NS 6-18

GRADUATE AERONAUTICAL LABORATORIES
CALIFORNIA INSTITUTE OF TECHNOLOGY

Instability of Cylindrical Shells Subjected to Axisymmetric Moving Loads

G. A. Hegemier

July 1965

Firestone Flight Sciences Laboratory

Guggenheim Aeronautical Laboratory

Karman Laboratory of Fluid Mechanics and Jet Propulsion

Pasadena

INSTABILITY OF CYLINDRICAL SHELLS SUBJECTED TO AXISYMMETRIC MOVING LOADS*

G. A. Hegemier

Research Fellow

Firestone Flight Sciences Laboratory
California Institute of Technology
Pasadena, California

* Parts of the study were supported by a National Aeronautics and Space Administration Research Grant.

ABSTRACT

31049

Using a Donnell type nonlinear theory and the stability in the small concept of Poincaré, the instability of an infinite-length cylindrical shell subjected to a broad class of axisymmetric loads moving with constant velocity is studied. Special cases of the general loading function include the moving ring, step and decayed step loads. The analysis is carried out with a double Laplace transform-functional difference technique. Numerical results are presented for the case of the moving ring load.

Author

ACKNOWLEDGMENTS

The author wishes to express his appreciation to Dr. E. E. Sechler for his encouragement and suggestions regarding this study and to Drs. J. K. Knowles, Y. C. Fung, and T. K. Caughey for their technical advice. Thanks goes also to Dr. K. W. Jacob of the Caltech Computing Center for help in obtaining the numerical results contained herein.

NOMENCLATURE

E, ν = elastic constants

h = shell thickness

a = shell radius

D = $Eh^3/12(1-\nu^2)$

β^4 = $(h/a)^2/12(1-\nu^2)$

P = surface load

ρ = mass density

V_L = load velocity

V_{co} = $\sqrt{E/\rho}\,[(h/a)/\sqrt{3(1-\nu^2)} + N_X^0/Eh]^{1/2}$

U, V, W = midsurface displacements

X, Y = axial and circumferential coordinates

$-N_X^0$ = initial axial compression of cylinder

F = stress function

w = W/a

w_s = axisymmetric response

x = X/a

θ = Y/a

t = $(T/a)(E/\rho)^{1/2}$

q = Pa/Eh

f = F/a^2Eh

N_x^* = N_X^0/Eh

M = $V_L/(E/\rho)^{1/2}$

$P_c, P_n, P_n^*, \Omega_n, \Omega_n^*$ = load parameters

ξ = $x - Mt$

τ = t

C_j, C_j^*, a_j, a_j^* = displacement constants (see (12))

ζ, η = perturbation quantities

n = number of circumferential half waves

ζ_n, η_n = coefficients of Fourier Series (15)

$\bar{\zeta}_n, \bar{\eta}_n$ = Laplace transform of ζ_n and η_n with respect to t

$\bar{z}_n$ = the two dimensional vector $\begin{Bmatrix} \bar{\zeta}_n \\ \bar{\eta}_n \end{Bmatrix}$

$\bar{\bar{z}}_n$ = Laplace transform of $\bar{z}_n$ with respect to ξ

p = first transform parameter

s = second transform parameter

$\underset{\sim}{L}_j$ = 2x2 matrices defined by equations (26)

$\underset{\sim}{A}_j$ = $\underset{\sim}{L}_o^{-1}\,\underset{\sim}{L}_j$

$\underset{\sim}{\Psi}$ = 2 dimensional vector (see (25))

$\underset{\sim}{\phi}$ = $\underset{\sim}{L}_o^{-1}\,\underset{\sim}{\Psi}$

$\Delta\underset{\sim}{L}_o$ = determinant of $\underset{\sim}{L}_o$

ρ_j = roots of $\Delta\underset{\sim}{L}_o = 0$

$\underset{\sim}{B}_j$ = $\prod_{q=1}^{8}(s-\rho_q)\underset{\sim}{A}_j$

$\underset{\sim}{R}_j(k)$ = $\underset{\sim}{B}_j(s)/\prod_{\substack{r=1 \\ r\neq k}}^{8}(s-\rho_r)$ when $s = \rho_k$

S_i = defined by equation (34)

G_i, G_i^* = integration constants

$b_1(r,i)$, $b_2(r,i)$ = defined by equation (37b)

$C_{r,i;j_1\ldots j_N}$ = defined below (37b)

$\underset{\sim}{A}$ = matrix defined in (44)

λ = load parameter

1. INTRODUCTION

Questions concerning the stability of thin shells subjected to moving loads occur frequently in the design and analysis of aerospace vehicles. Unfortunately the mathematical complexity encountered with the simplest of these problems is formidable and as a result few or no solutions of even the most idealized cases are available. The analysis to follow focuses on a small but significant subset of the general question, namely on the instability of thin elastic cylindrical shells loaded by a class of axisymmetric pressure distributions moving with constant velocity in the direction of the shell generatrix.

For a geometrically perfect cylindrical shell the response to an axisymmetric load moving with constant velocity will, of course, be axisymmetric. Several investigators [1-5] have examined this response in the light of linear shell theory. Under certain circumstances, however, these motions can be unstable with respect to nonsymmetric disturbances. Since such instabilities lead to either a buckling phenomena or finite nonsymmetric oscillations, they are of considerable interest.

In the present paper, as in [1-4], the problem is idealized by considering an infinite shell length and a steady-state form of the axisymmetric response (Note: Tang [4] also considered the initial value problem). The discussion is devoted to the question of stability of these steady-state motions.

Mathematically, the shell is modeled by a nonlinear set of partial differential equations. The response of the shell to the axisymmetric load is sought as a static solution of these equations in a coordinate system moving with the load. Such motions can be visualized as the limiting case of a transient problem in which the load is applied and brought up to speed from rest in some manner. The stability of this response is defined according to the classical concept of Poincaré, i.e., stability is defined on the basis of the boundedness of a nonaxisymmetric perturbed motion about the axisymmetric state. Under this concept the analysis reduces to a study of a set of perturbation or so-called variational equations ("equations aux variations" of Poincaré). These equations are linearized under the assumption of infinitesimal disturbances and stability in the small is considered. The usual difficulty regarding the existence of variable coefficients in the variational equations is overcome by use of a double Laplace transform-functional difference technique.

Because of the scope of the subject and space limitations, this paper concerns only sufficient conditions for instability and a method for determining an upper bound on the transition from stability to instability. As a numerical example the problem of a moving ring load is considered.

2. FORMULATION OF THE PROBLEM

The Equations of Motion

All motions of the shell will be referred to the undeformed shell as illustrated in Fig. 1. Employing a Donnell type theory [6], the equations of motion can be written as a set of two equations: one governing the radial equilibrium of the shell and the other being the condition of compatibility. In terms of the radial displacement, W, of the midsurface and a stress function, F, these are respectively:

$$\left.\begin{aligned} D\nabla^4 W &= P + \frac{\partial^2 F}{\partial Y^2}\frac{\partial^2 W}{\partial X^2} - 2\frac{\partial^2 F}{\partial X\partial Y}\frac{\partial^2 W}{\partial X\partial Y} + \frac{\partial^2 F}{\partial X^2}\frac{\partial^2 W}{\partial Y^2} + \frac{1}{a}\frac{\partial^2 F}{\partial X^2} - \rho h\frac{\partial^2 W}{\partial T^2} \\ \nabla^4 F &= Eh\left[\left(\frac{\partial^2 W}{\partial X\partial Y}\right)^2 - \frac{\partial^2 W}{\partial X^2}\frac{\partial^2 W}{\partial Y^2} - \frac{1}{a}\frac{\partial^2 W}{\partial X^2}\right] \end{aligned}\right\} \quad (1)$$

where F is related to the stress resultants by

$$N_X = \partial^2 F/\partial Y^2, \; N_Y = \partial^2 F/\partial X^2, \; N_{XY} = -\frac{\partial^2 F}{\partial X\partial Y}$$

Here D denotes the flexure rigidity, P the normal surface loading, ρ mass density, h shell thickness, a shell midsurface radius, and ∇^4 the biharmonic operator.

Since equations (1) are well known their derivation will not be discussed. The reader is referred to [6,7] for details. It should be noted, however, that the use of (1) requires that strains and rotations are small compared to unity and $\frac{V}{a} \ll \partial W/\partial Y$. This latter approximation is usually associated with Donnell and is valid if, upon deforming, the square of the number of circumferential waves, n, is large compared to unity. For thin shells n > 3 is usually sufficiently large. For the special case n = 0 (axisymmetric motions), Donnell's approximation is not involved since V and $\partial W/\partial Y$ are identically zero. Finally, it is evident that only radial inertia was included.

In the discussion it will be convenient to introduce the nondimensional quantities

$$\left.\begin{aligned} & w = W/a,\ f = F/a^2Eh,\ x = X/a,\ \theta = Y/a \\ & t = \frac{T}{a}\sqrt{E/\rho},\ q = \frac{Pa}{Eh} \end{aligned}\right\} \qquad (2)$$

Substitution of (2) into (1) yields

$$\left.\begin{aligned} & \beta^4\nabla^4 w = q + f_{\theta\theta} w_{xx} - 2f_{x\theta} w_{x\theta} + f_{xx}(1+w_{\theta\theta}) - w_{tt} \\ & \nabla^4 f = (w_{x\theta})^2 - w_{xx}(1+w_{\theta\theta}) \end{aligned}\right\} \qquad (3)$$

where $\beta^4 = (\frac{h}{a})^2/12(1-\nu^2)$, $\nabla^4(\ ) \equiv (\ )_{xxxx} + 2(\ )_{xx\theta\theta} + (\ )_{\theta\theta\theta\theta}$ and $(\ )_x$ denotes $\partial(\ )/\partial x$, etc.

