 R_2 \cdot \alpha \begin{pmatrix} R_1 << 1 \\ R_2 << 1 \end{pmatrix} .$$

In this case the output signal level could not be much larger than the noise level, so we may put S/N equal to unity,

(AII-30) $$2R_1 R_2 \cdot \alpha = 1$$

and it seems to be sufficient to treat only the case $R_1 = R_2$, i.e., the case of (4'). Thus

$$2R^2 \alpha = 2 \frac{\Delta T^2}{T_n^2} \alpha = 1,$$

therefore

(AII-31) $$\Delta T_{(4)} = \frac{F_e T_o}{\sqrt{2}} \sqrt{\frac{\omega_L}{\omega_I}} .$$