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AN APPROXIMATE METHOD FOR THE COMPUTATION OF SCATTERING BY CONDUCTING CYLINDERS WITH ARBITRARY CROSS-SECTION

by H. Y. Yee

Prepared under Grant No. NsG-381 *by*
UNIVERSITY OF ALABAMA
Huntsville, Ala.
for

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SUMMARY:

The scattering of electromagnetic waves by conducting cylinders with arbitrary cross sections are computed by an approximated method called the point-matching method. The theory is confirmed by low frequency scattering and numerical examples. However, this method is not applicable to the scattering by an infinite strip. The low frequency scattering shows also that for a known scattered field the cross-section of the scatterer can be found.

1. INTRODUCTION:

Scattering of a plane wave by conducting cylinders has been studied extensively. The method of separation is applicable to only a few simple geometries, namely, the circular cylinder, the elliptical cylinder, and the wedge. For cases where the method of separation fails, approximate methods must be used. Variational methods, low-frequency approximations, high-frequency approximations, and the approximate solution of an integral equation are applied successfully to many cases.[1-6] In this paper a new approximate method* (point-matching method) is introduced for computation of scattering by conducting cylinders with arbitrary cross-sections. The calculations necessary for this method are very simple, particularly when a digital computer is available.

In the following considerations, the point-matching method is applied to the oblique incidence for both parallel and perpendicular polarizations. Scattering by a circular cylinder and by a square cylinder are considered as examples to demonstrate the accuracy of this method. It is very interesting to note that at very low frequencies, the cross-section of the scatterer can be determined approximately for a specific parallel polarization scattered field. A limitation of the point-matching method is also discussed in detail.

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* A recent article by Mullin and co-authors in IEEE Trans. on Antenna and Propagation (Vol. AP-13, p. 141, January, 1965) adopted the same method to treat the same problem with extensive data. It was read by the author during the final preparation of this paper. More discussion about the applicability of this point-matching method will be presented in this report.

2. THEORY

Consider the oblique incidence of a plane wave on a perfectly conducting cylinder as shown in Fig. 1. Parallel polarization means that the electric field vector is parallel to the plane of incidence. If the electric field vector is perpendicular to the plane of incidence, it is called perpendicular polarization. An arbitrarily polarized wave can be resolved into these two components. The scattered field of a conducting cylinder is determined from a scalor wave equation

$$(\nabla^2 + k_o^2)\ \psi^s = 0 \tag{1}$$

where $k_o^2 = \omega^2 \mu_o \epsilon_o$, and ∇^2 is the Laplacian operator. The boundary conditions that the fields must fulfill are:

(i) The scattered field satisfies the radiation condition.

(ii) The total field (incident plus scattered) is subjected either to Dirichlet or Neumann boundary conditions at the surface of the cylinder.

Let the normalized incident field be expressed by

$$\psi^i = (\cos \alpha) \exp. \ [jk_1 r \cos(\theta - \theta_i) + jk_2 z] \tag{2}$$

where

$$k_1 = k_o \cos \alpha$$

$$k_2 = k_o \sin \alpha$$

the angle α is the angle of incidence, and θ_i is the angle between the x-z plane and the plane of incidence (see Fig. 1). Since the cylinder is assumed to have infinite length, the scattered field may be written as

$$\psi^s = \sum_{n=-\infty}^{\infty} A_n \phi_n \exp.\ (jk_2 z) \tag{3}$$

where the constants A_n are determined by the boundary condition at the periphery of cylinder ; ϕ_n, a known function of transverse position (transverse to z),is the particular solution of

$$\nabla_{\perp}^2 \phi_n + k_1^2 \phi_n = 0 \tag{4}$$

with specification of outgoing wave, where $\nabla_{\perp}^2$ represents the two dimensional Laplacian operator. Under these assumptions, the total field is given by

$$\psi = \psi^i + \psi^s \tag{5}$$

The boundary condition (ii) requires that

$$\psi(c) = 0 \tag{6a}$$

and

$$\left.\frac{\partial \psi}{\partial n}\right|_c = 0 \tag{6b}$$

for parallel polarization and perpendicular polarization respectively, where C denotes the contour described by the cross-section of the cylinder, and $\vec{n}$ is the unit vector normal to the surface. If the expansion coefficients A_n can be evaluated by Eqs. (6), then Eq. (3) is the solution for the scattered field. The remaining problem, to obtain the A_n's, is solved by requiring Eq. (5) to satisfy Eqs. (6).

Suppose that the series of Eq. (3) converges uniformly and the scattered field may be approximately expressed by 2N + 1 terms. That is

$$\psi^s = \sum_{n=-N}^{N} A_n \phi_n \exp(jk_2 z) \tag{7}$$

Equation (5) is then reduced to

$$\psi = \{ (\cos \alpha) \exp[jk_1 r \cos(\theta - \theta_i)] + \sum_{n=-N}^{N} A_n \phi_n \} \exp(jk_2 z) \tag{8}$$

The point-matching approximation requires Eq. (8) to fulfill the boundary condition at only 2N + 1 points around the periphery of the conducting cylinder provided that these points sufficiently describe the shape of the cross-section. Under these assumptions, from Eq. (6a) or (6b) results a system of 2N+1 inhomogeneous algebraic equations with 2N+1 unknowns A_n. This set of equations can be solved for values of A_n by algebraic methods or by a digital computer. The approximate solution is then complete for one specific frequency. The scattering cross-section, radiation pattern, echo area, etc., can be approximately evaluated from Eq. (7). Incidentally, Eq. (4) is known only in two cylindrical coordinate systems, namely, circular cylindrical and elliptical cylindrical. Hence, the function ϕ_n can be either a circular cylindrical wave function or an elliptical wave function. For circular cylindrical coordinate system, $\phi_n = H_n^{(2)}(k_1 r) \exp(jn\theta)$, where $H_n^{(2)}$ is the Hankel function of the second kind. In the following discussion only the circular cylindrical wave function is considered.

