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THEORY OF THE RESISTIVELY LOADED

TRAVELING WAVE V-ANTENNA [redacted]

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SUMMARY

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A preliminary report is presented for the essential parts of the theory of the resistively loaded travelling wave V-antenna. The principle of superposition and a new theory for long resonant antennas is applied to the prediction of the actual antenna current. As a first approximation, self and mutual impedances are derived. The radiation properties of the antenna are also presented for an assumed current. Machine calculations for the radiation properties have been verified by experiment. No numerical values for the actual antenna current and impedance are yet available for comparison with experiment.

Author

I. INTRODUCTION:

The antenna system devised for radio-astronomical observations in space consists of two V-antennas whose apex is located on the satelite. A travelling wave of current is produced in each V-antenna by a suitable resistive loading. The travelling wave V-antenna is capable of producing unilateral radiation patterns, an important requirement for space observation.

The measurement of sky temperatures by a satellite borne directive antenna is dependent on the effective area A of the particular antenna. The effective area may be computed from the radiation and circuit properties of the antennas [1]. The effective available area of the antenna is given by

$$A = \mathrm{Re}\left\{8\left|\frac{\cos\psi\, h_e(\theta)}{Z_1 + Z_o}\right|^2 \zeta_o\, Z_1\right\} \qquad (1)$$

where $h_e(\theta)$ = complex effective length

Z_1 = load impedance

Z_o = characteristic impedance of the antenna

$\zeta_o \doteq 120\,\pi$ ohms $\doteq$ 377 ohms

ψ = tilt angle of the electric field

Thus, in order to compute the effective area A in a particular direction, the values of the effective length $h_e(\theta)$ (a function of the radiation pattern)

1. S.S. Sandler, "Effective Area of Satellite-Borne Antennas for Radio Astronomy", Plan. Sp. Sci. 1963, Vol. 11, pp. 817-822

and the characteristic impedance Z_0 must be known. A finite length centre driven linear antenna is known to have a standing wave of current along its length. The essential characteristics of this type of antenna are:

(a) a relatively narrow bandwidth of less than two to one and

(b) a bilateral radiation pattern.

The strongly resonant standing wave antenna has a frequency dependent impedance which limits the bandwidth. Furthermore, axially symetric structure can produce no unilateral directivity. Another type of linear antenna is the Beverage or travelling-wave antenna. A proper resistive termination at the end of the element to ground produces a travelling-wave distribution of current. A simple physical picture of the process follows from elementary transmission line theory. The voltage generator sends a travelling wave down the antenna toward the resistive termination. From transmission line theory it is known that a line terminated in its characteristic impedance has no reflected waves along its length. Therefore the antenna is properly terminated in its "characteristic impedance" and no waves are reflected from the end. It is the reflected waves which normally combine with the incident waves to produce a standing wave.

A travelling wave is produced on the satellite antenna by a suitable resistive loading, one quarter wavelength from the end of the antenna. [2] A travelling wave is produced from the driving point to the resistive termination and a standing wave is produced on the remainder of the element. This type of antenna has some of the same advantages as the conventional

2. Edward E. Altschuler, "The Travelling-Wave Linear Antenna", Scientific Report No. 7 (series 2), Harvard University, May 5, 1960

travelling-wave antenna. For example it has a wide bandwidth and when properly combined with a second element, (i.e. as in a V-antenna) it is capable of yielding a unilateral radiation pattern.

II DRIVING POINT IMPEDANCE

The driving point impedance of the travelling wave V-antenna is given by a suitable superposition of standing wave antennas. Only resonant length standing wave antennas are involved in the superposition if the top section is resonant and the lower section antiresonant. Thus for the geometry of Figure 1

$$\begin{aligned} (h - h_1) &= m \; \lambda/4 \quad (m \text{ odd}) \\ h_1 &= m \; \lambda/2 \quad (m = 1,2,3, \ldots) \end{aligned} \tag{1}$$

A simpler approximation used for the driving point impedance is based on the computation of the self and mutual impedance. The self impedance is computed for the linear resistively loaded antenna, as though isolated. The superposition for this case is described by Altschuler [3]. The mutual impedance may be computed based on the E.M.F. method under the assumption that the form of the current remains essentially the same independent of the angle of the V-antenna. The fundamental difficulty here is to have a good representation of the currents and impedances of long linear antennas. Although the driving point impedance of long antennas had been obtained

3. Edward E. Altschuler, "The Travelling-Wave Linear Antenna", Scientific Report No. 7 (series 2), Harvard University, May 5, 1960

