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The Earth-Moon System in the Light of Recent Discoveries in Space Science

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Lecture 1: <u>The Physical Setting of Geodesy</u>.

The study of the figure of the earth has its historical roots in studies made by geodesists. These studies came from two sources: One was the detached scientific desire to know more about the figure of the earth which moved the Erathosthenes and Snell; the other was the practical urge to produce adequate maps which moved the Cassinis and Digges. The scientific motivation for the study of the earth is relatively easy to understand, but I should like to take your time to point out some of the practical reasons which have powerfully reinforced scientific motivations.

The practical surveyor is attempting to construct a map which will serve the ordinary purposes of daily life. For some of them, such as hiking or automobile travel, an accuracy of 1 percent is more than sufficient. For others, including the problem of artillery firing, the laying out of pipe lines, the emplacement of micro-wave antennae and the putting in of telephone lines an accuracy of a tenth of 1 percent would be desirable so far as the paper stability permits it. These accuracies would not by themselves justify the precision which is lavished on first order triangulation. It might appear possible to make relatively crude surveys and patch them together. In practice, however, it is found that this policy is extremely expensive and that it is far more satisfactory to have an underpinning of precise survey. What happens when you have a set of inaccurate maps is that in the compilation room the conflicts between the maps appear. For example, suppose that the maps are in error by 1

[redacted]

percent; then along the junction between two individual sheets you may have an error of a few tenths of an inch, which might be tolerable; but when you have joined together 20 or 30 such maps to form a loop or an area, then you find that there are discrepancies of many times this amount where the loops close. Since the mapping of even so small an area as France involves several hundred map sheets, this procedure is evidently very unsatisfactory.

Theoretically again one could go into the compilation room and say to the other compilers that they should distort their sheets in such a way as to produce a unified whole and that you don't care how they do it. If this is resorted to, then enormous waste and delays will ensue. The compilers will want to work on the area a little at a time. Left to themselves they will crowd all the errors into one area where they become intolerable, or they will start in two different areas and when these two areas join an intolerable discrepancy will be found. In the meantime endless discussions will rage among the compilers as to how this problem is to be met. Since the compilers are very numerous compared to the first order triangulators, the net loss is very large indeed.

Just prior to the German invasion of France in 1940 there was a conference among the allies about the problem of the adjustment of the Dutch, Belgian and German map and survey data to agreement with the French. The plan called for the recalculation of the Belgian and Dutch triangulation starting from French triangles. German triangulation was adjusted by applying blanket corrections to the latitudes and longitudes. Since these corrections left a discrepancy on the order of 11 meters between certain points of Holland and Germany, a graph was prepared.

This graph was intended to adjust not the map data but only the lists of surveyed points which were supplied to the artillery for their purposes.

When it is a matter of adjusting the triangulation between several countries, it is an enormous advantage if there exists a framework so precise that each of the several countries involved will accept it as superior to its own. The reason is that when a staff conference is held, each of the military officers in the conference is representing a group of civilian employees whom he cannot easily consult. A few of them may be sitting back of the conference table at his elbow, but the great majority are necessarily left at home. He cannot easily make concessions. The question of national pride is deeply involved. To adopt the proposal of another country when it is obviously unscientifically constructed and to distort one's native maps and surveys to fit it is felt as humiliating and is resisted. If, on the other hand, the proposal for survey unification is scientifically drawn and will represent an overall improvement in the survey situation even in the separate countries, then acceptance is much more readily secured.

Thus we see that <u>precision in survey is a tool of the high command</u>.

In securing survey precision one obstacle is more serious than any other and sets a limit to the precision that is reached. This obstacle is the crookedness of the path of light through the atmosphere. Let us remember that at the moment when we see the sun's lower limb touch the horizon, the whole of the sun would be below the horizon if there were no atmosphere. That is to say the refraction amounts to one-half a degree on long rays through the atmosphere. If we compare the curvature of some 1800 seconds of arc with the desired angular precision which is less than a single second of arc, we see the magnitude of the problem which the geodesists must face.

Using everything except the graph, the U.S. Army Map Service prepared a series of maps of Holland. The maps were compared with the coordinate lists. Since the Dutch maps were on the stereographic projection, there was felt to be some uncertainty about putting them into the framework of the Lambert projection used by the French. These worries became acute when it was discovered that the original Dutch stereographic coordinates could not be converted to satisfactory agreement with the British coordinate lists by the aid of information available to the U.S. In the meantime the invasion of France by the Germans resulted in the loss of important portions of the records. For several months, efforts continued at the Army Map Service and in the Corps of Engineers to discover some mathematical discrepancy which would explain the difference between the American coordinates lists and the British lists. During this time the printing of the maps was delayed. The discrepancy was finally explained when the British produced the graph, but the dislocation of the map production program had serious effects on the later conduct of the war. Had there been an orderly and well understood program, this delay would not have occurred.

It turns out that the only way of adjusting a whole series of maps to agreement with one another is to provide a precise framework for the area as a whole and to pin each map to that framework. Of course the framework itself must suffer arbitrary adjustments which are disguised as least-squares solutions, but the magnitude of the discrepancies which are tolerated here can be kept below the level which is detected by the compilers. As a result the inevitable squabbling about how those discrepancies are to be adjusted can be confined to a relatively small number of people. Here the sternest practicality indicates the need for precise survey data.

It is characteristic of geodesy that the method by which this problem is attacked is the use of the gravitational field of the earth. In the determination of height above sea level, in the determination of position on sea level, and in the exploration of the sea level surface itself, the geodesist takes advantage of the gravitational field of the earth to correct the errors arising from atmospheric refraction.

The first example is the measurement of height. When it is impossible to avoid it, vertical angles are sometimes measured between points whose relative elevation is to be found. The inevitable effects of the curvature of the ray are minimized so far as possible by measuring reciprocally over the line; that is, measuring the angular elevation of B as seen from A and the elevation of A as seen from B simultaneously. It turns out that this procedure eliminates the effect of the mean curvature over the line. It does not, however, eliminate higher order difficulties, and the angular accuracy which is attainable is on the order of one ten-thousandth or one twenty-thousandth of the distance. Here it will be noted that by referring the angles to the zenith at both ends of the line, some use was made of the earth's gravitational field.

A far more effective use arises when the line is cut up into a large number of small pieces and the relative elevations are determined section-by-section. The best instrument for this purpose is the spirit level. In practice, the surveyor puts the spirit level at the center of the small section which he is measuring, he sets the optical axis level and points first at the rod ahead and then at the rod in back or vice versa. Through his telescope he can read the height of the mark on the rod to which his telescope is pointing. The difference of the two rod readings is a very good approximation to the difference between the heights of the feet of the rods. The curvature of the ray is much less troublesome on a short section since its effects increase with the square of the distance. Thus a section one kilometer long cut into 100 meter bits will have only one-tenth the total amount of curvature that the whole kilometer piece would have had. Moreover, by measuring both forward and backward from the middle of the line, the surveyor is able to make the effects of curvature cancel on each separate line. The ray curves downward from the instrument toward the mark by the same amount in both cases. By this method of spirit leveling it is possible, for example, to determine the heights of points in the center of the United States with an accuracy of a few tenths of a meter referred to tide gauges on the coast. At a distance of a few thousand kilometers these tenths of a meter subtend angles of only a small fraction of a second of arc. We see that the curvature of the ray has in a certain sense been straightened out by continual reference to the direction of the vertical.

In measurements of horizontal position, again we find that the properties of the gravitational field are used. It turns out that the ray of light is curved in a direction perpendicular to the stratification of the atmosphere. This stratification is in nearly horizontal layers. If, therefore, the geodesist measures angles in the horizontal plane his angles will be nearly free of the effects of refraction. It turns out that on a day when vertical angles are distorted by many minutes of arc, the horizontal angles as measured will be accurate within a fraction of a second of arc.

Since the days of Pierre Bouguer, in the middle of the 18th century, it has been customary to represent the results of such angle measurements as these by supposing them to have been measured on an imaginary prolongation of the sea level surveys under the land. This prolongation is called the geoid. In order to bring the measured lengths into the same intellectual framework, it has been customary since the time of Bouguer to reduce the lengths to the values which they would have had if measured at sea level between the points vertically below the actual ends of the measured pieces. Thus the net result of an extensive triangulation measurement is the fixing of angles and lengths as if they had been measured on the geoid. They are accompanied at the same time by spirit leveling measurements which give heights above the geoid.

In all of the above the question of the exact form of the geoid is systematically ignored. For local surveys it is possible to get by with the assumption that the earth is flat. No significant distortions of horizontal angles will appear unless the triangle approaches an area of 100 square kilometers. For more extensive surveying up to the size of a state of the U.S., it is often possible to get by with the assumption that the earth is a sphere. Even in national surveys it is possible to make a precise computation assuming that the earth is an ellipsoid of revolution, but not troubling to get the exact parameters of the ellipsoid. These methods are perfectly adequate as long as the measurements are only those of horizontal angles or lengths along the surface, and as long as the results which are desired from the measurements are of the same kind. In particular, the heights which are wanted for the construction of dams or the laying of pipes or other hydraulic problems are of just this kind. The notion of the true form of the geoid is merely parasitic in most ordinary engineering applications of geodesy.

The mathematicians have been confronted with a situation which they thoroughly enjoy. The problem is to divise coordinate systems and methods of thought in which it will be possible to move about over the surface of the earth in the spirit of a two dimensional being who does not know that there is such a thing as up and down. The problem is one of great mathematical interest. Some of the most beautiful of the papers of Gauss concerned themselves with this problem, and the modern theory of relativity inherits its point of view and many of its mathematical techniques from Gauss, his pupil, Riemann, and his successors, the founders of tensor analysis.

The geophysicists never really liked this situation and were constantly endeavoring to find out something about the form of the geoid. They got very little support from the practical people until the modern age of the intercontinental ballistic missiles, the earth satellite and the space probe. For each of these, what is needed is the true x, y, z coordinate of the tracking station referred to the center of the earth. To convert the measurements made on the geoid to measurements referred to the center requires a knowledge of the shape of the geoid, and it is with this we will concern ourselves next.

The first approximation to the form of the geoid which is in practical use today is the assumption that it is an ellipsoid of revolution with a semi-major axis a, and a semi-minor axis b. Instead of giving b, it is more customary to give the quantity $\frac{a-b}{a}$ which is called f for flattening. The measurement of these two quantities was originally made by determining the radius of curvature at various latitudes. The first determination was made in the 18th century by the expedients of the French academy to Peru and Lapland. The method has remained in vogue with improvements right up to the work of Chovitz and Fischer on the Hough's spheroid in 1956. In recent times, however, there has been a tendency to rely on measurements of gravity for the determination of the flattening. There has also been a tendency to obtain the flattening from the relationship between the constant of precession and the hydrostatic theory. It turns out, in fact, that measurements of the radius of curvature do not give particularly reliable measures of both quantities a and f. Instead, they give a relation between the two.