3. PARALLEL POLARIZATION

The direction of propagation of the incident plane wave is at an angle of incidence α as shown in Fig. 1. The z-component of the normalized electric field vector which lies on the x-z plane is given by

$$E_z^i = (\cos\alpha)\ \exp.\ [jk_1 r \cos(\theta - \theta_i) + jk_2 z]$$

The z-component of the scattered electric field may be approximately written as

$$E_z^s = \sum_{n=-N}^{N} A_n H_n^{(2)}(k_1 r)\ \exp.\ (jn\theta + jk_2 z) \qquad (9)$$

If $(r_o, \theta_o), (r_1, \theta_1), (r_2, \theta_2) \ \ldots \ (r_{2N}, \theta_{2N})$ are the suitable points around the cross-section of the cylinder as shown in Fig. 2, the point-matching method requires

$$(\cos\ \alpha)\ \exp.\ [jk_1 r_m \cos(\theta_m - \theta_i)] + \sum_{n=-N}^{N} A_n H_n^{(2)}(k_1 r_m)\ \exp.\ (jn\theta_m) = 0 \qquad (10)$$

where m = 0, 1, 2, ... 2N. Eq. (10) can be solved for A_n algebraically.

4. PERPENDICULAR POLARIZATION

If the magnetic field vector of the incident wave lies on the x-z plane, the z-component of the magnetic field is assumed to be

$$H_z^i = Z_o^{-1}\ (\cos\ \alpha)\ \exp.\ [jk_1 r \cos(\theta - \theta_i) + jk_2 z]$$

where $Z_o = (\mu_o/\epsilon_o)^{1/2}$ is the intrinsic impedance of free space. The approximate expression for the z-component of the scattered magnetic field may be written as

$$H_z^s = \sum_{n=-N}^{N} B_n\ H_n^{(2)}\ (k_1 r)\ \exp.\ (jn\theta + jk_2 z) \qquad (11)$$

Similarly, if the points (r_o, θ_o), (r_1, θ_1), (r_2, θ_2), ..., (r_{2N}, θ_{2N}) are chosen around the cross-section for the point-matching method, the boundary condition of Eq. (6b) requires

$$(\cos\phi_m \frac{\partial}{\partial x} + \sin\phi_m \frac{\partial}{\partial y})\ \{ Z_o^{-1}\ (\cos\ \alpha)\ \exp.\ [jk_1 r \cos(\theta - \theta_i)] + \sum_{n=-N}^{N} B_n H_n^{(2)}\ (k_1 r)\ \exp.\ (jn\theta) \} \Bigg|_{\substack{r = r_m \\ \theta = \theta_m}} = 0 \qquad (12)$$

where

$$r^2 = x^2 + y^2$$

$$\theta = \tan^{-1}(y/x)$$

$$\vec{n}_m \cdot \hat{x} = -\cos\theta_m$$

$\hat{x}$ is the unit vector in the x-direction, $\vec{n}_m$ is the unit vector normal to the surface at points (r_m, θ_m) (see Fig. 2) and $m = 0, 1, 2, \ldots 2N$. Eq. (12) can be evaluated without difficulty at each point if the contour is a smooth curve, and the expansion coefficients B_n are determined as discussed previously. In case that the contour is made of broken lines as shown in Fig. 3, the corner points have significant effects on the scattered field. The normal unit vector at these points are taken along the bisector of the corner angle (see Fig. 3).

5. TOTAL SCATTERING CROSS-SECTION

In two dimensional problems, the total scattering cross-section is defined as the ratio of the total scattered power per unit axial length to the incident power density. Thus

$$\sigma = \left(\oint \mathrm{Re}\, \vec{S}^{\,s} \cdot d\vec{\ell} \right) / P^{i} \tag{13}$$

where the line integral is an arbitrary closed path encircling the cylinder. The scattering poynting vector $\vec{S}^{\,s}$ are defined as

$$\vec{S} = \frac{1}{2} \vec{E} \times \vec{H}^{*}$$

where the star denotes the complex conjugate quantity. The incident power density $P^{i} = 1/Z_o$. The line integral, in general, is more conveniently evaluated at r approaching infinity. By using Eqs. (9) and (11), the total scattering cross-sections are given by

$$\sigma_{/\!/} = (4/k_o \cos^2 \alpha) \sum_{n=-N}^{N} \left| A_n \right|^2 \tag{14}$$

$$\sigma_{\perp} = (4Z_o^2 / k_o \cos^2 \alpha) \sum_{n=-N}^{N} \left| B_n \right|^2 \tag{15}$$

for parallel and perpendicular polarization respectively. With known values of A_n and B_n, it is very simple to calculate the scattering cross-sections.

6. NORMAL INCIDENCE

Normal incidence is a special case of the oblique incidence. It is obtained simply by putting $\alpha = 0$ in all equations of sections 3 and 4. However, in parallel polarization, no transverse component (transverse to z) of electric field exists in the normal incidence while the transverse electric field appears in oblique incidence. Similarly, no transverse magnetic field exists in perpendicular polarization of normal incidence. In both cases, the expressions for the total scattering cross-section [Eqs. (14) and (15)] remain the same with $\alpha = 0$.

