AN INVISCID MODEL OF THE SOLAR WIND*

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ABSTRACT

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In this paper we consider an inviscid, thermal conductive, spherically symmetric model of the solar wind. The equations of mass, momentum and energy conservation are integrated simultaneously to give a unique solution for this solar wind model. The solution obtained smoothly passes through the critical radius and satisfies the condition $T = o$ as $r \rightarrow \infty$. The quantitative physical implication of the solution is discussed.

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1. INTRODUCTION

In the solar corpuscular radiation, fully ionized plasma is perpetually flowing outward from the sun in all directions. This continuous outflow of plasma is referred to as the solar wind. Parker (1958, 1961, 1963, 1964a, 1964b) has done an excellent pioneering work on the hydrodynamic theory of the solar wind. He used the hydrodynamic equations to describe the dynamical behavior of the spherically symmetric solar wind. In one of his recent papers, Parker (1964a) solved the mass and momentum equations for a model in which the temperature T is taken to be a simple monotonically decreasing continuous function $T(r)$ of radial distance from the sun. It was followed by another paper in which Parker (1964b) solved the energy conservation equation for the form of $T(r)$. His results have shown many general qualitative features of the solar wind. However, since the momentum and energy equations were treated separately, no quantitative result can be obtained from Parker's model to describe the actual physical process involving the simultaneous transfer of both energy and momentum. Showing the strong coupling effect between the variations of temperature and velocity, it is the purpose of this paper to find the solar wind solution by integrating the momentum and energy equations simultaneously. This approach has been attempted by Noble and Scarf (1963), but they are not able to find the exact numerical solution which must satisfy the flow conditions at the critical radius and approach $T = o$ at $r = \infty$. In this paper we can successfully find the exact simultaneous solution of the momentum and energy equations satisfying all required conditions, from this solution the quantitative values for the strength of the solar wind at the orbit of earth or any other planet of the solar system can be deduced.

In this paper we make the following assumptions: (i) The model of the solar wind is steady and is restricted to spherical symmetry about the sun. We consider the flow velocity is along the radial direction, and all flow properties (velocity V, pressure p , density ρ and temperature T) are functions of a single space coordinate r. The solution provided by this spherically symmetrical model will describe some proper average values of the velocity, temperature, and other flow properties over a spherical surface at constant r , then we can approximately describe the actual flow condition by these average values. (ii) The perfect gas law is valid, the specific heats are constant and the ratio of specific heats $\gamma = \frac{5}{3}$. (iii) The viscosity effects are neglected; and (iv) energy is transported by thermal conduction in which the thermal conductivity κ is proportional to $T^{5/2}$. Based on these assumptions we write the basic fluid equations in section 2. The equation system has a singular point which is the only tunnel the flow can smoothly pass from a region of low speed flow to the supersonic region. In order to integrate the momentum and energy equations simultaneously we have obtained in section 3 the solution of the equation system near the special smooth passage (at the critical radius); and in section 4 the asymptotic solution of the equation system for large radius. Then a <u>unique</u> solution for the inviscid model of the solar wind in dimensionless form is obtained in section

5; this solution is valid for all value of $r\,(>R_\odot)$ and satisfies (i) the condition of smooth transition at the critical radius, and (ii) the asymptotic solution of the equation system. In section 6, we will discuss the quantitative physical implication of this solution, and compare the result with the observed data.

2. BASIC EQUATIONS

The equation of continuity is

$$\frac{d}{dr}(\rho V r^2) = 0 \tag{1}$$

which can be integrated to $\rho V r^2 = \text{constant}$.

The equation of motion is

$$\rho V \frac{dV}{dr} = -\frac{d}{dr}\left(\frac{3}{5}\rho a^2\right) - \frac{\rho G M_\odot}{r^2} \tag{2}$$

where a is the speed of sound $\left(a^2 = \frac{5}{3} R T\right)$, G the gravitational constant, and $M_\odot$ the mass of the sun. Making use of the continuity equation $\rho V r^2 =$ constant, equation (2) can be written as

$$\frac{r}{V}\frac{dV}{dr} = \frac{\frac{6}{5}\left(a^2 - r a \frac{da}{dr}\right) - \frac{G M_\odot}{r}}{V^2 - \frac{3}{5} a^2} \tag{3}$$