The Loading Condition

The shell will be assumed loaded by an axial stress resultant, N_X^0 (positive in tension) and an axisymmetric lateral pressure distribution moving with velocity V_L and defined by:

$$P(\xi_1) = P_c\delta(\xi_1)+H(\xi_1)[P_o+\sum_{n=1}^{N}P_n e^{-\Omega_n \xi_1}]+H(-\xi_1)[P_o^*+\sum_{k=1}^{K}P_k^* e^{\Omega^*_k \xi_1}] \tag{4}$$

where $\xi_1 = X-V_L\ T$. Here N and K are finite, P_o and P_o^* are real constants, P_n, P_k^*, Ω_n, Ω_k^* are in general complex valued and Re $\Omega_n, \Omega_n^* > 0$. The quantities $\delta(\xi_1)$ and $H(\xi_1)$ are, respectively, the Dirac delta function and the Heaviside step function. Several examples of the type of loads that can be constructed from equation (4) are illustrated in Figs. 2a, b, c and d. They include the moving ring load, step load, decayed steps and general pulse (including internal pressure and axial compression) respectively. Many load distributions not falling directly into the above class can be closely approximated by (4). The coefficients can be determined by a collocation procedure or by minimizing the total square error between the actual load function and the approximation. The question of completeness of the exponential portions of (4) as N or K $\rightarrow \infty$ has been discussed by Erdelyi [20].

In the nondimensional form (2) the load velocity will be denoted by $M = V_L\ (E/\rho)^{-\frac{1}{2}}$ so that $q = q[a(\ x-Mt)]$.

The neglect of such quantities as rotary inertia, transverse shear deformation and longitudinal inertia in the present theory will necessitate a restriction on the magnitude of the load velocity. In this connection only the case where the load velocity is less than the minimum velocity for which axisymmetric sinusoidal wave trains can be propagated in the shell will be considered. This is equivalent to the restriction:

$$V_L < V_{co} = \sqrt{E/\rho}\ [\frac{h}{a}/\sqrt{3(1-\nu^2)} + N_X^0/Eh]^{\frac{1}{2}} \tag{5}$$

where the quantity N_X^0 will be considered only in compression, i.e., $N_X^0 \leq 0$.

For steel shells, with $N_X^0 = 0$, V_{co} lies between 400-2000 f.p.s. for a/h = 1000-40 respectively. The effect of $N_X^0 < 0$ is to lower these values.

The value of N_X^0 yielding $V_{co} = 0$ is the classical buckling load due to axial compression. For all compressive loads less than this value $V_{co} > 0$. Physically V_{co} marks a basic change in the character of the axisymmetric response. The necessity of the restriction (5) will be discussed later.

Axisymmetric Response

The response of the shell to the load (4) will be obtained as a solution of equations (2) of the form:

$$w(x,\theta,t) = w_s(x-Mt)$$
$$f_{xx}(x,\theta,t) = f_{s_{xx}}(x-Mt);\ f_{\theta\theta}(x,\theta,t) = f_{s_{\theta\theta}}(x-Mt);\ f_{s_{x\theta}} = 0 \tag{6}$$

Definition of Stability

Let us perturb the steady-state motion w_s and f_s by, respectively, the quantities $\zeta(x,\theta,t)$ and $\eta(x,\theta,t)$. If w_p and f_p denote the perturbed solutions we have

$$w_p = w_s + \zeta$$
$$f_p = f_s + \eta \tag{7}$$

Inserting w_p and f_p into the nonlinear equations (3) and neglecting powers of ζ and η above the first we obtain linear variational equations for ζ and η. We shall consider only those solutions of the variational equations which are regular as $|x| \to \infty$ for fixed θ and t. If all such solutions are bounded as $t \to \infty$ the shell will be said to be stable, otherwise unstable. More precisely: w_s and f_s are stable iff given an $\epsilon > 0$ and $t_o \exists\ \delta = \delta(\epsilon, t_o) \ni |\zeta(x,\theta,t_o)|$ and $|\eta(x,\theta,t_o)| < \delta$ implies $|\zeta(x,\theta,t)|$ and $|\eta(x,\theta,t)| < \epsilon$.

3. GENERAL ANALYSIS

The Axisymmetric Response

Although equations (2) are nonlinear, the present shell theory is such that (2) reduce to linear equations for $V = \partial/\partial\theta = 0$ if it is assumed that the pre-tension or compression of the cylinder, N_X^o is maintained as a constant at $X = \pm\infty$. In the interest of brevity we merely state the resulting axisymmetric equations (the reader is referred to [8] for details):

$$\beta^4 w_{s_{xxxx}} - N_x^* w_{s_{xx}} + w_s + w_{s_{tt}} = q[a(x-Mt)] + \nu N_X^* \tag{8}$$

At this point recall that the effects of longitudinal inertia, rotary inertia and shear deformation were neglected in (1) and hence (8). An estimate of the validity of these approximations can be made by referring to the works of Tang [4] and Jones and Bhuta [1]. A comparison of the phase spectrum of (8)(with zero right hand side) with the more exact theory of [4], which includes both rotary inertia and transverse shear deformation, indicates that (8) is in general a valid approximation only if the load velocity, V_L, is less than the cutoff velocity V_{co} given by (5). Further, [1] indicates the effects of longitudinal inertia are negligible for $V_L < V_{co}$. In view of these results we have placed the restriction (5) on V_L.

The axisymmetric response of the shell is obtained by solution of (8) under the condition that $w_s = w_s(x-Mt) = w_s(\xi)$. This leads to the following total differential equation for w_s:

$$\beta^4 w_s'''' + (M^2 - N_X^*) w_s'' + w_s = q(a\xi) + \nu N_X^* \tag{9}$$

where $(\ )' \equiv d/d\xi$.

Requiring only that the solutions of (9) be bounded as $|\xi| \to \infty$, one obtains $w_s(\xi)$ as:

$$w_s(\xi) = \int_{-\infty}^{\infty} g(\xi,\lambda)\,[q(a\lambda) + \nu N_X^*]\,d\lambda \tag{10}$$

where $g(\xi, \lambda)$ represents the Green's function of equation (9) and has the form:

$$\begin{aligned} g(\xi,\lambda) &= \sum_{i=1}^{2} g_i\, e^{-\alpha i(\xi-\lambda)} , \qquad \xi-\lambda > 0 \\ &= \sum_{i=1}^{2} g_i\, e^{+\alpha i(\xi-\lambda)} , \qquad \xi-\lambda < 0 \end{aligned} \tag{11a}$$

Here

$$g_{1,2} = \frac{1}{4(\bar{\beta}^4-\bar{M}^4)^{\frac{1}{2}}}\left[(\bar{\beta}^2 + \bar{M}^2)^{\frac{1}{2}} \pm i\,(\bar{\beta}^2 - \bar{M}^2)^{\frac{1}{2}}\right]$$

$$\alpha_{1,2} = \frac{1}{\bar{\beta}^2}\left[(\bar{\beta}^2 - \bar{M}^2)^{\frac{1}{2}} \pm i\,(\bar{\beta}^2 + \bar{M}^2)^{\frac{1}{2}}\right] \tag{11b}$$

$$\bar{\beta} = \sqrt{2}\,\beta, \quad \bar{M}^2 = M^2 - N_x^*$$

Evaluation of the integral under the assumption that the arguments of the exponentials in the loading function are not roots of the characteristic equation of (9) yields $w_s(\xi)$ formally as

$$\begin{aligned} w_s(\xi) &= C_o + \sum_{j=1}^{\ell} C_j\, e^{-\alpha_j\xi}, \quad \ell = N + 2; \quad \xi > 0 \\ &= C_o^* + \sum_{j=1}^{\ell^*} C_j^*\, e^{\alpha_j^*\xi}, \quad \ell^* = K + 2; \quad \xi > 0 \end{aligned} \tag{12}$$

where C_j, C_j^*, α_j and α_j^* are in general complex valued and Re $\alpha_j > 0$, Re $\alpha_j^* > 0$.

The Variational Equations

In the following, it will be convenient to introduce the transformation:

$$\xi = x - Mt, \quad \theta = \theta, \quad \tau = t \tag{13}$$

Application of (13) to (3) and substitution of the perturbed motions (7) into the resulting differential equations yields the following variational equations

$$\beta^4\nabla^4\zeta = (N_x^*-M^2)\,\zeta_{\xi\xi} + w_s''(\xi)\,\eta_{\theta\theta} + (\nu N_x^* - w_s(\xi))\,\zeta_{\theta\theta} + \eta_{\xi\xi} - \zeta_{\tau\tau} + 2M\zeta_{\xi\tau} \qquad (14)$$
$$\nabla^4\eta = -w_s''(\xi)\,\zeta_{\theta\theta} - \zeta_{\xi\xi}$$

where $\nabla^4(\ ) = (\ )_{\xi\xi\xi\xi} + 2(\ )_{\xi\xi\theta\theta} + (\ )_{\theta\theta\theta\theta}$ and primes denote differentiation with respect to ξ. In (14) powers of the perturbations higher than the first have been neglected.

In the following pages we construct a certain class of solutions of the variational equations (14) in the Laplace transform plane and outline a method whereby the transition from stability to instability can be obtained for zero load velocity and an upper bound on the transition for moving loads.