7. NUMERICAL EXAMPLES

The accuracy of the point-matching method discussed previously can be demonstrated by numerical examples. The first case considered is the parallel polarization of normal incidence. The scatterer is a circular conducting cylinder of radius a. This is not a trivial example because the boundary conditions are satisfied at only a finite number of points around the periphery. Table I lists the exact values of $\sigma/4a$ and those calculated by point-matching method for various values of N in Eq. (14). Equal spaces between points are chosen for

Table I. Scattering cross-sections ($\sigma/4a$) of a circular cylinder

$k_o a$ \ L	0	1	2	3	Exact
1	0.9324	1.4497	1.4780		1.4783
2	0.8795	0.9858	1.2829	1.3056	1.3065

the approximate calculation with the angle starting from $\theta = 0$. Note that the symmetry with respect to x-axis of the cross-sectional geometry and $\theta_i = 0$ reduce Eq. (10) to the following form:

$$(\cos \alpha) \exp . (jk_1 r_m \cos \theta_m) = \sum_{d=o}^{L} A_\ell \epsilon_\ell H_\ell^{(2)} (k_1 r_m) \cos \ell \theta_m = 0 \qquad (16)$$

where

$$\epsilon_\ell = \begin{cases} 1 & \text{if } \ell = 0 \\ 2 & \text{if } \ell \neq 0 \end{cases}$$

The expression for total scattering cross-section is changed to

$$\sigma = (4/k_o \cos^2 \alpha) \sum_{\ell=o}^{L} | A_\ell |^2 \epsilon_\ell \qquad (17)$$

Consequently, only the points on and above the x-axis are used in the calculation. Consider parallel polarization of normal scattering by a square cylinder with sides equal to 2a (see Fig. 4). Calculations are made for matching 3,5, and 9 points at $k_o a = 0.5$ and 1. Points are chosen in the order marked in Fig. 4. Values of $\sigma/4a$ computed by point-matching method are listed in Table II and compared with those calculated by Mei and Van Bladel.[6]

Table II. Scattering cross-sections ($\sigma/4a$) of a square cylinder.

$k_o a$ \ L	2	4	8	Mei and Van Bladel
0.5	2.264	2.120		2.0
1	2.093	1.8116	1.748	1.7

These results show that the point-matching method is applicable to calculate approximately the scattering cross-section of an arbitrarily shaped cylinder.

8. VERY LOW FREQUENCY SCATTERING

As shown previously, the point-matching method is applicable for approximately solving the problem of scattering of a plane wave by cylinders with arbitrary cross-sections. The approximation is introduced when the boundary condition requires Eqs. (6) to be fulfilled only at finite number of points. If Eq. (8) is carefully examined where the expansion coefficients A_n are calculated by point-matching method, one cannot find any point, other than the chosen points, satisfying the boundary condition. However, at very low frequencies a closed curve which approximately fulfills the requirement can be obtained for both polarizations. This contour may be recognized as the cross-section of the conducting cylinder at low frequencies. The details of the symmetrical case is given as follows:

Consider first the approximate cross-section of parallel polarization of normal incidence. If $kr_m << 1$ for all m in Eq. (16) the expansion coefficients A_ℓ of the scattered field are determined by

$$1 - j \sum_{\ell=o}^{L} A_\ell \epsilon_\ell N_\ell (kr_m) \cos \ell \theta_m = 0 \tag{18}$$

where N_ℓ is the Bessel function of second kind, (r_o, θ_o), (r_1, θ_1), ... (r_L, θ_L) are the chosen points. The cross-section of the scatter for the scattered field of Eq. (9) is given by

$$1 - j \sum_{\ell=o}^{L} A_\ell \epsilon_\ell N_\ell (kr) \cos \ell \theta = 0 \tag{19}$$

Similarly, the expansion coefficients B_ℓ of the perpendicular polarization scattered field for normal incidence are given by (Appendix A)

$$kr_m \{ F(x_m, y_m) + [\sin \theta_m - F(x_m, y_m) \cos \theta_m] [\sum_{\ell=o}^{L} B_\ell \epsilon_\ell N'_\ell(k.r_m) \cos \ell \theta_m]$$

$$-[\cos \theta_m + F(x_m, y_m) \sin \theta_m] [\sum_{\ell=o}^{L} \epsilon_\ell B_\ell N_\ell (kr_m) \sin \ell \theta_m] = 0 \tag{20}$$

where $F(x,y) = -\frac{\partial f(x,y)}{\partial x} \Big/ \frac{\partial f(x,y)}{\partial y}$, the curve of the original cross-section is given by

$$f(x,y) = 0 \tag{21}$$

The cross-section for scattered field of Eq. (11) is determined from

$$kr \left\{ \frac{dy}{dx} + \left[\sin\theta - \frac{dy}{dx}\cos\theta \right] \left[\sum_{\ell=o}^{L} B_\ell \, \epsilon_\ell \, N'_\ell (kr) \cos \ell\theta \right\} \right.$$

$$- \left[\cos\theta + \frac{dy}{dx} \sin\theta \right] \left[\sum_{\ell=o}^{L} B_\ell \, \epsilon_\ell \, N_\ell (kr) \sin \ell\theta \right] = 0 \tag{22}$$

However, Eq. (22) is not easy to solve and no details will be discussed.