Under the assumption that energy is transmitted outward from the base of the solar corona by thermal conduction only, the equation of energy conservation is

$$\rho V \frac{d}{dr}\left(\frac{3}{2} a^2 + \frac{V^2}{2}\right) = -\frac{1}{r^2}\frac{d}{dr}(r^2 q) - \frac{\rho G M_\odot}{r^2} V$$

where q represents the energy flux due to thermal conduction. Inasmuch as $\rho V r^2$ is constant, this equation can be integrated immediately to give

$$-q = \rho V \left(\frac{3}{2} a^2 + \frac{V^2}{2} - \frac{GM_\odot}{r} - h_0 \right)$$

Here h_0 is the constant of integration determined by the condition at $r =$ infinity. Introducing $q = -\varkappa \frac{dT}{dr} = -\frac{6}{5} \frac{\varkappa a}{R} \frac{da}{dr}$, the above equation can be written as

$$\frac{da}{dr} = \frac{5}{6} \frac{R \rho V}{a \varkappa} \left[\frac{3}{2} a^2 + \frac{V^2}{2} - \frac{GM_\odot}{r} - h_0 \right] \tag{4}$$

From equation (3) we can see that in general a barrier for the solution exists at $M^2 = 3/5$, where the denominator of equation (3) vanishes. We shall denote this critical velocity at $M^2 = 3/5$ by V^*, and the critical radial distance where $M^2 = 3/5$ occurs by r^*, and introduce the following dimensionless variables:

$$u = V/V^*, \quad A = a/V^*, \quad Z = r/r^* \tag{5}$$

and dimensionless parameters:

$$\alpha = \frac{5}{2} R r^* \rho^* V^* / k^*, \quad \Upsilon = GM_\odot / r^* V^{*2}, \quad H = h_0 / V^{*2} \tag{6}$$

In terms of these dimensionless quantities, we can write (3) and (4) as

$$\frac{z}{u} \frac{du}{dz} = \frac{(6/5)(A^2 - zA\, dA/dz) - \Upsilon/z}{u^2 - (3/5) A^2} \tag{7}$$

and

$$\frac{dA}{dz} = \frac{1}{3}\left(\frac{5}{3}\right)^{5/2} \frac{\alpha}{A^6 z^2}\left(\frac{3}{2}A^2 + \frac{u^2}{2} - \frac{\Upsilon}{z} - H\right) \tag{8}$$

These two ordinary differential equations will suffice to solve for the two dependent variables u and A. From (7) and (8) we can see that u and A are closely connected and in general the solution for u and A will be in the form

$$u = u\,(z, \alpha, \Upsilon, H) \quad and \quad A = A(z, \alpha, \Upsilon, H) \tag{9}$$

In the mathematical solution developed in the following sections, we will show that a unique solution exists when the parameters $\alpha = 0.25253$, $\Upsilon = 2.4185$ and $H = 4.7245$.

3. SOLUTION NEAR THE CRITICAL RADIUS

At the critical radius where $z = u = 1$ and $A^2 = 5/3$ the denominator of equation (7) vanishes. The smooth transition of the mass motion from a low Mach number flow $(M^2 < 3/5)$ in the region near the sun's surface to supersonic flow region is possible only when the condition that the numerator of (7) vanishes at the critical radius. From this condition and equation (8) we obtain that

$$\left(\frac{dA}{dz}\right)^* = -\frac{1}{2}\left(\frac{5}{3}\right)^{1/2}(\Upsilon - 2) \tag{10}$$

and

$$\alpha = \frac{5}{2}\,\frac{\gamma-2}{\gamma+H-3} \tag{11}$$

The explicit dependence of α on γ and H(Equation (11)) implies that the parameters in equation (9) reduces from three to two, i.e.,

$$u = u(z,\gamma,H) \quad and \quad A = A(z,\gamma,H) \tag{12}$$

However, since both the denominator and the numerator on the right hand side of (7) disappear simultaneously at the critical radius, l'Hospital's rule should be used to determine $(du/dz)^*$. The result is

$$\left(\frac{du}{dz}\right)^* = -B + \sqrt{B^2 + C} \tag{13}$$

where $$B = \frac{\alpha}{10}(\gamma + H - 2)$$

and $$C = \frac{\alpha}{2}(\gamma + H - 2)(\gamma - 2) + \frac{\alpha}{5}(9 - 4\gamma - 3H) + \frac{\gamma}{2}$$

Equations (10) and (13) represent the slope of the solution curves in the neighborhood of the critical radius.

