THE STABILITY OF SWIRLING FLOW
OF A VISCOUS CONDUCTING FLUID
IN THE PRESENCE OF A CIRCULAR MAGNETIC FIELD*

by

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CONTENTS

ABSTRACT

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A sufficient condition for the stability of a swirling flow in a circular magnetic field is established. A stronger sufficient condition for stability is also given on physical grounds and by an approximate mathematical proof. Detailed results for small spacing between the cylinders are given. It is shown that the stronger sufficient condition for stability is exact for small spacings. A new branch of solution which corresponds to negative critical Taylor number is calculated. An approximate solution for a positive α_0 is also given. A rather striking phenomenon is that there is a case which the unstable circulatory flow field is counterbalanced by the unstable circular magnetic field such that the combined field makes a stable flow. The dual roles of viscosity and magnetic diffusivity and their physical mechanism are also discussed.

Author

1. INTRODUCTION

The stability of swirling flow in a circular magnetic field was first investigated by Michael[1] who established a necessary and sufficient condition for the stability of fluids with zero viscosity and infinite electrical conductivity. Taylor[2] showed that in the case of flow of homogeneous non-conducting fluid between rotating cylinders, viscosity is always stabilizing. Thus,one is inclined to conjecture that Michael's criterion will also hold true in a dissipative medium of finite viscosity and magnetic diffusivity. It is found, however, that this is not so for the present case. The viscosity ν and the magnetic diffusivity η play the diffusive role as well as the dissipative one. This diffusive role can sometime very well be destabilizing. Therefore, the magnetic Prandtl number ν/η is important for the instability of real fluids. Accordingly, a new criterion is needed for the stability of a dissipative medium. This paper provides such a criterion.

Lai[3] studied a more complicated problem with the present one as a special case when the temperature variation was absent. He carried out some detailed calculations for the case of small spacing and found that both the viscosity and magnetic diffusivity have dual roles. However, he neglected one branch of solution which is quite pertinent to the present problem. Therefore, the results and conclusions derived from his calculations should be clarified. This paper contains such a clarification.

The instability situation here is almost exactly the same as in the case of gravitational instability discovered by Stommel, et al[4], and the case of thermal instability by Yih[5].

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1. D. H. Michael, Mathenatika 1, 45-50(1954).
2. G.I. Taylor, Phil. Trans. Roy, Soc. London A 223, 289-343(1923).
3. W. Lai, Phys. Fluids 5, 560-566(1962).
4. H. Stommel, A. B. Arons, and D. Blanchard, Deep-Sea Research 3, 152-153(1956).
5. C.-S. Yih, Phys. Fluids 4, 806-811(1961).

In the present case, the balance of the stabilizing and destabilizing effects of the centrifugal force and the centripetal magnetic force is through the diffusive effects of the momentum diffusivity ν and the magnetic diffusivity η. Therefore, it should be emphasized here that caution must be taken in applying the stability criteria derived from a non-dissipative medium to that of a real fluid, especially when two or more diffusive coefficients are involved in the problem.

The present problem was also studied by Edmonds[6], with Fermi boundary conditions of perfectly conducting cylinders. He has made a few calculations for some special cases. In the present investigation, the principle of the exchange of stabilities is assumed for the small spacing calculations. It is hoped that a subsequent paper will present some calculations for the case of overstability and of large spacing between cylinders.

GENERAL ANALYSIS

2. FORMULATION OF THE PROBLEM

For an incompressible, viscous, electrically conducting fluid, the governing equations are:

$$\frac{\partial \underset{\sim}{U}}{\partial t} + (\underset{\sim}{U}\cdot\nabla)\underset{\sim}{U} = -\nabla Q + (\underset{\sim}{H}\cdot\nabla)\underset{\sim}{H} - \nu\nabla\times(\nabla\times\underset{\sim}{U}), \tag{2.1}$$

$$\nabla\cdot\underset{\sim}{U} = 0, \tag{2.2}$$

$$\nabla\cdot\underset{\sim}{H} = 0, \tag{2.3}$$

$$\frac{\partial \underset{\sim}{H}}{\partial t} = \nabla\times(\underset{\sim}{U}\times\underset{\sim}{H} - \eta\nabla\times\underset{\sim}{H}), \tag{2.4}$$

6. F.N. Edmonds, Jr., Phys. Fluids <u>1</u>, 30-41(1958).

where $\underset{\sim}{U}$ is the fluid velocity, ν the kinematic viscosity, $\eta = 1/(4\pi \mu_1 \sigma_1)$ the magnetic diffusivity; μ_1 the magnetic permeability, and σ_1 the electrical conductivity. We have written $\underset{\sim}{H} = \underset{\sim}{H}_1(\mu_1/4\pi\rho)^{1/2}$ and $Q = |\underset{\sim}{H}|^2 + p/\rho + b_1$, where $\underset{\sim}{H}_1$ is the magnetic field intensity, p is fluid pressure, ρ is uniform density and b_1 is the potential per unit mass of the conservative body force. In the derivation of (1) and (4), it is assumed that the net charge density is zero, and that ν, σ_1 and μ_1 are constant.

We adopt the cylindrical coordinate system (r, θ, z). If axisymmetry is assumed, with (U_r, U_θ, U_z) and (H_r, H_θ, H_z) denoting the components of the velocity and of the normalized magnetic intensity (which is actually the Alfvén speed) in the radial r, the transverse θ and the axial z, directions, respectively, Eqs. (1) to (4) become

$$\frac{DU_r}{Dt} - \frac{U_\theta^2}{r} = -\frac{\partial Q}{\partial r} + \left(\frac{\mathcal{D}H_r}{\mathcal{D}t} - \frac{H_\theta^2}{r}\right) + \nu\left(\nabla^2 U_r - \frac{U_r}{r^2}\right), \tag{2.5}$$

$$\frac{DU_\theta}{Dt} + \frac{U_r U_\theta}{r} = \left(\frac{\mathcal{D}H_\theta}{\mathcal{D}t} + \frac{H_r H_\theta}{r}\right) + \nu\left(\nabla^2 U_\theta - \frac{U_\theta}{r^2}\right), \tag{2.6}$$

$$\frac{DU_z}{Dt} = -\frac{\partial Q}{\partial z} + \frac{\mathcal{D}H_z}{\mathcal{D}t} + \nu\nabla^2 U_z, \tag{2.7}$$

$$\frac{\partial(rU_r)}{\partial r} + \frac{\partial(rU_z)}{\partial z} = 0, \tag{2.8}$$

$$\frac{\partial(rH_r)}{\partial r} + \frac{\partial(rH_z)}{\partial z} = 0, \tag{2.9}$$

$$\frac{DH_r}{Dt} = \frac{\mathcal{D}U_r}{\mathcal{D}t} + \eta\left(\nabla^2 H_r - \frac{H_r}{r^2}\right), \tag{2.10}$$

$$\frac{DH_\theta}{Dt} - \frac{U_r H_\theta}{r} = \left(\frac{\mathcal{D}U_\theta}{\mathcal{D}t} - \frac{U_\theta H_r}{r}\right) + \eta\left(\nabla^2 H_\theta - \frac{H_\theta}{r^2}\right), \tag{2.11}$$

$$\frac{DH_z}{Dt} = \frac{\mathscr{D} U_z}{\mathscr{D} t} + \eta \nabla^2 H_z \ , \tag{2.12}$$

in which

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + U_r \frac{\partial}{\partial r} + U_z \frac{\partial}{\partial z}, \quad \frac{\mathscr{D}}{\mathscr{D} t} \equiv H_r \frac{\partial}{\partial r} + H_\theta \frac{\partial}{\partial z} ,$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} .$$

It may be readily verified that the governing equations admit the stationary solution

$$U_r = U_z = H_r = H_z = 0 \ , \ U_\theta = U_\theta(r) \ \text{and} \ H_\theta = H_\theta(r) \ , \tag{2.13}$$

provided

$$\frac{dQ}{dr} = \frac{U_\theta^2}{r} - \frac{H_\theta^2}{r} \ , \tag{2.14}$$

$$\nu\left(\nabla^2 U_\theta - \frac{U_\theta}{r^2}\right) \equiv \nu \frac{d}{dr}\left(\frac{dU_\theta}{dr} + \frac{U_\theta}{r}\right) = 0 \ , \tag{2.15}$$

and

$$\eta\left(\nabla^2 H_\theta - \frac{H_\theta}{r^2}\right) \equiv \eta \frac{d}{dr}\left(\frac{dH_\theta}{dr} + \frac{H_\theta}{r}\right) = 0 \ . \tag{2.16}$$

The general solutions of Eqs. (2.15) and (2.16) with $\nu \neq 0$ and $\eta \neq 0$ are

$$U_\theta(r) = A_1 r + B_1 / r = \Omega \ r \ , \tag{2.17}$$

and

$$H_\theta(r) = A_2 r + B_2 / r = L r \ , \tag{2.18}$$

where Ω and L are angular velocity and angular magnetic intensity respectively, and A_1 , B_1 , A_2 and B_2 are constants. Thus, the presence of a circular magnetic field does not affect the distribution of the swirling velocity which is permissible in the absence of the magnetic field; likewise, the reverse is also true. The constants A_1 and B_1 in the solution (2.17) are related to the angular velocities Ω_1 and Ω_2 of the two cylinders confining the fluid; they are given by

$$A_1 = -\Omega_1 \gamma^2 \frac{1-\mu/\gamma^2}{1-\gamma^2} \quad \text{and} \quad B_1 = \Omega_1 R_1^2 \frac{1-\mu}{1-\gamma^2} , \tag{2.19}$$

where

$$\mu = \Omega_2/\Omega_1 \quad \text{and} \quad \gamma = R_1/R_2 .$$

The constants A_2 and B_2 in (2.18) can also be related to the magnetic field at R_1 and R_2 in a similar manner, but it is more practical to express them in terms of electrical currents. Actually the magnetic field in (2.18) can be produced by electrical currents flowing in the axial direction. A simple and practical example is an axisymmetric axial current J confined to the inner cylinder and an axial current of uniform density J_o confined between the two cylinders. Thus we have

$$A_2 = 2\pi J_o (\mu_1/4\pi\rho)^{1/2} \quad \text{and} \quad B_2 = 2\pi R_1^2 (J_c - J_o)(\mu_1/4\pi\rho)^{1/2}, \tag{2.20}$$

where $Jc = J/\pi R_1^2$ is the average current density confined to the inner cylinder. It follows that

$$L_1 = 2\pi J_c (\mu_1/4\pi\rho)^{1/2} \quad \text{and} \quad L_2 = 2\pi \{ J_o + \gamma^2 (J_c - J_o) \} (\mu_1/4\pi\rho)^{1/2}. \tag{2.21}$$

A distrubance of the undistrubed state will give rise to small velocity components (u_r, u_θ, u_z), and a deviation of magnetic field denoted by (h_r, h_θ , h_z). Assuming that the various perturbations are axisymmetric and independent of θ , we obtained from (2.5)-(2.12) the linearized equations

$$\frac{\partial u_r}{\partial t} - \frac{2U_\theta u_\theta}{r} = -\frac{\partial q}{\partial r} - \frac{2H_\theta h_\theta}{r} + \nu \left(\nabla^2 u_r - \frac{u_r}{r^2} \right) , \tag{2.22}$$

$$\frac{\partial u_\theta}{\partial t} + u_r \left(\frac{\partial U_\theta}{\partial r} + \frac{U_\theta}{r} \right) - f \left(\frac{\partial H_\theta}{\partial r} + \frac{H_\theta}{r} \right) + \nu \left(\nabla^2 u_\theta - \frac{u_\theta}{r^2} \right) , \tag{2.23}$$

$$\frac{\partial u_z}{\partial t} = -\frac{\partial q}{\partial z} + \nu \nabla^2 u_z \ , \tag{2.24}$$

$$\frac{\partial h_r}{\partial t} = \eta \left(\nabla^2 h_r - \frac{h_r}{r^2} \right) , \tag{2.25}$$

$$\frac{\partial h_\theta}{\partial t} + u_r \left(\frac{\partial H_\theta}{\partial r} - \frac{H_\theta}{r} \right) = h_r \left(\frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r} \right) + \eta \left(\nabla^2 h_\theta - \frac{h_\theta}{r^2} \right) , \tag{2.26}$$

$$\frac{\partial h_z}{\partial t} = \eta \ \nabla^2 h_z \ , \tag{2.27}$$

$$\frac{\partial (r u_r)}{\partial r} + \frac{\partial (r u_z)}{\partial z} = 0 \ , \tag{2.28}$$

$$\frac{\partial (r h_r)}{\partial r} + \frac{\partial (r h_z)}{\partial z} = 0 \ , \tag{2.29}$$

in which $q = H_\theta h_\theta + p_1 / \rho$ with p_1 denoting the pressure perturbation.

From the form of Eqs. (2.25) and (2.27) we can conclude that h_r and h_z will eventually be damped out if they are not initially everywhere zero (see 7). Since the stability is characterized by undamped disturbances, we will, therefore, set h_r and h_z equal to zero without loss of generality.

By analysing the disturbance into normal modes, we seek solutions of the foregoing equations which are of the form

$$\{ u_r, u_\theta, u_z, h_\theta, q \} = \{ u(r), v(r), i w(r), g(r), \ell(r) \} \exp(pt + ikz). \tag{2.30}$$

For solutions of the form (2.30), Eqs. (2.22) - (2.29) become, upon eliminating the pressure q and axial velocity u_z

$$(DD_* - k^2 - c)(DD_* - k^2) u = \frac{2k^2}{\nu} \left(\frac{K}{r^2} v - L g \right) , \tag{2.31}$$

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7. C.-S. Yih, Phys. Fluids 2, 125-130 (1959).

$$(DD_* - k^2 - c)\, v = \frac{1}{\nu}\,\frac{DK}{r}\, u \,, \tag{2.32}$$

$$(DD_* - k^2 - P_r\, c)\, g = \frac{1}{\eta}\, r\,(DL)\, u \,, \tag{2.33}$$

where

$$\begin{aligned} D &= \frac{d}{dr} \,, \qquad D_* = \frac{d}{dr} + \frac{1}{r} \,, \\ K &= U_\theta r \,, \quad L = H_\theta / r \,, \quad c = p/\nu \,, \end{aligned} \tag{2.34}$$

and Pr is the magnetic Prandtl number ν/η . The boundary conditions for non-conducting walls are

$$u = Du = v = g = 0 \qquad at \quad r = R_1 \quad and \quad r = R_2 \,. \tag{2.35}$$

3. SUFFICIENT CONDITION FOR STABILITY

A sufficient condition for stability can be given for the present case on a mathematical ground, in the manner of Synge[8].