Once an ellipsoid has been assumed, the geodesists concern themselves with the deviations between the actual shape of the geoid and that of the assumed ellipsoid. Several methods of measuring these undulations of the geoid are in use.

In the first place, it is possible to make astronomic measurements of latitude and longitude along a triangulated arc. Each measurement of latitude and longitude amounts to a determination of the direction of the vertical at that point. When this is compared with the calculated direction of the vertical, the so called geodetic latitude and longitude, the differences which appear are called the deflection of the vertical or perhaps the deflection of the plumb, depending on whether we think of ourselves as looking upward or downward along the vertical. Each deflection of the vertical can be thought of as giving the slope of the geoid with respect to the ellipsoid at a particular point. If we combine these deflections, we can build up a picture of the height of the geoid above the ellipsoid in much the same way as a picture is built up of the form of the topography by clinometric measurements, i.e., measurements of the slope. The process is called astronomical leveling, and it is found that with a reasonable distribution of the astronomical stations, a precision of the order of a few meters can be reached. The weakness of this method lies in the fact that only relative heights are determined. An initial height above the ellipsoid must be quite arbitrarily assumed. Hayford arbitrarily assumed a height of +10 meters at Calais, Maine. It was also necessary to make a more or less arbitrary assumption about the place at which the slope of the geoid matches that of the ellipsoid. For the United States, the <u>average</u> slope of the geoid matches that of the ellipsoid very closely; for France the two are made equal for five astronomic stations near Paris; for England they are equated at the old Greenwich Observatory; for Spain at the observatory in Madrid, and so on.

Another method, having a different set of troubles, relies upon gravity. If gravity data were available for the whole earth then it would be possible, according to a theorem worked out by G.G. Stokes, to determine the gravitational potential at every point. The underlying idea can perhaps be put in the following way. The intensity of gravity as it is measured at any point depends essentially on the integrated mass in a unit column under the station. In its effect on the gravity meter, a layer which is at a depth of several kilometers has no less effect than one which is only a few meters down. The reason is that while a single gram would be much more effective when nearby than when far away, yet in terms of its contribution to the vertical component of gravity it is only the chunks which are within a reasonable angle from the vertical that matter. The amount of any layer which is within a cone of say 45° from the vertical will be proportional to the square of the distance from the station, and this increase in the amount of material balances the decrease in the effectiveness per gram, so that in a horizontally stratified earth the intensity of gravity is a fair measure of the column integral of the mass. As a consequence, it is possible in many cases to formulate the application of Stokes' principle by imagining the earth to consist of a shell with a surface distribution of matter which is proportional to the intensity of gravity at the point. The elaborate integrals which appear in Stokes' equation are, in fact, not much more than the expression of this idea.

It will be seen at once that the effectiveness of Stokes' theorem depends on a reasonably complete knowledge of the intensity of gravity over the earth. Any gaps in our knowledge will inevitably falsify the potential, not only as far as the absolute value of the slope is concerned, but even the shape of the geoid. On the whole, the dimensions of the form of the geoid from gravity are usually found to be more accurate in local details but less accurate in overall shape than the dimensions found by astronomical leveling.

The end result, therefore, of the geodetic surveys of the earth is a set of x, y, z coordinates in which we have superposed the measured heights and measured horizontal coordinates on a geoid whose general shape was found by the methods of astronomy and gravity. It is a long detour to get a simple result, and many modern geodesists have suggested that this detour is not really necessary. In particular, Brigadier Martin Hotine has suggested that surveyors should regard their measured angles in the same way that a photogrammetrist regards the angles which he can obtain from a single photograph. Hotine suggests that triangulation nets should be built up by the step-wise accumulation of sets of angles, using the same mathematics that are used in photogrammetry. The comparison is very instructive but, in fact, it is found that when Hotine's procedure is carried out, the results are inferior to those produced by ordinary techniques of calculation.

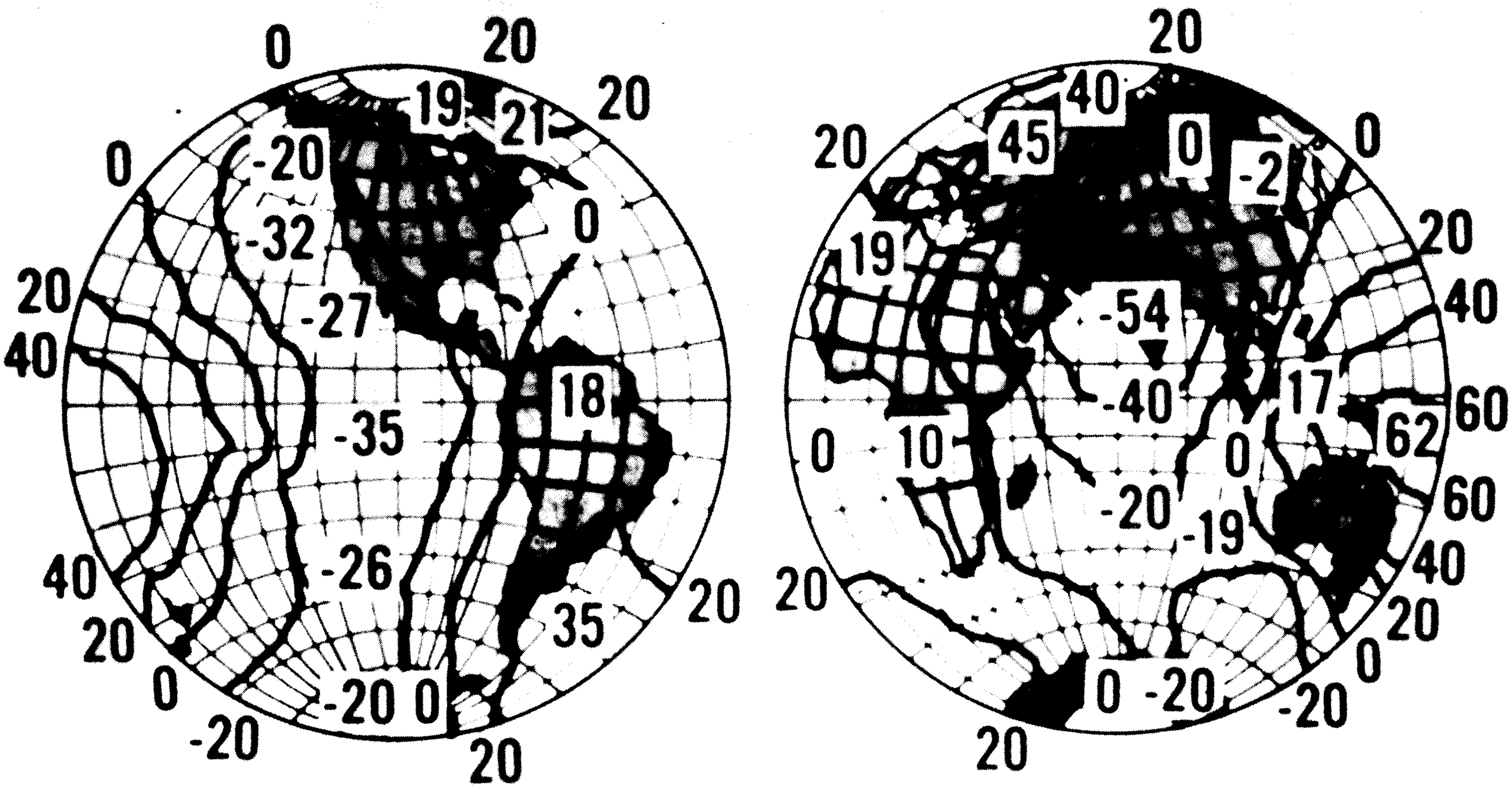
The reasons for the failure of three dimensional geodesy are twofold. First, in an ordinary photogrammetric survey most of the angles are nearly vertical, which means that the height of the aircraft is well determined. It also means that the refraction of light along the lines is relatively small. In the second place, the requirements for precision in photogrammetric surveys are much less than the requirements in geodetic surveys. As a consequence of these two facts, the photogrammetrist is justified in considering that any direction which he measures is in error by a small solid angle whose trace on the sphere is nearly circular. The geodesist, on the other hand, considers that his angles are likely to have errors in the vertical direction which are orders of magnitude larger than those in the horizontal direction. It is for this reason that the techniques of geodesy are so entirely alien to those of photogrammetry.

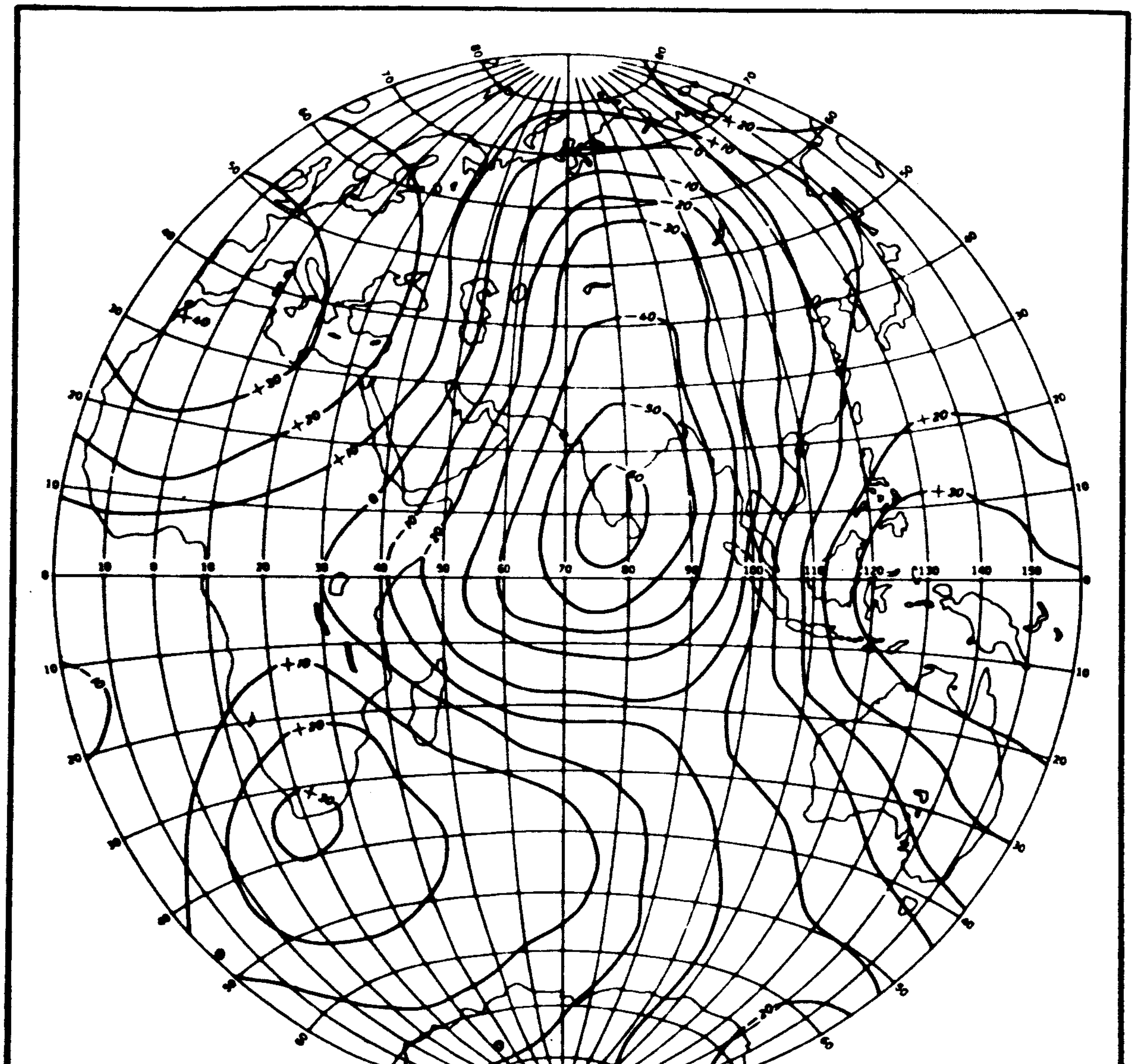
On the other hand, it is a consequence of this thought that when we observe targets which are very high above the earth, such as satellites instead of the conventional geodetic targets, which are lights around the horizon, then the mathematical situation in geodesy becomes very much like that in photogrammetry. Since the future is likely to bring us more high targets to observe on, and since the mathematics required to deal with these problems is much simpler than that required in the usual geodetic methods, it is likely that this whole fragile web of thought which I have been describing for you is one whose practical significance will become less every year.