4. ASYMPTOTIC SOLUTION

In this analysis we consider that the solar wind does not end at finite distance from the sun and that the equations (7) and (8) are valid for all values of z.

When z approaches infinity $u \rightarrow u_\infty = \sqrt{2H}$ and $A \rightarrow 0$ asymptotically. The asymptotic part of the exact solution of (7) and (8) for large values of z can be conveniently obtained by introducing $\epsilon = z^{-1/5}$. In terms of ϵ, equations (7) and (8) become

$$(u^2 - \tfrac{3}{5}A^2)\frac{du}{d\epsilon} + 6u\left(\frac{A^2}{\epsilon} + \frac{A}{5}\frac{dA}{d\epsilon}\right) - 5\gamma\epsilon^4 u = 0 \tag{14}$$

and

$$A^6 \frac{dA}{d\epsilon} + \left(\frac{5}{3}\right)^{7/2}\alpha\epsilon^4\left(\frac{3}{2}A^2 + \frac{u^2}{2} - \gamma\epsilon^5 - H\right) = 0$$

The formal expansions of A and u in powers of ϵ, satisfying the differential equations and the conditions at infinity, are

$$A = c_1\epsilon\,(1 + a_1\epsilon + a_2\epsilon^2 + a_3\epsilon^3 + a_4\epsilon^4 + a_5\epsilon^5 + \cdots)$$

and (15)

$$u = u_\infty - c_2\epsilon^2\,(1 + b_1\epsilon + b_2\epsilon^2 + b_3\epsilon^3 + b_4\epsilon^4 + b_5\epsilon^5 + \cdots)$$

where

$$c_1 = \sqrt{\tfrac{5}{3}}\,(3.5\alpha)^{0.2}, \quad c_2 = \frac{6}{u_\infty}(3.5\alpha)^{0.4}, \quad a_1 = b_1 = a_3 = 0,$$

$$a_2 = \frac{1}{59}\frac{c_2}{u_\infty}, \quad a_4 = \frac{44{,}711}{2{,}391{,}447}\frac{c_2^2}{u_\infty^2}, \quad a_5 = -\frac{1}{59}\frac{\gamma}{u_\infty^2},$$

$$b_2 = \frac{427}{708}\frac{c_2}{u_\infty}, \quad b_3 = -\frac{\gamma}{c_2 u_\infty}, \quad b_4 = \frac{21{,}359{,}580}{28{,}697{,}364}\frac{c_2^2}{u_\infty^2}, \quad b_5 = -\frac{401}{341}\frac{\gamma}{u_\infty^2}$$

5. NUMERICAL SOLUTION

When no explicit solution of (7) and (8) for all values of z is known, the most straight-forward method of obtaining the solution is by numerical integration. In sections 3 and 4, we have obtained the solution both near the singular point and at large z. For any given values of γ and H starting with the slope at $z = 1$, a simple point-by-point integration of equations (7) and (8) will yield the solution for A and u as function of z. We recall that A and u are functions of the independent variable z and the two parameters γ and H [equation (12).] We expect that the numerical solution passing the singularity point joint the asymptotic solution (15) at a suitable point. This condition is used in this section to determine the values of γ and H.

The features of the solution may best be seen from the plot of the solution curves in the u, A - plane. For a fixed value of H, numerical solutions are obtained for various values of γ, as shown in Fig. 1, with arrows indicating the direction of increasing z. From this figure we can see that only one particular curve could approach $u = u_\infty$ and $A = o$ at z = infinity. This means there is only one value of γ for each given H. In other words, only one parameter H remains independent. Therefore, from (12) we can write

$$u = u(z,H) \quad \text{and} \quad A = A(z,H) \tag{16}$$

It should be noted that under the steady state assumptions, although the solution curves on the u,A-plane (Fig. 1) may be discussed in terms of mathematical stability, this is in no way connected with any question of physical stability. To show the existence of a unique solution, it remains to verify that only for a particular value of H, the numerical solution can smoothly join the asymptotic solution. This is shown by plotting the numerical solution for different values of H in Fig. 2. In general the numerical solution cannot match the asymptotic solution, and this becomes possible only when H = 4.7245. Here we have shown the existence of exactly one solution

$$u = u(z) \quad and \quad A = A(z) \tag{17}$$

which satisfies all the required conditions. The values for $A(z)$ and $\dot{u}(z)$ are given in Table 1.