Multiplying Eq. (2.31) by ru* (complex conjugate is denoted by a superscript*) and integrating (by parts if necessary) with respect to r between R_1 and R_2, we have, upon utilization of the boundary conditions in (2.35),

$$I_2 + (2k^2 + c)\, I_1 + k^2 (k^2 + c)\, I_0 = \frac{2k^2}{\nu}\left\{\int_{R_1}^{R_2} \frac{K}{r}\, u^* v\, dr - \int_{R_1}^{R_2} L\, r\, u^* g\, dr\right\}, \tag{3.1}$$

8. J. L. Synge, Proc. Roy.Soc. A, 167, 250-256(1938).

where we have denoted certain positive definite integrals as follows:

$$I_0 = \int_{R_1}^{R_2} r|u|^2 dr \ , \quad I_1 = \int_{R_1}^{R_2} r|D_* u|^2 dr \ , \quad I_2 = \int_{R_1}^{R_2} r|DD_* u|^2 dr \ . \tag{3.2}$$

Similarly, multiplying (2.32) by $-2k^2 K v^* / DK$ and (2.33) by $2k^2 L g^* / Pr\, DL$, and integrating over the range, we obtain, respectively,

$$2k^2(k^2+c)\int_{R_1}^{R_2} \frac{K}{DK}|v|^2 dr - 2k^2\int_{R_1}^{R_2} \frac{K}{DK} v^* DD_* v\, dr = -\frac{2k^2}{\nu}\int_{R_1}^{R_2} \frac{K}{r} u v^* dr \ , \tag{3.3}$$

$$-2k^2\left(\frac{k^2}{Pr}+c\right)\int_{R_1}^{R_2} \frac{L}{DL}|g|^2 dr + \frac{2k^2}{Pr}\int_{R_1}^{R_2} \frac{L}{DL} g^* DD_* g\, dr = \frac{2k^2}{\nu}\int_{R_1}^{R_2} L u g^* r dr \ . \tag{3.4}$$

Let us add to (3.1), (3.3) and (3.4) their complex conjugates and then add them together, we obtain

$$E + c_r F + S + (W + W^*) = 0 \ , \tag{3.5}$$

where c_r is the real part of c and

$$E = I_2 + 2k^2 I_1 + k^4 I_0 > 0 \ , \tag{3.6}$$

$$F = I_1 + k^2 I_0 + k^2\int_{R_1}^{R_2} \mathcal{K}|v|^2 dr + k^2\int_{R_1}^{R_2} \mathcal{L}|g|^2 dr \ ; \tag{3.7}$$

$$S = k^4\int_{R_1}^{R_2} \mathcal{K}|v|^2 dr + \frac{k^4}{Pr}\int_{R_1}^{R_2} \mathcal{L}|g|^2 dr \ , \tag{3.8}$$

$$W = -\frac{k^2}{2}\int_{R_1}^{R_2} \mathcal{K} v^* DD_* v\, dr - \frac{k^2}{2Pr}\int_{R_1}^{R_2} \mathcal{L} g^* DD_* g\, dr \ , \tag{3.9}$$

with

$$\mathcal{K} = DK^2/(DK)^2 \quad \text{and} \quad \mathcal{L} = -DL^2/(DL)^2 . \tag{3.10}$$

From (3.9) it follows that

$$W + W^* = k^2\int_{R_1}^{R_2}\left\{\mathcal{K}\frac{vv^*}{r^2} + \mathcal{K}v'v^{*\prime} + \frac{1}{2}(vv^*)' r\frac{d}{dr}\left(\frac{\mathcal{K}}{r}\right)\right\}dr + \frac{k^2}{P_r}\int_{R_1}^{R_2}\left\{\mathcal{L}\frac{gg^*}{r^2} + \mathcal{L}g'g^{*\prime} + (gg^*)' r\frac{d}{dr}\left(\frac{\mathcal{L}}{r}\right)\right\}dr \tag{3.11}$$

since

$$K = A_1 r^2 + B_1 \quad \text{and} \quad L = A_2 + B_2/r^2 , \tag{3.12}$$

we have, upon utilizing the boundary conditions,

$$\int_{R_1}^{R_2}\frac{1}{2}(vv^*)' r\frac{d}{dr}\left(\frac{\mathcal{K}}{r}\right)dr = -\int_{R_1}^{R_2}(vv^*)'\frac{B_1}{A_1 r^2}dr = -\int_{R_1}^{R_2}(vv^*)'\frac{\mathcal{K}}{r}dr , \tag{3.13}$$

and

$$\int_{R_1}^{R_2}\frac{1}{2}(gg^*)' r\frac{d}{dr}\left(\frac{\mathcal{L}}{r}\right)dr = \int_{R_1}^{R_2}(gg^*)'\frac{A_2 r^2}{B_2}dr , = \int_{R_1}^{R_2}(gg^*)'\frac{\mathcal{L}}{r}dr . \tag{3.14}$$

Subsitution of (3.13) and (3.14) into (3.11) gives

$$W + W^* = k^2\int_{R_1}^{R_2}\mathcal{K}\left|v' - \frac{v}{r}\right|^2 dr + \frac{k^2}{P_r}\int_{R_1}^{R_2}\mathcal{L}\left|g' + \frac{g}{r}\right|^2 dr . \tag{3.15}$$

Now if the steady flow and magnetic field satisfy, respectively,

$$DK^2 \geqslant 0 \quad \text{and} \quad DL^2 \leqslant 0 , \tag{3.16}$$

then

$$F > 0 , \quad \mathcal{S} \geqslant 0 \quad \text{and} \quad W + W^* \geqslant 0 .$$

It follows from Eq. (3.5) that c_r is negative, and hence the steady motion is stable. Therefore, (3.16) is a sufficient condition for stability of the present case.

It is noted here that (3.16) is also a sufficient condition for stability of an infinite fluid. Since, when the inner cylinder is removed, the boundary conditions at r = o are u = v = g = o , if u, v, w and g are free from singularity there. Noting that the limite of integration now being from zero to R_2, and following the similar procedure,the condition (3.16) can indeed be obtained. If we also let $R_2 \to \infty$, the fluid becomes infinite and thus we have established the sufficient condition for stability of an infinite fluid.

For the present problem, two particular cases can readily be observed:

Case 1. (A) Zero-Magnetic Field (L = 0), or (B) Constant Angular Magnetic Intensity (L= constant)

In this case, (3.16) reduces to

$$\frac{d}{dr}(U_\theta r)^2 \geqslant 0 \;, \tag{3.17}$$

which is the Rayleigh's criterion[9] for stability. Here the constant angular magnetic field is neutrally stable, and does not affect the stability of circulatory motion.

Case 2. (A) Stationary Fluid (K = 0), or (B) Constant Angular Momentum (Pure Vortex, K = constant).

In this case, (3.16) reduces to

$$\frac{d}{dr}\left(\frac{H_\theta}{r}\right)^2 \leqslant 0 \;, \tag{3.18}$$

which is essentially the Yih's criterion[10] for stability. Yih has expressed his criterion, however, in terms of electrical currents which reads

$$J_c > J_o \qquad \text{or} \qquad J_c < -J_o\,(R_2^2 - R_1^2)/R_1^2 \;. \tag{3.19}$$

A careful examination revelas that (3.19) needs a supplementary condition

$$J_o > 0 \;. \tag{3.20}$$

Since the criterion (3.18) requires that for stability, the magnitude of the angular magnetic field must decrease outward monotomically. This condition imples that

9. Lord Rayleigh, Scientific paper 6, 447-453, Cambridge University Press (1916).
10. C.-S. Yih, J. Fluid Mech. 5, 436-444 (1959).

$$\frac{L_1}{L_2} > 1 \tag{3.21}$$

which leads to

$$-\frac{R_1^2}{R_2^2 - R_1^2} < \frac{J_o}{J_c} < 1 \ . \tag{3.22}$$

Indeed, criterion (3.22) is different from Yih's criterion (3.19) if $J_o < 0$. Henceforth, we will refer Yih's criterion to (3.22), or (3.19) plus (3.20).

4. A STRONGER SUFFICIENT CONDITION FOR STABILITY

In section 3, a sufficient condition for stability has been established which is the combined criterion of Rayleigh's and Yih's. This criterion is rather weak because it requires the stability of both velocity field and magnetic field in order to insure the stability of the whole field. Actually, stability is still possible when the stablizing effect of one field overcomes the destablizing effect of the other.

For fluids of zero viscosity and electrical resistivity, Michael[1] has established a necessary and sufficient condition for stability

$$r^{-3}\frac{d}{dr}K^2 - r\frac{d}{dr}L^2 > 0 \ . \tag{4.1}$$

comparing (4.1) with (3.16) it is readily confirmed that when the stablizing effect of one field overcomes the destablizing one, stability is indeed insured. We will now proceed to investigate the validity of the condition (4.1) when the medium is dissipative, although a dissipative medium usually tends to damp out small disturbances. Before we turn on resistivities, it will be advisable to know the physical background of the criterion (4.1).

When $H_r = Hz = \eta = \nu = 0$, Eqs. (2.5)-(2.12) then become

$$\frac{Du_r}{Dt} = -\frac{\partial Q}{\partial r} + \frac{U_\theta^2}{r} - \frac{H_\theta^2}{r} \ , \tag{4.2}$$

$$\frac{D}{Dt}(U_\theta r) = 0 \ , \tag{4.3}$$

$$\frac{Du_z}{Dt} = -\frac{\partial Q}{\partial z} \quad , \tag{4.4}$$

$$\frac{D}{Dt}\left(\frac{H_\theta}{r}\right) = 0 \quad . \tag{4.5}$$

Therefore, the angular momentum K ($=U_\theta r$) and the angular magnetic field L ($= H_\theta / r$) are conserved quantities for a given fluid element in this ideal case. And the motions in the radial and the axial directions take place as though U_θ and H_θ were absent and, instead, a body force $(U_\theta^2 - H_\theta^2)/r$ were acting in the radial direction.

Suppose now that a fluid element at r_1 with transverse velocity $U_{\theta 1}$ is displaced to a new position $r_2(=r_1 + \xi)$, while the velocity becomes $U_{\theta 12}$. Hence a small radial displacement ξ leads to a velocity discontinuity at r_2

$$U_{\theta 12} - U_{\theta 2} = \frac{K_1}{r_2} - \frac{K_2}{r_2} = -\left(\frac{dK}{dr}\right)_r \frac{\xi}{r_2} \quad . \tag{4.6}$$

According to our notation, this is the perturbation velocity u_θ at r_2, or $u_{\theta 2}$. Similarly, the magnetic intensity discontinuity at r_2 due to a small displacement ξ is

$$h_{\theta 2} = H_{\theta 12} - H_{\theta 2} = L_1 r_2 - L_2 r_2 = -\left(\frac{dL}{dr}\right)_r \xi r_2 \quad . \tag{4.7}$$

Since the body force is intrinsic property of fluid element while the total pressure gradient is determined by the surrounding environment. The body force $(U_\theta^2 - H_\theta^2)/r$, therefore, must be balanced by the pressure gradient as shown in Eq. (4.2), with the perturbation term Du_r/Dt neglected. If a ring of fluid at r_1 is displaced to $r_2 = r_1 + \xi$ (with $\xi > 0$), the body force becomes

$$\frac{U_{\theta 12}^2}{r_2} - \frac{H_{\theta 12}^2}{r_2} = \frac{(U_{\theta 2} + u_{\theta 2})^2}{r_2} - \frac{(H_{\theta 2} + h_{\theta 2})^2}{r_2} \quad . \tag{4.8}$$

The prevailing pressure gradient at r_2 is, by Eq. (4.2),

$$\frac{U_{\theta 2}^2}{r_2} - \frac{H_{\theta 2}^2}{r_2} \tag{4.9}$$

which is larger than the body force in (4.8) if

$$\frac{U_{\theta 2}^2}{r_2} - \frac{H_{\theta 2}^2}{r_2} > \frac{(U_{\theta 2} + u_{\theta 2})^2}{r_2}$$

or

$$\frac{U_{\theta 2}\, u_{\theta 2}}{r_2} < \frac{H_{\theta 2}\, h_{\theta 2}}{r_2} \quad . \tag{4.10}$$

Therefore the net restoring forces are proportional to the discontinuities u_θ and h_θ respectively. If the relations in (4.6) and (4.7) are substituted, (4.10) becomes

$$\Sigma \equiv r^{-3}\frac{d}{dr}K^2 - r\frac{d}{dr}L^2 > 0 \ . \tag{4.11}$$

In this case, the fluid element will be forced back to its initial position, and the motion is stable.

The above physical argument is essentially an extension of Von Karman's[11] to the conducting fluid. Rayleigh[9] treated these ideas from the point of view of the minimum energy. With the body force $(U_\theta^2 - H_\theta^2)/r$, we may associate with each fluid element a "potential energy" $\rho(K^2/r^2 + L^2r^2)/2$. This potential energy is clearly the kinetic and magnetic energy of circulatory flow and magnetic field. Therefore, in this case, the equilibrium is stable only if the potential energy is a minimum. The total energy (kinetic plus magnetic) associated with a given fluid ring at a distance r from the axis is proportional to

$$\frac{\rho}{2}(U_\theta^2 + H_\theta^2)\, r dr = \frac{\rho}{2}\left(\frac{K^2}{r^2} - L^2r^2\right) r dr \, .$$

Suppose now that two rings of fluid of equal areas ($r_1 dr_1 = r_2 d r_2$) are interchanged. The corresponding increment in the total energy is proportional to

$$\delta E \sim (U_{\theta 12}^2 + H_{\theta 12}^2 + U_{\theta 21}^2 + H_{\theta 21}^2) - (U_{\theta 1}^2 + H_{\theta 1}^2 + U_{\theta 2}^2 + H_{\theta 2}^2) \ , \tag{4.12}$$

where $U_{\theta 21}$ and $H_{\theta 21}$ are the velocity and magnetic intensity at r_1, originating from r_2. Hence the perturbation velocity and magnetic intensity at r_1 are

11. Th. von Karman, Proc. 4th Int. Congr. Appl. Mech., Cambridge, England, 54-91(1934).

$$u_{\theta 1} = U_{\theta 21} - U_{\theta 1} = \frac{K_2}{r_1} - \frac{K_1}{r_1} = \left(\frac{dK}{dr}\right)_r \frac{\xi}{r_1} , \tag{4.13}$$

$$h_{\theta 1} = H_{\theta 21} - H_{\theta 1} = L_2 r_1 - L_1 r_1 = \left(\frac{dL}{dr}\right)_r \xi r_1 . \tag{4.14}$$

Substitution into (4.12) gives

$$\begin{aligned} \delta E &\sim 2 \left(U_{\theta 2} u_{\theta 2} + H_{\theta 2} h_{\theta 2} + U_{\theta 1} u_{\theta 1} + H_{\theta 1} h_{\theta 1} \right) \\ &\sim \xi \left(\frac{1}{r_1^2} - \frac{1}{r_2^2} \right) \left(\frac{d}{dr} K^2 - r^4 \frac{d}{dr} L^2 \right) \end{aligned} \tag{4.15}$$

which is positive if

$$\Sigma \equiv r^{-3} \frac{d}{dr} K^2 - r \frac{d}{dr} L^2 > 0 . \tag{4.16}$$

Hence $\Sigma > 0$ makes the total energy a minimum and thus ensure stability.