It is still, however, the best way to obtain precise positions. Finally, its historic importance as the parent of differential geometry and so of the theory of relativity will give it a place in the hearts of mathematicians for years to come.

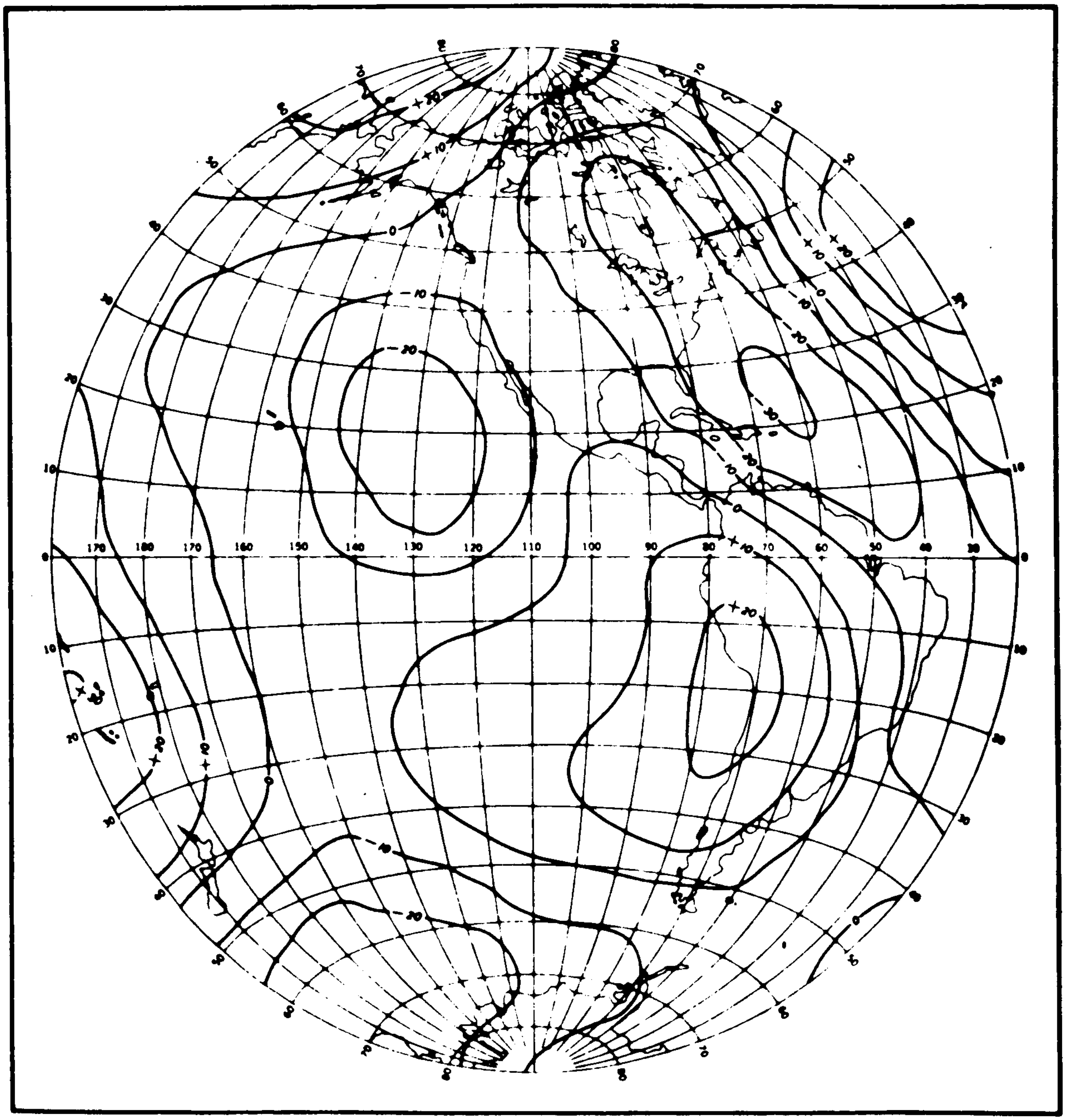
GEOID HEIGHTS IN METERS

REFERRED TO AN ELLIPSOID OF FLATTENING 1/298.24





Free Air Geoid
(Eastern Hemisphere)



Free Air Geoid
(Western Hemisphere)

Lecture 2 and Lecture 3: The Physical Significance of the Flattening of the Earth.

It was Newton who first pointed out that, as a consequence of the rotation of the earth, it was necessary to assume that the earth is flattened. He showed that, if the earth were not flattened, then the seas in the equatorial regions would be more than six miles deep, and the land would protrude in a corresponding way in polar regions. Newton calculated, on the basis of the assumption of a homogeneous earth, that the flattening f should be about 1/230. A few years later, Domenique Cassini announced that the remeasurement of the meridian on France from Dunkirk south toward the Pyrenees indicated that the length of a degree of latitude tended to increase as one went southward. If the earth were really flattened, then the length of a degree of latitude should have decreased going southward, as may be seen from Fig. 1.

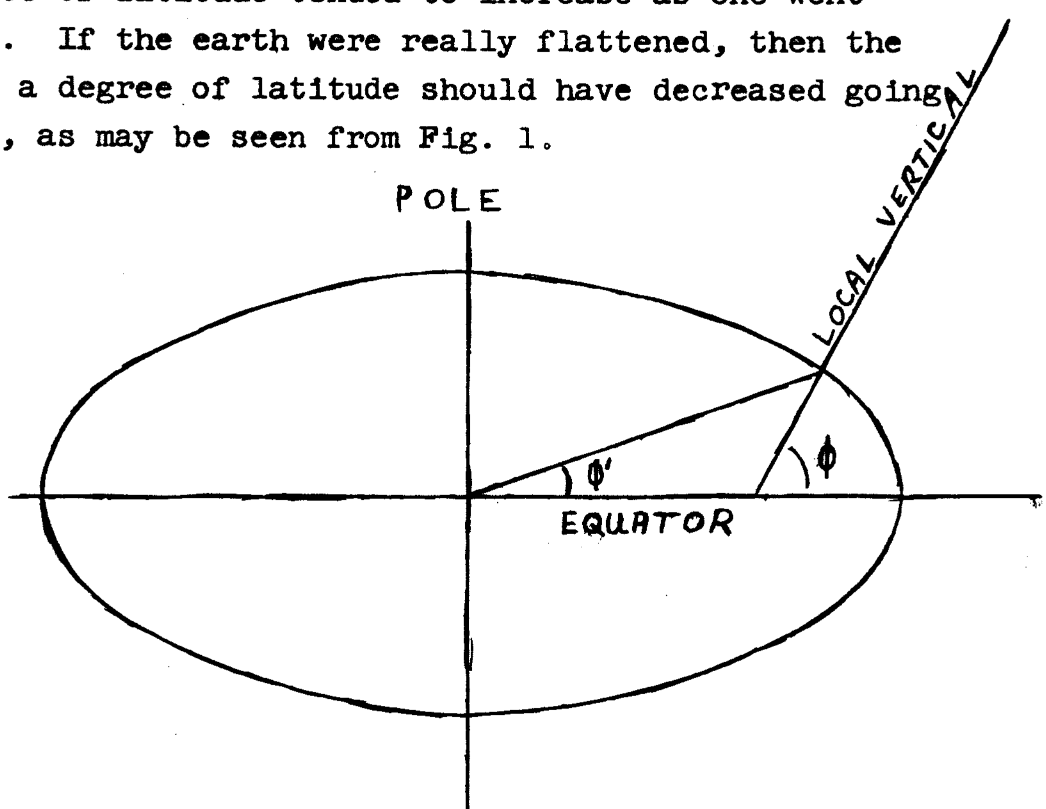


Fig. 1
Relation of Geocentric Latitude (ϕ') to Geodetic Latitude (ϕ)

It is to be remembered that latitudes and longitudes represent angles between the local vertical and the reference planes respectively of the equator and the meridian of Greenwich. If, on the other hand, they were geocentric angles, then the length of a degree of latitude would be greatest at the equator and least at the poles. The discrepancy between Newton's prediction and Cassini's observations led to a bitter quarrel between the French and the English mathematicians. The English scientists of that time were not quite sure of their position, as is witnessed by the fact that Newton chose to write in Latin, evidently not quite sure that the English language was here to stay. The quarrel has been caricatured by Swift in Gulliver's Travels. In the end, the above-mentioned measurements carried out by Maupertuis in Lapland (1736) and by Bouguer and de la Condamine (1735) in Peru showed that, in fact, Newton was right, and the earth was flattened rather than football-shaped.