6. QUANTITATIVE DISCUSSION

We now discuss the quantitative physical implication of the exact numerical solution. We shall first compare our exact solution with the observed densities in the corona (van de Hulst 1950, 1953, Michard 1954, Blackwell 1956). As shown in Fig. 3, we can see that if we choose $r^* = 7.5\ R_\odot$, the coronal density curve predicted by this model agrees very well with the observed data. When $r^* = 7.5\ R_\odot$ is fixed then using the relations

$$\gamma = \frac{GM_\odot}{r^* V^{*2}} = 2.4185 \quad and \quad M^{*2} = \frac{3V^{*2}}{5RT^*} = \frac{3}{5}$$

we can calculate the velocity and the temperature at the critical radius $V^* \cong 100$ km/sec and $T^* \cong 6.3 \times 10^5\ ^\circ K$. Using these calculated V*, T*, and the observed density $N_i^* = N_e^* = 2 \times 10^4\ cm^{-3}$ at $r^* = 7.5\ R_\odot$, we can calculate N, T, V and M as functions of radial distance from the center of the sun (Fig. 4). At the orbit of Earth the typical interplanetary conditions directly observed from vehicles in space (Shklovskii, Moroz and Kurt 1960; Gringauz, Bezruvkikh, Ozerov, and Rybchinskii 1960; Bridge, Dilworth, Lazarus, Lyon, Rossi and Scherb 1962, Neugebauer and Snyder 1962; Bonetti, Bridge, Lazarus, Rossi and

Scherb 1963) are of the order of $V \sim 300 - 600$ km/sec, $T \sim 10^5$ °K, and $N \sim 2-20$ ions/cm^3, while the conditions predicted by this theory are $V \sim 260$ km/sec, $T \sim 1.6 \times 10^5$ K, and $N = 10$ ions/ cm^3. Note that this theory can actually predict the strength of the solar wind at the orbit of any planet of the solar system, the quantitative results are given in Table 2. A more accurate theory can be obtained if the viscous effect is not neglected in the equations of momentum and energy conservation; the calculation will be published shortly.

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Table 1 Numerical Solution

Z = radial distance from sun / critical radius	Log Z	A = Speed of sound / critical velocity	U = gas velocity / critical velocity
1.0000×10^{-1}	-1.0	2.1990846	.00249263
1.5849	-0.8	1.9544054	.02448650
2.5120	-0.6	1.7483707	.12026203
3.9810	-0.4	1.5737168	.34068417
6.3098	-0.2	1.4231542	.65687788
1.0000×10^{0}	0.0	1.2909945	1.0000000
1.5849	0.2	1.1733288	1.3239839
2.5120	0.4	1.0677619	1.6103798
3.9810	0.6	.97258423	1.8553683
6.3098	0.8	.88646813	2.0615027
1.0000×10	1.0	.80839318	2.2333299
1.5849	1.2	.73751945	2.3758568
2.5120	1.4	.67308887	2.4938425
3.9810	1.6	.61448560	2.5913896
6.3098	1.8	.56116321	2.6720113
1.0000×10^{2}	2.0	.51260174	2.7386991
1.5849	2.2	.46837279	2.7938884
2.5120	2.4	.42807776	2.8395952
3.9810	2.6	.39132092	2.8775042
6.3098	2.8	.35775164	2.9089711
1.0000×10^{3}	3.0	.32738502	2.9351321
1.5849	3.2	.29950830	2.9571397
2.5120	3.4	.27408350	2.9754869
3.9810	3.6	.25089752	2.9908081
6.3098	3.8	.22974523	3.0036145
1.0000×10^{4}	4.0	.21044052	3.0143356
1.5849	4.2	.19282244	3.0233234
2.5120	4.4	.17674305	3.0308715
3.9810	4.6	.16205756	3.0372206
6.3098	4.8	.14864409	3.0425676
1.0000×10^{5}	5.0	.13639171	3.0470790
1.5849	5.2	.12519053	3.0508964
2.5120	5.4	.11494478	3.0541255
3.9810	5.6	.10557373	3.0568668
6.3098	5.8	.09699080	3.0591971