If the fluid has now non-zero resistivities, the situation is somewhat different. The viscosity ν and the magnetic diffusivity η are responsible for diffusion, in addition to dissipation, of momentum and magnetic intensity respectively. The diffusive effect is by no means always stabilizing. In fact, it can very well be destabilizing. We will consider the following two cases.

4.1 Inability Sets in as a Stationary Secondary Flow

When the resistivities of the medium are considered the linearized perturbation equations are, from (2.23) and (2.26),

$$\frac{\partial u_\theta}{\partial t} + \frac{u_r}{r} \frac{dK}{dr} = \nu \left(\nabla^2 - \frac{1}{r} \right) u_\theta , \tag{4.17}$$

$$\frac{\partial h_\theta}{\partial t} + u_r r \frac{dL}{dr} = \eta \left(\nabla^2 - \frac{1}{r} \right) h_\theta . \tag{4.18}$$

Consider a steady motion, with wave length λ along z-direction, which is so slow that diffusive effects cannot be ignored. Such a motion leads ultimately to a state in which $\frac{\partial u}{\partial t} = 0 = \frac{\partial h}{\partial t}$. In this steady state, (4.17) and (4.18) become

$$\frac{\xi}{\Delta t} \frac{1}{r} \frac{dK}{dr} = - \left(\frac{4\pi^2}{\lambda^2} - \frac{1}{r^2} \right) \nu u_\theta , \tag{4.19}$$

$$\frac{\xi}{\Delta t} r \frac{dL}{dr} = - \left(\frac{4\pi^2}{\lambda^2} - \frac{1}{r^2} \right) \eta h_\theta . \tag{4.20}$$

Comparing (4.19) and (4.20) with (4.6) and (4.7), we conclude that it is now νu_θ and ηh_θ that are proportional, respectively, to $-(dK/dr)\xi/r$ and $-(dL/dr)\xi r$. That is, because of the diffusive effects of these dissipative coefficients, u_θ and h_θ are multiplied, respectively, by quantities proportional to $1/\nu$ and $1/\eta$,

$$u_{\theta 2} \sim \frac{1}{\nu}\left(U_{\theta 12} - U_{\theta 2}\right) , \tag{4.21}$$

$$h_{\theta 2} \sim \frac{1}{\eta}\left(H_{\theta 12} - H_{\theta 2}\right) . \tag{4.22}$$

Accordingly, the criterion (4.11) becomes

$$\Phi = \frac{1}{\nu} r^{-3} \frac{d}{dr} K^2 - \frac{1}{\eta} r \frac{d}{dr} L^2 > 0 . \tag{4.23}$$

A similar physical argument due to Rayleigh will lead to the same conclusion. Therefore, we have established a sufficient condition for stability of stationary convective motion in a diffusive medium.

4.2 Instability Sets in as Oscillations of Increasing Amplitude (Overstability)

Suppose that, in a non-dissipative medium, the motion is in neutral oscillation. Then, the restoring force is balanced by the instability force in (4.10). Eqs. (4.17) and (4.18) without diffusion are

$$\frac{\partial u_\theta}{\partial t} + \frac{u_r}{r}\frac{dK}{dr} = 0 , \tag{4.24}$$

$$\frac{\partial h_\theta}{\partial t} + u_r r \frac{dL}{dr} = 0 . \tag{4.25}$$

If we now turn on the diffusion mechanism, diffusion then starts. The right sides of Eqs. (4.24) and (4.25) are no longer zero, and u_θ and h_θ will be modified accordingly. During any half-oscilation, the diffusion term in (4.17) reduces the inequalities of velocity u_θ from which the instability (or restoring) force arise by a fraction proportional to ν ; the magnetic diffusion term in (4.18) similarly reduces the disturbance of the magnetic field by a fraction proportional to η . Thus the total restoring force is largeras the material is pushed back to its equilibrium position than when it is traveling out, provided that

$$\frac{\nu u_{\theta 2} U_{\theta 2}}{r_2} > \frac{\eta h_{\theta 2} H_{\theta 2}}{r_2} \quad ,$$

or

$$\nu r^{-3} \frac{d}{dr} K^2 - \eta r \frac{d}{dr} L^2 < 0 \; . \tag{4.26}$$

In this case, the fluid element will oscillate with increasing amplitude, and the motion is unstable. Therefore, the sufficient condition for stability of this oscillatory motion is

$$\Psi = \nu r^{-3} \frac{d}{dr} K^2 - \eta r \frac{d}{dr} L^2 > 0 \; . \tag{4.27}$$

It is noted here that Michael's criterion (4.1) is actually a special case of (4.23) and (4.27) when $\nu \to 0$ and $\eta \to 0$, with $\nu/\eta \longrightarrow 1$; that is, the diffusive effects are equally inefficient. From the forms of (4.23) and (4.27) we may observe that only the diffusive effects have been included, since they only show the relative effectiveness of the diffusive mechanism between ν and η . Aside from being agents for momentum and magnetic diffusion, viscosity and magnetic diffusivity are always dissipative agents responsible for the eventual conversion of kinetic and magnetic energy into heat. Thus, as far as the dissipative roles are converned, the diffusive coefficients are always stabilizing. However, caution must be taken to distinguish diffusion from dissipation, which will be discussed in Section 9.

5. APPROXIMATE MATHEMATICAL PROOF

In the last section, a sufficient condition for stability has been established on physical grounds. In this section, a mathematical proof will be given, although an exact proof is still lacking at this time. Nevertheless, the following treatment is, in some way, quite illuminating and furnishes an independent proof of establishing the sufficient condition for stability.

5.1 Stationary-Secondary Flow

In this case, the 'principle of exchange of stabilities' is valid and the marginal state is marked by vanishing c. Eqs. (2.31) - (2.33) become, for c = 0,

$$(DD_* - k^2)^2 u = \frac{2k^2}{\nu}\left(\frac{K}{r^2} v - Lg\right), \tag{5.1}$$

$$(DD_* - k^2) v = \frac{DK}{\nu r} u, \tag{5.2}$$

$$(DD_* - k^2) g = \frac{rDL}{\eta} u. \tag{5.3}$$

If we let

$$v = \frac{DK}{\nu r} v_1 = \frac{2A_1}{\nu} v_1 , \tag{5.4}$$

$$g = \frac{rDL}{\eta} g_1 = -\frac{2B_2}{\eta r^2} g_1 , \tag{5.5}$$

Eqs. (5.2) and (5.3) then become

$$(DD_* - k^2) v_1 = u , \tag{5.6}$$

$$(DD_* - k^2)(r^{-2} g_1) = r^{-2} u . \tag{5.7}$$

If we now assume that

$$DD_*(r^{-2} g_1) \simeq r^{-2} DD_* g_1 \tag{5.8}$$

Eq. (5.7) then becomes

$$(DD_* - k^2) g_1 = u \tag{5.9}$$

The boundary conditions for v_1 and g_1 are

$$v_1 = g_1 = 0 \quad \text{at} \quad r = R_1 \quad \text{and} \quad r = R_2. \tag{5.10}$$

Since the differential equations and the boundary conditions are identical for v_1 and g_1 and the left sides of (5.6) and (5.9) are not identically zero, we conclude that

$$v_1 = \mathfrak{z}_1 \ . \tag{5.11}$$

We observe that the approximation in (5.8) is excellent if the spacing between cyclinders is small compared with the average radius, thus making r essentially a constant throughout the range. Having obtained the approximate relation in (5.11), we now go back to the exact equations (5.1) - (5.3). Since Eqs. (5-1)-(5.3) are the special case, with c =0, of Eqs. (2.31)-(2.33), the corresponding equation to Eq. (3.5) then is

$$E + S + (W + W^*) = 0, \tag{5.12}$$

where E, S and W are still given in Eqs. (3.6) - (3.9). If the relations in (5.11), (5.4) and (5.5) are substituted, we have

$$S = \frac{k^4}{\nu}\int_{R_1}^{R_2}\left(\frac{1}{\nu} r^{-3} DK^2 - \frac{1}{\eta} r DL^2\right)|v_1|^2 r\,dr, \tag{5.13}$$

$$W + W^* = \frac{k^2}{\nu}\int_{R_1}^{R_2}\left(\frac{1}{\nu} r^{-3} DK^2 - \frac{1}{\eta} r DL^2\right)\left|v_1' - \frac{v_1}{r}\right|^2 r\,dr. \tag{5.14}$$

It follows immediately that the Eq. (5.12) can be possible only if

$$\Phi \equiv \frac{1}{\nu} r^{-3} DK^2 - \frac{1}{\eta} r DL^2 < 0 \ . \tag{5.15}$$

In other words, if $\Phi > 0$, no marginal state is possible; that is, the motion is either entirely stable or entirely unstable when $\Phi > 0$. However, criterion (3.16) assures us that the motion is stable if $DK^2 > 0$ and $DL^2 < 0$. If we now start out with a stable motion which satisfies criterion (3.16), and let the flow and magnetic fields vary continuously. Accordingly, Φ will also vary continously. Since the governing equations are not singular with respect to Φ, DK^2 and DL^2, any velocity and magnetic fields corresponding to positive Φ can be obtained by this continuously varying process during which Φ always remains positive. Hence, the marginal state is never reached and thus, the motion is always stable. We, therefore, have rederived criterion (4.23) by an independent argument.

5.2 Overstability

In this case the instability sets in as an oscillatory motion. We assume now that, during the oscillatory motion, the diffusion term is not important in (2.32) and (2.33), thus we have

$$-cv \sim \frac{1}{\nu}\frac{DK}{r}u \;, \tag{5.16}$$

$$-cg \sim \frac{1}{\nu} r DL u \;. \tag{5.17}$$

With v and g related in this manner, Eq. (3.5) now becomes

$$E + S + (W + W^*) = 0 \;, \tag{5.18}$$

where E is still given by (3.6), and

$$S = \frac{k^4}{\nu^3}\int_{R_1}^{R_2}(\nu r^{-3} DK^2 - \eta r DL^2)\left|\frac{u}{c}\right|^2 r\,dr \;, \tag{5.19}$$

$$W + W^* = \frac{k^2}{\nu^3}\int_{R_1}^{R_2}(\nu r^{-3} DK^2 - \eta r DL^2)\left|\frac{u' - u/r}{c}\right|^2 r\,dr \;. \tag{5.20}$$

It follows that Eq. (5.18) can be possible only if

$$\Psi \equiv \nu r^{-3} DK^2 - \eta r DL^2 < 0 \;. \tag{5.21}$$

By a similar argument as that in § 5.1, we can conclude that the sufficient condition for stability in this case is

$$\Psi > 0 \;, \tag{5.22}$$

which is the same as criterion (4.27).

CASE OF SMALL SPACING

6. REDUCTION TO THE CASE OF SMALL SPACING

For small spacing of cylinders, we introduce the following dimensionless parameters:

$$\xi = (r - R_1)/d\,, \quad d = R_2 - R_1\,, \quad \bar{r} = r/R_1\,, \quad \zeta = z/d\,,$$
$$a = kd\,, \quad \sigma = cd^2\,, \quad \bar{K} = K/\nu\,, \tag{6.1}$$
$$\bar{\Omega} = \Omega/\Omega_1\,, \quad \bar{L} = L/L_1\,,$$

whence

$$\bar{K} = \bar{K}_1(\bar{A}_1\bar{r}^2 + \bar{B}_1)\,, \quad \bar{\Omega} = \bar{A}_1 + \frac{\bar{B}_1}{\bar{r}^2}\,, \quad \bar{L} = \bar{A}_2 + \frac{\bar{B}_2}{\bar{r}^2}\,,$$

with

$$\bar{K}_1 = \frac{\Omega_1 R_1^2}{\nu}\,, \quad \bar{A}_1 = \frac{A_1}{\Omega_1}\,, \quad \bar{B}_1 = \frac{B_1}{\Omega_1 R_1^2}\,, \quad \bar{A}_2 = \frac{A_2}{L_1} = \frac{J_0}{J_c}\,, \quad \bar{B}_2 = \frac{B_2}{L_1 R_1^2} = 1 - \frac{J_0}{J_c}\,.$$

Thus $\bar{A}_1 + \bar{B}_1 = 1$ and $\bar{A}_2 + \bar{B}_2 = 1$.

If we now make these substitutions in (2.31) - (2.33) and neglect higher powers of the small parameters d/R_1 , we obtain

$$(D^2 - a^2 - \sigma)(D^2 - a^2)u = 2a\left(\frac{d}{R_1}\right)\left\{R\bar{\Omega}v - S_\nu \bar{L} g\right\}\,, \tag{6.2}$$

$$(D^2 - a^2 - \sigma)v = 2a\bar{A}_1\left(\frac{R_1}{d}\right)Ru\,, \tag{6.3}$$

$$(D^2 - a^2 - P_r\sigma)g = -2a\,\frac{\bar{B}_2}{\bar{r}^2}\left(\frac{R_1}{d}\right)S_\eta u\,, \tag{6.4}$$

where

$$\begin{aligned} D &= \frac{d}{d\xi}\,, & \bar{r} &\simeq 1 + \frac{d}{R_1}\xi \\ \bar{\Omega} &\simeq 1 + \alpha_1\xi\,, & \alpha_1 &= \frac{\Omega_2}{\Omega_1} - 1 \\ \bar{L} &\simeq 1 + \alpha_2\xi\,, & \alpha_2 &= \frac{L_2}{L_1} - 1 \\ R &= \frac{\Omega_1 d^2}{\nu}\,, & S_\nu &= \frac{L_1 d^2}{\nu}\,, \quad S_\eta = \frac{L_1 d^2}{\eta}\,. \end{aligned} \tag{6.5}$$

If we let

$$\begin{aligned} v &= 2a\bar{A}_1\left(\frac{R_1}{d}\right)Rv_1\,, \\ g &= -2a\bar{B}_2\left(\frac{R_1}{d}\right)S_\eta g_1\,. \end{aligned} \tag{6.6}$$

Eqs. (6.2) - (6.4) become

$$(D^2 - a^2 - \sigma)(D^2 - a^2)u = 4a^2(R^2\bar{A}_1\bar{\Omega}v_1 + M^2\bar{B}_2\bar{L}g_1)\,, \tag{6.7}$$

$$(D^2 - a^2 - \sigma)v_1 = u\,, \tag{6.8}$$

$$(D^2 - a^2 - P_r\sigma) g_1 = u \,, \tag{6.9}$$

where M = $(S_\nu S_\eta)^{1/2}$ is the Hartman number. We will only consider the case that instability sets in as a stationary secondary flow. Hence, assuming that for the marginal stability σ is zero, Eqs. (6.7) - (6.9) become

$$(D^2 - a^2)^2 u = 4a^2 \{ \bar{A}_1 R^2 (1 + \alpha_1 \xi) v_1 + \bar{B}_2 M^2 (1 + \alpha_2 \xi) g_1 \} , \tag{6.10}$$

$$(D^2 - a^2) v_1 = u \,, \tag{6.11}$$

$$(D^2 - a^2) g_1 = u \,. \tag{6.12}$$

The boundary conditions for non-conducting wall are

$$u = Du = v_1 = g_1 = 0 \quad \text{at} \quad \xi = 0 \quad \text{and} \quad \xi = 1 \,. \tag{6.13}$$

It follows from Eqs. (6.11) and (6.12) and the boundary conditions (6.13) that

$$v_1 = g_1 \,. \tag{6.14}$$

Eqs. (6.10) - (6.12) then become

$$(D^2 - a^2)^2 u = -a^2 T (1 + \alpha_0 \xi) v_1 \,, \tag{6.15}$$

$$(D^2 - a^2) v_1 = u \,, \tag{6.16}$$

where

$$T = T_1 + T_2 \,, \tag{6.17}$$

$$\alpha_0 = \beta_1 \alpha_1 + \beta_2 \alpha_2 \,, \qquad \beta_1 = \frac{T_1}{T} \,, \quad \beta_2 = \frac{T_2}{T} \,, \tag{6.18}$$

$$T_1 = -4\bar{A}_1 R^2 = \frac{4\Omega_1^2 d^4}{\nu^2} \frac{\gamma^2 - \mu}{1 - \gamma^2} \,, \qquad T_2 = -4\bar{B}_2 M^2 = \frac{4\pi\mu_1 d^4}{\rho\nu\eta} J_c (J_o - J_c) , \tag{6.19}$$

in which T is the generalized Taylor number.