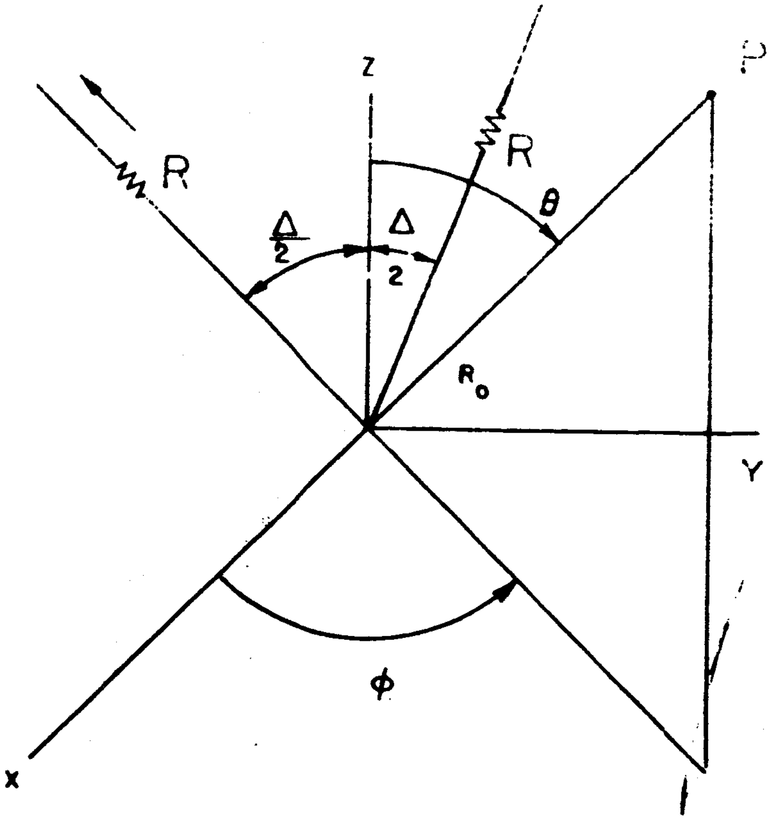

FIGURE 1a COORDINATE SYSTEM FOR V-ANTENNA

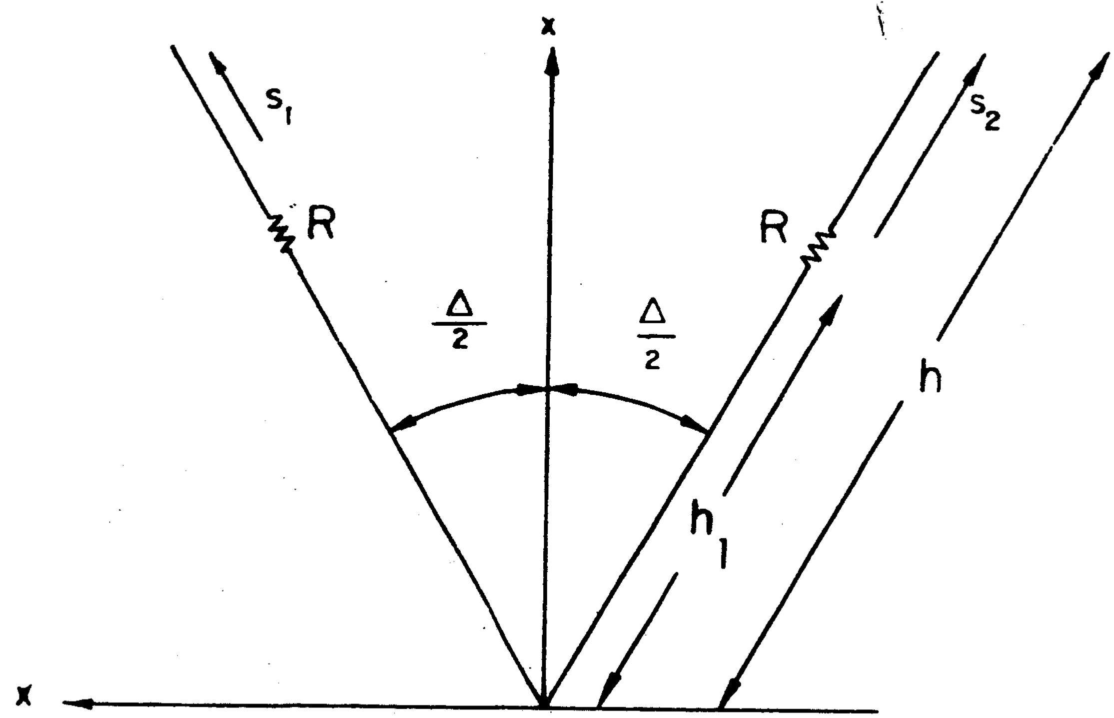

FIGURE 1b DETAIL OF V-ANTENNA

before, no accurate representation for the current was available. An accurate simple trogonometric representation for the currents was obtained from the integral equation. The theory is given in Appendix I. The values of Z_o as computed from this theory are given for some representative lengths and radii in the appendix. These values of Z_o agree remarkably well with the results quoted by King[4] and with the theory of Wu [5]. For example, for $\Omega = \quad n(\lambda/a) = 10$ and $\beta h = 3\pi/4$. King gives $Z_o = 127.6 + j43$ and the present theory gives $Z_o = 127.45 + j37.8$. A comparison of the currents is shown in Figure 2. The computation of the approximate mutual impedance is shown in Appendix II. The antenna current used in this approach is the current given by the linear resistively loaded travelling wave antenna. No theoretical numerical values are presently available for the driving point impedance of the V-antenna.

III RADIATION PATTERNS

The general form for the radiation functions is given in Appendix II. Computations have been performed based on a perfect travelling wave up to the terminating resistances and an equal amplitude standing wave along the end section. A comparison of some machine computations with the experimental results of Izuka at Harvard is shown in Figure 3. The agreement is very good and further improvement will appear with the use of an improved current form.

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4. R.W.P. King, The Theory of Linear Antennas, Harvard University Press, Cambridge, Massachusetts, 1956

5. T.T. Wu, Theory of the Dipole Antenna and the Two-Wire Transmission Line, Journal of Math. Phys., Vol.2, No.4, pp. 550-574, July-August 1961