From the latter part of the 18th century on, it became clear that the measured value of the flattening of the earth was inconsistent with the idea that the earth is homogeneous. The values were much nearer to 1/300 than to the value of 1/230 which would have been required if the earth had been homogeneous.

In the early stages of the measurements, it was quite enough to measure the flattening without specific reference to the surface that was involved; later on, after the introduction of the idea of the geoid, it became clear that the best surface to discuss was the sea level surface of the earth. Once the idea had been introduced, it was possible to give a rather precise meaning to the idea of the flattening of the earth, and to calculate the expected value on various assumptions about the interior.

A considerable number of particular hypotheses were discussed: the possibility that the earth was homogeneous, the possibility that it consisted of a nucleus which contained nearly all of the mass plus a sort of atmosphere, and the possibility of various smooth distributions of density which would interpolate between these. A very important result was shown by Radau about 1880, namely, that the predicted value of the flattening of the earth depended on a moment of inertia around the polar axis, and that all distributions of density having the same moment of inertia would have almost the same flattening. The error of this assumption is in the fourth significant figure, provided that the density always decreases outward. Thus the kernel of the problem of predicting the flattening of the earth is the problem of the calculation of the flattening of a body whose polar moment of inertia C is given.

The theory of this calculation will be given below. For the moment it is important to view this problem as it was seen up to 1958. During that time, the problem of determining the earth's flattening was thought to be best treated by thinking of three unknowns. These were the polar moment of inertia C, the difference between C and the axial moment of inertia A, i.e., the quantity C-A, and the hydrostatic value of the flattening f. From hydrostatic theory, as mentioned, it was possible to find an equation between C and f. From the theory of the luni-solar perturbations, it was possible to determine the quantity $(C-A)/C$, to which the name the dynamical flattening was given, and the symbol H. In addition, it was known that the quantity $(C-A)/Ma^2 = J_2$ was equal to $\frac{2}{3}(f-\frac{1}{2}m)$, where m is the ratio of centrifugal force at the equator to gravity at the equator. This relation is somewhat approximate, since there are small higher-order terms of the order of a fraction of a percent, but it is also purely mathematical, and depends in no way on assumptions about hydrostatic equilibrium. This equation related C-A to f, but it should be noted that the f here is the real flattening of the earth and not necessarily the one predicted by hydrostatic theory. Before 1958, it was customary to make the assumption that the real f equaled the hydrostatic f. One then had three equations among the three unknowns, and the solution was possible. In recent years, the determination of J_2 directly from satellite orbits has furnished a new equation in this problem. At the same time, the recognition that the hydrostatic flattening is not necessarily equal to the actual flattening means that we have a new unknown, and so the system is now more complicated than before; we have four equations with four unknowns. The point which is not clear from the older discussions is that the hydrostatic flattening of the earth depends only on the assumed value of the polar moment of inertia. This is directly determinable now, since we can measure $(C-A)/Ma^2$ and also $(C-A)/C$; the quotient of these is evidently C/Ma^2. From this, the hydrostatic flattening is directly determinable. I repeat, formerly it was impossible to obtain C/Ma^2 with adequate accuracy unless one made the auxiliary assumption that the hydrostatic and the actual flattening were equal. Thus it is the older situation which is complicated and the newer one which is simple.

I shall now give the theory of the relation between C/Ma^2 and f, the flattening, as it would be in a plastic or liquid body. I shall follow Jeffreys' theory as stated in The Earth (Cambridge University Press, 1952 or 1958 edition). My excuse for giving you a long commentary on section 4.03 of his book, which covers only 8 pages, is that I have found these pages very difficult. Since there are 2 errors on these pages which appear in the 1952 edition and were reprinted in the 1958 edition, it is just possible that I am not the only person who has had trouble reading these pages. (Since 1958, both errors have been spotted by others beside myself.)

My equations will be numbered in accordance with his; those with letters following are interpolated.

The theory of the interior of the earth starts from the assumption that the earth is in hydrostatic equilibrium. That is to say that it is in equilibrium under the action of forces which cause no motion and which produce pressures acting equally in all directions, as in a fluid. Under these circumstances, we will expect that the density will be stratified in layers such that the surfaces of constant density will also be surfaces of constant potential. The result is intuitively obvious -- it means only that a fluid seeks its level. If there were a place where the density above an equipotential surface exceeded the density below it, then the heavier fluid above would tend to displace the lighter fluid below the surface. The point can be proved analytically, but it is one which is too simple to be worth such a discussion.

It is important to remember that the potential which is involved here is not the true gravitational potential of the body, but rather the geopotential. The difference is the centrifugal forces which arise from the rotation of the body. These forces are included in the geopotential, on exactly the same footing as the true gravitational force. Once again, this is a matter of ordinary experience; the force which we call gravity in daily life is 99 percent the real gravitational attraction of the earth, but the remaining fraction is the force of the earth's rotation. The difference is quite perceptible in ordinary life. The flow of the Mississippi requires about one foot per mile, which is a small fraction of the difference between the geopotential and the earth's true gravitational potential. The maximum inclination between surfaces of true gravitational potential and geopotential is of the order of 5 or 10 minutes of arc.

We shall follow Jeffreys in this derivation and designate the density by the symbol ρ, and the geopotential by the symbol Ψ. The surfaces of the constant Ψ will be surfaces of constant ρ. We consider a homogeneous, nearly spherical body whose surface is given by the equation

$$r = a(1 + \sum_{n=1}^{\infty} \epsilon_n S_n), \qquad (2)$$

according to Jeffreys, where S_n is a surface harmonic, a is the earth's mean radius, and ϵ_n is a small numerical coefficient (Fig. 2). Notice that Jeffreys has written this equation as a single summation over n; this is merely a convenience to avoid the ugliness of a double summation. In fact, the S_n's must be considered as functions not only

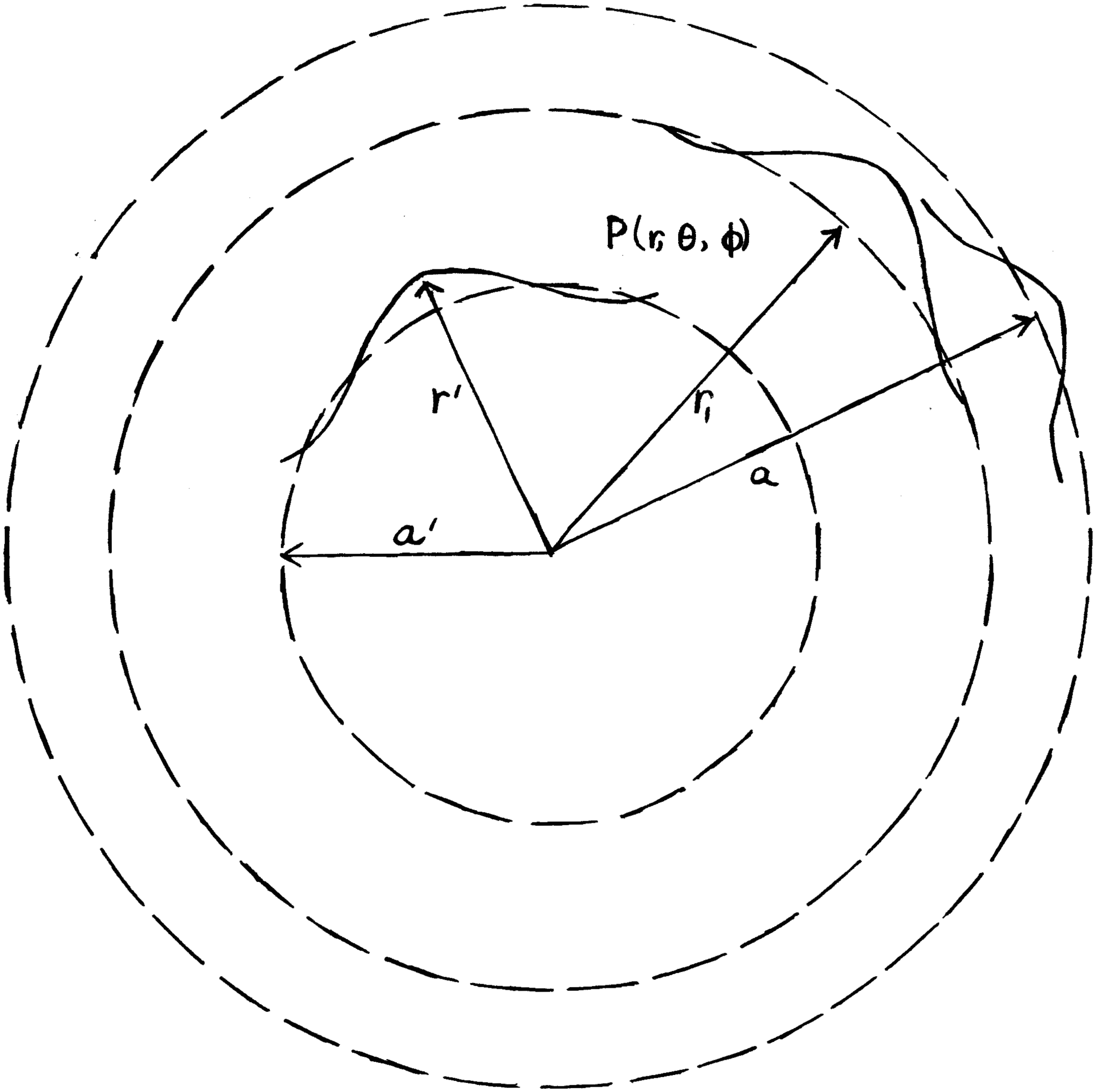


Fig. 2

of the degree n of the harmonic but also of its order m. Since we shall get rid of all these harmonics except S_2 at an early stage in the game, it is not important to distinguish between tesseral and zonal harmonics, and hence m may be omitted.

We now consider the gravitational potential due to this body. In calculating the potential, Jeffreys makes the assumption that all of the ϵ's are so small that we can neglect second order terms. Under these circumstances, we can represent the attraction of the body as that of a sphere combined with the attraction of an infinitely thin surface distribution of matter painted on the outside of the sphere. What is neglected here is the fact that a real 3-dimensional bulge would attract, not toward a point right on the sphere, but toward a point half way up through the bulge. The neglect of second order terms is fully justified for all harmonics except the second. In the case of the second harmonic, quadratic terms have been calculated by Darwin. They represent an enormous increase in the difficulty of the computation without any real increase in the accuracy with which the computation represents physical reality. The effects of lack of fluidity in the earth are large enough so that the use of second order terms is not justified even for the second harmonic.