The boundary conditions are still specified by Eq. (6.13). Comparing Eqs. (6.15) and

(6.16) with that treated by Chandrasekhar [12], we see that they are identical except the definitions of T and α_0 . In his case, T is the Taylor number $-4\bar{A}_1 R^2$ and α_0 is the value of $(\Omega_2/\Omega_1)-1$. Chandrasekhar found the positive critical value of T for a variety of values of α_0 , all corresponding to negative values of α_0 . Yih[5,10] and Lai[3] have calculated the critical values of T for some positive values of α_0, however, one branch of solution which corresponds to negative α_0 and negative critical T has been entirely neglected. Equations (6.15) and (6.16) with the boundary conditions (6.13) are now solved numerically for both positive and negative critical T corresponding to the entire range of α_0 .

7. NUMERICAL RESULTS AND THE SUFFICIENT CONDITION FOR STABILITY

To solve the present characteristic value problem, we have followed the method derived by Chandrasekhar [12,13]. Table I includes the critical Taylor number Tc, and the associated wave number a, for the entire range of α_0. The coefficient $\mathcal{C}_m$ ($= C_m/(m^2\pi^2+a^2)^2$) for the marginal state in the expansion for u is also given[13]. The (Tc, α_0) and the $[a(\text{Tc}), \alpha_0]$ - relationships are further illustrated in Figs. 1 and 2. Fig. 3 shows the profile of the velocity u for various α_0 , from which the corresponding cell pattern can easily be obtained.

7.1 (A) Zero-Magnetic Field (L = 0), or (B) Constant Angular Magnetic Intensity (L = constant).

In this case, α_2 and T_2 are zero, thus α_0 and T reduce to α_1 and T_1 . According to (6.5) we have

$$\alpha_1 = -\bar{B}_1\left(1-\frac{R_1^2}{R_2^2}\right) \simeq (\bar{A}_1 - 1)\left(1-\frac{R_1^2}{R_2^2}\right) \simeq \bar{A}_1\left(1-\frac{R_1^2}{R_2^2}\right), \tag{7.1}$$

which follows that

12. S. Chandrasekhar, Mathematika 1, 5-13 (1954).
13. S. Chandrasekhar, Hydrodynamic and hydromagnetic stability, Oxford University Press, 298-324(1961).

$$T_1 > 0 \quad \text{corresponds to} \quad \alpha_1 < 0 \,,$$

and

$$T_1 < 0 \quad \text{corresponds to} \quad \alpha_1 > 0 \,. \tag{7.2}$$

Therefore, in this case, the Taylor number exists only in the second and fourth quadrants in the ($\alpha_o - T$) plane as shown in Fig. 4. Here no critical Taylor number exists in the fourth guadrant, hence, positive α_o corresponds to stability. This is in consistant with the Rayleigh's criterion, which states that the flow is stable if

$$\frac{K_2}{K_1} > 1 \,. \tag{7.3}$$

(7.3) is equivalent to

$$\overline{A}_1 \left(\frac{R_2^2}{R_1^2} - 1 \right) > 0 \,, \tag{7.4}$$

which leads to

$$T_1 < 0 \,,$$

and by (7.2)

$$\alpha_1 > 0 \,.$$

Thus according to Rayleigh criterion no negative critical Taylor number exists for $\alpha_1 > 0$.

8.2 (A) Stationary Fluid (K = 0), or (B) Constant Angular Momentum (Pure Vortex, K = constant)

In this case α_1 and T_1 are zero, thus α_o and T reduce to α_2 and T_2. According to (6.5) we have

$$\alpha_2 = - \overline{B}_2 \left(1 - \frac{R_1^2}{R_2^2} \right) \,, \tag{7.5}$$

which follows that

$$T_2 > 0 \quad \text{corresponds to} \quad \alpha_2 > 0 \,,$$

and

$$T_2 < 0 \quad \text{corresponds to} \quad \alpha_2 < 0 \,. \tag{7.6}$$

Therefore, in this case, the Taylor number exists only in the first and third quadrants in the (α_o-T) plane as shown in Fig. 4. Here no critical Taylor number exists for $\alpha_o > -1$ in the third quadrants, hence, $0 > \alpha_o > -1$ corresponds to stability. This is in consistant with Yih's criterion, which states that the fluid is stable if

$$\frac{L_1}{L_2} > 1 \quad . \tag{7.7}$$

(7.7) is equivalent to

$$\alpha_2 (1 + \alpha_2) < 0 \quad , \tag{7.8}$$

which leads to

$$0 > \alpha > -1 \quad ,$$

and by (7.6)

$$T_2 < 0$$

Therefore, according to Yih's criterion, no negative critical Taylor number exists for $0 > \alpha > -1$.

7.3 General Case ($T \neq 0$, $T_2 \neq 0$)

Combine the above two cases we can conclude that, in this general case, the Taylor number exists in all four quadrants of the (α_0-T) plane as shown in Fig. 4. Since we have defined

$$\Phi \equiv \frac{1}{\nu} r^{-3} \frac{d}{dr} K^2 - \frac{1}{\eta} r \frac{d}{dr} L^2 \ ,$$

it follows that

$$\Phi = \frac{4\nu}{d^4} (R^2 \bar{A}_1 \bar{\Omega} + M^2 \bar{B}_2 \bar{L} \bar{r}^{-2}) . \tag{7.9}$$

At the inner wall, $\bar{r} = 1$, (7.9) becomes

$$\Phi_1 = \frac{4\nu}{d^4} (R^2 \bar{A}_1 + M^2 \bar{B}_2) . \tag{7.10}$$

We should observe that (7.10) is exact which does not involve any small spacing approximations. Comparing (7.10) with (6.17), we obtain

$$\Phi_1 = - \frac{\nu}{d^4} T \ . \tag{7.11}$$

Now, for small spacing approximation, Eq. (7.9) becomes

$$\Phi = - \frac{\nu}{d^4} T (1 + \alpha_0 \xi) . \tag{7.12}$$

From (7.11) and (7.12) we can conclude that

$$T < 0 \quad \text{and} \quad \alpha_0 > -1 \tag{7.13}$$

if $\Phi > 0$ for the entire range of ξ. In this case, the flow is stable since, from the previous argument, no negative critical Taylor number exists for the range $\alpha_0 > -1$. Therefore, we have established exactly the criterion (4.23) for stability for the case of spacing spacing.

An interesting fact should be observed that for the non-magnetic couette flow, if the small spacing result yields that the flow is stable, then the large spacing result will always predict a stable one. But the reverse is not always true. This point is demonstrated in Fig. 5. In this case, we have defined

$$T = -\frac{4A_1B_1}{\nu^2}R_2^2 = \frac{4\Omega_1^2 R_1^4}{\nu^2}\,\frac{(1-\mu)(1-\mu/\gamma^2)}{(1-\gamma^2)^2},$$
$$\kappa = -\frac{A_1R_2^2}{B_1} = \frac{1-\mu/\gamma^2}{1-\mu}.$$

The critical Taylor numbers, both for large and small spacings, are due to Chandrasekhar[13]. We can see that Tc for large spacing is always larger than that for small spacing in this particular case. Therefore, as far as the sufficient condition for stability is concerned, the criterion based on the small spacing results is always on the safe side for this non-magnetic couette flow. A general theory is still lacking at the present time. The physical reasoning seems to be that the existence of the boundary walls makes the flow less stable and hence, lower critical Taylor number. Although, the generalization of the above conclusion to the magnetic case is not obvious, yet it serves an independent evidence that the stability criterion (4.23) will also be valid for large spacing as well.

8. APPROXIMATE SOLUTIONS FOR $\alpha_0 \geqslant -1$

From the numerical results given in Table 1, it appears that the formula

$$\mathrm{Tc} = \frac{C}{\alpha_0 + 2} \tag{8.1}$$

gives a very good approximation to the true values for $\alpha_0 \geqslant -1$, where $C \simeq 3400$. Moreover, for the same range of α_0, the wave number ($\simeq 3.12$) and the coefficient $\mathcal{G}_3/\mathcal{G}_1 (\simeq -0.001146)$ at which instability sets in, hardly seem to depend on α_0. This is not a surprising phenomenon at all, because, for $\alpha_0 \geqslant -1$ which corresponds to the case that the two cylinders rotate in the same direction in Taylor's problem, the governing equations of the present case closely resemble that of the simple Benárd problem.

For the present purpose, it is convenient to translate the origin of the coordinate system to be midway between the two cylinders. Thus let

$$\xi = x + \frac{1}{2} , \tag{8.2}$$

so that the limits of x are $\pm$ 1/2. In terms of x, Eq. (6.15), after substitution of (6.16), becomes

$$(D^2 - a^2)^3 v_1 = -a^2 S(1 + \epsilon x) v_1 , \tag{8.3}$$

where

$$S = T(\alpha_o + 2)/2 \quad \text{and} \quad \epsilon = 2\alpha_o/(\alpha_o + 2) , \tag{8.4}$$

and the boundary conditions are

$$v_1 = (D^2 - a^2) v_1 = D(D^2 - a^2) v_1 = 0 \quad \text{for} \quad x = \pm 1/2 . \tag{8.5}$$

When $\epsilon = 0$, the characteristic value problem presented by Eqs. (8.3) and (8.5) reduces to the one for the simple Benárd problem (for the case of two rigid boundaries) when formulated in terms of the amplitude Θ of the fluctuations in the temperature.

We first observe that according to the results of Benárd problem the lowest value of S, when $\epsilon = 0$, is 1708 and occurs for $a = 3.117$. By (8.4) the corresponding value of T is 3416/ $(2 + \alpha_o)$; and this is exactly the same as (8.1). Therefore, formulas (8.1) with C = 3416 appears as the zero-order term in a perturbation series. To obtain the high-order terms, we expand the various quantities in powers of ϵ. Thus, we write

$$v_1 = \Theta_o + \epsilon v_1^{(1)} + \epsilon^2 v_1^{(2)} + \cdots ,$$

and

$$S = S_o + \epsilon S^{(1)} + \epsilon^2 S^{(2)} + \cdots . \tag{8.6}$$

Chandrasekhar[13] has obtained a solution, to order ϵ^2, which reads

$$T_c = \frac{3416}{\alpha_0+2}\left\{1 - 7.61\times10^{-3}\left(\frac{\alpha_0}{\alpha_0+2}\right)^2\right\} . \qquad (8.7)$$

This formula gives Tc = 3390.0 and 65.228 for α_0=-1.0 and 50.0 respectively; these values should be compared with 3390.3 and 65.234 given by the "exact" calculations. When α_0 =-1.5, corresponding to $\epsilon = -6$, (8.7) gives Tc = 6364 which has an error less than one per cent from the "exact" solution, Tc = 6417. The above comparisons show that the series expansions in (8.6) converge very rapidly. When $\alpha_0 \rightarrow +\infty$, corresponding to $\epsilon = 2$, (8.7) gives

$$\mathrm{Tc} = \frac{3390}{\alpha_0+2} . \qquad (8.8)$$

In view of the fact that for $\alpha_0 > -1$ the first order formula (8.7) gives values which differ from the results of more exact calculations by less than one per cent, we can easily show that the principle of the exchange of stabilities is approximately valid for that range of α_0. Chandrasekhar[13] showed, to the first order in ϵ, that this principle is valid for $0 > \alpha_0 > -1$. His conclusion can, indeed, be extended to include the range of positive α_0. We observe that for $\alpha_0 < -1$, the possibility of overstability should not be excluded. In fact, overstability can be an important one when the condition (4.27) is violated. As we can see that when ν and η are very small yet the magnetic Prandtl number ν/η is very large or very small there are conditions that can very well be favorable for the on set of overstability. Chandrasekhar has pointed out that it would be particularly worthwhile to explore the case α_0=- 2.

9. DUAL ROLES AND COUNTERBALANCE FIELDS

The dual roles of viscosity and magnetic diffusivity have been discussed by Yih[5] and Lai[3]. However, with the additional branch of solution[14] taken into consideration here, their dual roles are more prominent and interesting.

With Tc substituted for T in (6.17), solution of (6.17) and (6.18) yields the parametric relationship between T_{1c} and T_{2c}:

14. This branchhas been neglected by the previous authors. Therefore, some modification and additional computations are needed for their works. This will be given in a subsequent paper.

$$T_{1c} = \frac{\alpha_0 - \alpha_2}{\alpha_1 - \alpha_2} T_c \quad , \tag{9.1}$$

$$T_{2c} = \frac{\alpha_1 - \alpha_0}{\alpha_1 - \alpha_2} T_c \quad . \tag{9.2}$$

For a given pair (α_1, α_2), values of α_0 are assumed, and the corresponding values of Tc read off from Table I, and T_{1c} and T_{2c} are then computed. Figs. 6 and 7 are stability charts for various pairs of (α_1, α_2). We should note that, from (7.2), negative α_1 corresponds to positive T_1 only and positive α_1 corresponds to negative T_1 only. Likewise, from (7.6), positive and negative α_2 correspondsto positive and negative T_2 respectively. In the figures, the stable regions are indicated by arrows. We observe that the region which contains the origin is always stable.