Table 1 Numerical Solution

$Z = \frac{\text{radial distance from sun}}{\text{critical radius}}$	Log Z	$A = \frac{\text{Speed of sound}}{\text{critical velocity}}$	$U = \frac{\text{gas velocity}}{\text{critical velocity}}$
1.0000×10^6	6.0	.08912985	3.0611932
1.5849	6.2	.08192611	3.0628886
2.5120	6.4	.07531871	3.0643305
3.9810	6.6	.06926089	3.0655856
6.3098	6.8	.06367599	3.0666837
1.0000×10^7	7.0	.05856591	3.0675840
1.5849	7.2	.05382247	3.0684074
2.5120	7.4	.04950765	3.0691440
3.9810	7.6	.04549311	3.0696931
6.3098	7.8	.04173049	3.0702420
1.0000×10^8	8.0	.03826551	3.0707911

Table 2 -- The strength of the solar wind at the orbit of any planet of the solar system

	Velocity Km/sec.	Temperature °K	Particle Density ions/cm^3	Mach Number
Mercury's orbit	230	2.3×10^5	72	2.85
Venus's orbit	250	1.9×10^5	20	3.50
Earth's orbit	260	1.6×10^5	10	3.84
Mar's orbit	270	1.4×10^5	4	4.31
Jupiter's orbit	280	8.5×10^4	3	5.8
Saturn's orbit	290	6.3×10^4	0.1	6.7