Series Representation

We begin by representing the functions ζ and η by the following Fourier series

$$\left.\begin{aligned} \zeta(\xi,\theta,\tau) &= \sum_{n=0}^{\infty} \zeta_n(\xi,\tau)\cos n\theta \\ \eta(\xi,\theta,\tau) &= \sum_{n=0}^{\infty} \eta_n(\xi,\tau)\cos n\theta \end{aligned}\right\} \qquad (15)$$

By use of (14) one obtains the following set of coupled partial differential equations governing ζ_n and η_n for each integer $n = 0,1,2,\ldots$

$$\beta^4\zeta_{n_{\xi\xi\xi\xi}} + (M^2-N_x^*-2n^2\beta^4)\zeta_{n_{\xi\xi}} + (\beta^4 n^4 - n^2 w_s + n^2\nu N_x^*)\zeta_n + \zeta_{n_{\tau\tau}} - 2M\,\zeta_{n_{\xi\tau}} = \eta_{n_{\xi\xi}} - n^2 w_s''\,\eta_n \qquad (16)$$
$$\eta_{n_{\xi\xi\xi\xi}} - 2n^2\,\eta_{n_{\xi\xi}} + n^4\eta_n = n^2 w_s''\zeta_n - \zeta_{n_{\xi\xi}}$$

Laplace Transform

Next a Laplace transform of (16) with respect to τ is performed. This yields in a matrix formulation:

$$\begin{bmatrix} \beta^4 & 0 \\ 0 & 1 \end{bmatrix} \bar{z}_n'''' + \begin{bmatrix} M^2-2n^2\beta^2-N_x^* & -1 \\ 1 & -2n^2 \end{bmatrix} \bar{z}_n'' + \begin{bmatrix} -2Mp & 0 \\ 0 & 0 \end{bmatrix} \bar{z}_n'$$

$$\begin{bmatrix} \beta^4 n^4 - n^2(w_s - \nu N_x^*) + p^2 & n^2 w_s'' \\ -n^2 w_s'' & n^4 \end{bmatrix} \bar{z}_n = \begin{Bmatrix} p\zeta_n(\xi,0) - 2M\zeta_{n_\xi}(\xi,0) + \zeta_{n_\tau}(\xi,0) \\ 0 \end{Bmatrix} \quad (17)$$

where z_n is the two dimensional vector:

$$z_n \equiv \begin{Bmatrix} \zeta_n \\ \eta_n \end{Bmatrix} \quad (18)$$

and $\bar{z}_n$ is defined by

$$\bar{z}_n(\xi,p) \equiv \int_0^\infty e^{-p\tau} z_n(\xi,\tau)\, d\tau \;, \quad \tau > 0, \quad \text{Re } p > C \quad (19)$$

From the regularity conditions on ζ and η at $\xi = \pm\infty$ we have in addition the requirement

$$\bar{z}_n(\xi,p) \text{ remain bounded as } \lim |\xi| \to \infty, \text{ Re } p > C \quad (20)$$

The terms on the right hand side of (17) represent the initial conditions of the problem or the form of the initial disturbance. For the present discussion we will consider as the initial disturbance a delta function in velocity located at $\xi = 0$, and having the form

$$\zeta(\xi,\theta,0) = 0$$
$$\zeta_\tau(\xi,\theta,0) = \delta(\xi)\cos n\theta \quad (21a)$$

which in turn implies

$$\zeta_n(\xi,0) = 0, \quad \zeta_{n_\tau} = \delta(\xi) \quad (21b)$$

The solution of equations (17) subject to the regularity condition (20) and the initial conditions (21) is the solution of the boundary value problems in the domains $-\infty < \xi < 0$ and $0 < \xi < \infty$ consisting of the solution to (17) with zero right hand side, the condition (20), the continuity relations

$$\left[\frac{d^{\ell}\,\bar{\underline{z}}_n}{d\xi^{\ell}}\right]_{\xi=0^-}^{\xi=0^+} = 0, \quad \ell = 0,1,2\,. \tag{22}$$

and a jump condition:

$$\left[\frac{d^3\bar{\underline{z}}_n}{d\xi^3}\right]_{\xi=0^-}^{\xi=0^+} = \begin{Bmatrix} 1/\beta^4 \\ 0 \end{Bmatrix} \tag{23}$$

Second Laplace Transform

We shall now construct the solution to the set of total differential equations (17). We begin by noting the form of the variable coefficients. Since $w_s(\xi)$ consists of a finite sum of exponentials (see (12)) one observes that the variable coefficients of (17) also consist of a sum of exponentials. In view of this it is possible to perform a Laplace transform of (17) with respect to ξ. We shall consider, for the present, only the interval $0 < \xi < \infty$ and a unilateral transform will be applied. Inversion will yield a solution for $\xi > 0$ from which the solution for $\xi < 0$ is easily deduced.

Denoting the transform of $\bar{z}_n$ by

$$\bar{\bar{\underline{z}}}_n(s) = \int_0^{\infty} \bar{\underline{z}}_n(\xi)\, e^{-s\xi}\, d\xi, \quad \xi > 0, \ \mathrm{Re}\, s > b \tag{24}$$

one obtains the transformed version of (17) in the form:

$$\underline{L}_o\, \bar{\bar{\underline{z}}}_n(s) = \sum_{j=1}^{\ell} L_j\, \bar{\bar{\underline{z}}}_n(s+a_j) + \underline{\Psi}(s) \tag{25}$$

where $\underline{L}_o, \underline{L}_1, \ldots, \underline{L}_{\ell}$ represent the following 2 x 2 matrices:

$$\underset{\sim}{L}_o = \begin{bmatrix} \beta^4 s^4 + s^2(M^2 - N_x^* - 2n^2\beta^4) - 2Mps & -s^2 \\ \quad + \beta^4 n^4 - n^2(C_o - \nu N_x^*) + p^2 & \\ s^2 & (s^2 - n^2)^2 \end{bmatrix} \tag{26a}$$

$$\underset{\sim}{L}_j = n^2 C_j \begin{bmatrix} 1 & -a_j^2 \\ a_j^2 & 0 \end{bmatrix} \tag{26b}$$

and $\underset{\sim}{\Psi}$ is a two dimensional vector containing initial data at $\xi = 0^+$. Its form is not pertinent to the discussion.

Premultiplying equation (25) by $\underset{\sim}{L}_o^{-1}$, the inverse of $\underset{\sim}{L}_o$, we obtain:

$$\bar{\bar{\underset{\sim}{z}}}_n(s) = \sum_{j=1}^{\ell} \underset{\sim}{A}_j(s)\, \bar{\bar{\underset{\sim}{z}}}_n(s+a_j) + \underset{\sim}{\phi}(s) \tag{27}$$

where

$$\underset{\sim}{\phi}(s) = \underset{\sim}{L}_o^{-1}\, \underset{\sim}{\Psi} \quad , \quad \underset{\sim}{A}_j(s) = \underset{\sim}{L}_o^{-1}\, \underset{\sim}{L}_j \tag{28}$$

Equations (27) are a system of linear functional difference equations with variable coefficients.[1] Our next task is to obtain a suitable solution to these equations. Note first that the variable coefficients of (27) possess the property:

$$\lim_{\xi \to \infty} w_s(\xi) = C_o$$

$$\lim_{\xi \to \infty} w_s''(\xi) = 0$$

Thus, it is not surprizing that the solutions of (17) are of exponential order, i.e.,

$$|\bar{z}_n(\xi)| \le a\, e^{b\xi}$$

where a and b are constants and $|\bar{z}_n|$ denotes the norm of $\underset{\sim}{\bar{z}}_n$ and is defined by:

$$|\bar{z}_n| = |\bar{\zeta}_n| + |\bar{\eta}_n|$$

[1] For a discussion of the relationship between the Laplace transform and difference equations and the solution of difference equations, see [10-12].

However, this implies

$$|\bar{\bar{z}}_n(s)| = |\int_0^\infty e^{-s\xi} \bar{z}_n \, d\xi| \le \int_0^\infty e^{-\text{Re } s\xi} |\bar{z}_n| d\xi \le \frac{a}{\text{Re } s - b} \text{ for Re } s > b.$$

Therefore the following quiescent condition on the second transform must be satisfied:

$$\lim_{\text{Re } s \to \infty} \bar{\bar{z}}_n (s) = 0 \tag{29}$$

The quiescent requirement (29) is sufficient to render the second transform unique, or more specifically, the solution of the difference equation unique. This follows from the fact that all solutions of the homogeneous counterpart of (27), (representing the difference between any two particular solutions) are unbounded as Re $s \to \infty$. (See [8] for details). Thus on the basis of (29) only the trivial solution of the homogeneous equation can be accepted. There is thus a unique particular solution of (27) to be found.

The desired particular solution can be constructed by the method of ascending continued fractions [11] and has the form:

$$\begin{aligned}\bar{\bar{z}}_n(s) = \underset{\sim}{\phi}(s) + \sum_{j_1=1}^{\ell} \underset{\sim}{A}_{j_1}(s)\underset{\sim}{\phi}(s+\alpha_{j_1}) + \sum_{j_1,j_2=1}^{\ell} \underset{\sim}{A}_{j_1}(s)\underset{\sim}{A}_{j_2}(s+\alpha_{j_1})\underset{\sim}{\phi}(s+\alpha_{j_1}+\alpha_{j_2}) + \\ + \sum_{j_1,j_2,j_3=1}^{\ell} \underset{\sim}{A}_{j_1}(s)\underset{\sim}{A}_{j_2}(s+\alpha_{j_1})\underset{\sim}{A}_{j_3}(s+\alpha_{j_1}+\alpha_{j_2})\underset{\sim}{\phi}(s+\alpha_{j_1}+\alpha_{j_2}+\alpha_{j_3}) + \dots\end{aligned} \tag{30a}$$

Equation (30a) can be written in closed form as:

$$\bar{\bar{z}}_n(s) = \underset{\sim}{\phi}(s) + \sum_{N=1}^{\infty} \prod_{k=1}^{N} \sum_{j_k=1}^{\ell} \underset{\sim}{A}_{j_k}(s + \sum_{q=0}^{k-1} \alpha_{j_q})\underset{\sim}{\phi}(s + \sum_{r=1}^{N} \alpha_{j_r}) \tag{30b}$$

where $\alpha_{j_o} \equiv 0$.