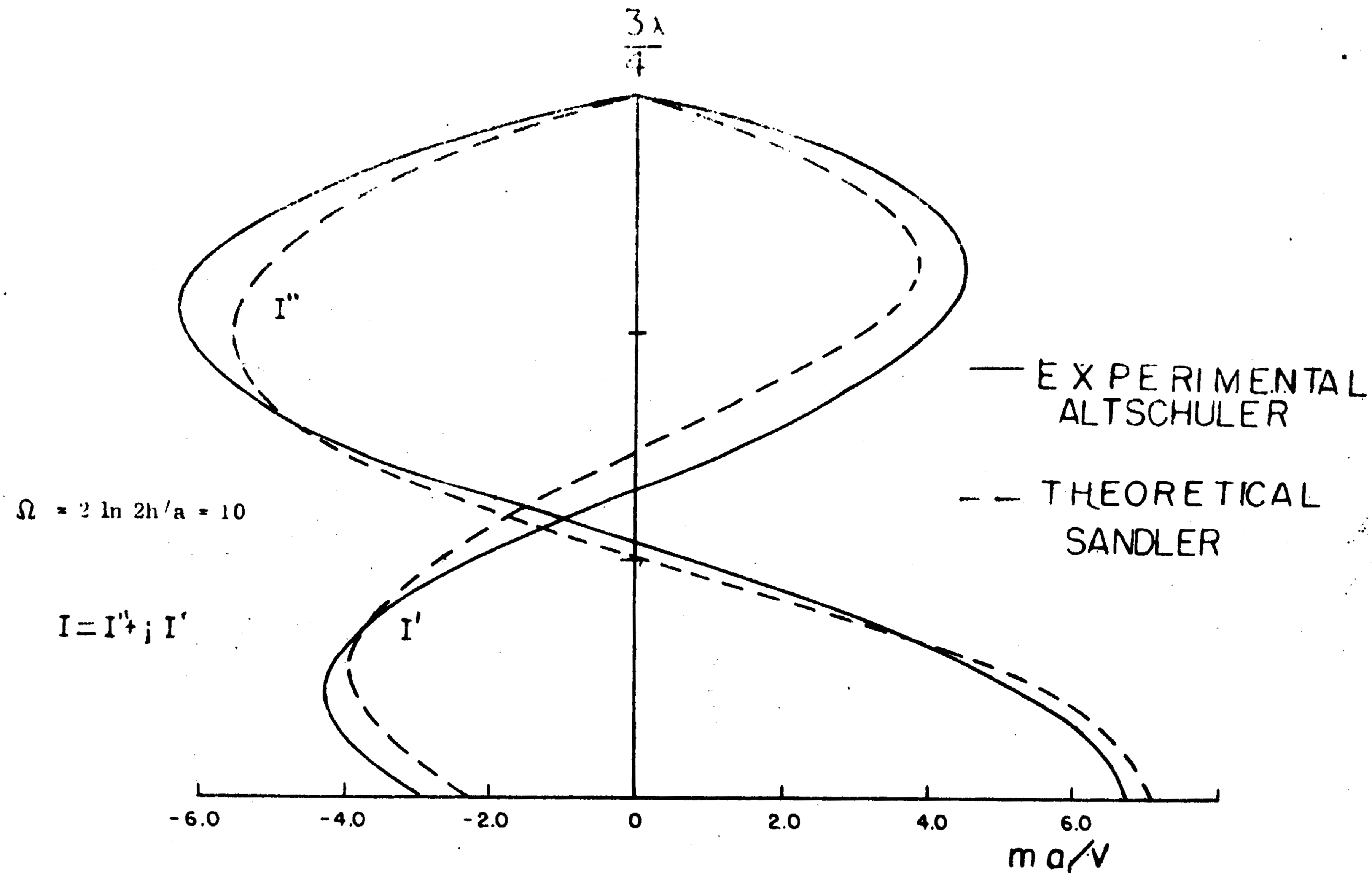

FIGURE 2 DISTRIBUTION OF CURRENT FOR $h = 3\lambda/4$

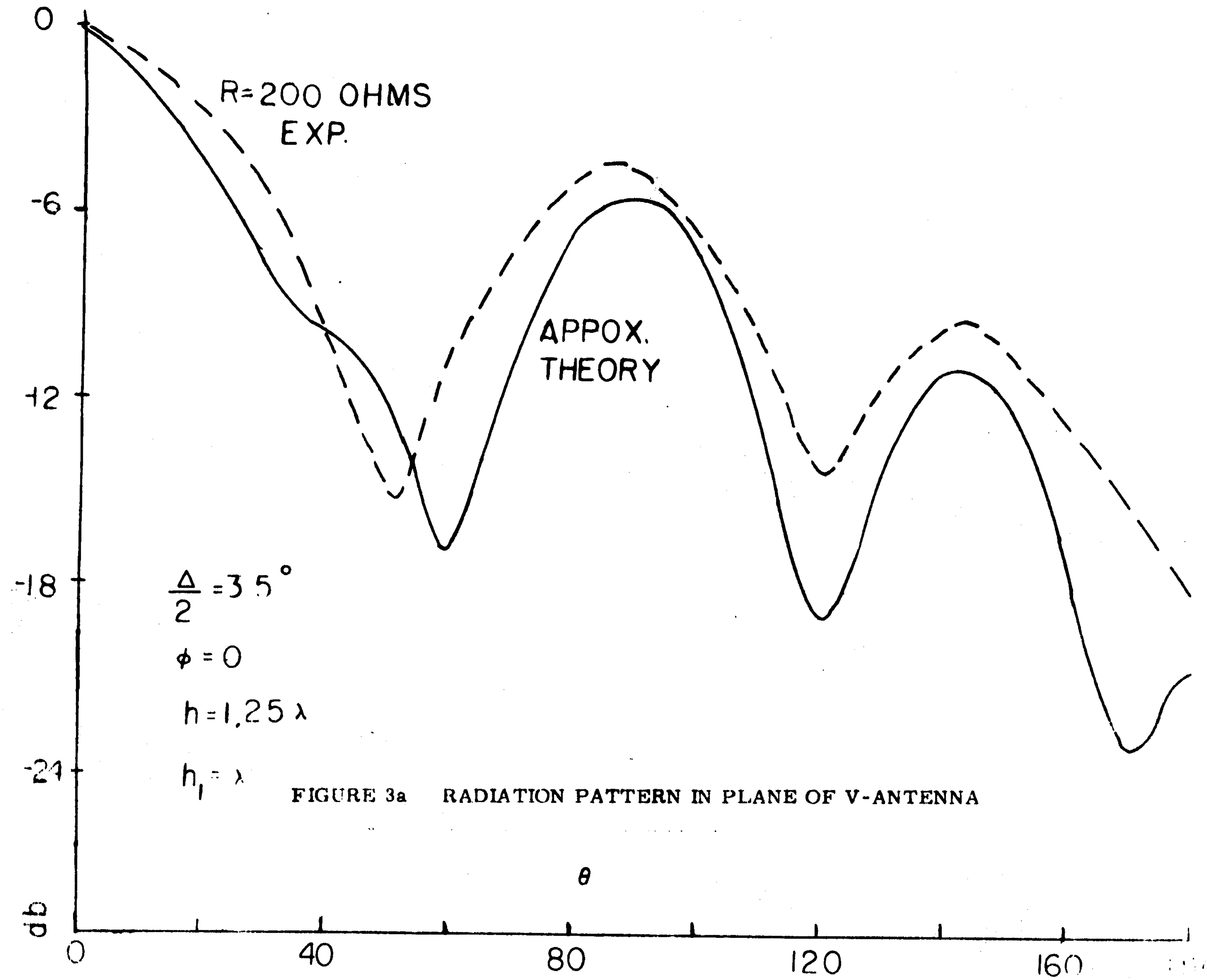

FIGURE 3a RADIATION PATTERN IN PLANE OF V-ANTENNA

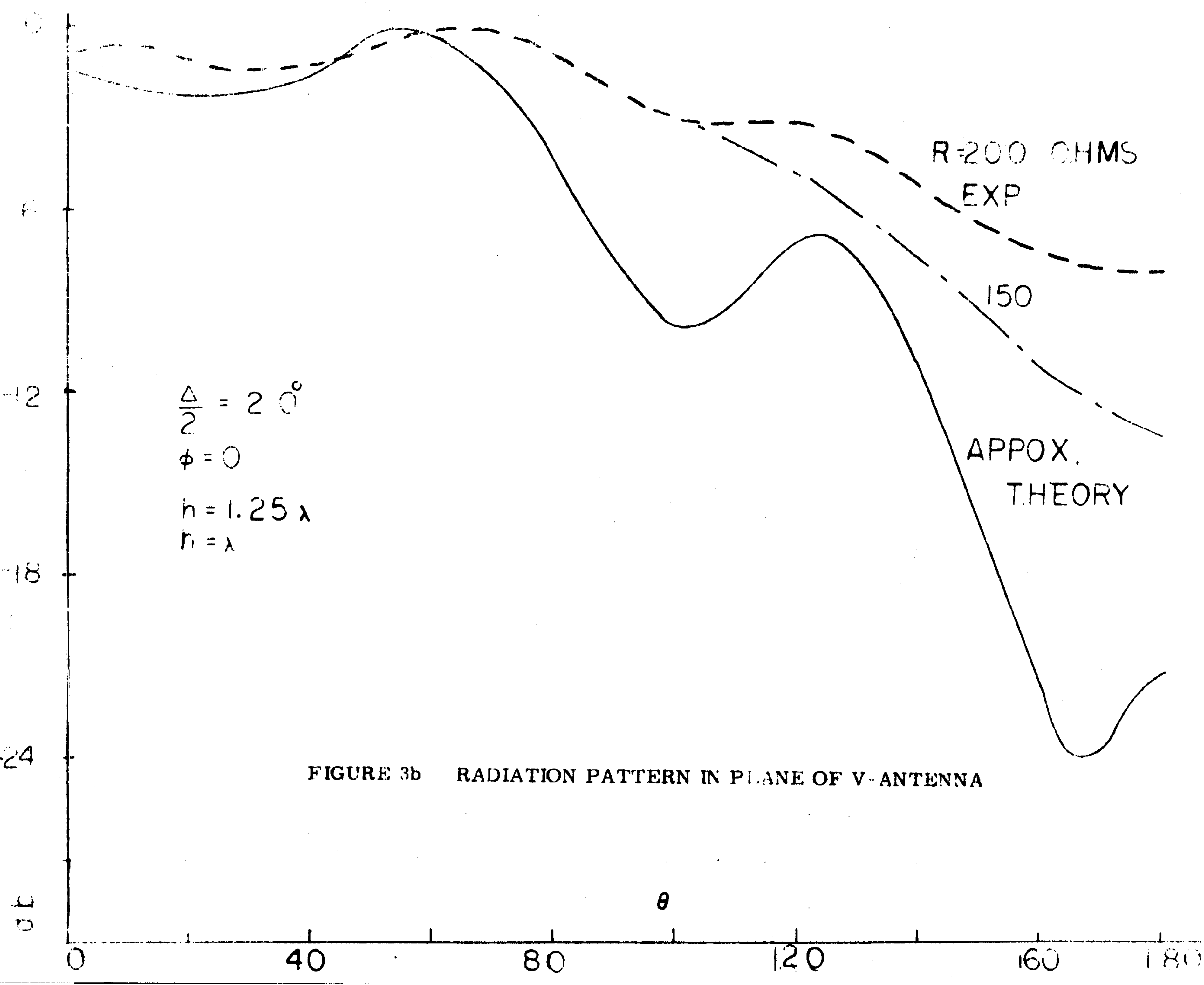

FIGURE 3b RADIATION PATTERN IN PLANE OF V-ANTENNA

APPENDIX I

DRIVING POINT IMPEDANCE AND CURRENT OF RESONANT ANTENNAS

INTRODUCTION

An extensive treatment of the centre driven cylindrical antenna based on the King-Middleton iterative procedure is given in King [6]. The currents and the driving point impedances are given for antennas with half lengths generally less than a wavelength. A few experimental results are given for long antennas [7] but theoretical impedances and currents were not computed for the longer lengths. An improved iteration procedure by King [8] yields a simpler expression for the currents and impedances, but is still limited to antennas with half-lengths less than $1\frac{1}{2}\ \lambda$. The currents and impedances of long antennas were derived by Wu [9] based on a Wiener-Hopf technique for solving the integral equation. No simple form for either the current or impedance is presented in this paper.