As an example, the parallel polarization of normal incidence on a square conducting cylinder with sides equal to 2a is considered. If only six points on the surface are matched with the boundary condition as indicated in Fig. 5, then the z-component of the scattered electric field is given by

$$E_z^s = \sum_{n=o}^{2} A_n H_n^{(2)} (kr) \cos n\theta \tag{23}$$

where $A_o = (-\cos\varsigma) / H_o^{(2)} (\sqrt{2}\,\varsigma)$

$$A_1 = -j\sqrt{2}\, \sin\varsigma / H_1^{(2)} (\sqrt{2}\,\varsigma)$$

$$A_2 = [H_o^{(2)}(\sqrt{2}\,\varsigma) - H_o^{(2)}(\varsigma)\cos\varsigma] / [H_o^{(2)}(\sqrt{2}\,\varsigma) H_2^{(2)}(\varsigma)]$$

$$\varsigma = k_o a$$

The total z-component of electric field in the neighborhood of the cylinder at very low frequencies is found to be

$$E_z = 1 + [\ell n\, k_o r - 0.1159 + 0.3466(a/r)^2 \cos 2\theta] \;/\; (\ell n\, k_o a + 0.2306) \qquad (24)$$

where $k_o r$, $k_o a$ are assumed very small in comparison with unity. The closed contour indicated by solid line in Fig. 5 is the boundary of the fields satisfying Dirichlet boundary condition. Therefore, Eq. (23) is actually the approximate scattered field of the parallel polarization normal incident on the conducting cylinder with cross-section as shown in Fig. 5 at low frequency. Observe that this is too big for the approximate cross-section of the square.

If the solution is obtained by matching at eight points as shown in Fig. 6, then the cross-section of the conducting cylinder (as indicated by the solid line in Fig. 6) at low frequencies satisfies the boundary condition. By comparing Fig. 6 with Fig. 5, obviously, the former is a better approximation. Hence, it can be concluded that the more points chosen for point-matching method, the better the approximate solution for the total scattering cross-section. This is in agreement with the numerical results at higher frequencies.

The above discussion suggests that the point-matching method can be applied to find the cross-section of a conducting cylinder corresponding to a specified scattered field. That is, assuming that the scattered z-component electric field is given by

$$E_z^s = \sum_{n=-N}^{N} A_n H_n^{(2)}(k_o r)\, e^{jn\theta} \qquad (25)$$

the total field is then expressed as

$$E_z = \exp(jk_o r \cos\theta) + \sum_{n=-N}^{N} A_n H_n^{(2)}(k_o r)\, e^{jn\theta} \qquad (26)$$

Let Eq. (26) vanish at 2N + 1 points, the expansion coefficients A_n are then determined. At very low frequencies, the contour described by equating Eq. (26)

to zero is the cross-section of the scattering conducting cylinder, and Eq. (21) is the corresponding scattered field. A table of these pairs can be made for synthesis problem.

9. LIMITATION ON THE POINT-MATCHING METHOD

In the previous discussion, the point matching method was applied to conducting cylinders enclosing the z-axis. The energy associated with a finite region is always finite. However, this method is not applicable to cases where the z-axis is either on or outside the enclosed surface of the conducting cylinder, for the energy associated with a finite region around the z-axis is infinity. For example, consider the parallel polarization normal scattering of a plane wave by an infinite strip of width 2a as shown in Fig. 7(a). The incident electric field is given by

$$E_z^i = \exp\,(jk_o r \cos\,\theta\,)$$

If the total field satisfies the Dirichlet boundary condition at only four points, namely, $(a, -\pi/2)$, $(a/2, -\pi/2)$, $(a/2, \pi/2)$, and $(a, \pi/2)$ the equation which approximately determines the contour of the cross-section is found to be as follows

$$\cos 2\,\theta \;=\; [\xi^2 \; n \;(\xi^3/2)]\; / \,\ell n2 \tag{27}$$

where $\xi a = r$. There are two closed contours, one enclosed by the other, satisfy Eq. (27) as shown in Fig. 7(b). The scattered field found under the above assumptions is caused by the outer contour. Obviously this geometry has a larger effect on scattering than that of a strip. The total scattering cross-section of this object at $k_o a = 0.01$ is 48.94a. This is too big to compare with that ($\sigma = 31.05a$) of the strip calculated by Bonwkamp.[7]

10. CONCLUSION AND REMARKS

The point-matching method for computation of scattering of a plane wave by conducting cylinders with arbitrary cross-section has been examined. The theory was confirmed by considering low-frequency scattering and numerical examples. This method is particularly convenient when a digital computer is available. The inapplicability of this method to the scattering by a strip was discussed also. Low frequency scattering suggests that for a known scattering field, the cross-sectional shape of the scatterer may be obtained.

If the incident wave is from the close by line sources, this method is applicable also. To do this, the plane wave expression in Eq. (2) is replaced by the wave function radiating from the sources. The rest of the equations are changed accordingly.

The point-matching method can be extended to the three-dimensional problem. This can be done by expressing the scattered field in a series of spherical Hankel functions with specification of outgoing wave. However, the point-matching steps are complicated and tedious.

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APPENDIX A

If the contour described by the cross-section is represented by Eq. (21), then

$$\frac{dy}{dx} = F(x,y) \tag{A-1}$$

and

$$\tan \phi = - 1/F(x,y) \tag{A-2}$$

The partial derivatives of r and θ with respect to x and y are given by

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \cos \theta \\
\frac{\partial r}{\partial y} &= \sin \theta \\
\frac{\partial \theta}{\partial y} &= \frac{1}{r} \sin \theta \\
\frac{\partial \theta}{\partial y} &= \frac{1}{r} \cos \theta
\end{aligned} \tag{A-3}$$

Substituting Eqs. (A-2) and (A-3) into Eq. (12) of the symmetrical cross-section yields Eq. (20) for normal incidence. Replacing $F(x,y)$ by $\frac{dy}{dx}$, r_m by r, and θ_m by θ, one obtains Eq. (22).