For the potential outside the body, Jeffreys gives

$$U_0 = \frac{4}{3}\pi f \rho a^3\left(\frac{1}{r} + \sum_{n=1}^{\infty} \frac{3}{2n+1} \frac{a^n}{r^{n+1}} \epsilon_n S_n\right), \tag{3}$$

where f is the absolute constant of gravitation. This equation may be derived from W.D. MacMillan, <u>Theory of the</u>

Potential, p. 395, Equation 1:

$$V = \sum_{m=0}^{\infty} S_m(\phi_0, \theta_0)\left(\frac{a}{r}\right)^{m+1}.$$

Here V is the potential; m is Jeffreys' n; ϕ_0, θ_0 are the coordinates of the point at which the potential is being evaluated; and $S_m(\phi_0, \theta_0)$ is a surface harmonic, multiplied by its coefficient, defined by the following equation for the surface density σ:

$$\sigma = \frac{1}{4\pi a} \sum_{m=0}^{\infty} (2m+1) S_m(\phi, \theta),$$

where ϕ, θ are the coordinates of any point. In this case, the mass distribution corresponding to the m^{th} harmonic will be

$$\sigma_m = \frac{2m+1}{4\pi a} S_m(\phi, \theta).$$

For Jeffreys, this surface distribution of mass is produced by additional thickness of the homogeneous body. It is thus

$$\sigma_n = \rho a \,\epsilon_n S_n.$$

Equating σ_n to σ_m,

$$\rho a \,\epsilon_n S_n \cdot \frac{4\pi a}{2m+1} = S_m(\phi, \theta).$$

Substituting in MacMillan's equation for V, and multiplying by f (which MacMillan takes equal to unity), we obtain, for the n^{th} term

$$\frac{4}{3}\pi f \rho a^3 \cdot \frac{3}{2n+1} \frac{a^n}{r^{n+1}} \epsilon_n S_n,$$

as for Jeffreys. In (3), the first term is nothing but the Newtonian attraction of a sphere.

For the interior attraction, Jeffreys gives the following equation:

$$U_1 = \frac{4}{3}\pi f \rho a^3 \left(\frac{3a^2-r^2}{2a^3} + \sum_{n=1}^{\infty} \frac{3}{2n+1} \frac{r^n}{a^{n+1}} \epsilon_n S_n \right). \tag{4}$$

This equation is obtainable from MacMillan's equation,

$$V = \sum_{m=0}^{\infty} S_m(\phi_0, \theta_0)\left(\frac{r}{a}\right)^m \qquad \text{if } r < a,$$

with the same substitutions for σ except for the first term inside the parentheses. The first term represents the potential at a point in the interior of a sphere. It consists of two contributions. The first is that due to the portion of the sphere interior to the point in question, which is clearly

$$\frac{4}{3}\pi f \rho r^3,$$

where r is the radius from the center of the sphere to the point in question. The potential due to the portion of the sphere outside the point in question is given by MacMillan (p. 38):

$$2\pi\rho f(a^2-r^2),$$

and the combined effect is

$$\frac{4}{3}\pi\rho f a^3 \frac{(3a^2-r^2)}{2a^3},$$

which is the first term inside the parentheses of Jeffreys' Equation 4. We now consider a heterogeneous body. The density is constant and equal to ρ' over a surface given by Jeffreys' Equation 5:

$$r' = a'(1 + \sum \epsilon_n S_n), \tag{5}$$

where ρ' and ϵ_n are functions of a'. In order to keep straight the varying meanings and kinds of radii which are involved in this situation, let us look at Fig. 2. First we have a, which is the mean radius of the outer surface of the body. It is thus approximately the semi-major axis of the earth. Next we have a', which is the mean radius of any interior surface. We can describe a point of the equal density surface by giving r and S_n, since S_n will contain the angular variables. The mean radius of that surface which passes through the interior point $P(r,\theta,\phi)$, where the potential is to be found, is defined by Jeffreys as r_1.

To calculate the potential, Jeffreys proceeds to take the difference between two homogeneous bodies, one having the outer surface corresponding to the density ρ, and the other having a surface equal to

$$\rho' + \Delta\rho'.$$

The external potential is therefore clearly given by Equation 6:

$$U_0 = \frac{4}{3}\pi f \int_0^a \rho' \frac{\partial}{\partial a'}\left(\frac{a'^3}{r} + \sum \frac{3}{2n+1} \frac{a'^{n+3}}{r^{n+1}} \epsilon_n S_n\right) da'. \tag{6}$$

The quantity ρ' is not differentiated because the gravitational attraction of the thin spherical shell is proportional to the difference in radius da' between its two sides, but is proportional to ρ' itself and not to $d\rho'$. The integration is extended over a' up to a rather than to ∞, clearly because beyond a there is no density.

For an internal point, we calculate the potential U_1 in two parts. The first term is due to the matter which is interior to the point under consideration. For this, an explanation exactly like Equation 6 applies, except that the integral extends only up to the mean radius r_1 through the point in question. For matter external to the point, we differentiate and integrate Equation 4 in an entirely similar way:

$$U_1 = \frac{4}{3}\pi f \int_0^{r_1} \rho' \frac{\partial}{\partial a'}\left(\frac{a'^3}{r} + \sum \frac{3}{2n+1} \frac{a'^{n+3}}{r^{n+1}} \epsilon_n S_n\right) da'$$
$$+ \frac{4}{3}\pi f \int_{r_1}^{a} \rho' \frac{\partial}{\partial a'}\left(\frac{3}{2}a'^2 + \sum \frac{3}{2n+1} \frac{r^n}{a'^{n-2}} \epsilon_n S_n\right) da'. \quad (7)$$

Note that in these differentiations and integrations, the only variable is a'; r is the radius to the point P at which the potential is being evaluated; r_1 is the mean value of r on the equipotential through P, i.e.,

$$r = r_1(1 + \sum \epsilon_n S_n).$$

To obtain Ψ, the geopotential, we must add the contribution from the centrifugal force. Thus

$$\Psi = U + \frac{1}{2}\omega^2 r^2 \cos^2\phi' = U + \frac{1}{3}\omega^2 r^2 + \frac{1}{2}\omega^2 r^2\left(\frac{1}{3} - \sin^2\phi'\right). \quad (8)$$

Let us note that, after Equation 8, Jeffreys mentions that he can ignore the difference between ϕ and ϕ'. The next sentence, which discusses the behavior of ρ and Ψ over the equipotential surfaces, contains the word "then", which appears to refer back to the remark about ϕ and ϕ'. I have been unable to make sense out of this relation, and I believe that the sentence about ϕ and ϕ' is simply misplaced. In fact, Jeffreys continues to use ϕ' until after his Equation 12. The justification for ignoring the difference is the fact that trigonometric functions of

ϕ' occur only with the small coefficient ω^2 or one of the epsilons.

Jeffreys proceeds to point out that, in his Equations 7 and 8, Ψ can be a function only of r_1. This is because the value of r_1 is constant over an equipotential surface. In particular, Ψ cannot be a function of the S_n's, which are functions of the coordinates ϕ, λ. Jeffreys next defines $\bar{\rho}$, the mean density in the body, by means of his Equation 9, namely

$$M = 4\pi \int_0^a \rho' a'^2 da' = \frac{4}{3}\pi a^3 \bar{\rho}. \qquad (9)$$

He then defines the mean density ρ_0 within a surface whose mean radius is r_1 by Equation 10, namely

$$\rho_0 = \frac{3}{r_1^3} \int_0^{r_1} \rho' a'^2 da'. \qquad (10)$$

Jeffreys then proceeds to substitute for $1/r$ in his Equations 7 and 8. It is important to notice that r has small coefficients except in the first term. In this term, therefore, we must retain first order of small quantities. Elsewhere we can replace r by r_1. We notice also that r can be taken out from under the integral sign and from the differentiation, since both of these refer to the running variable a' rather than to the point at which the potential is being evaluated.

The quantities ϵ_n and ρ' are to be regarded as functions of a'. In obtaining Equation 11, namely

$$\frac{4}{3}\pi f\Bigg[\frac{1-\sum\epsilon_n S_n}{r_1}\int_0^{r_1} 3\rho' a'^2 da' + \sum\frac{3}{2n+1}S_n\left\{\frac{1}{r_1^{\,n+1}}\int_0^{r_1}\rho' d(a'^{n+3}\epsilon_n)\right.$$

$$\left. + r_1^{\,n}\int_{r_1}^{a}\rho' d\left(\frac{\epsilon_n}{a'^{n-2}}\right)\right\} \tag{11}$$

$$+ \frac{1}{3}\omega^2 r_1^{\,2} + \frac{1}{2}\omega^2 r_1^{\,2}\left(\frac{1}{3} - \sin^2\varphi'\right)\Bigg] = \text{function of } r_1 \text{ only,}$$

Jeffreys has twice preferred to replace expressions of the form $\frac{\partial f}{\partial a'}da'$ by df. The function of r to which (11) is equated is

$$\Psi - M = \Psi - \frac{4}{3}\pi f\int_{r_1}^{a}\rho' d\left(\tfrac{3}{2}a'^2\right).$$

Since the left hand side of Equation 11 must be constant for a given r_1, the coefficients of all of the S_n's, where n is greater than or equal to 1, must vanish because the S_n's contain the angle variables. If their coefficients did not vanish, then the left hand side of the equation would depend on the angle variables. Moreover, because of the orthogonality properties of the S_n's, no combination of the S_n's could have the same effect as one of them. Hence, it is not possible to arrange the coefficients in such a way that the variation of one S_n is covered up by the others. The only way to make the whole left hand side of (11) independent of the angle variables is to make the coefficient of each S_n equal to zero. When we do so, we get Equation 12 after dividing through by $4\pi f$:

$$-\frac{\epsilon_n}{r_1}\int_0^{r_1}\rho' a'^2 da' + \frac{1}{2n+1}\left\{\frac{1}{r_1^{\,n+1}}\int_0^{r_1}\rho' d(a'^{n+3}\epsilon_n)\right.$$

$$\left. + r_1^{\,n}\int_{r_1}^{a}\rho' d\left(\frac{\epsilon_n}{a'^{n-2}}\right)\right\} = 0, \tag{12}$$

except in the case when S_n is $(\frac{1}{3} - \sin^2\phi)$, when we get an extra term,

$$-\tfrac{1}{8}\omega^2 r_1{}^2/\pi f,$$

on the right hand side. The right side is therefore written

$$(0, -\tfrac{1}{8}\omega^2 r_1{}^2/\pi f).$$

We next multiply (12) through by $r_1{}^{n+1}$ and replace r_1 by r:

$$-r^n \epsilon_n \int_0^r \rho' a'^2 da' + \frac{1}{2n+1}\left\{\int_0^r \rho' d(a'^{n+3}\epsilon_n) + r^{2n+1}\int_r^a \rho' d\left(\frac{\epsilon_n}{a'^{n-2}}\right)\right\} = 0. \qquad (12a)$$

We now consider the variation of the potential with distance from the center of the earth, so that we regard r as a variable. In differentiating the integrals, it is important to remember that the integral for a general function f(a') is

$$\frac{d}{da'}\int_a^r f(a')da' = f(r).$$

With this in mind, Equation 12a is differentiated as follows:

$$\left\{ -nr^{n-1}\epsilon_n - r^n \frac{d\epsilon_n}{dr}\right\} \int_0^r \rho' a'^2 da' - r^n \epsilon_n \rho r^2$$

$$+ \frac{1}{2n+1}\left\{\rho(n+3)r^{n+2}\epsilon_n + r^{n+3}\rho\frac{d\epsilon_n}{dr}\right.$$

$$+ (2n+1)r^{2n}\int_{r_1}^{a} \rho' d\left(\frac{\epsilon_n}{a'^{n-2}}\right) - r^{2n+1}\left[\rho\frac{d\epsilon_n}{dr} \cdot \frac{1}{r^{n-2}}\right.$$

$$\left.\left. - \rho\epsilon_n \cdot (-n+2)\frac{1}{r^{n-1}}\right]\right\} = -\frac{5\omega^2 r^4}{8\pi f}.$$

In writing this equation, we must keep in mind that ρ is the value of ρ' when $r = a$. This equation simplifies Jeffreys' Equation 13 when we combine the two terms in the second bracket which depend on $d\epsilon_n/dr$, and note that the sum of three terms in $\epsilon_n \rho r^{n+2}$ is zero.