For some applications, particularly problems in superposition, it is desirable to have a simple representation for the element current and the corresponding driving point impedance. For linear antennas of resonant

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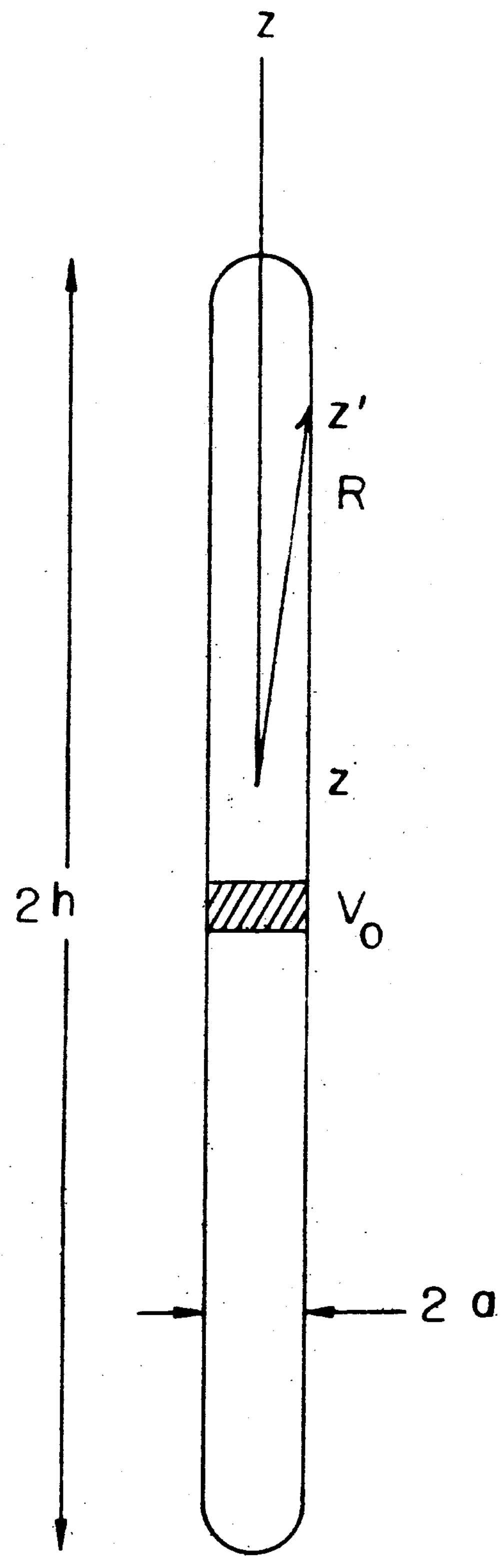
6. R.W.P. King, The Theory of Linear Antennas, Harvard University Press, Cambridge, Massachusetts, 1956.

7. R.W.P. King, op. cit page 140

8. R.W.P. King, "Linear Arrays, Currents, Impedances and Fields", Trans. I.R.E. AP-7; S440 (December 1959)

9. T.T. Wu, Theory of the Dipole Antenna and the Two-Wire Transmission Line, Journal of Math. Phy., Vol.2, No.4, pp. 550-574, July-August 1961



ISOLATED CYLINDRICAL ANTENNA DRIVEN BY A SLICE GENERATOR length a simple trogonometric representation of the element currents is possible. This simple trogonometric representation is used to solve the Hallén integral equation for the restricted case of resonant length antennas.

An examination of the experimentally detemined current distribution on resonant and antiresonant antennas [10] shows that the current on a resonant antenna is of nearly constant amplitude. Furthermore, the form of the current is nearly trigonometric except near the end. No such simple form is shown by the currents on the antiresonant antenna. As a check on the results of this approximate theory, the driving point impedance will be compared to the results of Wu's theory and the current distribution with the experimental results of Altschuler.

<u>Formulation of the Problem</u>

The Hallen integral equation for the current in linear cylindrical antenna driven by a delta function generator V_o at the centre is given by

$$\int_{-h}^{h} I_z(z') \; K(z,z') = -j \, \frac{4\pi}{\zeta_o} \left(C \cos \beta z + \frac{1}{2} V_o \sin \beta|z|\right) \qquad (1)$$

$$\text{where } K(z,z') = e^{-j\beta R}/R \qquad (2)$$

$$R = \sqrt{(z-z')^2 + a^2}\,, \qquad \zeta_o \doteq 120\pi \text{ ohms} \qquad (3)$$

10. E. Altschuler, Ph.D., Disseration, Harvard University, 1960

The constant C in (1) is determined in the usual case by evaluating the integral equation at $z = \pm h$. The actual current is determined in the King-Middleton method by successive iteration based on a zeroth order assumption for the current. Two iterations are usually required. In the King quasi-zeroth order solution [11] a separate iteration is performed for each of two separate current distributions which compose the total current. The experimentally determined current distributions of Altschuler show that the current on a resonant antenna is essentially trigonometric, except near the ends. One simple representation for the current $I_z(z)$ is

$$I_z(z) = -jA \cos \beta (z-\lambda/8) + B \cos \beta z + f(z) \qquad (4)$$

where A is real B is complex. (5)

The f(z) in (4) is constructed such that the current vanishes at $z = \pm h$. This f(z) function is required since the shifted cosine distribution in (4) does not vanish at $z = \pm h$. However, the sifted cosine distribution is valid over the range $0 \leq z \leq (h-\lambda/8)$ and the function f(z) is only non-zero in the range $(h-\lambda/8) \leq z \leq h$.

The first step in the solution is the rearrangement of the right hand side of (1) in a form which resembles the current, thus,

11. R. King, "Linear Arrays; Currents, Impedances and Fields", Trans. I.R.E. AP-7, S440, (December, 1959).

$$\int_{-h}^{h} I_z\,(z')\,K(z,z')\,dz' = -j\,\frac{4\pi}{\zeta_o}\left[C_1\cos\beta z + \frac{V_o}{\sqrt{2}}\cos\beta(z-\lambda/8)\right] \qquad (6)$$

The constant $V_o/\sqrt{2}$ was determined by making the right hand sides of (1) and (6) equivalent with the assumption in (6) of the different trogonometric forms. Now as a trial function the current in (4) is substituted in (6). Here it is important to note the functional behaviour of the resulting integrals on the left hand side of (6). The only integral which behaves like cos $\beta(z-\lambda/8)$ is the real part

$$\operatorname{Re}\int_{-h}^{h}\cos\beta(z'-\lambda/8)\,K(z,z)\,dz' \sim \cos\beta(z-\lambda/8) \qquad (7)$$

The only term on the right hand of (6) which behaves like cos $\beta(z-\lambda/8)$ has a coefficient $V_o/\sqrt{2}$. Thus equating coefficients of the shifted cosine function on both sides of (6) the A coefficient of (6) is given by

$$A = \frac{4\pi}{\zeta_o}\;\frac{V_o}{\sqrt{2}\;\Psi_{CR}(\lambda/8)} \qquad (8)$$

where

$$\Psi_{CR}(\lambda/8) = \operatorname{Re}\left[\operatorname{Cos}\,(\pi/4)\;C_a(h,\lambda/8) + \sin\left(\frac{\pi}{4}\right)\;S_a(h,\lambda/8)\right] \qquad (9)$$

and

$$C_a(h,z) = \int_{o}^{h}\cos\beta z'\left(\frac{e^{-j\beta R_1}}{R_1} + \frac{e^{-j\beta R_2}}{R_2}\right)dz' \qquad (10)$$

$$S_a(h,z) = \int_0^h \sin \beta z' \left(\frac{e^{-j\beta R_1}}{R_1} + \frac{e^{-j\beta R_2}}{R_2} \right) dz' \tag{11}$$

$$R_1 = \sqrt{(z-z')^2 + a^2} \qquad R_2 = \sqrt{(z+z')^2 + a^2}$$

a = antenna radius.