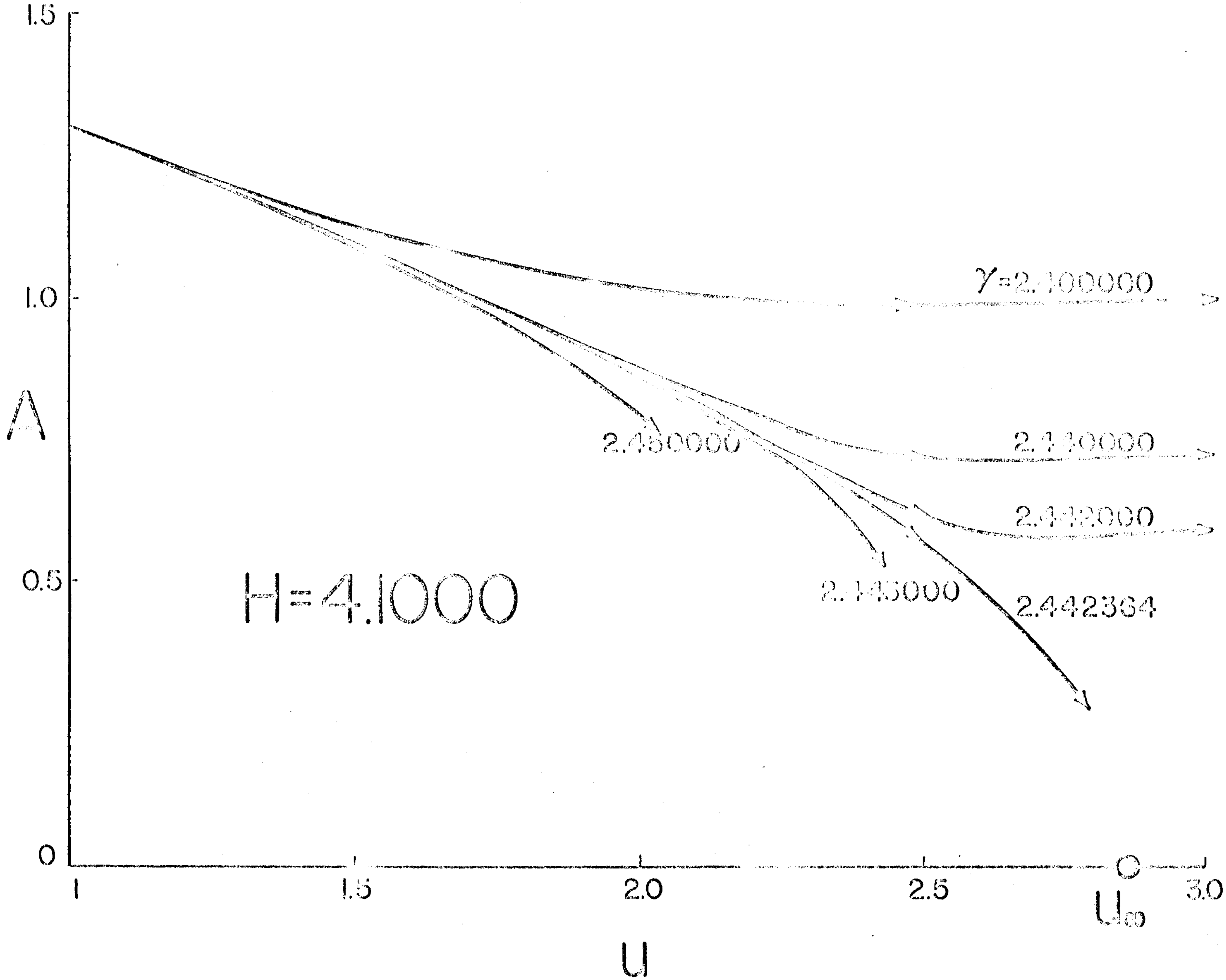

Fig 1 Solution curves for a fixed value of H and varying values of γ .