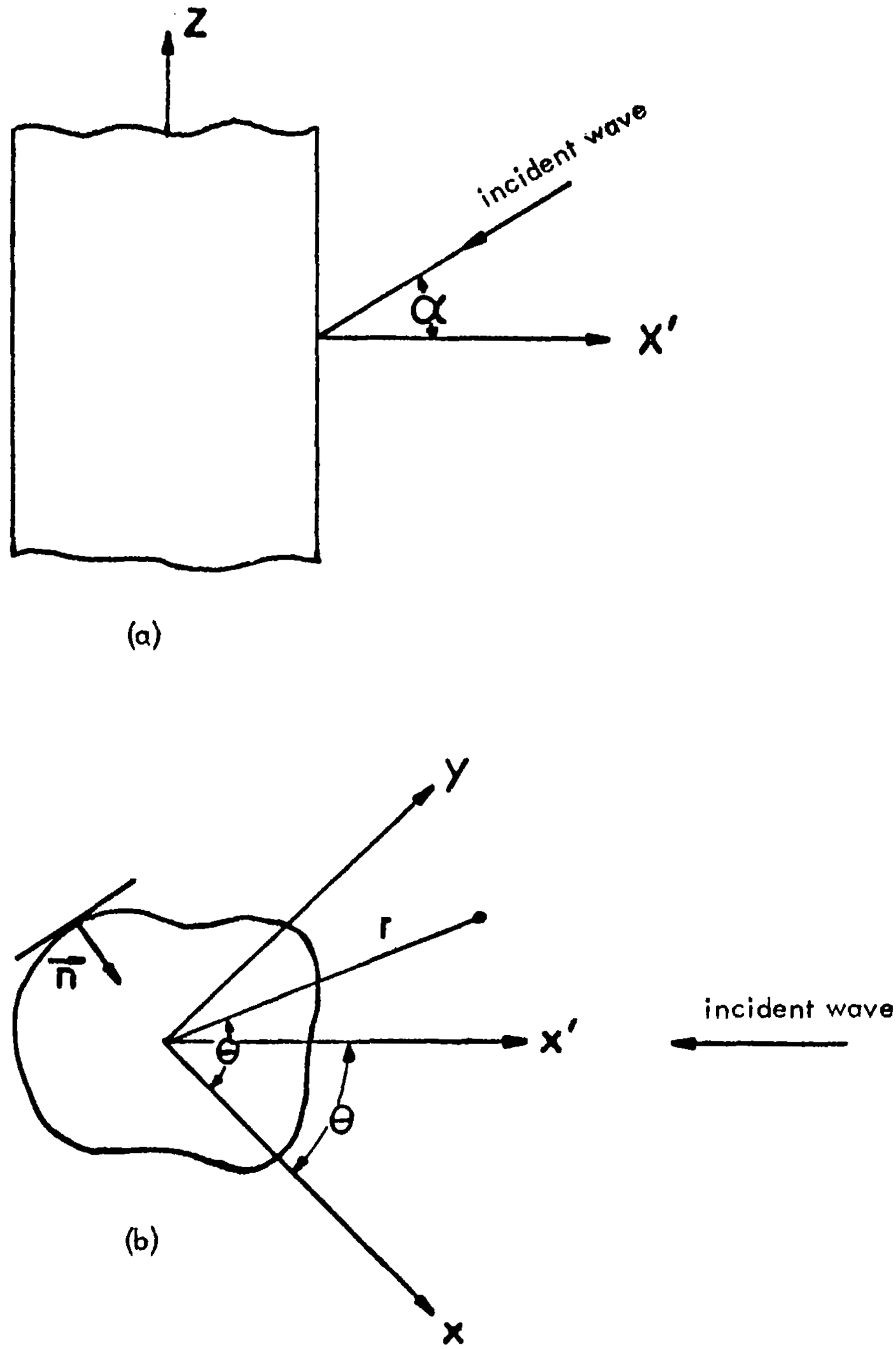


Fig. 1 - Cylindrical scatterer and cylindrical coordinate system.

(a) Side view
(b) Cross-sectional view

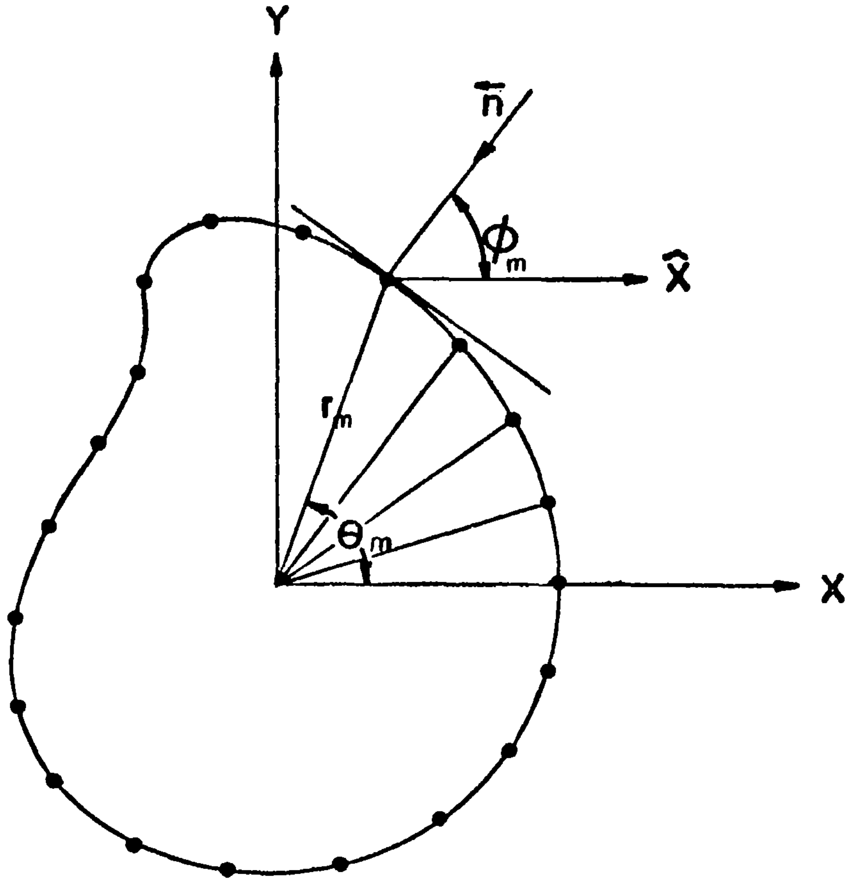


Fig. 2 - Boundary condition for point-matching method.