The $f(z)$ function is chosen such that the continuation of the shifted cosine current in the range $(h-\lambda/8) \leq z \leq h$ is a cosine current which vanishes at the end, thus

$$f(z) = -jA \left\{ \frac{\cos \beta(h-\lambda/4) \cos \beta z}{\cos \beta(h-\lambda/8)} - \cos \beta(z-\lambda/8) \right\} \tag{12}$$

or

$$f(z) = -jA \left(a_o \cos \beta z + b_o \sin \beta z \right) \tag{13}$$

where

$$a_o = \frac{\cos \beta(h-\lambda/8)}{\cos \beta(h-\lambda/4)} - \cos (\beta \lambda/8)$$

$$b_o = -\sin(\beta \lambda/8) = -\sin(\pi/4) \tag{14}$$

The B coefficient of (4) is determined by evaluating the integral equation at $z = h$. The integral equation at $z = h$ is

$$-jA \int_{-h}^{h} \cos \beta(z-\lambda/8)\, K(h,z')\, dz' + B \int_{-h}^{h} \cos \beta z'\, K(h,z')\, dz'$$

$$-j \left(\int_{h-\lambda/8}^{h} + \int_{h}^{-(h-\lambda/8)} \right) (a_o \cos \beta z + b_o \sin \beta z)\, K(h,z')\, dz' =$$

$$-j \frac{4\pi}{\zeta_o} \frac{V_o}{\sqrt{2}} \cos \beta(h-\lambda/8) \qquad (15)$$

or in a succint form

$$-jA\, \Psi_c(h) + B\, \Psi_c(h) - jA\, \Psi_c^1 = -jA\, \Psi_{CR}(\lambda/8) \cos \beta(h-\lambda/8) \qquad (16)$$

where

$$\Psi_c(h) = \cos\left(\frac{\pi}{4}\right) C_a(h,h) + \sin\left(\frac{\pi}{4}\right) S_a(h,h)$$

$$\Psi_c(h) = C_a(h,h)$$

$$\Psi_c^1 = a_o \,[C_a(h,h) - C_a(h-\lambda/8)] \qquad (17)$$

$$+ b_o\, [S_a(h,h) - S_a(h-\lambda/8,h)]$$

The B coefficient is given by (16) or

$$b = \frac{-jA[\Psi_c(h) + \Psi_c^1 - \Psi_{CR} \cos \beta(h-\lambda/8)]}{\Psi_c(h)}$$

The complete current distribution (4) is now determined and the driving point impedance and admittance are found from (4) with z set equal to zero.

APPENDIX II

THE MUTUAL IMPEDANCE OF A V-ANTENNA WITH A GIVEN CURRENT

An approximate value for the mutual impedance Z_{12} of two linear elements with an assumed current is given by the E.M.F. method, thus

$$Z_{12} = -\int_{\Sigma} E_{z2}\, I_{z1}\, ds \tag{1}$$

where E_{z_2} is the electric field at the surface of antenna 1 and $I_z^{\,1}$ is the current along antenna 1. The integration is performed along the length of antenna 1. Note that a cylindrical antenna is assumed which has no current variation in the transverse direction. The geometry is shown in Figure 1. The electric field may be derived from the vector potential, or,

$$E = \frac{-j\beta_o^{\,2}}{\omega} \quad \vec{\nabla} \times \vec{\nabla} \times \vec{A} \tag{2}$$

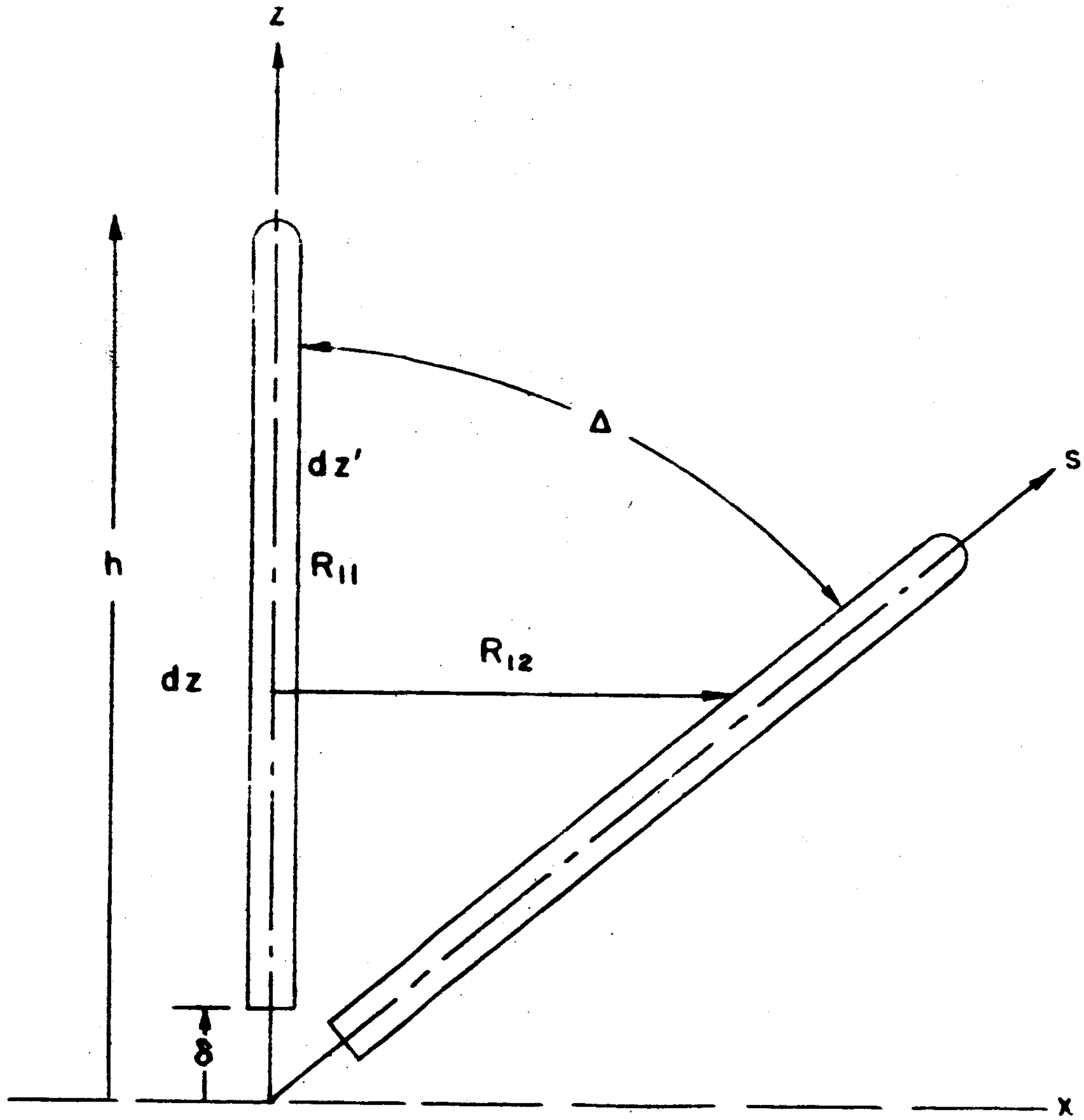
where

$$A = \text{vector potential}$$

$$\beta_o = \frac{2\pi}{\lambda_o} \tag{3}$$

$$\lambda_o = \text{free space wavelength.}$$

Since antenna 2 is in the x-y plane the cartesian components of the vector potential are



APPENDIX II FIGURE 1   GEOMETRY FOR V-ANTENNA

$$A_z = \frac{\cos\Delta}{4\pi}\,\mu_o \int_{\delta}^{h} I(z')\,\frac{e^{-j\beta_o R_{12}}}{R_{12}}\,dz' \tag{4}$$

$$A_x = \frac{\sin\Delta}{4\pi}\,\mu_o \int_{\delta}^{h} I(z')\,\frac{e^{-j\beta_o R_{12}}}{R_{12}} \tag{5}$$

where

$$R_{12} = \sqrt{\zeta^2 + a^2} \tag{6}$$

$$\zeta^2 = z^2 + x^2 - 2z\,z'\cos\Delta - 2\,x\,z'\sin\Delta + z'^2 \tag{7}$$

The z-component of the electric field along element 1 is given by (2) with (4) and (5), or

$$E_{z_2} = \frac{-j\beta_o^2}{\omega}\left(\frac{\partial^2 A_x}{\partial x\partial z} - \frac{\partial^2 A_z}{\partial x^2}\right)_{x=0} \tag{8}$$