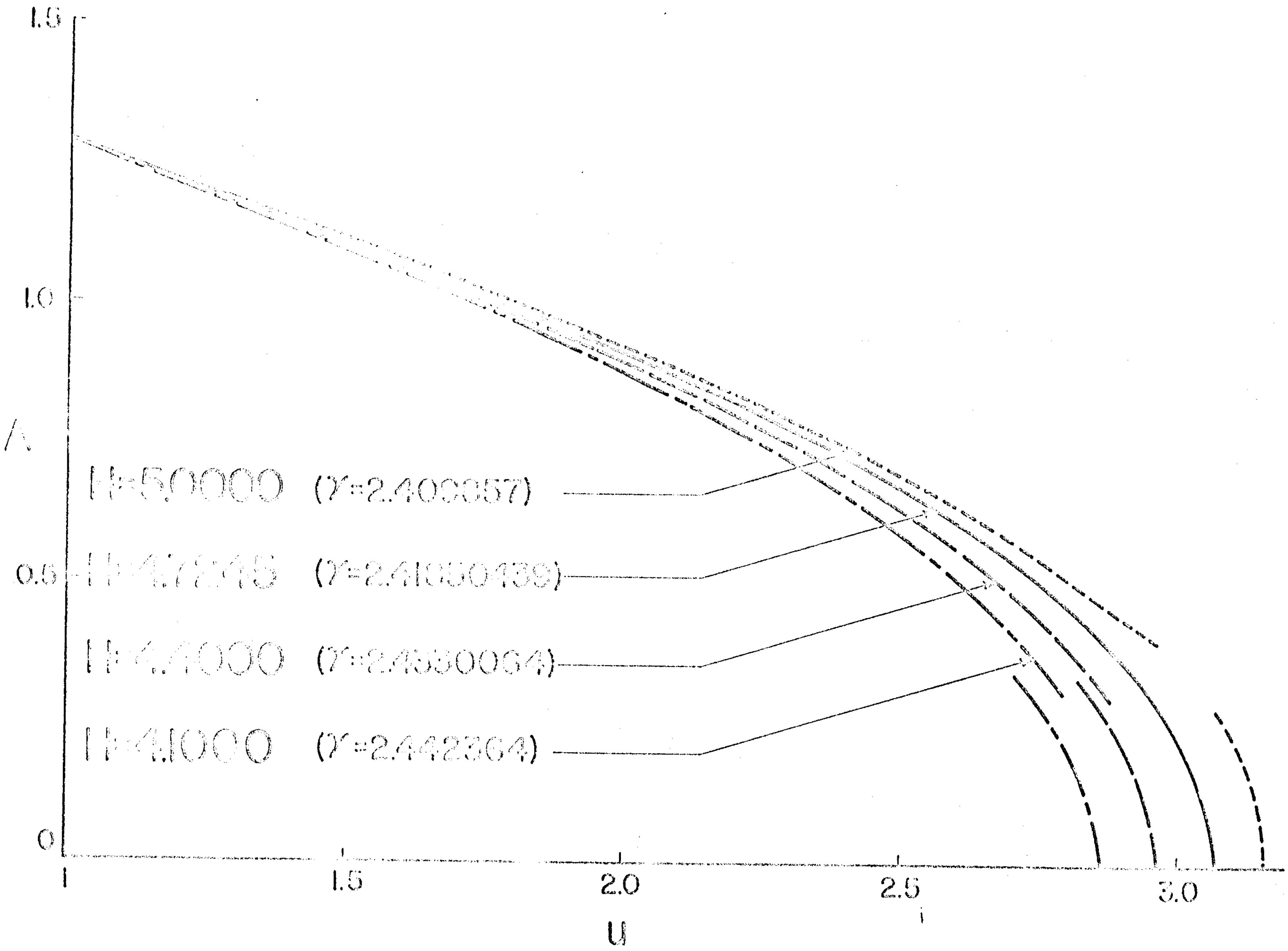

Fig 2 Solution curves for different values of H .

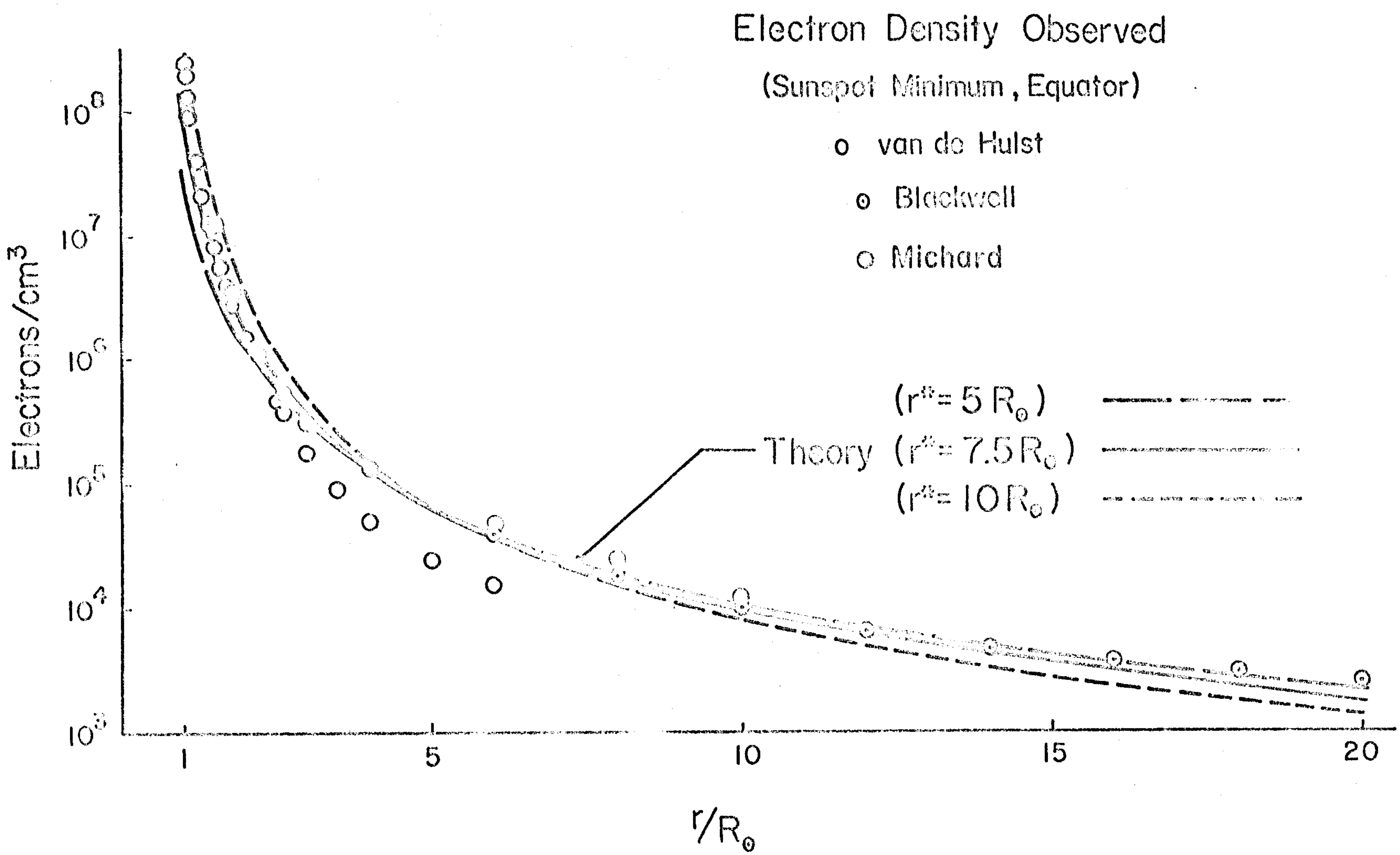

Fig 3 Comparison between the theoretical density curves and the observed densities.