Making these substitutions, we arrive at Jeffreys' Equation 13, which includes both integrals and derivatives:

$$-\left(r^n \frac{d\epsilon_n}{dr} + nr^{n-1}\epsilon_n\right)\int_0^r \rho' a'^2 da' + r^{2n}\int_r^a \rho' \frac{d}{da'}\left(\frac{\epsilon_n}{a'^{n-2}}\right)da' =$$

$$\left(0,\ -\frac{5\omega^2 r^4}{8\pi f}\right). \tag{13}$$

We now divide by r^{2n} and get Equation 13a:

$$-\left(\frac{1}{r^n}\frac{d\epsilon_n}{dr} + \frac{n}{r^{n+1}}\epsilon_n\right)\int_0^r \rho' a'^2 da' + \int_r^a \rho' \frac{d}{da'}\left(\frac{\epsilon_n}{a'^{n-2}}\right)da' =$$

$$\left(0,\ -\frac{5\omega^2}{8\pi f}\right). \tag{13a}$$

We differentiate with respect to r and note, as before, the effect of variable limits of integration. We further note that ρ is the value of ρ' at $a' = r$. This gives Equation 13b:

$$-\left(-\frac{n}{r^{n+1}}\frac{d\epsilon_n}{dr} + \frac{1}{r^n}\frac{d^2\epsilon_n}{dr^2} - \frac{n(n+1)}{r^{n+2}}\epsilon_n + \frac{n}{r^{n+1}}\frac{d\epsilon_n}{dr}\right)\int_0^r \rho' a'^2 da'$$

$$-\left(\frac{1}{r^n}\frac{d\epsilon_n}{dr} + \frac{n}{r^{n+1}}\epsilon_n\right)\rho r^2 - \rho\left[\frac{d\epsilon_n}{dr}\cdot\frac{1}{r^{n-2}} + \epsilon_n\frac{(-n+2)}{r^{n-1}}\right] = 0. \tag{13b}$$

In constructing this equation, we did not differentiate under the integral sign in the last term because all quantities there are regarded as functions of a'. We multiply through by $-r^n$, and this gives (13c),

$$\left(\frac{d^2\epsilon_n}{dr^2} - \frac{n(n+1)}{r^2}\epsilon_n\right)\int_0^r \rho' a'^2 da' + \left(\frac{d\epsilon_n}{dr} + \frac{n\epsilon_n}{r}\right)\rho r^2$$

$$+ \rho\left[r^2\frac{d\epsilon_n}{dr} - r\epsilon_n(n-2)\right] = 0, \tag{13c}$$

which simplifies into Jeffreys' Equation 14:

$$\left(\frac{d^2\epsilon_n}{dr^2} - \frac{n(n+1)}{r^2}\epsilon_n\right)\int_0^r \rho' a'^2 da' + 2\left(\frac{d\epsilon_n}{dr} + \frac{\epsilon_n}{r}\right)\rho r^2 = 0. \tag{14}$$

Now, from Equation 10, it is easy to see that

$$\int_0^r \rho' a'^2 da' = \tfrac{1}{3}r^3\rho_0.$$

Substituting for the integral and dividing through by $r^3/3$, we have Jeffreys' Equation 15, which is the famous equation of Clairaut:

$$\rho_0\left(\frac{d^2\epsilon_n}{dr^2} - \frac{n(n+1)}{r^2}\epsilon_n\right) + \frac{6\rho}{r}\left(\frac{d\epsilon_n}{dr} + \frac{\epsilon_n}{r}\right) = 0. \tag{15}$$

The equation of Clairaut was obtained in 1743. In the intervening two centuries, a great deal has been found out about the possible solutions of this equation subject to the restriction that the density decreases steadily downward. There are two reasons to think that this will happen: First, the denser materials would tend to sink in fluid equilibrium; second, materials which are at a lower level are under high pressure and, therefore, will be somewhat compressed. It follows that the mean density ρ_0 within a given surface will also be greater than the local density ρ, except at the center where $\rho_0 - \rho \longrightarrow$ zero.

We suppose that for small values of r, ϵ_n varies like r^p. Then, substituting in Clairaut's equation, we have

$$\rho_0\left[p(p-1)r^{p-2} - \frac{n(n+1)}{r^2}r^p\right] + \frac{6\rho}{r}\left(pr^{p-1} + \frac{r^p}{r}\right) = 0. \qquad (15a)$$

Dividing by r^{p-2} and also by ρ, which equals ρ_0 at the center of the earth, we have a quadratic equation in p:

$$p(p-1) - n(n+1) + 6p + 6 = 0. \qquad (16)$$

This equation is solved by the usual processes, giving either

$$p = n-2 \quad \text{or} \quad p = -n-3. \qquad (17)$$

Of the two solutions, we can discard p = -n-3, since in this case the solution would be proportional to $r^{-n-2}S_n$. As n goes from +1 to ∞, the exponent on r would be negative. Such a solution would go to ∞ at the center of the earth, and is therefore impossible. If, therefore, for

$p = n-2$, we take $n = 1$, then

$$\epsilon_n = kr^{-1} \tag{17a}$$

$$\frac{d\epsilon_n}{dr} = -\frac{k}{r^2}$$

$$\frac{d^2\epsilon_n}{dr^2} = \frac{2k}{r^3} .$$

Substituting, we find that, for this case, Clairaut's Equation 15 holds identically for arbitrary density functions ρ and ρ_0. The radial displacement is proportional to S_1 regardless of the distance from the center, and this, in turn, implies a rigid body displacement which needs not be further considered.

If $n = 2$, then ϵ_n is neither infinite nor zero near the center. For this border line case, a special treatment is needed because n-2 vanishes, and hence the previous treatment leads to constant ellipticity. We let

$$1 - \frac{\rho}{\rho_0} = Hr^k$$

hold for small r. In this equation, H must be positive so that the density may increase as r increases, and k must be positive to avoid an infinite value of the density at the center. We further suppose

$$\epsilon_2 = A + Br^s. \tag{18}$$

We substitute in Equation 15, and find (18a):

$$\rho_0\left[Bs(s-1)r^{s-2} - \frac{6(A+Br^s)}{r^2}\right] + \frac{6\rho}{r}\left(Bsr^{s-1} + \frac{A+Br^s}{r}\right) = 0. \tag{18a}$$

In this equation, we note that

$$\frac{6A}{r^2}(\rho - \rho_0) = -\frac{6A\rho_0}{r^2}\left(1 - \frac{\rho}{\rho_0}\right) = -6A\rho_0 Hr^{k-2}. \tag{18b}$$

We also can transform the terms whose coefficient is $6\rho B$:

$$\rho(6Bs + 6B)r^{s-2} = \left[\rho_0 - \rho_0\left(1 - \frac{\rho}{\rho_0}\right)\right](6Bs + 6B)r^{s-2}$$

$$= \rho_0(6Bs + 6B)r^{s-2} - \rho_0 Hr^{k+s-2}(6Bs + 6B). \tag{18c}$$

The second term in (18c) disappears because it is of an order higher than r^{s-2}. The remaining terms of (18a) are all multiplied by ρ_0, so that we find

$$Bs(s+5)r^{s-2} - 6AHr^{k-2} = 0. \tag{19}$$

Equation 19 can only be true if $s = k$. In this case, (19a) will hold:

$$Bk(k+5) = 6AH. \tag{19a}$$

Since k is positive, B must have the sign of AH. H, however, is positive, so that B has the sign of A. Therefore, ϵ_2 must increase numerically with r.

Finally, if n is greater than 2, then ϵ_n behaves like r^{n-2} for a small r. We thus say that ϵ_n increases numerically with r in all non-trivial cases for points near the center of the earth.

If the ϵ_n's should not continue to increase all the way to the surface, then we would come to a place where

$$\frac{d\epsilon_n}{dr} = 0.$$

Then the following would hold (Jeffreys' Equation 20):

$$\frac{d^2\epsilon_n}{dr^2} = \left\{n(n+1) - \frac{6\rho}{\rho_0}\right\}\frac{\epsilon_n}{r^2}. \tag{20}$$

Since $n(n+1)$ is positive and is at least 6, it follows that the right hand side of (20) is at least $6(1 - \rho/\rho_0)$, which is positive, since ρ is always less than ρ_0. Hence, the first derivative of epsilon will have the sign of ϵ_n and, therefore, ϵ_n would immediately increase again in absolute value.