The final integral for the mutual impedance is

$$Z_{12} = \frac{-j\beta_o^{2}}{\omega}\int_{\delta}^{h} I_z(z)\left(\frac{\partial^2 A_x}{\partial x\partial z} - \frac{\partial^2 A_z}{\partial x^2}\right)_{x=0} dz$$

$$= \frac{-j\beta_o^{\,2}}{\omega}\ \mu \int_{\delta}^{h} dz\ \ \sin\left\{\Delta \frac{\partial^2}{\partial x \partial z} \int_{\delta}^{h} I(z')\ \frac{e^{-j\beta_o\, R_{12}}}{R_{12}}\ dz'\right.$$

$$\left. - \cos\ \frac{\Delta\partial^2}{\partial x^2} \int_{\delta}^{h} I(z')\ \frac{e^{-j\beta_o\, R_{12}}}{R_{12}}\ dz' \right\}_{x\,=\,0} I_z(z)\ dz$$

APPENDIX III

THE RADIATION FIELD OF THE TRAVELLING WAVE V ANTENNA

The coordinate system for the V-antenna is shown in Figs. 1a and 1b. Instead of directly computing the far field and then the radiation functions it is simpler to compute the total vector potential, since it is in the direction of the antenna current.

The general form for the vector potential in the far field A^r is given by King [12].

$$\vec{A}^r = \hat{s}\, A^r_s = \hat{s}_1\, A^r_{s_1} + \hat{s}_2\, A^r_{s_2} \tag{1}$$

where s = direction of the linear element.

$$A^r_s = C\, \beta_o \int_{-h}^{h} f(s')\, e^{j\beta_o(\hat{R}_o \cdot \vec{s}')}\, ds' \tag{2}$$

$I(s) = I_o\, f(s)$

$I(s)$ = current along the antenna

The unit vectors are denoted by carrots and unnormalized vectors by arrows.

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12. R. King, "Fundamental Electromagnetic Theory", Dover Publications, New York, 1963

In (2) the vectors in cartesian coordinates are:

$$\hat{R}_o = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\hat{\theta}\,z \tag{3}$$

$$\hat{s}_1 = \sin\frac{\Delta}{2}x + \cos\frac{\Delta}{2}\hat{z}$$

$$\hat{s}_2 = -\sin\frac{\Delta}{2}\hat{x} + \cos\frac{\Delta}{2}\hat{z}$$

$$\vec{s}_1{}' = z\tan\frac{\Delta}{2}\hat{x} + z\,\hat{z} \tag{4}$$

$$\vec{s}_2{}' = -z\tan\frac{\Delta}{2}\hat{x} + z\,\hat{z}$$

The specialized forms for the vector potentials for a given current distribution are

$$A^r_{s_1} = C\,\beta_o \int f(s_1{}')\; e^{j\beta z(\sin\theta\cos\phi\tan\frac{\Delta}{2} + \cos\theta)}\; dz\,\sec\left(\frac{\Delta}{2}\right) \tag{5}$$

$$A^r_{s_2} = C\,\beta_o \int f(s_2{}')\; e^{j\beta z(-\sin\theta\cos\phi\tan\frac{}{2} + \cos\theta)}\; dz\,\sec\frac{\Delta}{2} \tag{6}$$

As a first approximation to the antenna current it is assumed that a travelling wave distribution of current exists for $0 \leq z \leq h_1$ and a standing wave distribution of current for $h_1 \quad z \quad h$.

Thus,

$$\left.\begin{aligned} f(s_1') &= e^{j\beta z \sec \frac{\Delta}{2}} \\ f(s_2') &= -e^{-j\beta z \sec \frac{\Delta}{2}} \end{aligned}\right\} \quad z \leq h \tag{7}$$

$$\left.\begin{aligned} f(s_1') &= \sin \beta(h - z \sec \tfrac{\Delta}{2}) \\ f(s_2') &= -\sin \beta(h - z \sec \tfrac{\Delta}{2}) \end{aligned}\right\} \quad h_1 \leq z \leq h \tag{8}$$

The final forms for the vector potentials are

$$A^r_{s_1} = C\beta \int_{0}^{h_1 \cos \frac{\Delta}{2}} \exp \left[j\beta z\left(-\sec \frac{\Delta}{2} + \sin \theta \cos \phi \tan \frac{\Delta}{2} + \cos \theta\right) \right] dz \sec\left(\frac{\Delta}{2}\right)$$

$$+ C\beta \int_{h_1 \cos \frac{\Delta}{2}}^{h_1 \cos \frac{\Delta}{2}} \sin \left[\beta(h - z \sec \tfrac{\Delta}{2})\right] \exp \left[\sin \theta \cos \phi \tan \frac{\Delta}{2} + \cos \theta\right] dz \sec\left(\frac{\Delta}{2}\right) \tag{9}$$

$$A^r_{s_2} = C\beta \int_0^{h_1 \cos\frac{\Delta}{2}} \exp\left[j\beta z\left(-\sec\frac{\Delta}{2} - \sin\theta\cos\phi\tan\frac{\Delta}{2} + \cos\theta\right)\right] dz \sec\left(\frac{\Delta}{2}\right)$$

$$-C\beta \int_{h_1 \cos\frac{\Delta}{2}}^{h_1 \cos\frac{\Delta}{2}} \sin\left[\beta\left(h - z\sec\frac{\Delta}{2}\right)\right] \exp\left[j\beta z\left(-\sin\theta\cos\phi\tan\frac{\Delta}{2} + \cos\theta\right)\right] dz \sec\left(\frac{\Delta}{2}\right) \qquad (10)$$

The radiation functions are computed from the θ and ϕ components of the vector potential, A_θ and A_ϕ where

$$A_\theta = (\hat{\theta} \cdot \hat{s}_1)\, A^r_{s_1} + (\hat{\theta} \cdot \hat{s}_2)\, A^r_{s_2} \qquad (11)$$

$$A_\phi = (\phi \cdot \hat{s}_1)\, A^r_{s_1} + (\hat{\phi} \cdot \hat{s}_2)\, A^r s_2 \qquad (12)$$

$$\hat{\theta} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z} \qquad (13)$$

$$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y} \qquad (14)$$

$$(\hat{\theta} \cdot \hat{s}_1) = \sin\frac{\Delta}{2}\cos\theta\cos\phi - \cos\frac{\Delta}{2}\sin\theta \qquad (15)$$

$$(\hat{\theta} \cdot \hat{s}_2) = -\sin\frac{\Delta}{2}\cos\theta\cos\phi - \cos\frac{\Delta}{2}\sin\theta \qquad (16)$$

$$(\hat{\phi} \cdot \hat{s}_1) = -\sin\phi\sin\frac{\Delta}{2} \qquad (17)$$

$$(\hat{\phi} \cdot \hat{s}_2) = \sin \phi \sin \frac{\Delta}{2} \tag{18}$$

The directivity D is given by

$$D = 4\pi K(0,0)/T \tag{19}$$

where

$$T = \int_0^{\pi} \int_0^{\pi} K^2(\theta,0/) \sin \theta \, d\theta \, d\phi \tag{20}$$

and

$$K^2(\theta,\phi) = A_\theta A_\theta^* + A_\phi A_\phi^* \qquad (* = \text{complex conjugate})$$

The ratio of the total power in a given cone angle to the total power over all space is given by:

$$G = \frac{\int_0^{2\pi} \int_0^{\theta_1} K^2(\theta,\phi) \sin \theta \, d\theta \, d\phi}{T}$$

APPENDIX IV

APPROXIMATE THEORY FOR INFINITE LENGTH V-ANTENNA

A approximate relation for the driving point impedance of the antenna will be obtained. This approximate relation includes the major electric coupling time between the two parts of the V-antenna and excludes the higher order magnetic coupling term. The approximation assumes that the resistively loaded element is perfectly matched. Thus, on the section which includes the driving point, only travelling waves of current are assumed to exist. Then the antenna could be infinite in length as far as the driving point is concerned. This follows from the fact that with no reflected waves, the driving point can have no knowledge of effects on other parts of the element.