Since both T_1 and T_2 contain ν, in order to separate the effects of ν and η, the value of $T_2/|T_1|^{1/2}$ is plotted against T_1 in Fig. 8. The ordinate of the curves is

$$T_2 \,/ |T_1|^{1/2} \qquad or \qquad -2\bar{B}_2 L_1^2 d^2 / (|\bar{A}_1|^{1/2} \Omega_1 \eta) \tag{9.3}$$

which is independent of the viscosity. Figure 9 shows an enlargement of the fourth quadrant of the stability chart in Fig. 8. If we hold all other parameters fixed and vary ν, the flow will go through various unstable and stable regions as the horizontal dotted line a - f in Fig. 9 shows. In the region ab, the flow is unstable for the given pair (α_1=-2,α_2=-2). When ν keeps on increasing, the flow will go through a marginal state to the stable region bc and then go through another marginal state to the unstable region cde and eventually go, through another marginal state, to the stable region ef where the viscous dissipation is predominant. On the other hand, if we hold all other parameters fixed and vary η the flow will also go through various regions as the vertical dotted line $\bar{a}$-$\bar{d}$ shows. In the region $\bar{a}\,\bar{b}$, the flow is unstable for the pair (-2,-2). When η keeps on increasing, the flow will go through the stable region $\bar{b}\,\bar{c}$ and end at the unstable region $\bar{c}\,\bar{d}$, where the magnetic dissipation prevails. The above processes are also shown in Fig. 4 and later in Table II. In Fig. 10 there shows an unstable region (b) which is otherwise stable for the circulatory flow or magnetic field alone. An interesting phenomenon is that there is a region (a) where the unstable circulatory flow field is counterbalanced by the unstable circular magnetic field such that the combined fields make a stable flow. This phenomenon appears to be rather striking at the first sight. It is, however, not striking at all but a natural consequence of the criteria (3.16) and (4.23) if we take a close look at the stabilizing and destabilizing mechanism of the flow field.

For the case $\alpha_1 = \alpha_2 < -1$, the basic flow and magnetic fields are of the same shape as shown in Fig. 11. For the basic velocity field alone, the flow is unstable. The origin of the instability lies in the violation of the Rayleigh's criterion in the layers between ab. A sketch of the cell patterns when instability sets in is also shown. The disturbance does not extend much beyond the nodal surface. Therefore, the nodal surface is essentially the dividing surface between the stable and unstable regions. Although, the region bc is stable, its stabilizing influence is not effective at all to stabilize the flow in the unstable region ab. In fact, in this case, the viscosity plays a predominant role in stabilizing or postponing the onset of instability in region ab. Similarly, for the basic magnetic field alone, the field is also unstable. This time, the origin of the instability lies in the violation of the Yih's criterion in the layers between bc. The region ab is stable, yet its stabilizing influence is not effective to stabilize the motion in the unstable region bc. However, when both the velocity and magnetic fields are present, the stabilizing influence of the velocity field in bc is quite effective to stabilize the unstable magnetic field in the same region. Likewise, the stabilizing influence of the magnetic field in region ab is quite effective to stabilize the unstable velocity field in the same region. Thus, the combining fields can make a stable flow.

As a summary of the roles of the various agents in all previous discussions, two diagrams and two tables, from the energy conversion point of view between the basic fields and the disturbances, have been introduced. Fig. 12 is a simplified energy flow diagram for the non-dissipative case. The u_r - disturbance sets in first which will give rise to u_θ - and h_θ - disturbances. Therefore, we can regard u_r as energy receiving and distributing agent. u_θ and h_θ act like energy converting agents which, together with the basic flow and magnetic fields, determine the direction and the rate at which the energy is being transferred. When dissipation comes in, the mechanism is modified accordingly as shown in Fig. 13. Along with Fig. 13, Table II is introduced which shows the predominant term in the different regions shown in the previous example in Fig. 9. The energy transfered between the left side and the right side of the nodal surface is negligibly small; hence, the major energy transfer consists of three parts at each side. The flow is unstable when the inflow (to u_r-disturbance) term is predominant, and it is stable when the outflow term is predominant. We can see that dissipation of u_θ due to ν affects the energy converting efficiency of u_θ, and dissipation of h_θ due to η affects its energy converting efficiency in the same manner. It is this dissipation of u_θ and h_θ that modifies their energy transfering mechanism. We have called this modification as "diffusion" in the previous discussions. This diffusive effect, however, can never bring about instability if only one field is present. But it does reduce the degree of stability. Fig. 13 shows clearly that when magnetic field is stable at one side and unstable at the other side, then if η is increased, the diffusive action makes one side more stable while the other side less stable. The viscosity plays the diffusive role in the same manner. In Table II, the viscosity and magnetic diffusivity in the regions a - d and $\bar{a}$ - $\bar{d}$ essentially plays the same role — diffusive role. It is the competion of the diffusive effectiveness between ν and η in the right or left side of the nodal surface that determines the stability. However, the magnetic diffusivity does not play a corresponding part as the viscosity does in the region ef , where viscous dissipation is predominant. Therefore, as far as the velocity disturbance energy is concerned, the magnetic dissipation hardly has any effect on it; its only effect is through the diffusive mechanism. Since it is the velocity disturbances that determine the stability, we can conclude that the magnetic diffusivity essentially plays a single role only — a diffusive role in the energy conversion mechanism of velocity disturbances. However, this diffusive role alone, together with the counterbalance fields can have an effect of simultaneous stabilization and destabilization.

In closing the present discussion, a comparison of the present roles of viscosity and magnetic diffusivity with that of viscosity associated with Tollmein - Schlichting waves is briefly summarized in Table III. It is clear from the table that they are distinctly different from one another. However, one feature is common to all these three agents that they can have simultaneously stabilizing and destabilizing effects. In both cases, the viscosity plays dual role and has dual effect. But the magnetic diffusivity should not be qualified for the dual role, it only plays a single role with dual effect.

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TABLE I. The critical Taylor numbers T_c and associated constants for various values of α_o.[a]

α_o	a_c	T_c	$\mathcal{E}_2/\mathcal{E}_1$	$\mathcal{E}_3/\mathcal{E}_1$	$\mathcal{E}_4/\mathcal{E}_1$	$\mathcal{E}_5/\mathcal{E}_1$	$\mathcal{E}_6/\mathcal{E}_1$
0.0	3.117	1.7079×10^3	0	-0.001144	0		
0.5	3.117	1.3659×10^3	-0.000795	-0.001144	+0.000004		
1.0	3.118	1.1377×10^3	-0.001325	-0.001145	+0.000006		
2.0	3.119	8.5235×10^2	-0.001986	-0.001145	+0.000009		
3.0	3.120	6.8132×10^2	-0.002383	-0.001146	+0.000011		
4.0	3.121	5.6740×10^2	-0.002648	-0.001146	+0.000012		
5.0	3.122	4.8610×10^2	-0.002838	-0.001147	+0.000013		
10.0	3.123	2.8317×10^2	-0.003309	-0.001147	+0.000015		
20.0	3.125	1.5430×10^2	-0.003611	-0.001148	+0.000016		
50.0	3.126	6.5234×10^1	-0.003820	-0.001148	+0.000017		
$+\infty$	3.127	3390/(+ 2)					
-1.00		$-\infty$					
-1.50	6.10	-1.9122×10^5	-0.41001	+0.061602	-0.001279	-0.000927	+0.000267
-1.60	5.43	-9.9621×10^4	-0.29049	+0.023515	+0.001123		
-1.70	4.96	-5.8885×10^4	-0.20709	+0.008411	+0.001172		
-1.80	4.61	-3.8072×10^4	-0.14771	+0.002446	+0.000875		
-1.90	4.29	-2.6120×10^4	-0.10326	-0.000004	+0.000594		
-2.00	4.00	-1.8677×10^4	-0.071389	-0.000929	+0.000391		
-2.25	3.49	-0.9438×10^4	-0.031633	-0.001259	+0.000155		
-2.50	3.30	-0.5853×10^4	-0.019342	-0.001217	+0.000091		
-3.00	3.20	-3.2074×10^3	-0.011882	-0.001181	+0.000055		
-3.50	3.17	-2.1895×10^3	-0.009264	-0.001169	+0.000042		
-4.00	3.16	-1.6587×10^3	-0.007965	-0.001165	+0.000036		
-5.00	3.14	-1.1155×10^3	-0.006597	-0.001153	+0.000030		
-10.00	3.13	-4.2202×10^2	-0.004956	-0.001149	+0.000022		
$-\infty$		0					
-0.50	3.118	2.2753×10^3	+0.001325	-0.001145	-0.000006		
-1.00	3.127	3.3903×10^3	+0.003973	-0.001149	-0.000018		
-1.50	3.20	6.4148×10^3	+0.011882	-0.001181	-0.000055		
-2.00	4.00	1.8677×10^4	+0.071389	-0.000929	-0.000391		
-2.50	5.06	4.619×10^4	+0.2281	0.0116	-0.0012		
-3.00	6.10	9.558×10^4	+0.4099	+0.0616	+0.00128	-0.000927	-0.000267
-3.50	7.10	1.771×10^5	+0.5804	+0.1560	+0.01777	-0.000459	+0.000819
-4.00	8.14	3.025×10^5	+0.7499	+0.2846	+0.0626	+0.006064	+0.001045
$-\infty$	$2.035\alpha_o$	$1182\,\alpha_o^4$					

[a] values of positive Tc for negative α_o from Ref. 13.

TABLE II. Engergy balance sheet for the u_r -disturbance

	Region	Left side energy in $(1/\nu)$		Left side energy out $(1/\eta)$	Left side energy out (ν)	Right side energy in $(1/\eta)$		Right side energy out $(1/\nu)$	Right side energy out (ν)	State
ν increase ↓	a	P						P		unstable
	b		=					P		marginal
	c			P	P		=			marginal
	d			P	P	P				unstable
	e			P	P		=			marginal
	f				P				P	stable
η increase ↓	$\bar{a}$			P		P				unstable
	$\bar{b}$			P			=			marginal
	$\bar{c}$		=					P	P	marginal
	$\bar{d}$	P						P	P	unstable

P: predominant term
$(1/\eta)$: magnetic diffusion
$(1/\nu)$: viscous diffusion
(ν): viscous dissipation

TABLE III. Comparison of the roles and effects of the dissipative agents for Poiseville flow and magnetic Couette flow

type of flow	agent	role		effect	condition
Boundary layer or Poiseuille flow	ν	dissipation	—	stabilizing	
		conversion	—	destabilizing	solid boundary
Magnetic Couette flow	ν	dissipation	—	stabilizing	
		diffusion	—	stabilizing and destabilizing	counterbalance fields
	η	diffusion	—	stabilizing and destabilizing	counterbalance fields

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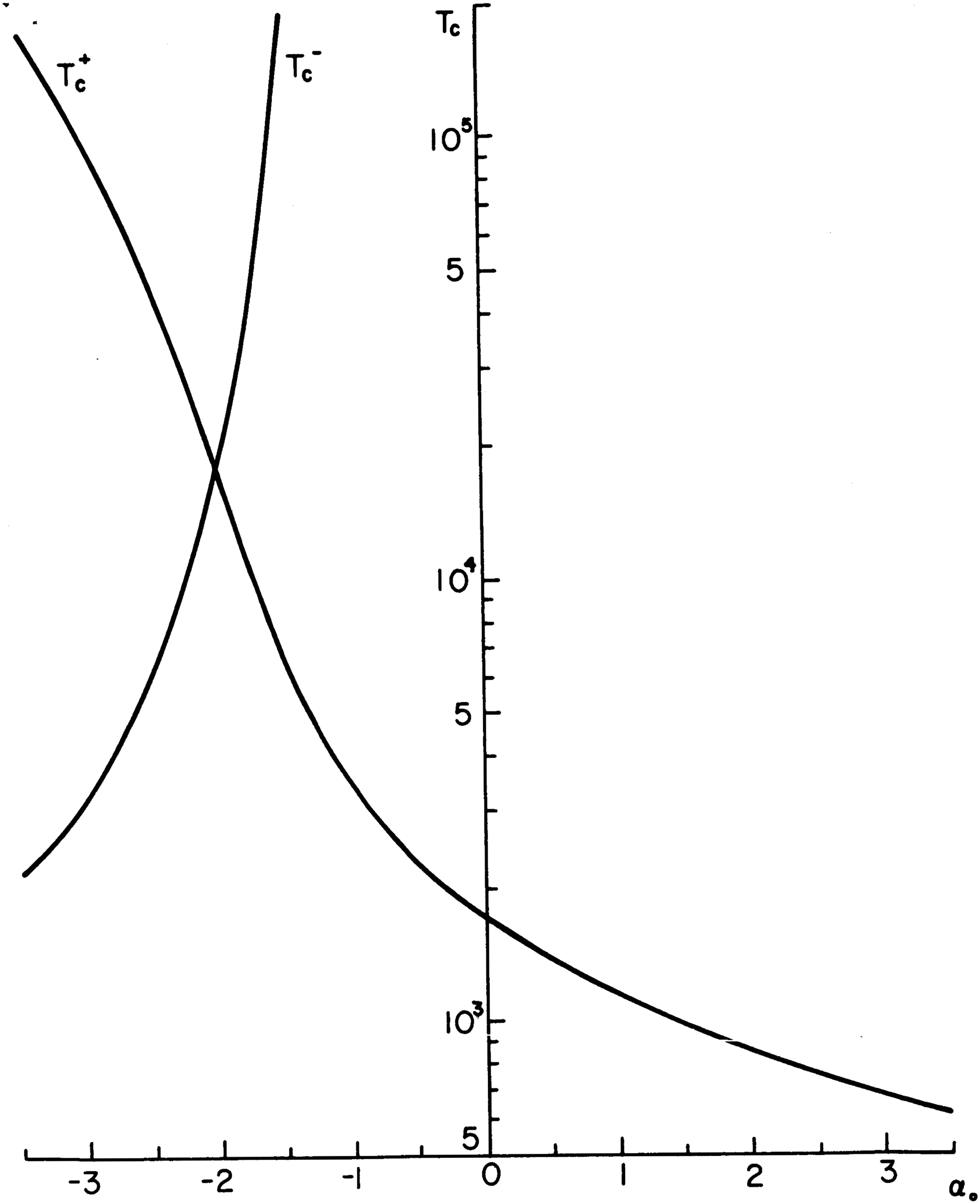


Figure 1. The critical Taylor number T_c for the onset of instability as a function of α_o. T_c^+ is the positive branch and T_c^- is the negative branch.