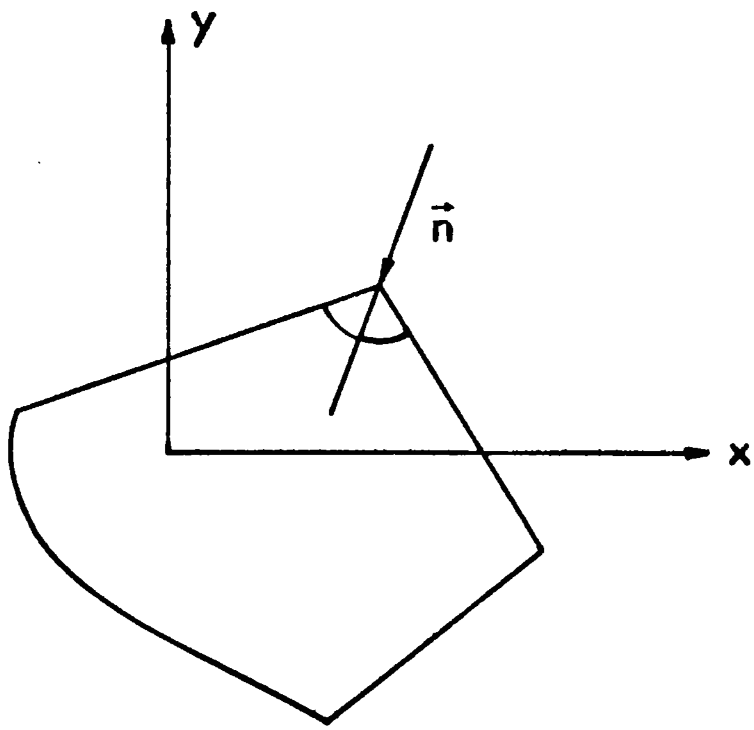


Fig. 3 - The unit vector normal to the corner on the surface.

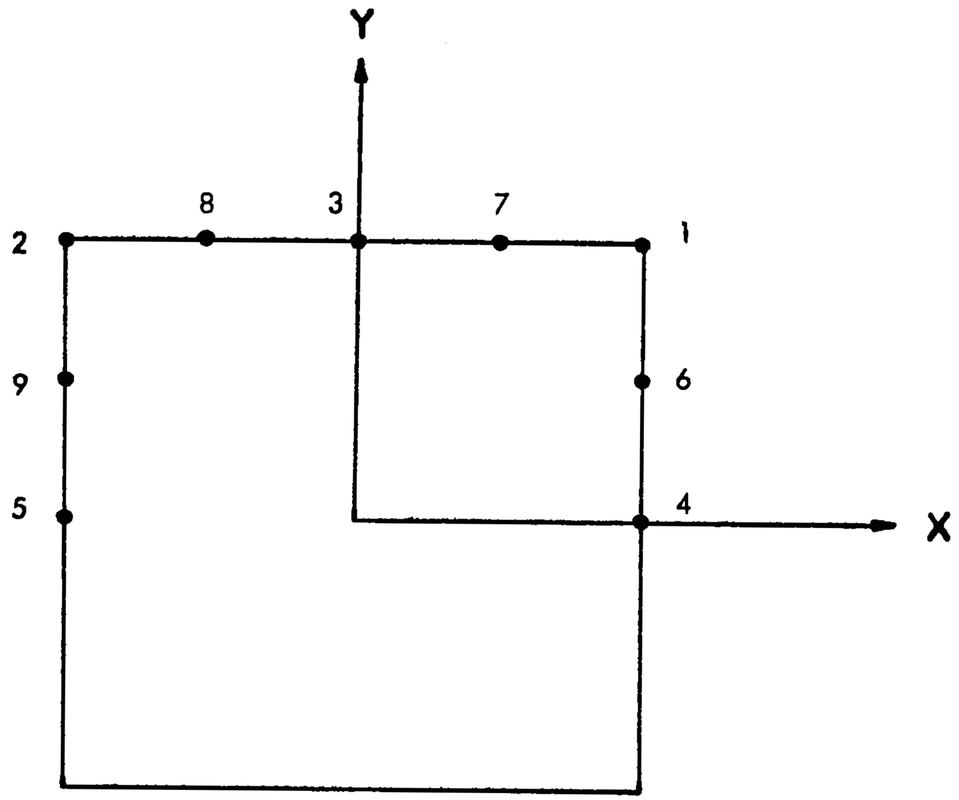


Fig. 4 - The points chosen for calculation of a square cylinder.