The integral equation for the V-antenna shown in Figure 3 has been derived in King [13]

$$\int_{\delta}^{h} I_z(z') \frac{e^{-j\beta R_1}}{R_1} - \cos\Delta \int_{\delta}^{h} I_z(z') \frac{e^{-j\beta R_{12}}}{R_{12}} dz' =$$

$$\frac{2\pi}{\zeta_o} V_o e^{-j\beta z} J_v(z) \qquad (1)$$

where

$$R_{12} = \sqrt{(z' - z_v)^2 + a_v^2}$$

$$R_1 = \sqrt{(z' - z)^2 + a^2}$$

13. R.W.P. King, "The Theory of Linear Antennas", Harvard University Press, Cambridge, Mass., p. 381 et seq.

$$z_v = z \cos \Delta$$

$$a_v^2 = z^2 \sin^2\Delta + a^2$$

$J_v(z)$ = higher order coupling term

The driving point impedance z_o is given by

$$Z_o = \frac{V_o}{I_z(\delta)} = Y_o \qquad (2)$$

where

Y_o = driving point impedance

$I_z(\delta)$ = antenna current evaluated at $z = \delta$

In this approximation for the solution of (1) it will be assumed that

(a) the travelling wave antenna is perfectly matched

(b) the form of the current is $I_z(z) = I_o\, e^{-j\beta z}$

(c) the higher order coupling term is negligible ($J_v \doteq o$)

With the integration extended to $h = \infty$ the simplified integral equation is

$$\int_{\delta}^{\infty} I_z(z') \frac{e^{-j\beta R_1}}{R_1} dz' - \cos\Delta \int_{\delta}^{\infty} I_z(z') \frac{e^{-j\beta R_{12}}}{R_{12}} dz' =$$

$$\frac{j2\pi}{\zeta_o} V_o \; e^{-j\beta z} \qquad (3)$$

Since Y_o is computed from (3) with $z = \delta$, the integral equation reduces to a linear algebraic equation. The value of $Z_o(\delta)$ is

$$Z_o \cong \frac{\zeta_o}{\partial \pi} \left\{ [-\text{Ci } \beta a_{\delta_1} + j(\text{Si } \beta a_{\delta_1} - \frac{\pi}{2})] \right. \tag{4}$$

$$\left. -\cos \Delta \ [-\text{Ci } \beta a_{\delta_2} + j(\text{Si } \beta a_{\delta_2} - \frac{\pi}{2})] \right\}$$

where

$$\text{Ci } x = \int_x^{\infty} \frac{\cos t}{t} \, dt$$

$$\text{Si } x = \int_o^{x} \frac{\sin t}{t} \, dt$$

For $\delta = 0$ and $(\beta a)^2 << 1$, (4) reduces to

$$Z_o(0) \doteq \frac{\zeta_o}{\pi} \sin^2 \frac{\Delta}{2} \ [\ell n \frac{1}{\beta a} - \ell n \ \gamma' - j \frac{\pi}{2}] \tag{5}$$

or

$$Y_o(0) \doteq \frac{\pi}{\zeta_o} \cos^2 \frac{\Delta}{2} \ \frac{(\ell n \frac{1}{\beta a} - \ell n \ \gamma') + j \frac{\pi}{2}}{(\ell n \frac{1}{\beta a} - \ell n \gamma')^2 + \frac{\pi^2}{4}} \tag{6}$$

TABLES OF DRIVING POINT IMPEDANCES AND CURRENT COEFFICIENTS

10 = Ω

h/λ	A	B_r	B_i	Z_r	Z_i
0.75	3.5749	7.2114	0.38783	127.45	37.821
1.25	4.0401	6.1527	1.4650	154.62	34.975
1.75	4.4219	5.5123	2.1599	175.99	30.898
2.25	4.7602	5.0537	2.6909	194.40	25.970

12.5 = Ω

h/λ	A	B_r	B_i	Z_r	Z_i
0.75	2.5918	7.3190	-0.73341	121.67	42.659
1.25	2.8272	6.3443	0.11631	144.86	42.991
1.75	3.0078	5.7689	0.61066	162.14	42.615
2.25	3.1590	5.3668	0.95877	176.38	41.900
2.75	3.2912	5.0601	1.2287	188.78	40.971
3.25	3.4103	4.8131	1.4506	199.80	39.886
3.75	3.5195	4.6068	1.6400	209.95	38.673
4.25	3.6210	4.4298	1.8063	219.38	37.351
4.75	3.7165	4.2750	1.9550	228.26	35.931
5.25	3.8069	4.1373	2.0902	236.70	34.420

TABLES OF DRIVING POINT IMPEDANCES AND CURRENT COEFFICIENTS

$\Omega = 15$

h/λ	A	B_r	B_i	Z_r	Z_i
0.75	2.0307	7.4310	-1.3443	118.05	44.166
1.25	2.1800	6.5047	-0.57249	139.05	45.190
1.75	2.2793	5.9679	-0.15563	154.05	45.621
2.25	2.3675	5.5943	0.13200	166.13	45.794
2.75	2.4388	5.3134	0.34310	176.29	45.831
3.25	2.5053	5.0878	0.51420	185.24	45.776
3.75	2.5587	4.9035	0.64993	193.14	45.664
4.25	2.6137	4.7441	0.77134	200.46	45.500
4.75	2.6692	4.6036	0.88172	207.33	45.293
5.25	2.7172	4.4811	0.97634	213.66	45.058
5.75	2.7595	4.3727	1.0590	219.55	44.800
6.25	2.8027	4.2735	1.1371	225.20	44.515
6.75	2.8331	4.1878	1.1991	230.30	44.227
7.25	2.8631	4.1088	1.2572	235.17	43.920
7.75	2.9138	4.0260	1.3312	240.50	43.555
8.25	2.9302	3.9637	1.3714	244.65	43.242
9.25	3.0047	3.8315	1.4836	253.89	42.474
9.75	3.0249	3.7784	1.5217	257.79	42.111
10.25	3.0723	3.7153	1.5830	262.55	41.655
10.75	3.0972	3.6652	1.6226	266.45	41.251