The vector function (30) above formally satisfies the difference equation (27) and the quiescent requirement (29). Further, the component series for the vector $\bar{\bar{z}}_n$ are absolutely and uniformly convergent with respect to s and represent analytic functions of s when $s \in R$, where the region R of the complex s-plane is defined by

$$|s-(\rho_i - m_1\alpha_1 - m_2\alpha_2 - \ldots - m_\ell \alpha_\ell)| \geq \epsilon > 0$$

$$m_j = 0,1,2,\ldots \; ; \; j = 1,2,\ldots,\ell \,.$$

Here ρ_i are the roots of the polynomial $\Delta \underline{L}_o(s) = 0$ where $\Delta \underline{L}_o$ denotes the determinant of $\underline{L}_o$. The singularities of $\bar{\bar{\underline{z}}}_n(s)$ are isolated poles, located at

$$s = \rho_i - m_1\alpha_1 - m_2\alpha_2 - \ldots - m_\ell\alpha_\ell \,,$$

$$m_j = 0,1,2,\ldots, \; j = 1,2,\ldots,\ell$$

If all problem parameters are fixed, including p, these poles lie a finite distance to the right of Re s = 0 and $\bar{\underline{z}}_n(s)$ is regular for Re $s > C_1$ = constant. Additional details and a proof of convergence can be found in [8].

Inversion of the s-transform

We will now invert the series (30) term by term, assuming the roots ρ_i are non-repeated (these points to be discussed later). Consider first the definitions:

$$\underline{B}_j(s) = \frac{\Delta \underline{L}_o}{\beta^4} A_j(s) = \prod_{q=1}^{8} (s-\rho_q)\underline{A}_j(s)$$

$$\underline{\Phi}(s) = \prod_{q=1}^{8} (s-\rho_q)\underline{\phi}(s) \tag{31}$$

The $N^{\underline{th}}$ term of equation (30b) is composed of the product

$$\frac{\underline{B}_{j_1}(s)}{\Pi(s-\rho_q)} \cdot \frac{\underline{B}_{j_2}(s+\alpha_{j_1})}{\Pi(s-\rho_q+\alpha_{j_1})} \cdots \frac{\underline{B}_{j_N}(s+\alpha_{j_1}+\ldots+\alpha_{j_{N-1}})}{\Pi(s-\rho_q+\alpha_{j_1}+\ldots+\alpha_{j_{N-1}})} \cdot \frac{\underline{\Phi}(s+\alpha_{j_1}+\ldots+\alpha_{j_N})}{\Pi(s-\rho_q+\alpha_{j_1}+\ldots+\alpha_{j_N})}$$

where Π denotes $\prod_{q=1}^{8}$. In the $N^{\underline{th}}$ term, each factor is inverted separately by the residue theorem. The inversion of the entire term is then obtained by repeated use of the convolution integral. After some manipulation the following series is obtained for $\bar{z}_n(\xi,p)$:

$$\bar{z}_n(\xi,p) = \sum_{i=1}^{8} \Big\{ e^{\rho_i \xi} I + \sum_{j_1=1}^{\ell} \sum_{k_1=1}^{8} R_{j_1}(k_1) \int_0^{\xi} \exp[(\rho_i - \alpha_{j_1})(\xi-\xi_1)+\rho_{k_1}\xi_1]\, d\xi_1 +$$

$$+ \sum_{j_1,j_2=1}^{\ell} \sum_{k_1,k_2=1}^{8} R_{j_1}(k_1) R_{j_2}(k_2) \int_0^{\xi}\int_0^{\xi_1} \exp[(\rho_i - \alpha_{j_1} - \alpha_{j_2})(\xi-\xi_1)+(\rho_{k_1}-\alpha_{j_1})(\xi_1-\xi_2)+\rho_{k_2}\xi_2]\, d\xi_2 d\xi_1$$

$$+ \ldots \tag{32}$$

$$+ \sum_{j_1,\ldots,j_N=1}^{\ell} \sum_{k_1,\ldots,k_N=1}^{8} R_{j_1}(k_1)\ldots R_{j_N}(k_N) \underbrace{\int_0^{\xi}\ldots\int_0^{\xi_N}}_{N\text{-fold}} \exp[(\rho_i - \sum_{q=1}^{N}\alpha_{j_q})(\xi-\xi_1) +$$

$$+(\rho_{k_1} - \sum_{q=1}^{N-1}\alpha_{j_q})(\xi_1-\xi_2) + (\rho_{k_2} - \sum_{q=1}^{N-2}\alpha_{j_q})(\xi_2-\xi_3)+\ldots+\rho_{k_N}\xi_N]\, d\xi_N \ldots d\xi_1$$

$$+ \ldots \Big\} Q_i \; ; \quad \xi > 0$$

where

$$R_j(k) \equiv \frac{B_j(\rho_k)}{\prod_{\substack{r=1 \\ r\neq k}}^{8} (\rho_k - \rho_r)} \tag{33}$$

The vectors Q_i, which contain unknown information concerning $\bar{\zeta}_n$ and $\bar{\eta}_n$ and their derivatives (up to the 3rd) at $\xi = 0^+$, can be regarded as arbitrary constants to be evaluated later from the boundary conditions. The elements of each ith vector, however, are not independent but are related through the differential equations (17). By direct substitution[2] of (32) into (17) (with zero right-hand side) one finds the differential equations are formally satisfied for each $i = 1, 2, \ldots, 8$ if the elements of Q_i are related according to:

[2] This cumbersome task can be accomplished by writing $C_j = \epsilon \bar{C}_j$ and observing that (32) is a power series in ϵ. Equation (34) then guarantees that (17) is satisfied for each order of ϵ.

$$\underset{\sim}{Q}_i \equiv \begin{Bmatrix} Q_i^{(1)} \\ Q_i^{(2)} \end{Bmatrix} = \begin{Bmatrix} S_i \\ 1 \end{Bmatrix} G_i \tag{34}$$

where $S_i = -\dfrac{(\rho_i^2 - n^2)^2}{\rho_i^2}$ and G_i are arbitrary scalar constants.

Each value of $i = 1, 2, \ldots, 8$ in equation (32) represents a linearly independent solution to (17) when the ρ_i are non-repeated. Therefore (32) represents the general solution to the homogeneous part of (17) for $\xi > 0$ when the ρ_i are non-repeated.

The function $\overline{\underset{\sim}{z}}_n(\xi, p)$ for $\xi < 0$ is easily obtained by inspection from equation (32). One need only replace ℓ, a_j, C_j and ρ_i in (32) by ℓ^*, $-a_j^*$, C_j^* and ρ_i^* respectively where

1) ℓ^*, a_j^* and C_j^* are obtained from equation (12)

2) ρ_i^* are the roots of $\Delta \underset{\sim}{L}_o = 0$ with C_o replaced by C_o^*.

We will indicate these changes by "starring" all quantities where changes occur. Then, for $\xi < 0$, $\overline{\underset{\sim}{z}}_n$ has the form (32) with the changes:

$$\rho_i \to \rho_i^*, \ \underset{\sim}{R}_j(k) \to \underset{\sim}{R}_j^*(k), \ a_j \to -a_j^*, \ G_i \to G_i^*$$

The unknown constants G_i and G_i^* are determined from the regularity condition (20), the continuity requirement (22), and the jump condition (23). Consider first the continuity and jump relations. Using the identity[3]

$$\sum_{k=1}^{N} \rho_k^m \Big/ \prod_{\substack{q=1 \\ q \neq k}}^{N} (\rho_k - \rho_q) = 0 \quad \text{when } N > m+1 \text{ and } \rho_q \text{ are non-repeated,}$$

one can show the series (32) and its equivalent for $\xi < 0$ possess the following property at $\xi = 0^+$ and $\xi = 0^-$:

[3] This is easily verified by expanding the leading term in partial fractions.

Equations (40) represent 8 equations in the eight unknowns, G_i(i=1 to 8). However, it can be shown that only 4 (one row) equations are linearly independent. With the vector components $b_1(r,i)$ and $b_2(r,i)$ as defined in (37b), these 4 equations are

$$\sum_{i=1}^{8} b_2(r,i)G_i = 0, \quad r = 1 \text{ to } 4 \tag{41}$$

In a similar manner, (20) can be satisfied for $\xi \to -\infty$ only if

$$\sum_{i=1}^{8} b_2^*(r,i)\, G_i^* = 0, \quad r = 5 \text{ to } 8 \tag{42}$$

where b_2^* is obtained from b_2 by the parametric changes described previously.