Our next step is to show that the ϵ_n's should be zero except for n = 1 and n = 2. In Equation 12, if we put $r_1 = a$, then the integral from r_1 to a vanishes. We also substitute from Equation 9 for

$$\int_0^a \rho a'^2 da' = \tfrac{1}{3}a^3\bar{\rho}, \tag{20a}$$

and Equation 12 becomes

$$-\epsilon_{na} \cdot \tfrac{1}{3}a^2\bar{\rho} + \frac{1}{2n+1} \cdot \frac{1}{a^{n+1}}\int_0^a \rho' d(a'^{n+3}\epsilon_n) =$$

$$(0, -\tfrac{1}{8}\omega^2a^2/\pi f). \tag{21}$$

We denote the integral in Equation 21 by I. We assume that ϵ_n is positive; then, integrating by parts, we get

$$I = \rho_a \epsilon_{na} a^{n+3} - \int_{a'=0}^{a} a'^{n+3}\epsilon_n d\rho'. \tag{22}$$

Here the subscript "a" indicates values taken at the surface. Since ρ' is a decreasing function of a', it follows that $d\rho'$ is negative; the integral in Equation 22 is therefore negative:

$$I > \rho_a \epsilon_{na} a^{n+3}. \tag{23}$$

On the other hand, since ϵ_n is a positive, increasing function of a', it is always less than the boundary value ϵ_{na} unless n = 1. Here Jeffreys says that ϵ_n does not change. Actually, it has been pointed out to me that it must increase without limit near the center, but this case is trivial.

$$-\int_{a'=0}^{a} a'^{n+3} \epsilon_n d\rho' < -\epsilon_{na} \int_{a'=0}^{a} a'^{n+3} d\rho' \tag{23a}$$

Substituting (23a) in (22), we have

$$I < \epsilon_{na}\left(\rho_a a^{n+3} - \int_{a'=0}^{a} a'^{n+3} d\rho'\right). \tag{23b}$$

The right hand side of (23b) represents the result of integrating by parts the expression

$$\epsilon_{na} \int_{a'=0}^{a} \rho' da'^{n+3}.$$

We replace ρ' by $\bar{\rho} + (\rho' - \bar{\rho})$:

$$\epsilon_{na} \int_{a'=0}^{a} \rho' da'^{n+3} = \epsilon_{na}\left[\bar{\rho} a^{n+3} + \int_{a'=0}^{a} (\rho' - \bar{\rho}) da'^{n+3}\right]. \tag{23c}$$

To evaluate the integral, let

$$d(a'^{n+3}) = (n+3)a'^{n+2}da' = \frac{n+3}{3}a'^{n}da'^{3}. \tag{23d}$$

Therefore,

$$\epsilon_{na}\int_{a'=0}^{a}\rho' da'^{n+3} = \epsilon_{na}\left[\bar{\rho}^{n+3} + \frac{n+3}{3}\int_{a'=0}^{a}(\rho'-\bar{\rho})a'^{n}da'^{3}\right]. \tag{24}$$

For n = 0, the last integral vanishes because the differential da'^3 weights the integral in proportion to the volume. In this case, the integral of $\rho'-\bar{\rho}$ must vanish by the definition of mean density.

In general, because of the fact that $\bar{\rho}$ is a volume average of ρ', it will be true that the integral of $(\rho'-\bar{\rho})$ multiplied by any constant and taken from 0 to a will be 0. In particular, if we choose a_0 for the level where $\rho' = \bar{\rho}$, then, since ρ' is a decreasing function $(\rho'-\bar{\rho}) > 0$ under this level, i.e., for $a' < a_0$, and $(\rho'-\bar{\rho}) > 0$ above this level. Then the product

$$(\rho'-\bar{\rho})(a'^{n} - a_0^{n})$$

will be negative for any power of n greater than 0, since, for all such powers, the power of the greater number is greater. Hence,

$$\int_{a'=0}^{a}(\rho'-\bar{\rho})a'^{n}da'^{3} = \int_{a'=0}^{a}(\rho'-\bar{\rho})(a'^{n} - a_0^{n})da'^{3} < 0.$$

Therefore, the integral in (24) is negative. Since the remaining term is necessarily positive, the integral can only decrease the whole expression, so that

$$I < \epsilon_{na}\bar{\rho}a^{n+3}.$$

Using (23), we see that the quantity I can, in fact, be bracketed between the limits

$$\epsilon_{na}\,\rho_a a^{n+3} < I < \epsilon_{na}\bar{\rho}\, a^{n+3}.$$

All the above assumes that ϵ is positive. If it is negative, the inequalities are reversed, and hence, whether ϵ is positive or negative,

$$I = \theta\, \epsilon_{na}\bar{\rho}\, a^{n+3},$$

where $0 < \theta < 1$. Going back to (21), therefore,

$$\epsilon_{na} a^2 \bar{\rho} \left(-\frac{1}{3} + \frac{\theta}{2n+1}\right) = \left(0,\ -\frac{\omega^2 a^2}{8\pi f}\right). \tag{25}$$

If the right hand side is 0, this equation cannot be satisfied for $n > 1$, since, in that case, the parenthesis on the left must be less than 0. Its coefficient is composed of quantities which also cannot vanish except at the center of the earth. Hence, for all n except n = 2, the ϵ_n must be 0 (to the first order) throughout the earth. No harmonics except the second degree zonal harmonics will exist.

With respect to the second degree zonal harmonic, for which the right hand side is negative, the value of ϵ_{na} must be positive. This, however, implies that ϵ_{na} is positive everywhere, since we have found that the ϵ_n's must increase steadily from the center. Jeffreys summarizes these results as follows:

"<u>On the hydrostatic theory the radius of a surface of</u>

constant density contains no harmonics other than that representing the ellipticity; the ellipticities increase all the way from the centre to the surface, and the surface is oblate."

Returning to Clairaut's Equation 15, for n = 2, we set

$$\epsilon_2 \equiv \epsilon = r^3\lambda. \tag{26}$$

Its derivatives are:

$$\frac{d\epsilon}{dr} = 3r^2\lambda + r^3\frac{d\lambda}{dr}$$

$$\frac{d^2\epsilon}{dr^2} = 6r\lambda + 6r^2\frac{d\lambda}{dr} + r^3\frac{d^2\lambda}{dr^2}.$$

Substituting these in Clairaut's Equation 15,

$$\rho_0\left(6r\lambda + 6r^2\frac{d\lambda}{dr} + r^3\frac{d^2\lambda}{dr^2} - 6r\lambda\right)$$

$$+ \frac{6\rho}{r}\left(3r^2\lambda + r^3\frac{d\lambda}{dr} + r^2\lambda\right) = 0.$$

Dividing through by $\rho_0 r^3$, we get

$$\frac{d^2\lambda}{dr^2} + \frac{6}{r}\frac{d\lambda}{dr} + \frac{24}{r^2}\frac{\rho}{\rho_0}\lambda + \frac{6}{r}\frac{\rho}{\rho_0}\frac{d\lambda}{dr} = 0,$$

i.e.,

$$\frac{d^2\lambda}{dr^2} + 6\left(\frac{\rho}{\rho_0} + 1\right)\frac{1}{r}\frac{d\lambda}{dr} + \frac{24\rho}{\rho_0}\frac{\lambda}{r^2} = 0. \tag{27}$$

We note that for small r, ϵ_n behaves like r^p, where $p = n-2$. For $n = 2$, this means that ϵ behaves like a constant and hence, from (26), λ must behave like r^{-3}. It follows that λ initially decreases. It cannot afterwards increase, since at the minimum,

$$\frac{d\lambda}{dr} = 0,$$

and we would also have

$$\frac{d^2\lambda}{dr^2} = -\frac{24\rho}{\rho_0 r^2}\lambda ,$$

and thus the second derivative would necessarily have the opposite sign from λ. But λ is positive. Hence,

$$\frac{d^2\lambda}{dr^2}$$

would necessarily be negative, and thus λ must decrease all the way from the center to the surface.

In (13), we put n = 2; then $S_2 = \frac{1}{3} - \sin^2\varphi'$. We consider conditions at the surface where r = a; then the second term disappears because of the coincidence of the limits of integration, and the integral in the first term is, from Equation 9, replaced by $\frac{1}{3}\overline{\rho}a^3$. Then

$$-\frac{1}{3}\overline{\rho}a^3\left[a^2\left(\frac{d\epsilon}{dr}\right)_{r=a} + 2a\,\epsilon_a\right] = -\frac{5\omega^2 a^4}{8\pi f}. \tag{28}$$

To the first order, we can say that

$$m = \frac{\omega^2 a^3}{fM} = \frac{\omega^2}{\frac{4}{3}\pi f\overline{\rho}},$$

i.e., very roughly, the centrifugal force at the equator divided by the intensity of gravity, and then the right hand side of (28) becomes

$$\frac{5}{6}ma^4\bar{\rho}.$$

We multiply through by $-3/a^3\rho$, and get

$$a\left(\frac{d\epsilon}{dr}\right)_a + 2\epsilon_a = \frac{5}{2}m. \tag{30}$$

It turns out that, at this point, it is advantageous to introduce a new dependent variable η, which is defined by

$$\eta = \frac{d\ \log\epsilon}{d\ \log\ r} = \frac{r}{\epsilon}\frac{d\epsilon}{dr}. \tag{31}$$

The derivatives of ϵ are

$$\frac{d\epsilon}{dr} = \frac{\eta\epsilon}{r};\ \frac{d^2\epsilon}{dr^2} = \left(\frac{1}{r}\frac{d\eta}{dr} + \frac{\eta^2-\eta}{r^2}\right)\epsilon. \tag{32}$$

When these are substituted in Equation 15, we get

$$\rho_0\epsilon\left(\frac{1}{r}\frac{d\eta}{dr} + \frac{\eta^2-\eta}{r^2} - \frac{6}{r^2}\right) + 6\rho\epsilon\left(\frac{\eta}{r} + \frac{1}{r}\right) = 0. \tag{32a}$$

We multiply this through by $r^2/\epsilon\rho_0$, and obtain

$$r\frac{d\eta}{dr} + \eta^2 - \eta - 6 + (\eta+1)\frac{6\rho}{\rho_0} = 0. \tag{33}$$

In order to eliminate ρ in Equation 33, we start from Equation 10:

$$\rho_0 = \frac{3}{r^3}\int_0^r \rho' a'^2 da', \tag{33a}$$

which, on differentiation, yields

$$\frac{1}{3}\frac{d}{dr}(\rho_0 r^3) = \rho r^2$$

and

$$\frac{1}{3}\frac{d}{dr}(\rho_0 r^3) = \frac{1}{3}\frac{d\rho_0}{dr}\cdot r^3 + \rho_0 r^2 = \rho r^2. \tag{33b}$$

Dividing by $\rho_0 r^2$, we find

$$\frac{1}{3}\frac{r}{\rho_0}\frac{d\rho_0}{dr} + 1 = \frac{\rho}{\rho_0}. \tag{34}$$

When this is substituted in (33), we get

$$r\frac{d\eta}{dr} + \eta^2 + 5\eta + \frac{2r}{\rho_0}\frac{d\rho_0}{dr}(1+\eta) = 0. \tag{35}$$

Now it turns out that the expression $\rho_0 r^5 \cdot \sqrt{1+\eta}$ is of great importance in this theory. We shall transform the equation so as to put it in these terms. Our first step is to differentiate this expression logarithmically, which gives

$$\frac{\frac{d}{dr}\left\{\rho_0 r^5\sqrt{1+\eta}\right\}}{\rho_0 r^5\sqrt{1+\eta}} = \frac{1}{\rho_0}\frac{d\rho_0}{dr} + \frac{5}{r} + \frac{1}{2(1+\eta)}\frac{d\eta}{dr}. \tag{36}$$