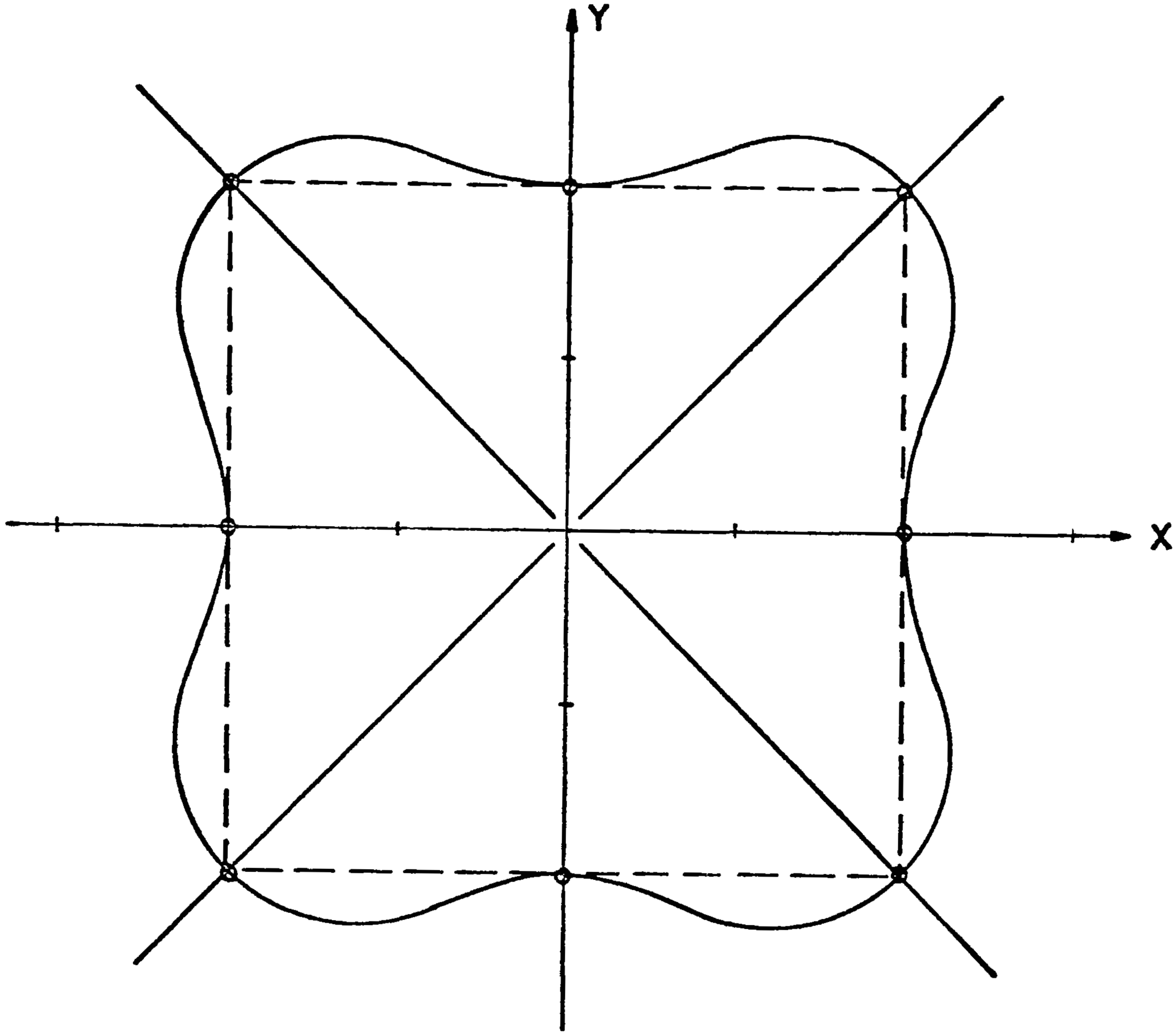


Fig. 6 - The scatterer cross-section of the eight-point approximation for square cylinder at low frequencies.