The constants G_i and G_i^* can now be determined from equations (36), (41) and (42). In matrix form we have

$$\underset{\sim}{A}\,\underset{\sim}{g} = \underset{\sim}{e} \tag{43}$$

where $\underset{\sim}{A}$ is a 16 x 16 matrix, the elements of which are given by:

$$\underset{\sim}{A} \equiv \left[\begin{array}{c|c}
\begin{array}{l} A_{r,i} = b_2(r,i) \\ r = 1 \text{ to } 4,\ i = 1 \text{ to } 8 \end{array} & \begin{array}{l} A_{r,i+8} = 0 \\ r = 1 \text{ to } 4,\ i = 1 \text{ to } 8 \end{array} \\ \hline
\begin{array}{l} A_{m+4,i} = \rho_i^{m-1} S_i \\ m = 1 \text{ to } 4,\ i = 1 \text{ to } 8 \end{array} & \begin{array}{l} A_{m+4,i+8} = -\rho_i^{*\,m-1} S_i^* \\ m = 1 \text{ to } 4,\ i = 1 \text{ to } 8 \end{array} \\ \hline
\begin{array}{l} A_{m+8,i} = \rho_i^{m-1} \\ m = 1 \text{ to } 4,\ i = 1 \text{ to } 8 \end{array} & \begin{array}{l} A_{m+8,i+8} = -\rho_i^{*\,m-1} \\ m = 1 \text{ to } 4,\ i = 1 \text{ to } 8 \end{array} \\ \hline
\begin{array}{l} A_{r+8,i} = 0 \\ r = 5 \text{ to } 8,\ i = 1 \text{ to } 8 \end{array} & \begin{array}{l} A_{r+8,i+8} = b_2^*(r,i) \\ r = 5 \text{ to } 8,\ i = 1 \text{ to } 8 \end{array}
\end{array}\right] \tag{44}$$

where, as usual, the first subscript refers to the row and the second the column.

The quantities $\underset{\sim}{g}$ and $\underset{\sim}{e}$ represent the following 16 dimensional vectors

$$\underset{\sim}{g} = \begin{Bmatrix} G_1 \\ \vdots \\ G_8 \\ G_1^* \\ \vdots \\ G_8^* \end{Bmatrix}, \quad \underset{\sim}{e} = \begin{Bmatrix} e_1=0 \\ \vdots \\ e_7=0 \\ e_8=1/\beta^4 \\ e_9=0 \\ \vdots \\ e_{16}=0 \end{Bmatrix} \tag{45}$$

Premultiplying (43) by $\underset{\sim}{A}^{-1}$, we obtain $\underset{\sim}{g}$ as

$$\underset{\sim}{g} = \underset{\sim}{A}^{-1}\,\underset{\sim}{e} \tag{46}$$

The solution is now complete. Next we discuss a few properties of the series for $\bar{\underset{\sim}{z}}_n$.

Properties of $\bar{\underset{\sim}{z}}_n\,(\xi, p)$ and Remarks Related to the s-Inversion

The assumption was made, upon inverting $\bar{\bar{\underset{\sim}{z}}}_n(s)$, that the roots ρ_i and ρ_i^* were not repeated. For all points in the p-plane such that $\frac{\partial \Delta \underset{\sim}{L}_o(s,p)^i}{\partial s} \neq 0$ the roots of $\Delta \underset{\sim}{L}_o(s,p) = 0$ are non-repeated. It can be shown that $\frac{\partial \Delta \underset{\sim}{L}_o(s,p)}{\partial s} = \Delta \underset{\sim}{L}_o(s,p) = 0$ occurs only at branch points of the roots as a function of p in the p-plane.

Now, let us define the region R_1 of the complex p-plane by:

(1) $|p-p_b| \geq \epsilon_1 > 0$, where p_b are branch points of the roots $\rho_i(p)$ and $\rho_i^*(p)$ in the p-plane.

(2) $|p-p_\Delta| \geq \epsilon_2 > 0$, where p_Δ are zeros of the determinant of $\underset{\sim}{A}$, $\Delta \underset{\sim}{A}$.

If $p \in R_1$ and $-\infty < -B < x < B < \infty$, B = arbitrary constant, the series (32) and its counterpart for $\xi < 0$ are absolutely and uniformly convergent with respect to both ξ and p. Since the series obtained by an n^{th} term by term ξ-derivative possesses the same property of uniform convergence with respect to ξ, our differentiation of the series was justified.

If appropriate branch cuts within the region R_1 are made to render the roots analytic functions of p, then each term of the series will be an analytic function of p. The uniform convergence with respect to p then indicates $\bar{z}_n(\xi, p)$ is an analytic function of p when $p \in R_1$. The points p_Δ for which the determinant of A vanishes represent poles of $\bar{z}_n$. The points p_b are possible branch points of $\bar{z}_n$. Additional details and proofs of convergence, etc., can be found in [8].

Instability Conditions

The vector $\bar{z}_n$ need not be inverted to obtain stability information. Indeed the boundedness of z_n is governed entirely by the location and type of singularities of $\bar{z}_n$ in the p-plane. From the theory of the Laplace transform [8,10,14] z_n will be unbounded as $\tau \to \infty$ if $\bar{z}_n$ possesses singularities of any type in Re $p > 0$. To demonstrate instability, therefore, it is only necessary to show a singularity exists in Re $p > 0$.

The possible singularities of $\bar{z}_n$ in Re $p > 0$ consist of branch points and poles. It will suffice to consider just the poles, which occur only when ΔA vanishes. To gain insight into the problem let us proceed from the static case. If $M = 0$ the shell and loading represent a conservative system and therefore the energy method and the present dynamic method are equivalent [15]. If one calculates the potential energy of the system according to the present shell theory, assuming ζ and η are virtual displacements from the loaded state, then equations (14) with $\partial/\partial\tau = 0$ are the result of requiring that the second variation of the potential energy vanish (a necessary condition for the transition from stability to instability). Equations (15) and (32) with $p = 0$ represent a solution to these equations which is completed by requiring that ζ and η be regular as $|\xi| \to \infty$ and continuity of ζ and η and their derivatives with respect to x up to and including the third at $x = 0$. Application of these conditions leads to the eigenvalue problem:

$$\underset{\sim}{A}\,\underset{\sim}{g} = 0 \tag{47}$$

whereby a solution exists iff $\Delta \underset{\sim}{A} = 0$. This, however, implies the transition from stability to instability according to the dynamic method occurs at $p = 0$ in the p-plane. This zero can be expected to move into Re $p > 0$ for above critical values.

For $M \neq 0$ the situation is of course more complex since the system is nonconservative and the variational equations are non-selfadjoint. We note first that if $p = 0$, the parameter M occurs everywhere in the combination $M^2 - N_x^*$. It therefore has the same effect as an axial compression of the cylinder. Since $\Delta \underset{\sim}{A}$ ($p = 0$, $M = 0$) possesses the same properties as $\Delta \underset{\sim}{A}$ ($p = 0$, $M \neq 0$), save an effective change in N_x^*, one deduces by analogy to the static problem that a zero of $\Delta \underset{\sim}{A}$ will appear at $p = 0$ for some set of load parameters. This zero can be expected to move into Re $p > 0$ for increased load magnitudes indicating an unstable shell.

The above discussion indicates a method whereby one can obtain 1) the transition from stability to instability for $M = 0$ and 2) an upper bound on the transition for $M > 0$. This is accomplished by 1) selecting all shell and load parameters and a value of n, 2) selecting a characteristic load parameter, say λ, which is a function only of C_j and C_j^* for $j > 0$[4], 3) truncating the series (37b), numerically plotting $\Delta \underset{\sim}{A}$ for $p = 0$ versus λ, selecting that value of λ for which a zero first occurs and minimizing with respect to n. That this load is an upper bound for $M > 0$ can be easily demonstrated by numerically showing that the zero moves into Re $p > 0$ for larger λ values.

For purposes of the above calculation it can be shown that it is sufficient to group the roots ρ_i and ρ_i^*[5] according to their real parts, denoting

[4] Since the roots ρ_i and ρ_i^* do not depend on C_j, C_j^* for $j>0$ they need not be recalculated for each λ if λ is selected in this manner.

[5] Explicit formulas for the roots ρ_i and ρ_i^* for the case $M = 0$ can be found in [16].

those with positive real parts as $i = 1$ to 4 and those with negative real parts as $i = 5$ to 8. The order of any roots with zero real parts is not important.

4. NUMERICAL EXAMPLE

Let us consider as a numerical example the case of a ring load moving with constant velocity. Here the load is defined by

$$P = P_c\ \delta(X - V_L\ T),\ N_X^o = 0 \text{ or } q = \frac{P_c}{Eh}\ \delta(\xi),\ N_x^* = 0 \tag{48}$$

and is illustrated in Fig. 2a. The axisymmetric response for this load has the form (11) if the constants g_j are multiplied by P_c/Eh, i.e.,

$$C_j = C_j^* = \frac{P_c}{Eh}\ g_j\ ,\ j = 1, 2;\ C_j = C_j^* = 0,\ j = 0 \text{ and } j > 2$$
$$\alpha_j = \alpha_j^* \text{ given by (11) for } j = 1, 2. \tag{49}$$

If we select as the parameter λ the quantity P_c/Eh, the matrix $\underset{\sim}{A}$ (44) can be written

$$\underset{\sim}{A} = \underset{\sim}{K}_o + \lambda \underset{\sim}{K}_1 + \lambda^2 \underset{\sim}{K}_2 + \lambda^3\ \underset{\sim}{K}_3 + \ldots \tag{50}$$

where the matrices $\underset{\sim}{K}_j$ do not depend on λ. They are obtained from (44) by grouping terms of like λ powers.

Truncating the series (50) and setting $p = 0$, $\Delta \underset{\sim}{A}$ was numerically evaluated by use of a digital computer. A Reguli Falsi method was employed to determine the minimum λ for which a zero of $\Delta \underset{\sim}{A}$ occurred at $p = 0$ ($\Delta \underset{\sim}{A}$ was found to be real valued for $p = 0$). In the neighborhood of this value of λ (50) was found to converge quite rapidly for a wide range of shell parameters ($a/h = 100$ To 1000). For all cases where $M < .95$ the correction due to the retention of more than three terms of (50) was found to be negligible.