In terms of this logarithmic derivative, we evaluate $d\eta/dr$ and get

$$\frac{d\eta}{dr} = 2(1+\eta)\frac{\frac{d}{dr}\left\{\rho_0 r^5\sqrt{1+\eta}\right\}}{\left\{\rho_0 r^5\sqrt{1+\eta}\right\}} - \frac{10(1+\eta)}{r}$$

$$- \frac{1}{\rho_0}\cdot 2(1+\eta)\frac{d\rho_0}{dr}. \tag{36a}$$

When this is substituted in (35),

$$2r(1+\eta)\,\frac{\frac{d}{dr}\left\{\rho_0 r^5\sqrt{1+\eta}\right\}}{\rho_0 r^5\sqrt{1+\eta}} - 10(1+\eta) - \frac{2r(1+\eta)}{\rho_0}\frac{d\rho_0}{dr}$$

$$+\ \eta^2 + 5\eta + \frac{2r}{\rho_0}\frac{d\rho_0}{dr}(1+\eta) = 0. \qquad (36b)$$

When this equation is simplified, it gives

$$\frac{2\sqrt{1+\eta}}{\rho_0 r^4}\frac{d}{dr}\left\{\rho_0 r^5\sqrt{1+\eta}\right\} = 10(1+\eta - \tfrac{1}{2}\eta - \tfrac{1}{10}\eta^2) \qquad (37)$$

or

$$\frac{d}{dr}\left\{\rho_0 r^5\sqrt{1+\eta}\right\} = \frac{10(1+\frac{1}{2}\eta - \frac{1}{10}\eta^2)}{\sqrt{1+\eta}}\,\frac{\rho_0 r^4}{2}. \qquad (37a)$$

If we set

$$\Psi(\eta) = \frac{1+\frac{1}{2}\eta - \frac{1}{10}\eta^2}{\sqrt{1+\eta}}, \qquad (39)$$

then

$$\frac{d}{dr}\left\{\rho_0 r^5\sqrt{1+\eta}\right\} = 5\rho_0 r^4\Psi(\eta). \qquad (38)$$

Jeffreys notes that this equation is due to Radau (1885). The point of introducing Ψ is that it is effectively a constant within the earth. By logarithmic differentiation, we can obtain from Ψ the expression

$$\frac{1}{\Psi}\frac{d\Psi}{d\eta} = \frac{\frac{1}{2} - \frac{2}{10}\eta}{1+\frac{1}{2}\eta - \frac{1}{10}\eta^2} - \frac{1}{2}\frac{1}{1+\eta}$$

$$= \frac{1}{20}\,\frac{\eta(1-3\eta)}{(1+\frac{1}{2}\eta - \frac{1}{10}\eta^2)(1+\eta)}. \qquad (40)$$

Note that Jeffreys botched here, writing 10 instead of 1/10 for the coefficient of η^2 in the second parenthesis of the denominator. (He mentioned this in a letter to me about a year ago. Let us hope that I don't make any worse botches!)

Clearly, Ψ has a maximum or minimum at $\eta = 0$ and at $\eta = 1/3$. Near $\Psi = 0$, the logarithmic derivative of Ψ is increasing with η, since the numerator is nearly η and the denominator is nearly 1. Hence, at this point, we have a minimum of Ψ. At $\Psi = 1/3$, on the other hand, we must have a minimum, since this point is a simple 0 of Ψ, and since there is no discontinuity of the function or its derivative in this period. N.B. The roots of $1 + \eta/2 - \eta^2/10$ are at $\eta = 2.5 \pm i3.87/2$. Both roots are complex.

If we return for a moment to the quantity ϵ, we find that, since ϵ/r^3 is a decreasing function, its logarithmic derivative $\frac{1}{\epsilon}\frac{d\epsilon}{dr} - \frac{3}{r}$ will be less than 0. Therefore, $\eta > 3$. If we actually substitute the values at the surface of the earth, namely $M = 1/288$ and $\epsilon_a = 1/298.2$, we find that $\eta_a = 0.58$. Values of η are then as in the following table due to Jeffreys, with slight modifications:

η =	0	1/3	0.58	3
$\Psi(\eta)$ =	1.00000	1.00074	0.99961	0.8 .

Note that Jeffreys has 0.99928 for $\eta = 0.57$; this is another botch, which someone had already told him about in 1960. For $r = 0$, $\eta = 0$. We see that Ψ is very nearly constant. Its maximum value exceeds unity by less than

1 part in 1,000 and, at the surface, it is sunk below unity by less than 1 part in 1,000. We have not entirely excluded the possibility that η may make a wide excursion beyond the values that it reaches at the center and the surface of the earth. This is, however, very improbable and, unless this happens, we can say to an accuracy of about 1 part in 1,000 that

$$\frac{d}{dr}\left\{\rho_0 r^5\sqrt{1+\eta}\right\} = 5\rho_0 r^4, \tag{42}$$

which is clearly an enormous simplification of Equation 37. Now we would like to express these results in terms of the moment of inertia. For a homogeneous sphere, the moment of inertia is known to be $\frac{2}{5}Ma^2$, or

$$\frac{8}{15}\pi\rho a^5.$$

Differentiating, the moment of inertia of a thin spherical shell is

$$\frac{8}{3}\pi\rho r^4\Delta r,$$

and that for a non-homogeneous sphere is therefore

$$C = \frac{8}{3}\pi\int_0^a \rho r^4 dr. \tag{43}$$

To bring this in terms of ρ_0 and its derivative, we first note that the derivative of ρ_0 in (33a) is

$$\frac{d\rho_0}{dr} = -9r^{-4}\int_0^r \rho a'^2 da' + \frac{3}{r^3}\rho r^2 = -\frac{3\rho_0}{r} + \frac{3\rho}{r}.$$

Then, multiplying by r^5, we find

$$r^5\frac{d\rho_0}{dr} = -3r^4\rho_0 + 3r^4\rho .$$

We can now replace ρ by saying

$$\frac{8}{3}\pi\int_0^a \rho r^4 dr = \frac{8}{9}\pi\int_0^a 3r^4\rho dr = \frac{8}{9}\pi\int_0^a\left(3r^4\rho_0 + r^5\frac{d\rho_0}{dr}\right)dr, \tag{43a}$$

which follows Jeffreys' Equation 43.

We now integrate the second term of (43a) by parts:

$$\int_0^a r^5\frac{d\rho_0}{dr}dr = r^5\rho_0\Big|_0^a - 5\int_0^a r^4\rho_0 dr = a^5\overline{\rho} - 5\int_0^a r^4\rho_0 dr. \tag{43b}$$

We combine the second term of (43b) with the first term in the bracket of (43a) to get Jeffreys' Equation 44:

$$C = \frac{8}{9}\pi\left\{\overline{\rho}a^5 - 2\int_0^a r^4\rho_0 dr\right\} . \tag{44}$$

But, integrating (42), we have (45):

$$\int_0^a \overline{\rho_0}r^4 dr = \frac{1}{5}\overline{\rho}\; a^5\sqrt{1+\eta_a}. \tag{45}$$

And when (45) is substituted into (44), we get (46):

$$C = \frac{8}{9}\pi\overline{\rho}a^5\left\{1 - \frac{2}{5}\sqrt{1+\eta_a}\right\}, \tag{46}$$

or, in terms of the mass,

$$\frac{C}{Ma^2} = \frac{2}{3}\left\{1 - \frac{2}{5}\sqrt{1+\eta_a}\right\}. \tag{47}$$

In view of (30), the Equation 31 can be rewritten in the form

$$\eta_a = \frac{5m}{2\epsilon_a} - 2. \tag{50}$$

When (50) is substituted into (47), we get Kaula's equation, which shows a direct relation between the moment of inertia of the earth and the hydrostatic value of the flattening:

$$\epsilon_a \doteq \frac{10m}{4 + 25\left(1 - \frac{3}{2}\frac{C}{Ma^2}\right)^2}.$$

Numerical evaluation of Kaula's equation, or the equivalent pair of equations from Jeffreys, yields approximately 1/300 for the hydrostatic value of the flattening of the earth. If account is taken of some second order corrections whose theory has been discussed by Sir George Darwin, and which are summarized in the chapter by Spencer Jones in Volume 2 of Kuiper's series on The Solar System, it is found that the hydrostatic value of the flattening is near 1/299.8.

It is worthwhile to insist on the subtleties which are involved here, because they mean that the hydrostatic flattening is less than the actual flattening. The value which has previously been spoken of as the hydrostatic flattening, namely, 1/297.3, is greater than the actual flattening. If it were really true that the hydrostatic flattening were greater than the actual flattening, it would be very difficult to furnish an explanation. In the actual case when it is less, there is an equally embarrassing superfluity of explanations. Conceivably, the difference is due to the melting of the polar ice caps and some lag in the restoration of isostasy especially, perhaps, in Antarctica. Again, it is conceivable that the discrepancy is a consequence, in some way, of the fact that the polar caps are colder than the equator. It turns out that the temperature difference continues to exist for a surprisingly great distance into the earth.

Since we are dealing with quantities of the order of 1 part in 100,000, it is clear that even a very moderate temperature difference may seriously affect the earth's flattening. Again, because of the fact that the laws of heat transport by conduction are irreconcileable with the kind of thermal stratification which is implied by the theory of hydrostatic equilibrium, there will be some necessary distortions of hydrostatic equilibrium in a rotating body, as was first pointed out by vonZeipel. Finally, and in my opinion most plausible, there is the explanation of G.K. MacDonald (personal communication, 1960) to the effect that the excess bulge around the equator is the resultant of a retardation in the earth's rotation over the past millions of years. I do not think that any of these explanations can be excluded in a satisfactory way, with the possible exception of the melting of the polar ice caps. Kaula has made some computations along this line which indicate that it is numerically inadequate. I am inclined to think that the most plausible explanation, if we must choose one, is the retardation of the earth's rotation, for which there exists independent evidence.

In any case, it is important to notice that the flattening is a direct function of the polar moment of inertia. If we are given another functional relationship between these two quantities, such as that provided by the luni-solar precession which yields the quantity $(C-A)/C$, then we are able to solve for the hydrostatic flattening. The solution does not depend in any way on what the actual value of the flattening is. If we know C within 1 part in 10,000, then we can calculate the value of the hydrostatic flattening to approximately the same accuracy. On the other hand, an error of 1 part in 10,000 in the actual value of C would upset the observed value of the flattening by the totally unacceptable amount of 10 units in the reciprocal of the flattening. Thus, the presently observed values of the actual flattening are better than are needed to make a satisfactory calculation of the hydrostatic flattening.