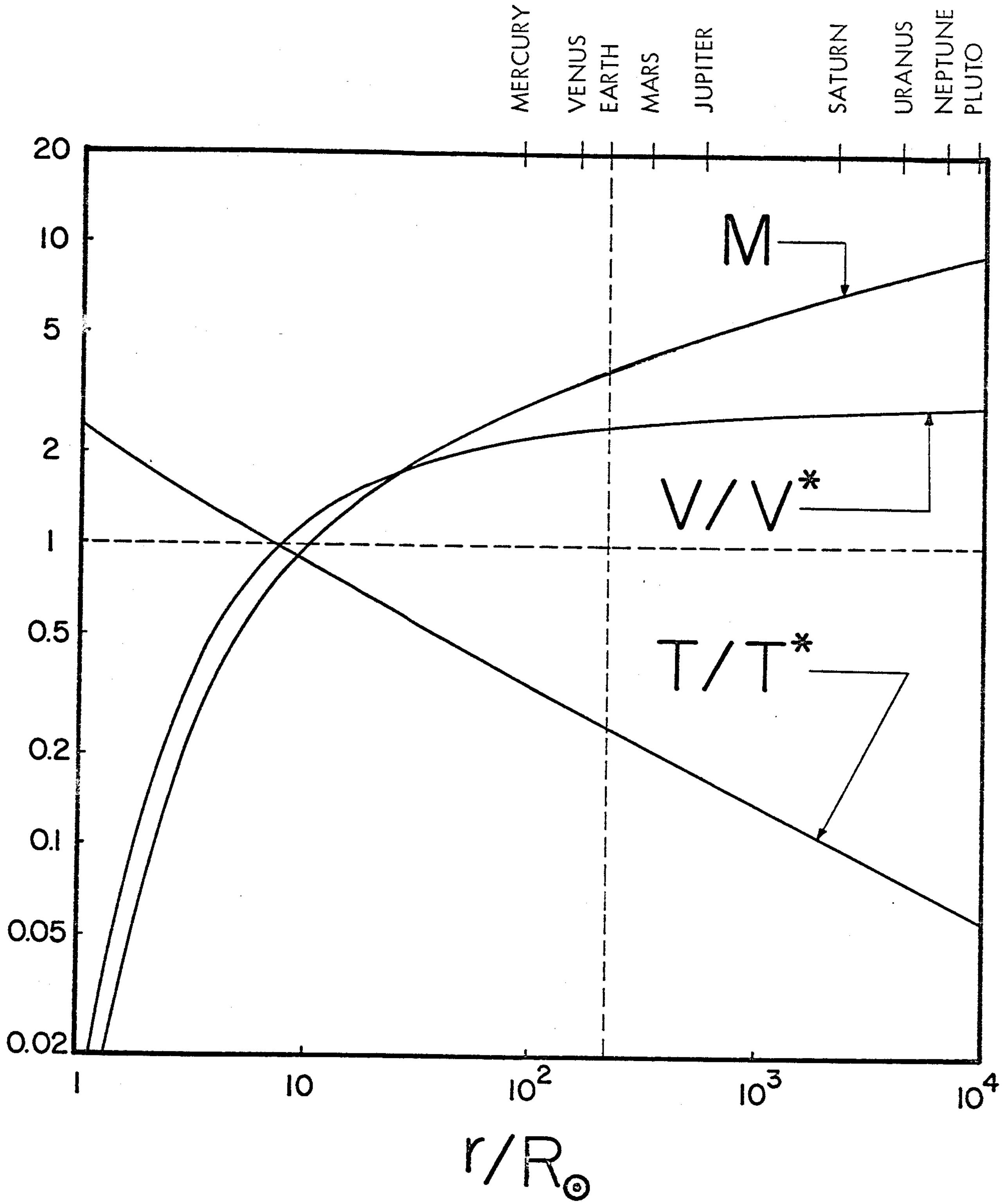

Fig 4 M, v / v*, T / T* as functions of the radial distance from the center of the sun.