The behavior of the minimum eigenvalue as a function of the number of circumferential waves, n and velocity, V_L, is illustrated in Fig. 3a for the case $a/h = 100$ and $\nu = 0.3$. For each value of n a curve similar to that of Fig. 3b can be constructed. Fig. 3b[6] is the minimum envelope of all such curves and represents an upper bound on the transition from stability to instability. The shell is unstable for all loads above the solid line. This was verified numerically by selecting a small positive real p value and showing that a zero of $\Delta \underset{\sim}{A}$ occurred in $Re\ p > 0$ for loads above this line. For all a/h values in the range of 100-1000 the form of the curve in Fig. 3b was found to remain essentially invariant.

For M or $V_L = 0$, the results obtained were the buckling load for an infinite shell subject to a uniform radial line load. Below a comparison is made with existing analyses on the subject for $a/h = 100$. Fig. 4 indicates the behavior of this buckling load as a function of a/h.

Present theory:	$P_c/Eh = 3.91 \times 10^{-4}$
Brush [18], long finite shell:	" $= 4.20 \times 10^{-4}$
Hahne [19], " " " :	" $= 4.61 \times 10^{-4}$

The agreement is quite good. No comparison can be made for the dynamic case since analyses on the subject apparently do not exist.

An interesting result can be observed from Fig. 3b. Prisekin [2] suggested that since the amplitude of the axisymmetric response varies inversely with $[1-(V_L-V_{co})^2]^{1/2}$ the transition from stability to instability should be proportional to this quantity. Our results, however, indicate this transition should lie below or at a curve which is very closely approximated by $1-(V_L/V_{co})^2$.

Concluding Remarks

A method for determining (1) the transition from stability to instability

---

[6] Axial compression can easily be incorporated into these results by replacing V_L/V_{co} by $\left[(V_L/V_{co})^2 - (N_x^0/h)\left(\frac{a}{h}\right)\sqrt{3(1-\nu^2)}\right]^{1/2}$ where in both cases V_{co} is computed from (5) with $N_x^0 = 0$.

for a class of axisymmetric static pressure distributions and (2) an upper bound on this transition when the distribution moves with constant velocity has been discussed. Utilizing the method the case of a moving ring load was considered. This example indicated a marked decrease in stability as the load velocity approached the minimum velocity for which axisymmetric sinusoidal wave trains can be propagated in the shell.

Naturally one would like to interpret the results of the analysis in terms of shell buckling. As with any infinitesimal stability analysis, however, care must be exercised in this respect. For example, states that are found stable by an infinitesimal analysis may actually be unstable if finite disturbances are considered. In the present case one cannot differentiate between instabilities that lead to buckling or those that merely lead to finite nonsymmetric oscillations.

References

1. J. P. Jones and P. G. Bhuta, "Response of Cylindrical Shells to Moving Loads," Journal of Applied Mechanics, vol. 31, March, 1964, pp. 105-111.

2. V. L. Prisekin, "The Stability of a Cylindrical Shell Subjected to a Moving Load" (in Russian), Izvestiza Akademii Nauk SSSR, Otdelenie Tekhnicheskikh Nauk (Mekhanika i Mashinostroenic), No. 5, 1961, pp. 133-134.

3. P. Mann-Nachbar, "On the Role of Bending in the Dynamic Response of Thin Shells to Moving Discontinuous Loads," Journal of the Aerospace Sciences, vol. 29, 1962, pp. 648-657.

4. Sing-Chih Tang, "Dynamic Response of a Thin-Walled Cylindrical Tube under Internal Moving Pressure," Ph.D. dissertation, The University of Michigan, January 1963.

5. P. G. Bhuta, "Transient Response of a Thin Elastic Cylindrical Shell to a Moving Shock Wave," Journal of the Acoustical Society of America, vol. 35, January 1963, pp. 25-30.

6. L. H. Donnell, "A New Theory for the Buckling of Thin Cylinders under Axial Compression and Bending," Trans. ASME, vol. 56, November 1934, pp. 795-806.

7. Y. C. Fung and E. E. Sechler, "Instability of Thin Elastic Shells," Structural Mechanics, Proc. of First Symposium on Naval Structural Mechanics, Pergamon Press, 1960.

8. G. A. Hegemier, "The Stability of Thin Cylindrical Shells Subjected to a Class of Axisymmetric Moving Loads," GALCIT SM, California Institute of Technology, 1965 (To be Published).

9. S. H. Crandall, "The Timoshenko Beam on an Elastic Foundation," Proceedings of the Third Midwestern Conference on Solid Mechanics, Ann Arbor, Mich., 1957.

10. B. Van Der Pol and H. Bremmer, Operational Calculus Based on the Two-Sided Laplace Integral, Cambridge University Press, 1950.

11. L. M. Milne-Thomson, The Calculus of Finite Differences, Macmillan and Co., 1933.

12. K. G. Vallev, "On Linear Differential Equations with Exponential Coefficients and Stationary Delays of the Argument," PMM, vol. 26, No. 3, 1962.

13. P. M. Naghdi and R. M. Cooper, "Propagation of Elastic Waves in Cylindrical Shells, Including the Effects of Transverse Shear and Rotating Inertia," Journal of the Acoustical Society of America, vol. 28, 1956, pp. 56-63.

14. Y. C. Fung, The Theory of Aeroelasticity, GALCIT Aeronautical Series, John Wiley and Sons, Inc., 1955.

15. Hans Ziegler, "On the Concept of Elastic Stability," Advances in Applied Mechanics, vol. 4, 1956, pp. 351-403.

16. W. Nachbar, "Characteristic Roots of Donnell's Equations with Uniform Axial Prestress." Journal of Applied Mechanics, vol. 29, No. 2, p. 434, June 1962.

17. D. O. Brush and F. A. Field, "Buckling of a Cylindrical Shell under a Circumferential Band Load," Journal of the Aero/Space Sci., vol. 26, No. 12, pp. 825-830, December 1959.

18. B. O. Almroth and D. O. Brush, "Buckling of a Finite-Length Cylindrical Shell Under a Circumferential Band of Pressure," Journal of the Aero/Space Sciences, vol. 28, No. 3, March 1961.

19. H. V. Hahne, "A Stability Problem of a Cylindrical Shell Subject to Direct and Bending Stresses, unpublished Ph.D. Thesis, Stanford University, July 1954.

20. A. Erdelyi, "Note on an Inversion Formula for the Laplace Transform," Journal of the London Mathematical Society, vol. 18, 1943, pp. 72-77.