TABLES OF DRIVING POINT IMPEDANCES AND CURRENT COEFFICIENTS

$\Omega = 20$

h/λ	A	B_r	B_i	Z_r	Z_i
0.75	1.5538	7.7538	-1.8449	114.93	44.787
1.25	1.2673	6.7994	-1.4939	130.90	46.011
1.75	1.7776	6.1353	-0.66232	148.46	46.442
2.25	1.8774	5.7644	-0.36056	159.78	46.791
2.75	1.8893	5.5123	-0.20474	168.27	47.030
3.25	1.9607	5.2883	-0.027740	176.48	47.192
3.75	1.8451	5.1786	-0.051147	180.71	47.314
4.25	1.9608	4.9959	0.12856	188.23	47.395
4.75	2.0291	4.8518	0.25161	194.54	47.443
5.25	1.9030	4.8079	0.18367	196.51	47.494
5.75	2.0308	4.6627	0.34656	203.37	47.517
6.25	1.4552	4.8456	-0.15010	194.84	47.410
6.75	2.1343	4.4658	0.51500	213.35	47.497
7.25	1.8142	4.5487	0.24915	209.05	47.506
7.75	1.8402	4.4783	0.30116	212.69	47.499
8.25	1.5599	4.5559	0.068672	208.74	47.390
8.75	2.1757	4.2177	0.65866	227.21	47.395
9.25	2.2459	4.1377	0.74409	232.03	47.331
9.75	1.6887	4.3580	0.25267	219.23	47.359
10.25	2.0808	4.1314	0.63017	232.42	47.320
10.75	2.3003	3.9885	0.84798	241.52	47.144
11.25	1.4398	4.3650	0.078437	218.95	47.133
11.75	2.1777	3.9768	0.76650	242.29	47.119
12.25	2.1508	3.9573	0.75576	243.59	47.097
12.75	2.1448	3.9299	0.76383	245.45	47.019
13.25	2.3090	3.8211	0.92537	253.03	46.838
13.75	1.3745	4.2484	0.089711	225.65	46.856
14.25	1.9273	3.9530	0.60251	243.95	46.919
14.75	2.4114	3.6906	1.0511	262.71	46.558
15.25	2.2001	3.7702	0.87046	256.76	46.668
15.75	1.7460	3.9705	0.47050	242.86	46.737
16.25	2.4040	3.6240	1.0733	267.93	46.327
16.75	2.4196	3.5948	1.0963	270.28	46.212
17.25	2.0307	3.7662	0.75439	257.10	46.527
17.75	1.1969	4.1603	0.013197	231.10	46.281
18.25	1.6893	3.8976	0.46462	247.87	46.419
18.75	2.2741	3.5882	0.99731	270.84	46.096
19.25	2.5049	3.4553	1.2117	282.02	45.665
19.75	2.4983	3.4415	1.2132	283.24	45.545
20.25	2.2241	3.5614	0.97398	273.07	45.904
20.75	1.6651	3.8251	0.48045	253.03	46.103

TABLES OF DRIVING POINT IMPEDANCES AND CURRENT COEFFICIENTS

$\Omega = 30$

h/λ	A	B_r	B_i	Z_r	Z_i
0.75	1.5383	7.5532	-1.8649	114.85	44.894
1.25	0.021413	7.2185	-2.7393	120.93	46.143
1.75	1.7662	6.1379	-0.67492	148.35	46.496
2.25	1.18677	5.7594	-0.37838	159.73	47.121
2.75	1.8777	5.5075	-0.22515	168.20	47.425
3.25	1.9501	5.2821	-0.48288	176.44	47.674
3.75	1.8233	5.1781	-0.081991	180.46	47.789
4.25	1.9459	4.9904	0.10218	188.13	48.017
4.75	2.0113	4.8485	0.22351	194.37	48.054
5.25	1.8499	4.8179	0.11943	195.65	48.271
5.75	2.0033	4.6601	0.30534	203.04	48.415
6.25	2.1456	5.4857	-0.15010	169.35	46.808
6.75	2.1208	4.4561	0.48626	213.37	48.523
7.25	1.7648	4.5555	0.18558	208.19	48.552
7.75	1.7126	4.5204	0.16379	209.95	48.637
8.25	1.4398	4.6051	-0.050322	206.06	47.807
8.75	2.1502	4.2120	0.61712	226.98	48.676
9.25	2.2318	4.1248	0.71157	232.19	48.779
9.75	1.5962	4.3823	0.14461	217.24	48.782
10.25	2.0212	4.1388	0.55317	231.26	48.948
10.75	2.2832	3.9841	0.81981	241.39	48.148
11.25	1.2349	4.4433	-0.13524	214.03	48.577
11.75	2.1529	3.9674	0.72178	242.19	48.870
12.25	2.0940	3.9608	0.67879	242.54	49.103
12.75	2.0823	3.9387	0.68423	244.12	48.852
13.25	2.2903	3.8132	0.89138	253.02	48.311
13.75	1.0991	4.3603	-0.19176	218.55	48.565
14.25	1.8627	3.9611	0.51725	242.57	48.983
14.75	2.3953	3.6703	1.0094	263.30	49.096
15.25	2.1349	3.7768	0.78489	255.37	49.005
15.75	1.3098	4.1678	0.053232	229.85	48.143
16.25	2.3776	3.6076	1.0209	268.21	49.092
16.75	2.4012	3.5759	1.0526	270.83	48.874
17.25	1.9723	3.7655	0.66905	256.06	49.342
17.75	6.2127	4.4219	-0.54506	215.47	47.966
18.25	1.5476	3.9469	0.31029	243.74	48.417
18.75	2.2410	3.5729	0.93522	270.94	49.243
19.25	2.4872	3.4310	1.1648	282.98	48.986
19.75	2.4734	3.4503	1.1874	282.35	45.957
20.25	2.1321	3.5756	0.85804	270.74	49.189
20.75	6.7869	4.2930	-0.44666	222.57	48.038

TABLES OF DRIVING POINT IMPEDANCES AND CURRENT COEFFICIENTS

$\Omega = 40$

h/λ	A	B_r	B_i	Z_r	Z_i
0.75	1.5383	7.5526	-1.8653	114.85	44.904
1.25	4.2107	7.2243	-2.7616	120.77	46.171
1.75	1.7662	6.1308	-0.68159	148.40	46.727
2.25	1.8677	5.7591	-0.37866	159.73	47.132
2.75	1.8777	5.5068	-0.22588	168.21	47.456
3.25	1.9501	5.2816	-0.048798	176.44	47.697
3.75	1.8233	5.1760	-0.084222	180.49	47.894
4.25	1.9459	4.9895	0.10119	188.14	48.067
4.75	2.0113	4.8456	0.22066	194.42	48.208
5.25	1.8499	4.8170	0.11848	195.67	48.323
5.75	2.0033	4.6597	0.30495	203.05	48.438
6.25	1.4810	5.4929	-1.5251	169.02	46.931
6.75	2.1208	4.4547	0.48485	213.41	48.613
7.25	1.7648	4.5540	0.18396	208.22	48.648
7.75	1.7126	4.5196	0.16290	209.97	48.690
8.25	1.4398	4.5934	-0.064773	206.24	48.621
8.75	2.1502	4.2093	0.61432	227.05	48.874
9.25	2.2318	4.1230	0.70973	232.24	48.916
9.75	1.5962	4.3815	0.14358	217.25	48.846
10.25	2.0212	4.1377	0.55210	231.29	49.025
10.75	2.2832	3.9731	0.80892	241.76	49.017
11.25	1.2349	4.4419	-0.13714	214.06	48.689
11.75	2.1529	3.9639	0.71822	242.30	49.153
12.25	2.0940	3.9599	0.67790	242.56	49.173
12.75	2.0823	3.9341	0.67946	244.26	49.234
13.25	2.2903	3.8028	0.88112	253.41	49.201
13.75	1.0991	4.3590	-0.19362	218.57	48.677
14.25	1.8627	3.9575	0.51322	242.67	49.296
14.75	2.3953	3.6694	1.0085	263.34	49.182
15.25	2.1349	3.7731	0.78109	255.51	49.335
15.75	1.3098	4.1584	0.040611	230.04	48.989
16.25	2.3776	3.6058	1.0192	268.29	49.260
16.75	2.4012	3.5717	1.0487	271.02	49.263
17.25	1.9723	3.7645	0.66793	256.10	49.439
17.75	6.2122	4.4201	-0.54819	215.49	48.140
18.25	1.5476	3.9373	0.29829	244.01	49.330
18.75	2.2410	3.5709	0.93320	271.03	49.441
19.25	2.4872	3.4286	1.1626	283.11	49.222
19.75	2.4734	3.4170	1.1562	284.10	49.286
20.25	2.1321	3.5719	0.85431	270.90	49.550
20.75	6.7851	4.2902	-0.45157	222.60	48.324