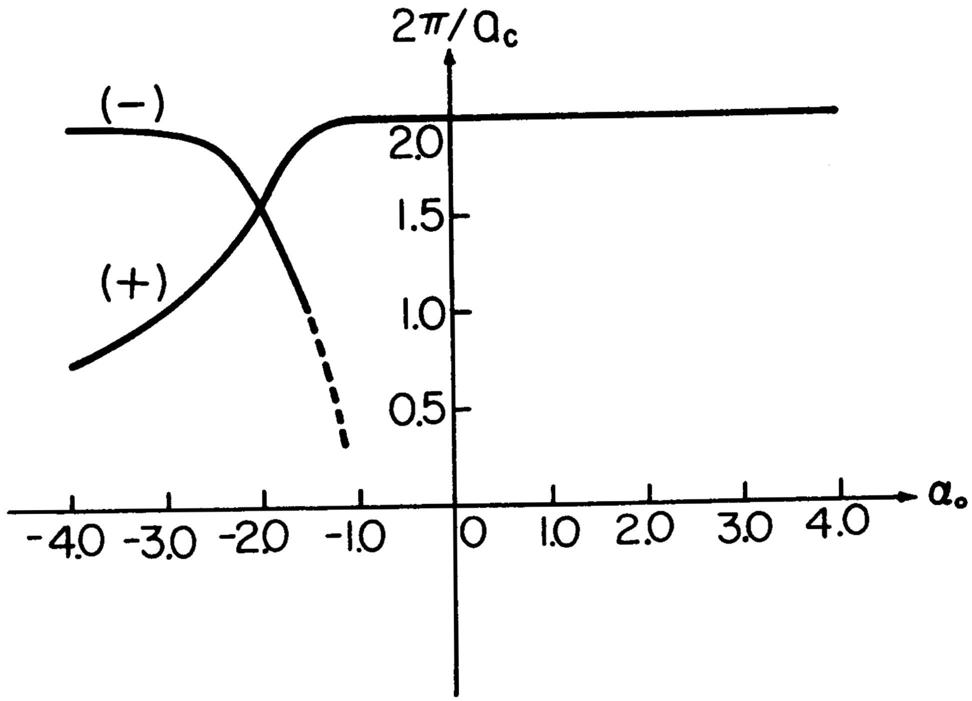


Figure 2. The wavelength of the disturbance (in units of the gap width d) manifested at onset of instability as a function of α_o. (+) corresponds to T_c^+ branch and (-) corresponds to T_c^- branch.

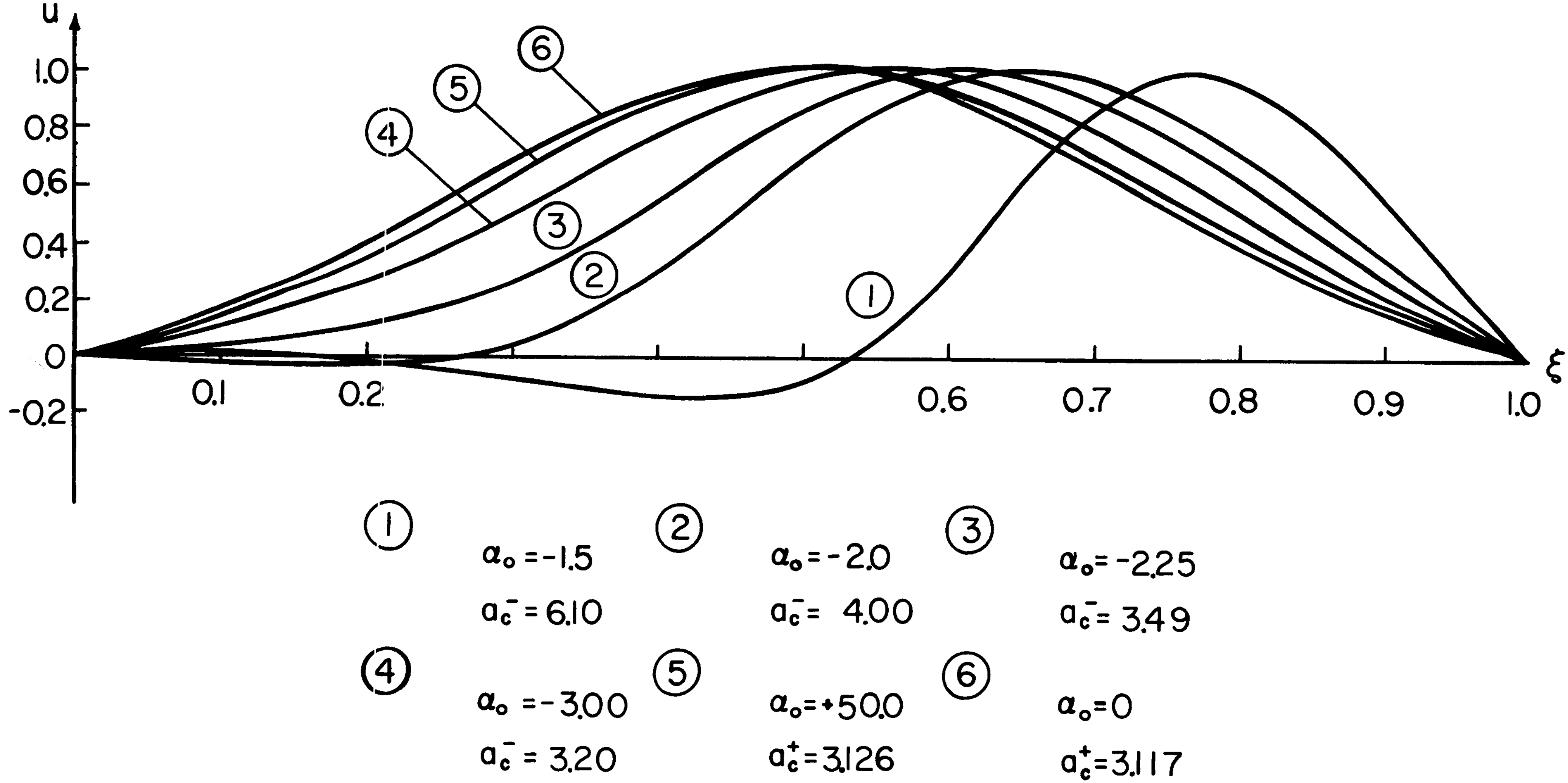
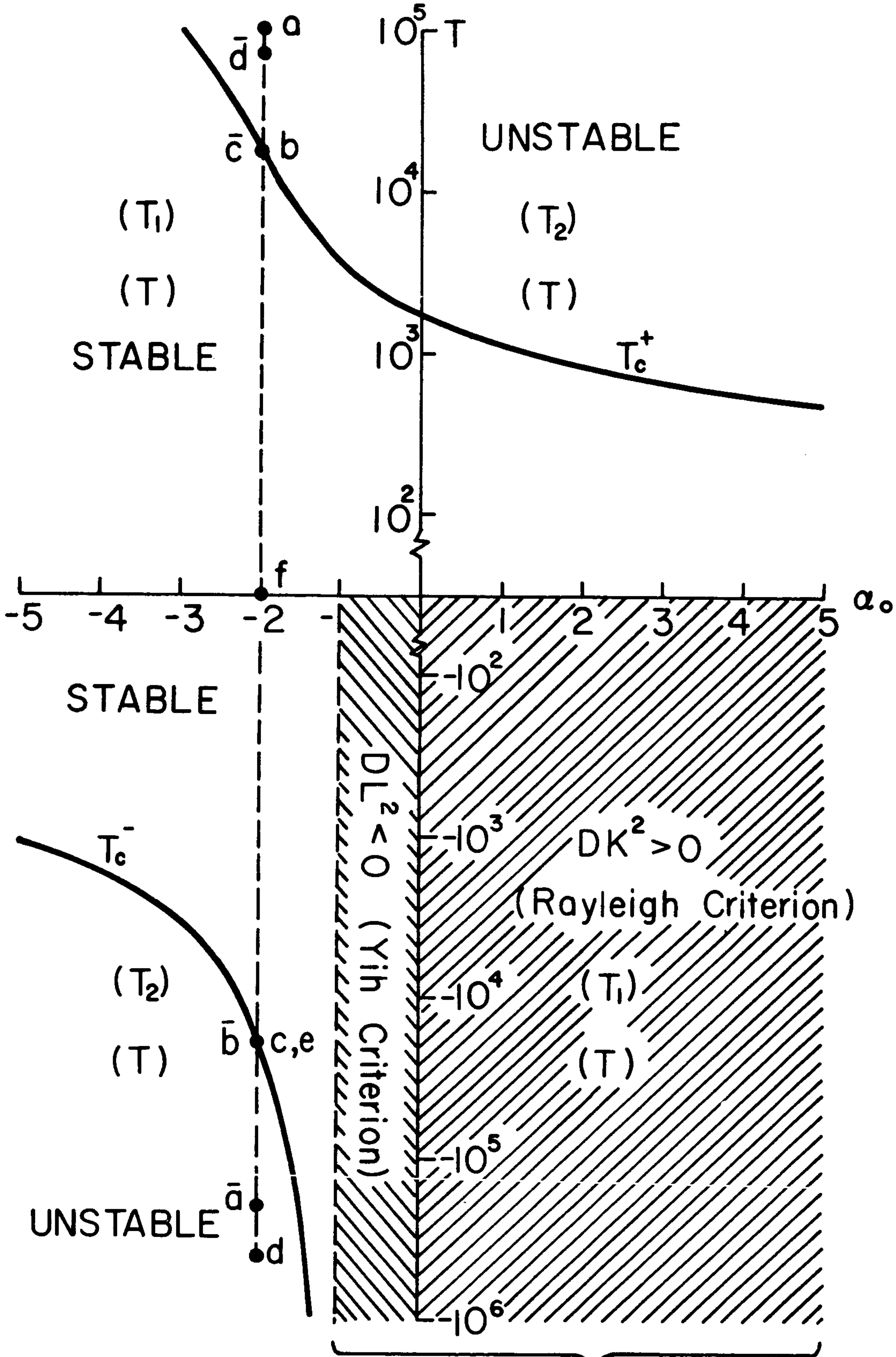


Figure 3. The velocity profile u at the onset of instability for various α_o.



$$\frac{r^{-3}}{\nu} DK^2 - \frac{r}{\eta} DL^2 > 0$$

(Criterion (4.23))

Figure 4. T - α_o relationship. The regions with shaded lines are stable regions by respective stability criterion.

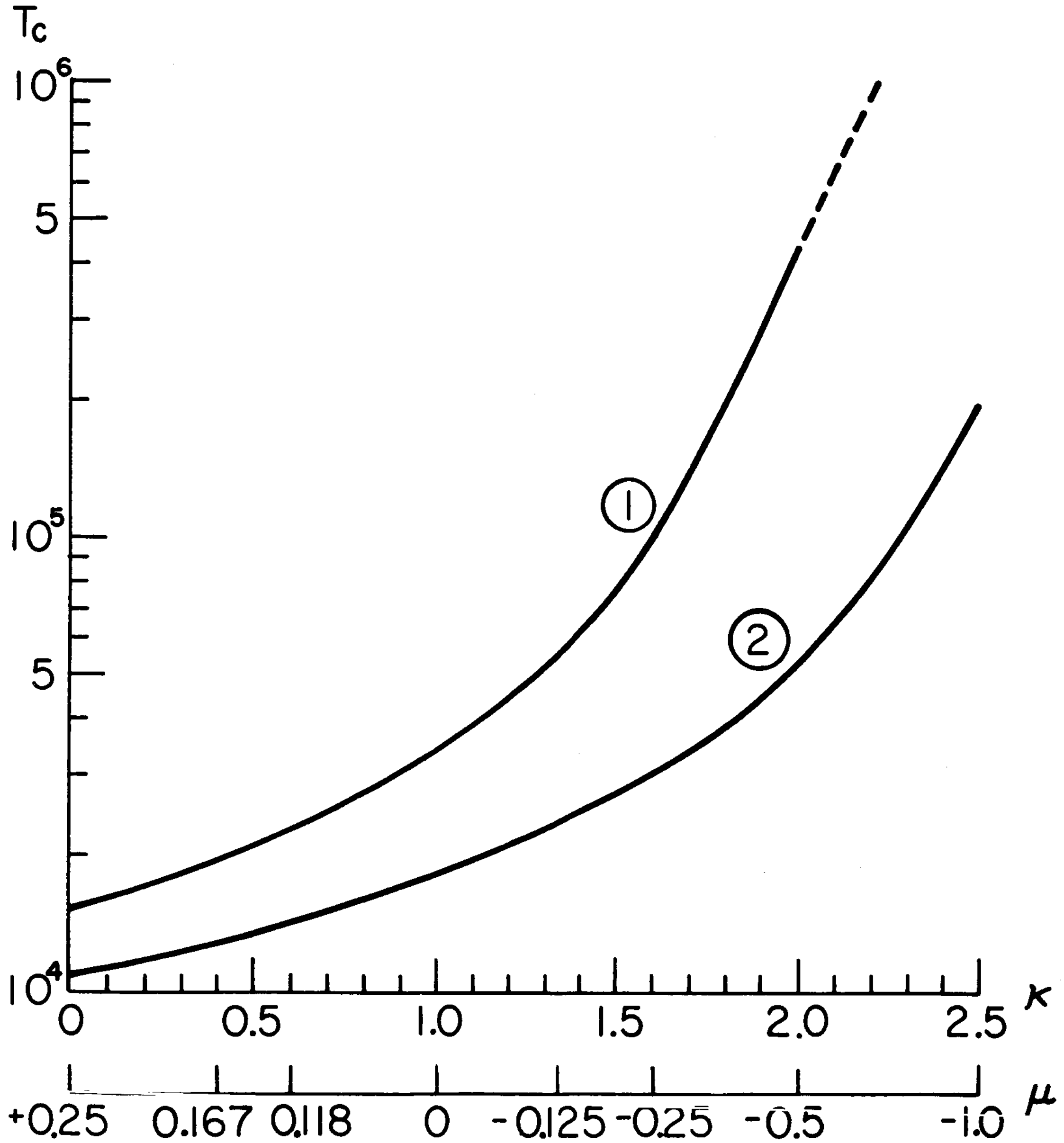


Figure 5. Comparison of the critical Taylor number T_c from the small spacing result (curve 2) and the large spacing result (curve 1).