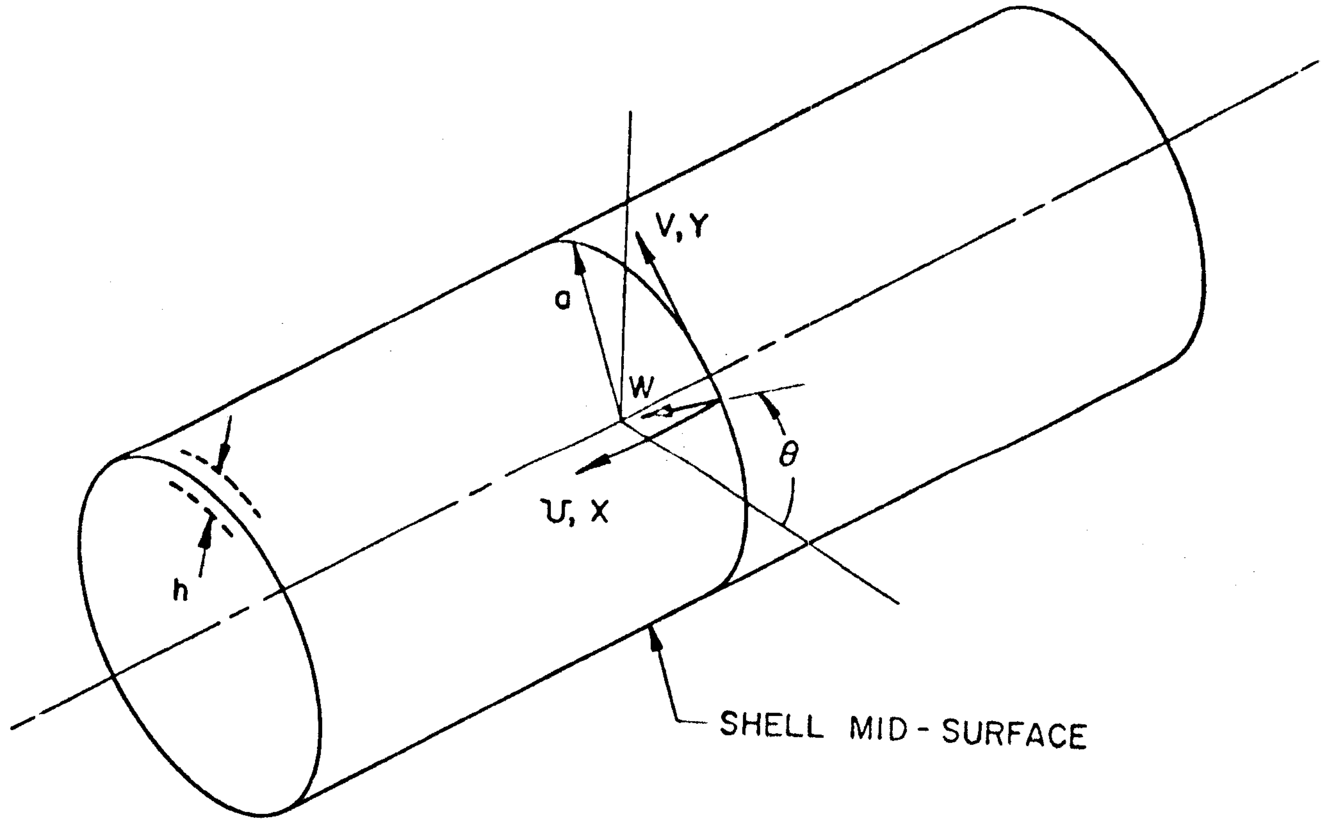

Fig. 1 Coordinate System

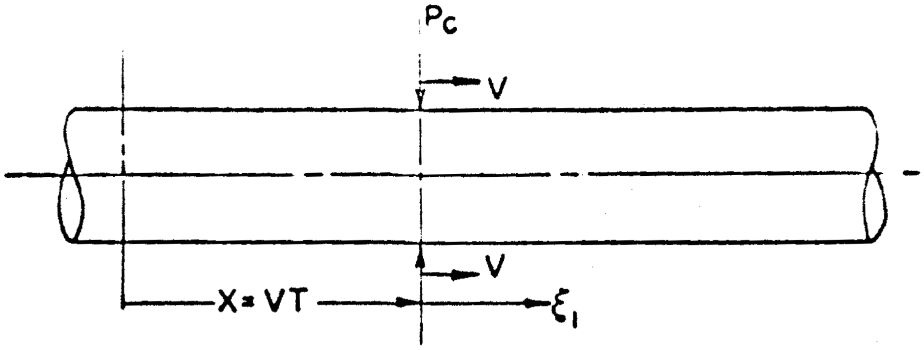

Fig. 2a: Moving Concentrated Load: $P(\xi_1) = P_c\ \delta(\xi_1)$

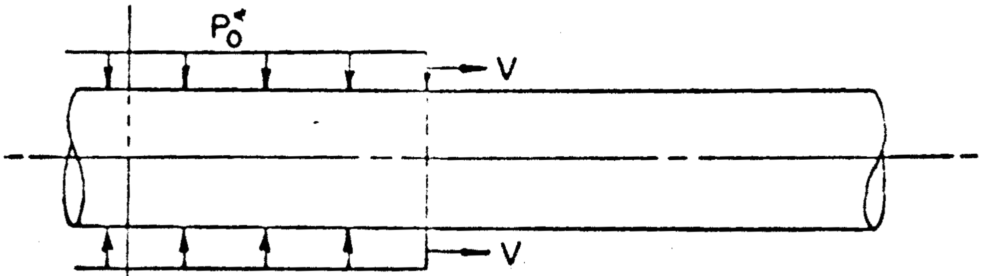

Fig. 2b: Moving Step Load: $P(\xi_1) = P^*_o\ H(-\xi_1)$

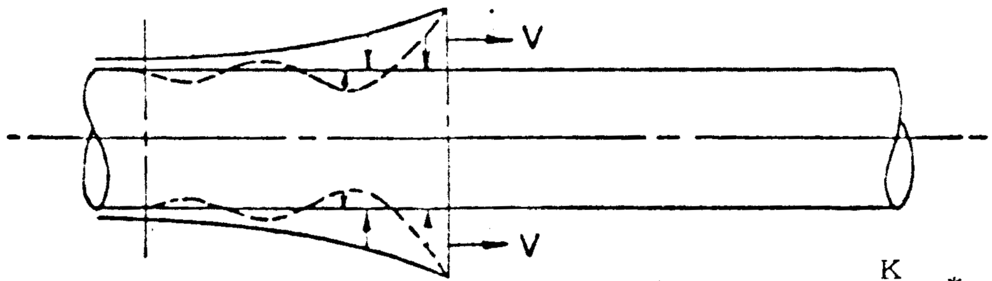

Fig. 2c: Moving Decayed Step Loads: $P(\xi_1) = H(-\xi_1) \sum_{k=1}^{K} P^*_k \exp[\Omega^* \xi_1]$

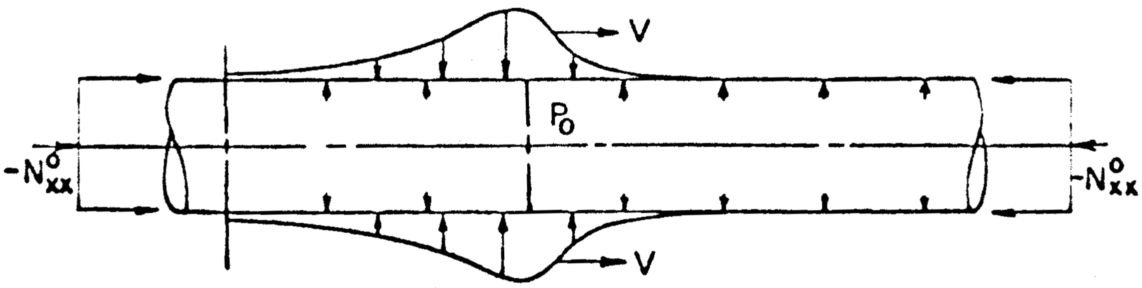

Fig. 2d: General Moving Pulse with Internal Pressure and Axial Compression: $P(\xi_1) = -P_o + H(\xi_1) \sum_{n=1}^{N} P_n \exp[-\Omega_n \xi_1]$

$$+H(-\xi_1) \sum_{k=1}^{K} P^*_k \exp[\Omega^*_k \xi_1]$$

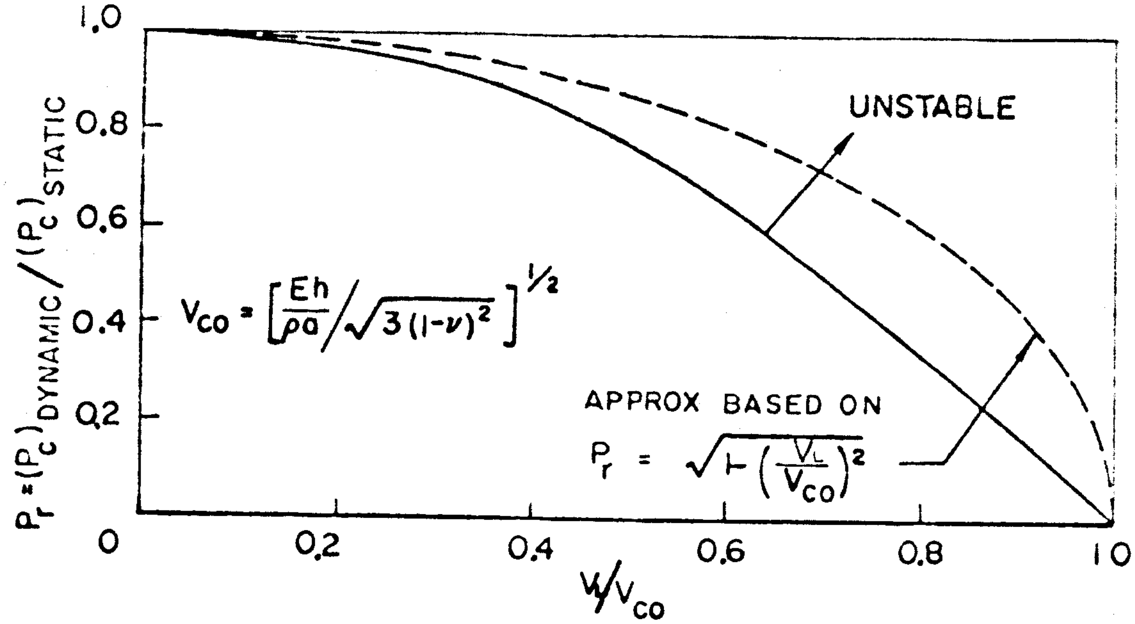

Fig. 3b: Critical Load versus Velocity, $N_x^o = 0$.

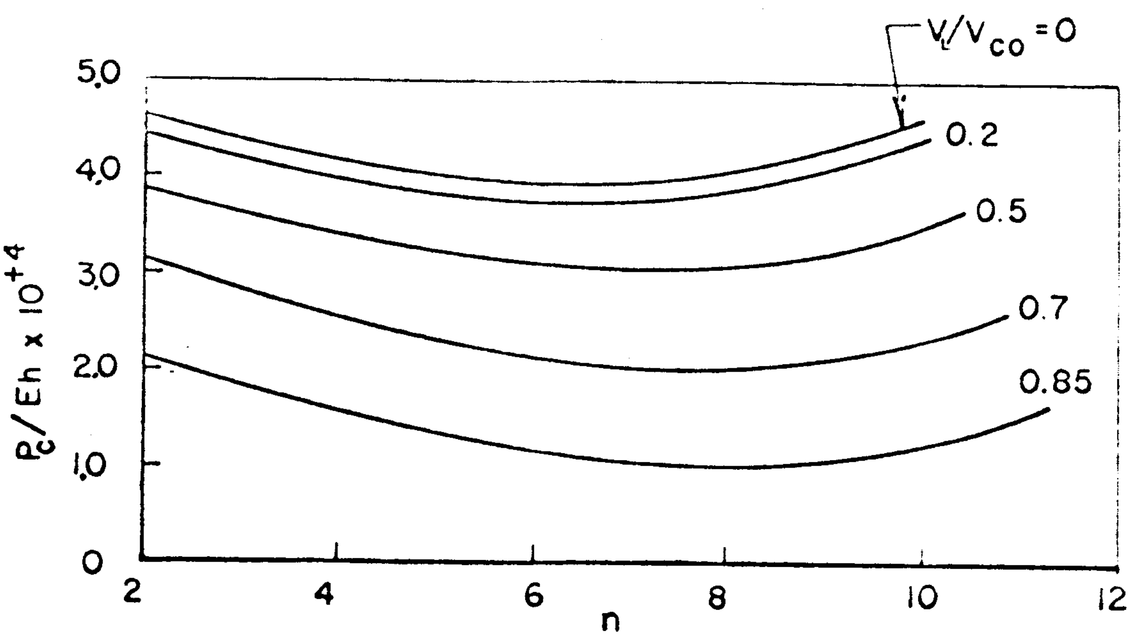

Fig. 3a: Load Parameter versus no. of circumferential half waves n, for a/h = 100, $\nu = 0.3$, $N_x^o = 0$.