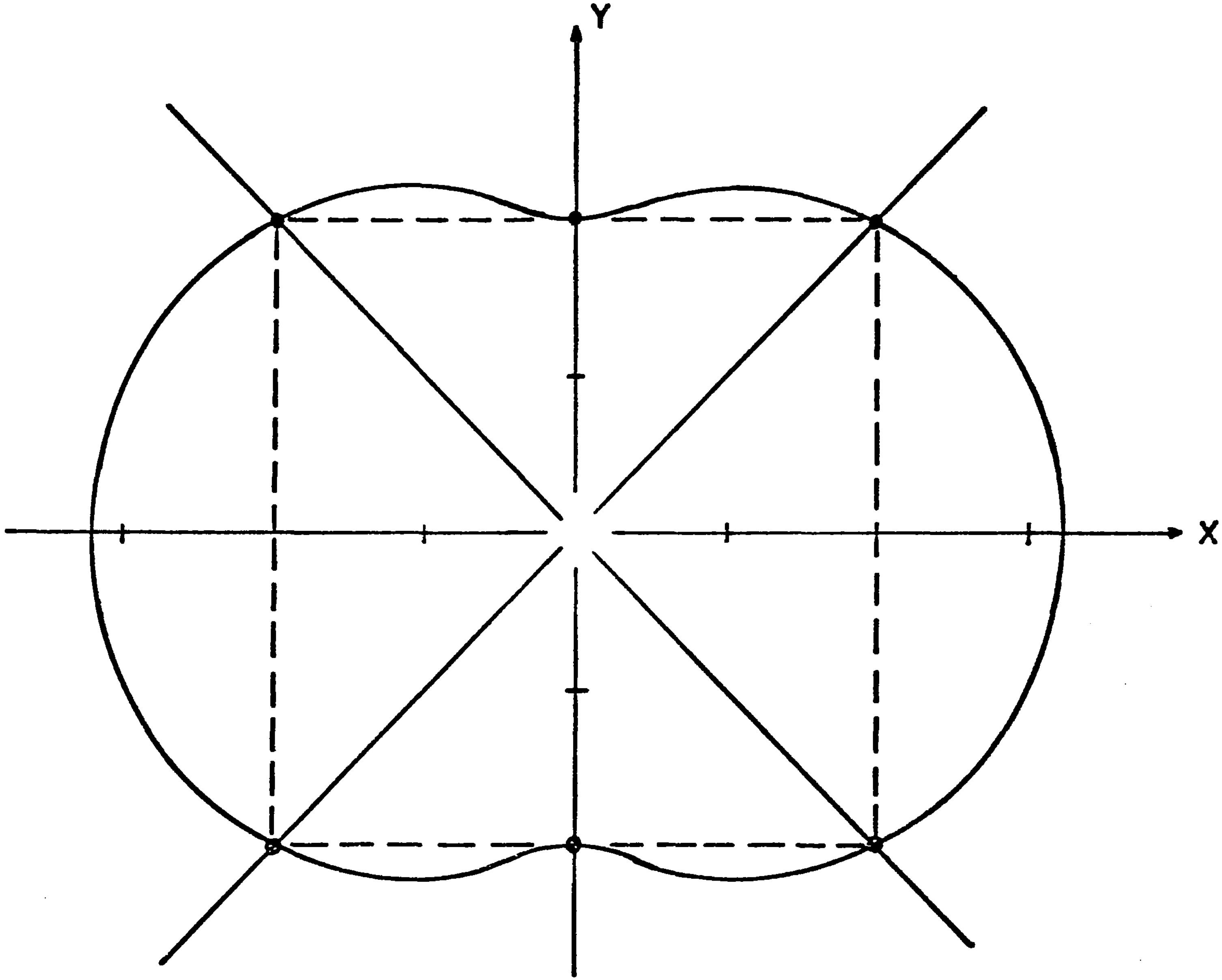
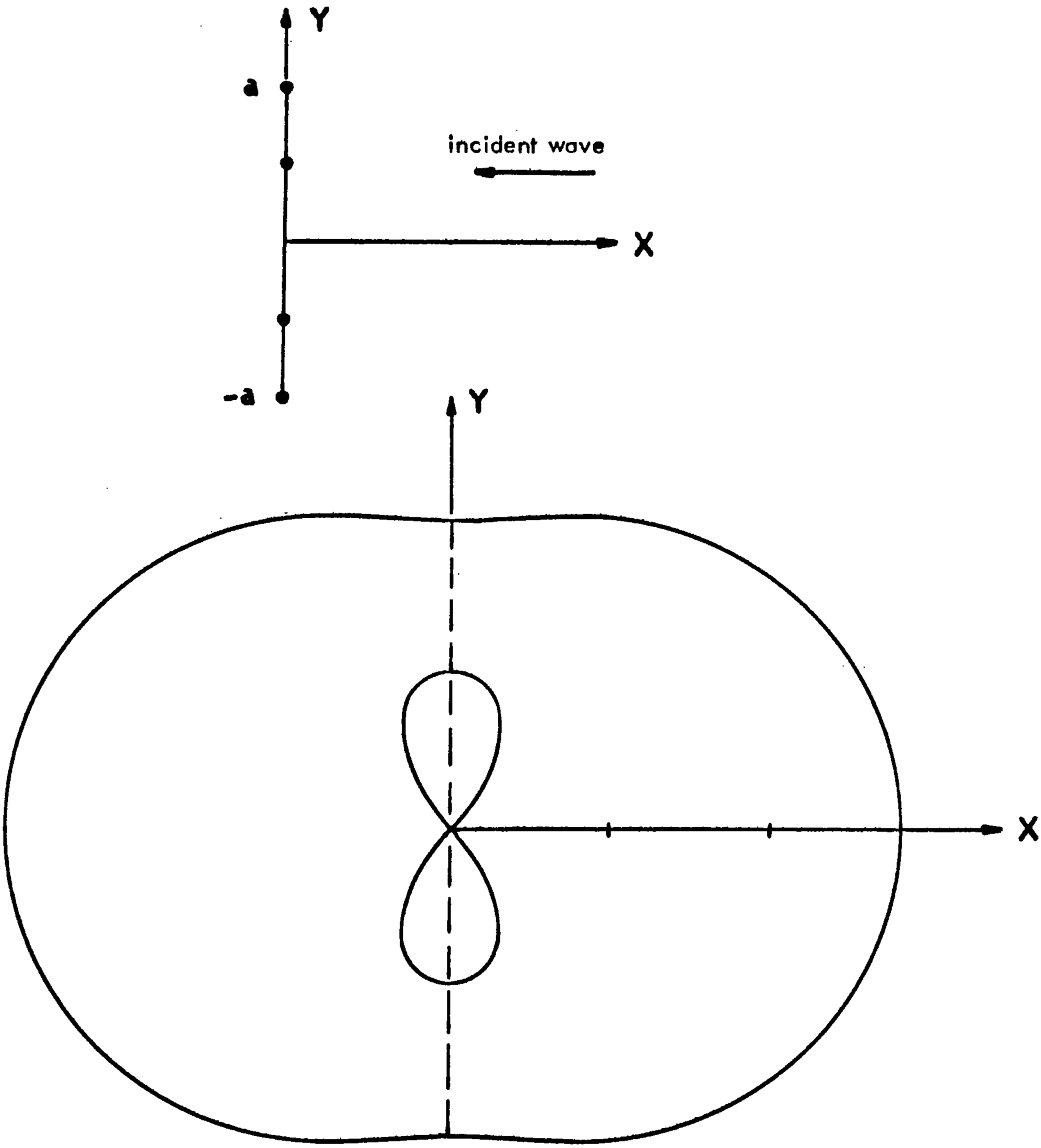


Fig. 5 - The scatterer cross-section of the six-point approximation for square cylinder at low frequencies.



(b)

Fig. 7 - (a) The cross-sectional view of the infinite strip and the coordinate system.
(b) The corresponding scatterer cross-section of the point-matching method at low frequency.

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