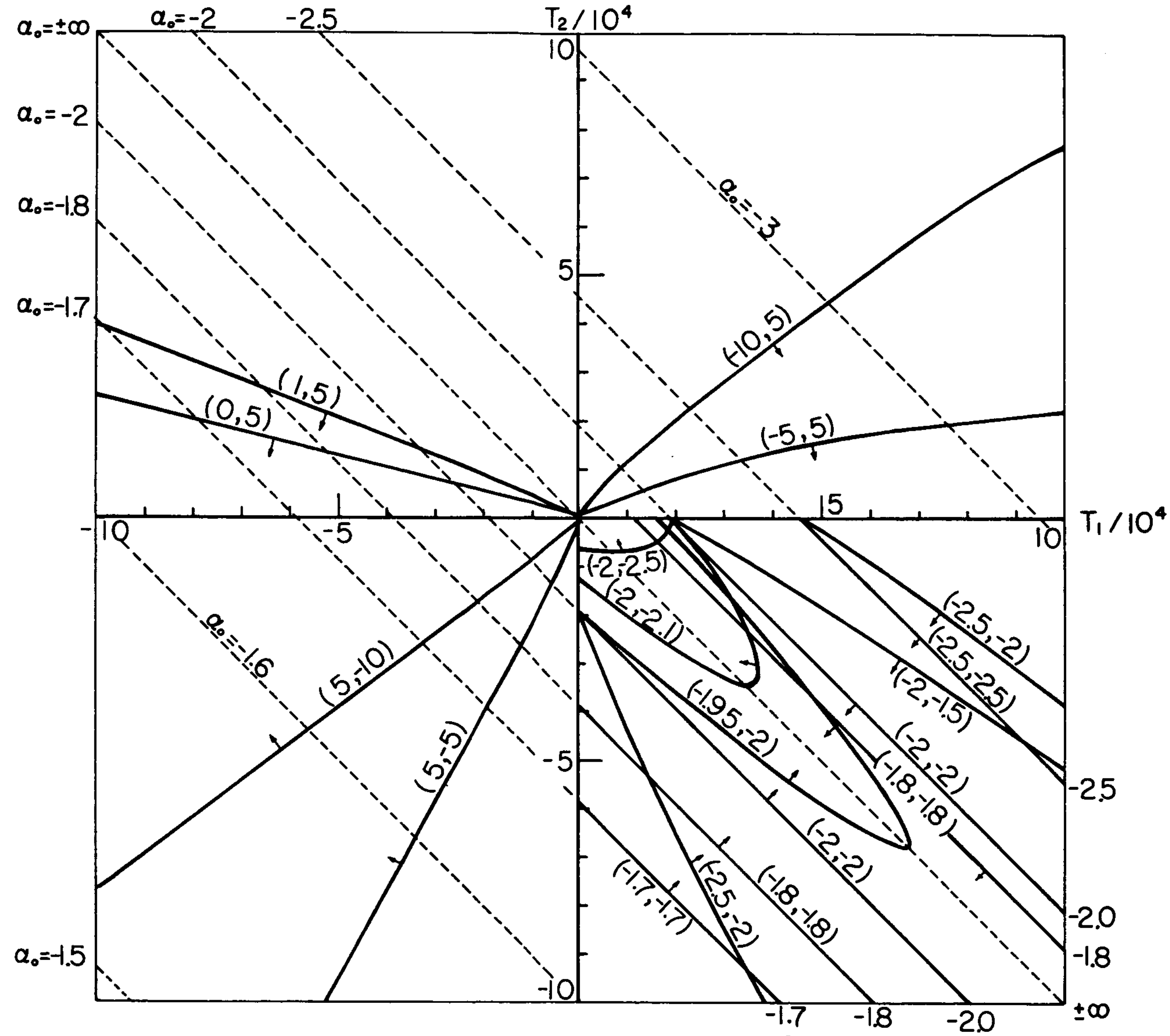


Figure 6. Stability chart for the case of small spacing. $T = -4\bar{A}_1 R_1^2$, $T_2 = -4\bar{B}_2 M^2$. For each pair (α_1, α_2), the region indicated by the arrow corresponds to stability, and the region at the other side corresponds to instability.

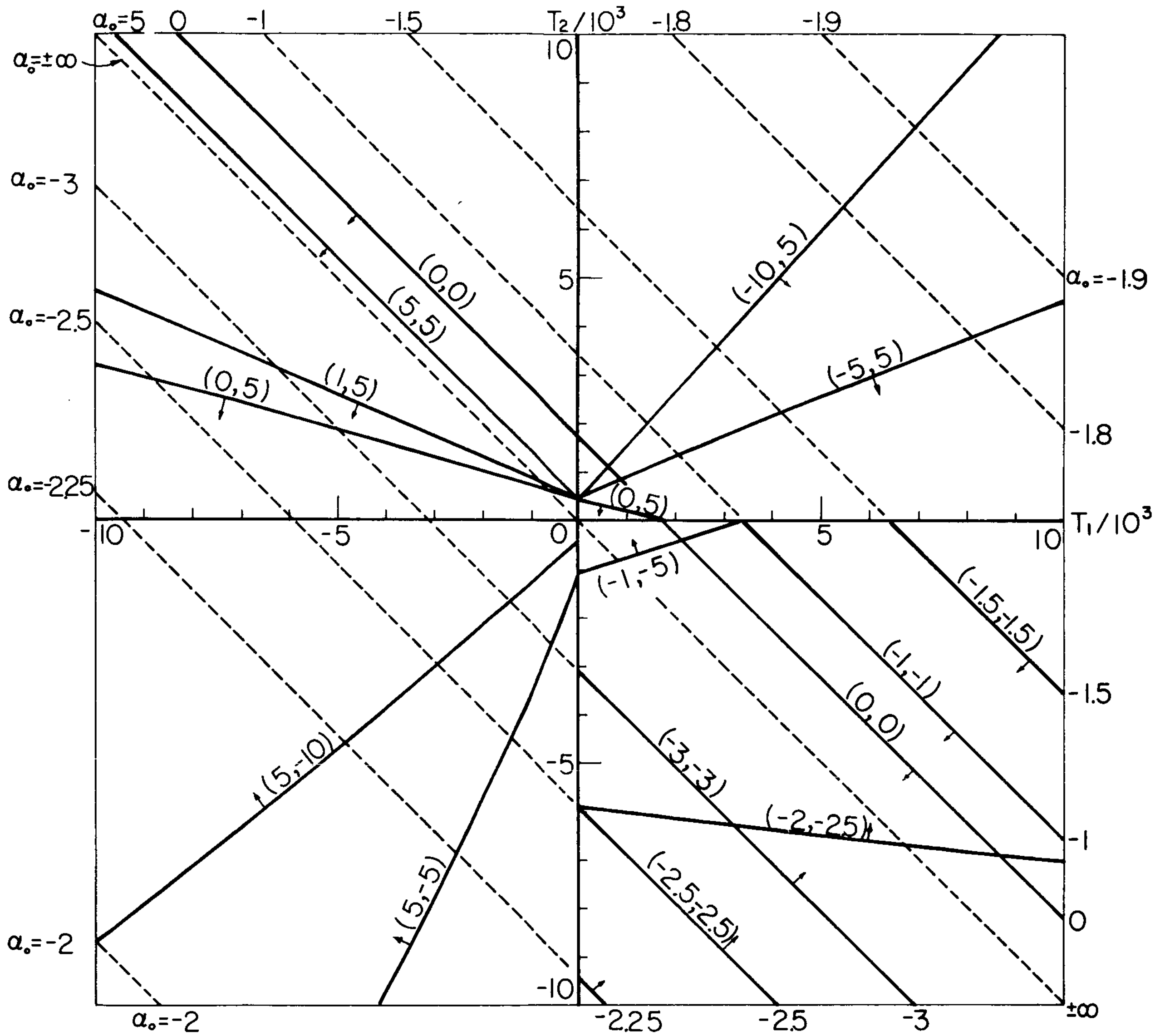


Figure 7. Stability chart for the case of small spacing. The symbols and notations have the same meanings as in Figure 6.

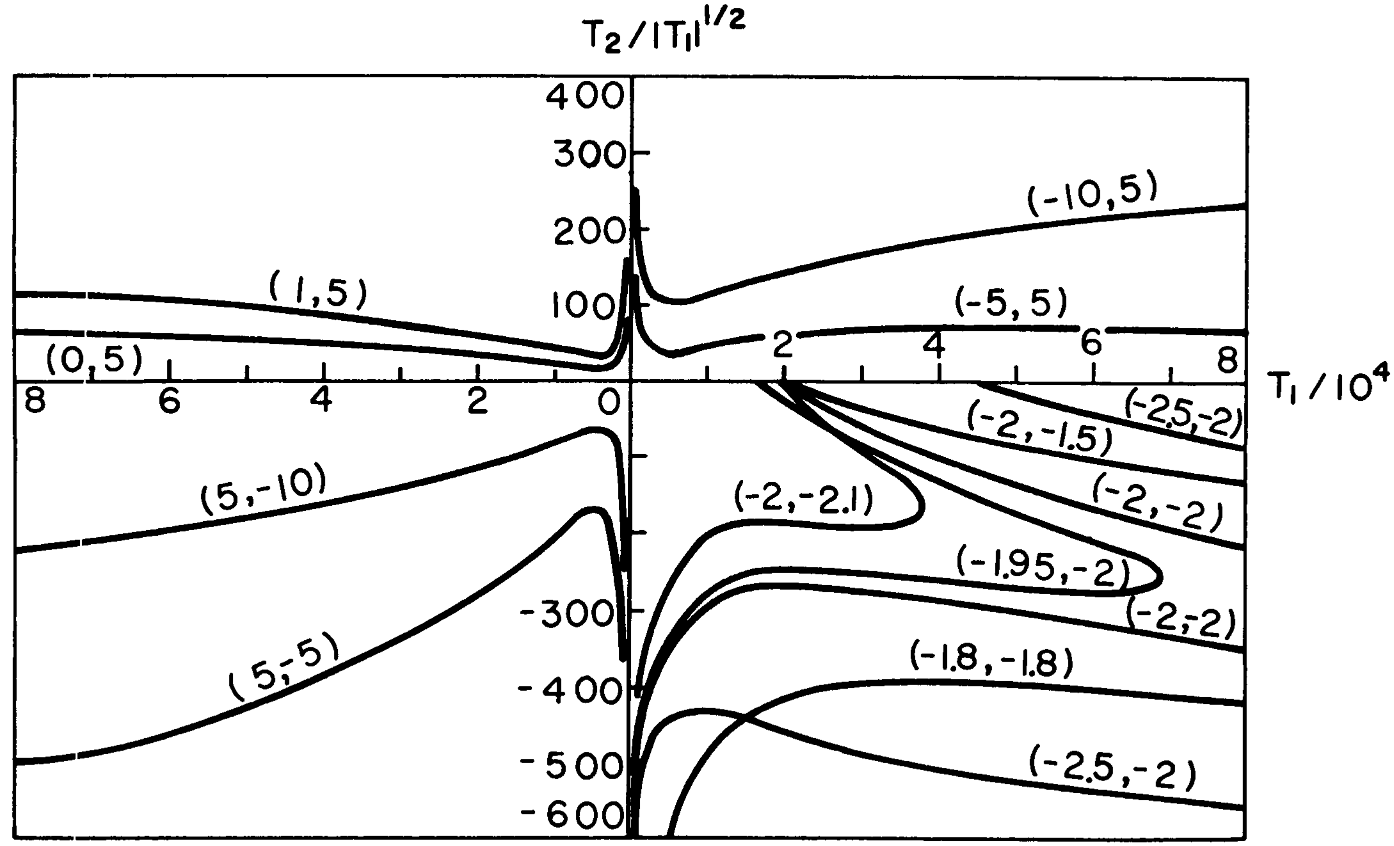


Figure 8. Stability chart for the case of small spacing. The symbols and notations have the same meanings as in Fig. 6. $T_2/|T_1|^{1/2} = -2\bar{B}_2 L_1^2 d^2/(|A_1|^{1/2}\Omega, \eta)$

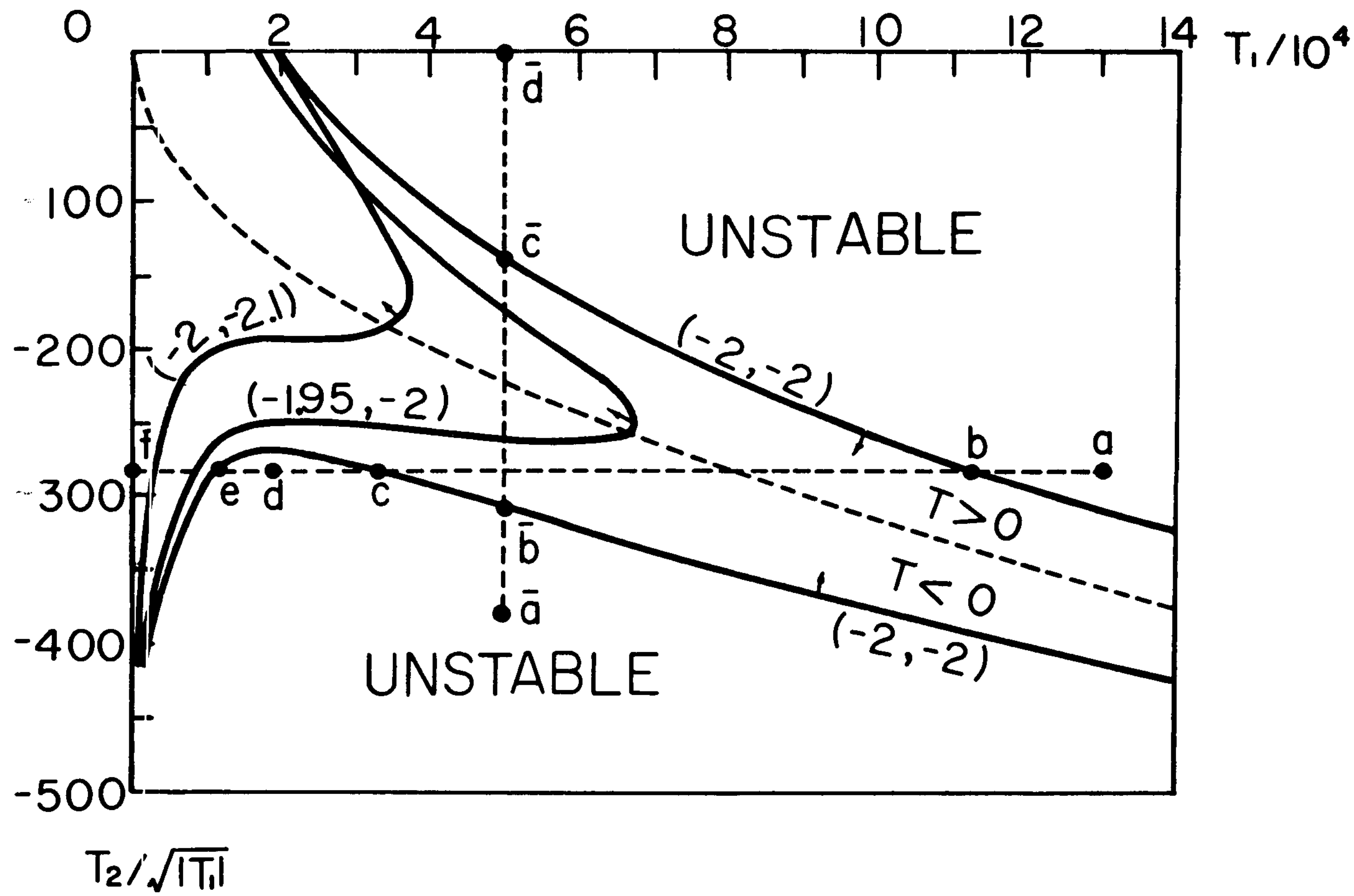


Figure 9. Enlargement of the fourth quadrant of the stability chart in Figure 8.