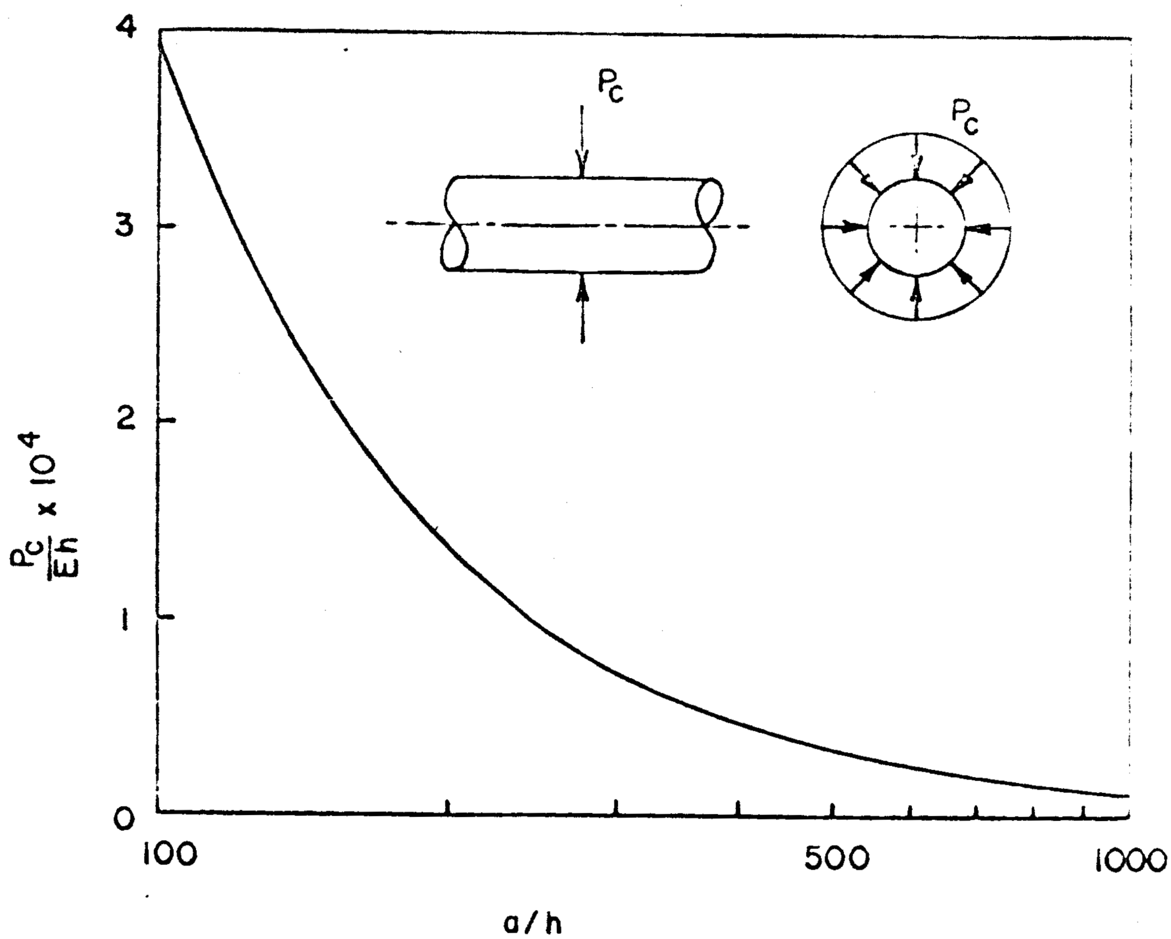

Fig. 4 Dependence of Static Buckling Load on a/h; $\nu = 0.3$, $N_x^o = 0$.

$$\left.\begin{aligned} \frac{d^m}{d\xi^m}\,\bar{\underset{\sim}{z}}_n(0^+) &= \rho_i^m \begin{Bmatrix} S_i \\ 1 \end{Bmatrix} G_i \qquad \text{for } m < 7 \\ \frac{d^m}{d\xi^m}\,\bar{\underset{\sim}{z}}_n(0^-) &= \rho_i^{*m} \begin{Bmatrix} S_i^* \\ 1 \end{Bmatrix} G_i^* \qquad \text{for } m < 7 \end{aligned}\right\} \tag{35}$$

Application of (22) and (23) with use of (35) yield

$$\begin{aligned} &\sum_{i=1}^{8} \rho_i^m G_i - \rho_i^{*m} G_i^* = 0, \ m = 0,1,2,3 \\ &\sum_{i=1}^{8} \rho_i^m S_i G_i - \rho_i^{*m} S_i^* G_i^* = 0, \ m = 0,1,2 \\ &\sum_{i=1}^{8} \rho_i^3 S_i G_i - \rho_i^{*3} S_i G_i^* = 1/\beta^4 \end{aligned} \tag{36}$$

To apply the condition (20) it is necessary to obtain from (32) the limiting form of $\bar{\underset{\sim}{z}}$ for large ξ. One finds

$$\underset{(\xi\to\infty)}{\bar{\underset{\sim}{z}}_n \sim} \sum_{r=1}^{8} e^{\rho_r \xi} \sum_{i=1}^{8} \begin{Bmatrix} b_1(r,i) \\ b_2(r,i) \end{Bmatrix} G_i \tag{37a}$$

where

$$\begin{aligned} \begin{Bmatrix} b_1(r,i) \\ b_2(r,i) \end{Bmatrix} \equiv \Bigg[& I\delta_{ri} + \sum_{j_1=1}^{\ell} \underset{\sim}{R}_{j_1}(r) C_{r,i;j_1} + \\ & + \sum_{j_1,j_2=1}^{\ell} \sum_{k_1=1}^{8} \underset{\sim}{R}_{j_1}(r) \underset{\sim}{R}_{j_2}(k_1) \cdot C_{r,k_1;j_1} \cdot C_{r,i;j_1 j_2} + \dots \\ & + \sum_{j_1,\dots,j_N=1}^{\ell} \sum_{k_1,\dots,k_{N-1}=1}^{8} \underset{\sim}{R}_{j_1}(r) \underset{\sim}{R}_{j_2}(k_1) \underset{\sim}{R}_{j_3}(k_2) \dots \underset{\sim}{R}_{j_N}(k_{N-1}) \cdot \\ & \cdot C_{r,k_1;j_1} \cdot C_{r,k_2;j_1 j_2} \cdot \\ & \cdot C_{r,k_3;j_1,j_2,j_3} \dots C_{r,k_{N-1};j_1,j_2 \dots j_{N-1}} \cdot C_{r,i;j_1,j_2,\dots,j_N} + \dots \Bigg] \begin{Bmatrix} S_i \\ 1 \end{Bmatrix} \end{aligned} \tag{37b}$$

In the above, δ_{ri} is the Kronecker delta and the quantities $C_{r,i;j_1,\ldots,j_N}$ are defined by:

$$C_{r,i;j_1,j_2,\ldots,j_N} = 1/\rho_r-\rho_i+\alpha_{j_1}+\alpha_{j_2}+\ldots+\alpha_{j_N} \quad \text{if } \rho_r-\rho_i+\alpha_{j_1}+\ldots+\alpha_{j_N} \neq 0$$
$$= 0 \quad \text{"} \quad \text{"} \quad = 0$$

Equation (37a) indicates that the growth or decay of $\bar{z}_n$ as $\xi \to \infty$ depends entirely on the sign of Re ρ_r, $r = 1,2,\ldots,8$. To ascertain if (20) can be satisfied it is necessary to consider C large in (19) and determine the large $|p|$ behavior of the roots $\rho_i(p)$. This is accomplished by noting that the equation $\Delta \underline{L}_o = 0$ is satisfied by the asymptotic series:

$$s = \sqrt{p}\,[s_o + \frac{s_1}{\sqrt{p}} + \frac{s_2}{(\sqrt{p})^2} + \frac{s_3}{(\sqrt{p})^3} + \ldots] \tag{38}$$

Equations governing the coefficients, s_n, are obtained by substituting (38) into $\Delta\underline{L}_o = 0$ and equating terms of the same p-order. Solution of those equations for the leading terms of (38) yield the following asymptotic values:

$$\rho_1,\rho_1^* \sim \frac{\sqrt{p}\,(1+i)}{\sqrt{2}\,\beta}\;;\; \rho_2,\rho_2^* \sim \frac{\sqrt{p}(1-i)}{\sqrt{2}\,\beta}\;;\; \rho_3,\rho_3^* \sim n$$
$$\rho_4,\rho_4^* \sim n\;;\; \rho_5,\rho_5^* \sim -\frac{\sqrt{p}\,(1+i)}{\sqrt{2}\,\beta}\;;\; \rho_6,\rho_6^* \sim -\frac{\sqrt{p}\,(1-i)}{\sqrt{2}\,\beta}\,, \tag{39}$$
$$\rho_7,\rho_7^* \sim -n\;;\; \rho_8,\rho_8^* \sim -n$$

where the designations as to root number were purely arbitrary.

Identifying the roots by their asymptotic values above and selecting the positive branch of $\sqrt{p}$ in the p-plane, it is evident that (20) can be satisfied only if

$$\sum_{i=1}^{8} \begin{Bmatrix} b_1(r,i) \\ b_2(r,i) \end{Bmatrix} G_i = 0 \qquad \text{for } r = 1 \text{ to } 4. \tag{40}$$