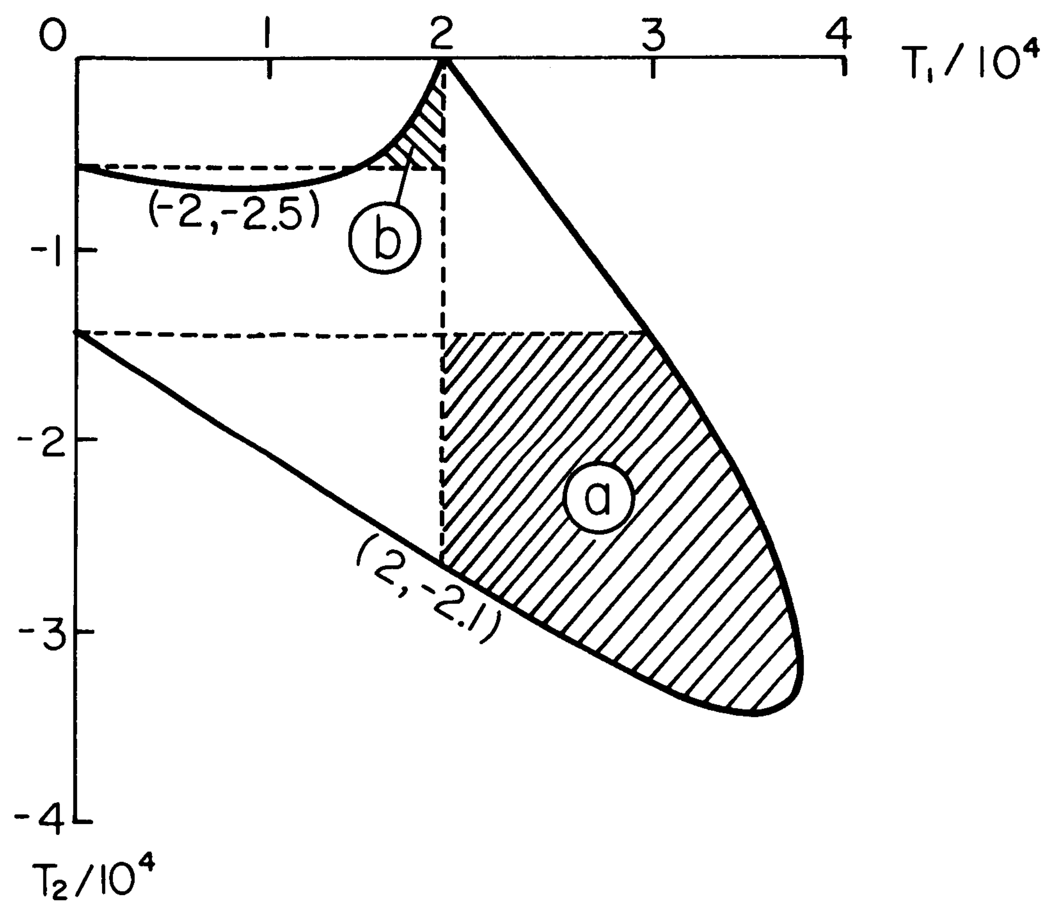


Figure 10. Enlargement of the Fourth quadrant of the stability chart in Figure 6. (a) stable region; (b) unstable region.

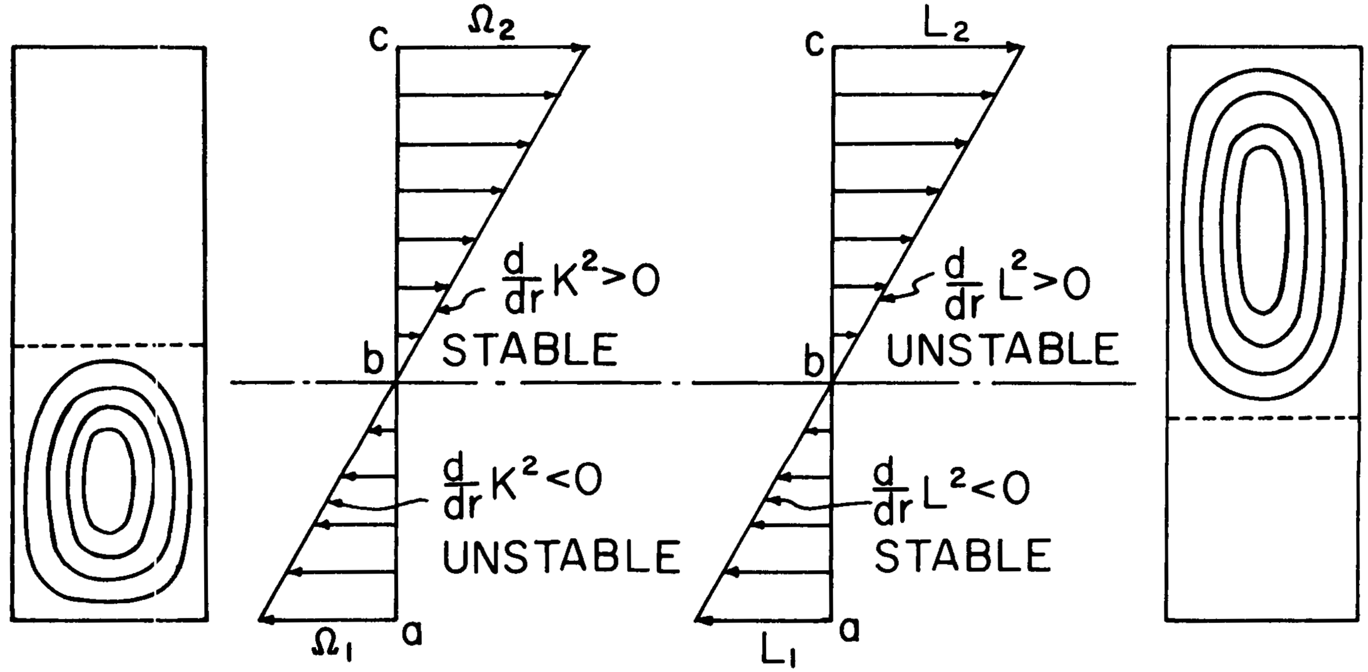


Figure 11. Basic flow and magnetic fields.

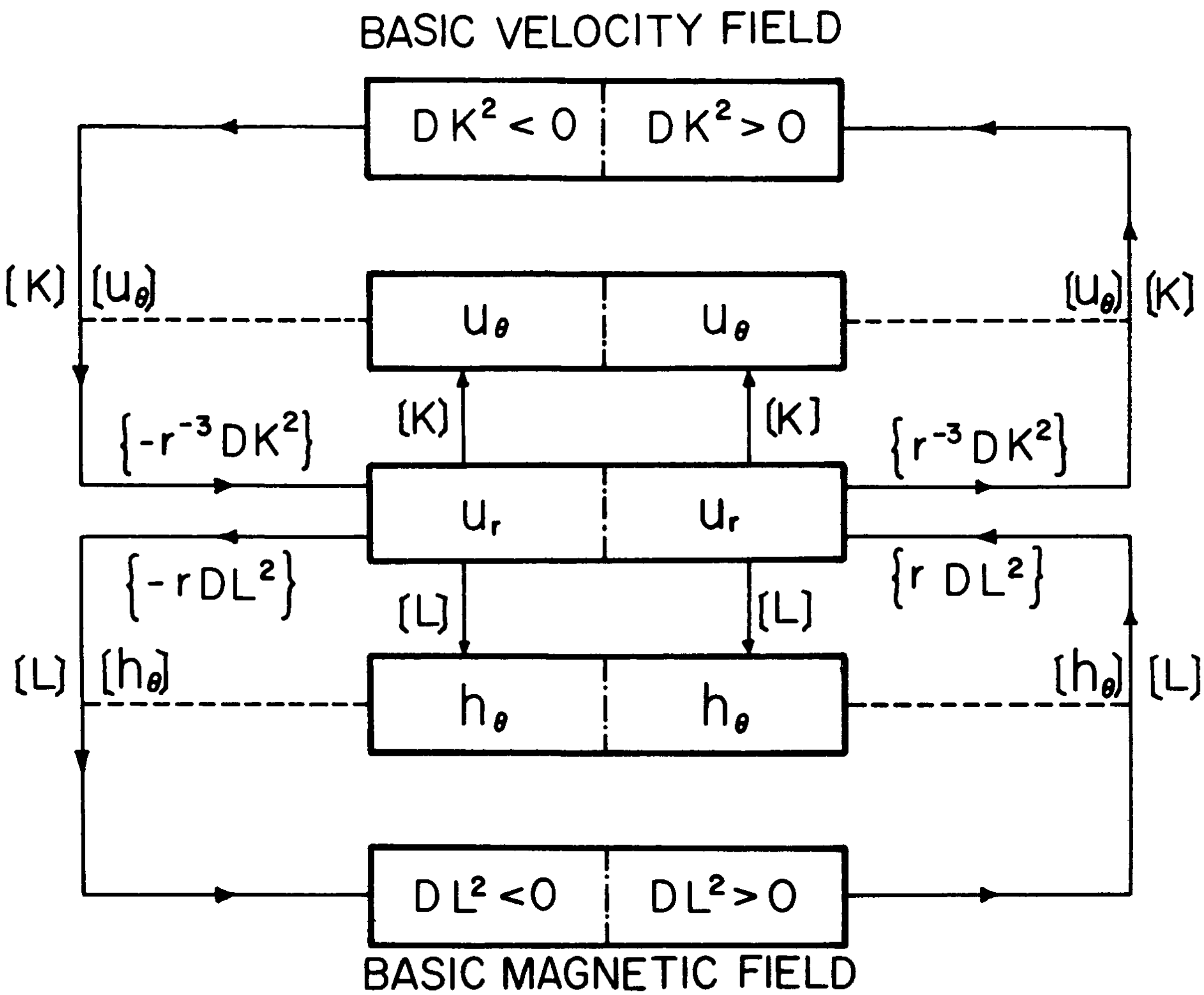


Figure 12. Energy balance diagram

[ ] - energy converting agent { } - energy converting rate

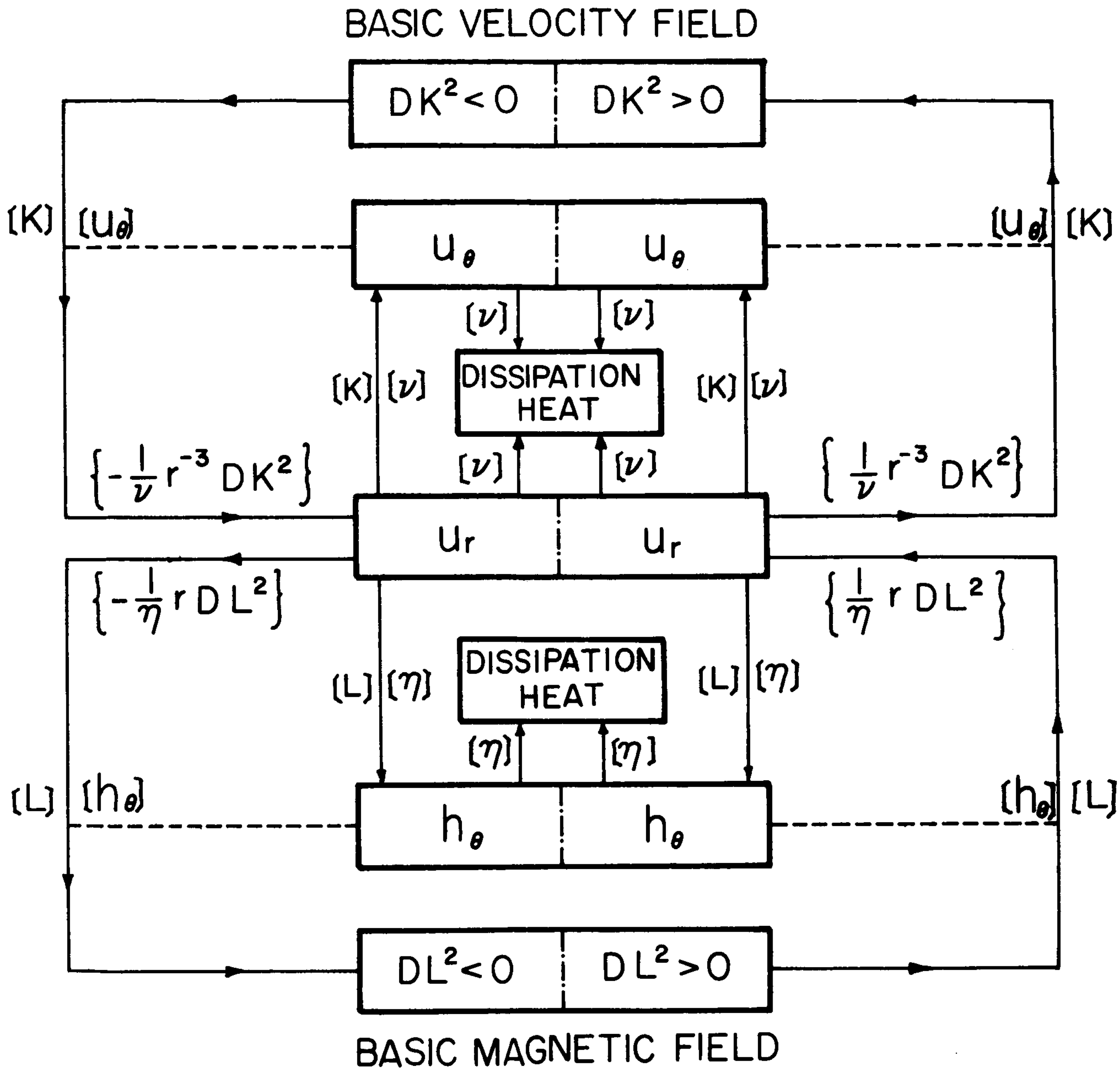


Figure. 13 Energy balance diagram

[ ] - energy converting agent { } - energy converting rate