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1. Introduction

Many physical problems are described well by linear differential equations of order N*.

$$\sum_{n=0}^{N} a_n(x)u^{(n)}(x) = 0 \quad , \qquad u^{(n)} = \frac{d^n u}{dx^n} \tag{1}$$

The initial value problem, of course, has N independent solutions $u_i(i=1, 2...N)$ and the desired physical solution is determined by taking the appropriate linear combination of these

$$u(x) = \sum_{i=1}^{N} A_i u_i(x)$$

which, in a well determined problem, satisfies N specified boundary conditions. If the u_i can be obtained analytically, the solution of the problem is formally trivial, but more often they can not and numerical techniques must be resorted to.

If none of the u_i grow too rapidly in the interval of integration, the initial value problem for (1) can be solved by standard numerical methods, but in many physical problems one or more of the u_i may increase so rapidly with x that they can not be eliminated or controlled by means of boundary conditions which are subject to roundoff error.

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*The mathematical development for the analogous problem of N first order equations is given in Section 5.

For instance the linearized equation for vertical wave propagation in a vertically inhomogeneous gas is a second order equation with two solutions corresponding to the two directions of propagation. If the effects of viscosity are added, however, the equation becomes fourth order. Naturally the two original solutions are altered somewhat, but also there must necessarily be two completely new solutions. Physically these two new solutions are the "viscous waves" for the two directions of propagation. A "viscous wave" is of no importance physically because it is so drastically attenuated. However a "viscous wave" is very important computationally because (approached from the other direction) it grows catastrophically. In other words the numerical procedure interprets part of every roundoff error as the remnant of a dying "viscous wave" and proceeds to calculate what it must have come from. If the effects of thermal conductivity are also added to the equation, its order is again increased by two and two more solutions (one growing in each direction) are possible.

Thus in many physical problems it has been impossible by straightforward numerical techniques to obtain from the appropriate linearized equation those solutions that are the only ones observed physically. It is the purpose of this paper to describe a method of numerically obtaining these so-called subdominant solutions.

2. Definitions and Restrictions

The method to be developed applies to complex equations and solutions as well as real ones, but it does not work for equations where dominant solutions have zeros. Thus in general the solutions must be expressible as $u = \exp(f(x))$ where $f(x)$ is a finite complex function.

A solution u_1 will be said to dominate another solution u_2 in the interval from x_o to x_1 if:

$$\left|\frac{u_2(x_1)}{u_2(x_o)}\right| < \varepsilon \left|\frac{u_1(x_1)}{u_1(x_o)}\right| \tag{2}$$

where ε is the accuracy required of the numerical result. Note that the direction of integration as well as the length of the interval is inherent in this definition.

It is easy to determine whether there is a solution which, for a given direction of integration, dominates all other solutions in a range $x_o<x<x_1$. Starting with arbitrary boundary conditions at x_o, integrate the equation beyond x_1, obtaining a solution $u_{1a}(x)$. Repeat this for a second set of boundary conditions obtaining u_{1b}. If $u_{1a}(x_1) = cu_{1b}(x_1)$ where c is a constant for all $x>x_1$, the solution just found dominates all others.

It may be that there is no single solution which dominates all others, but rather a subset of M solutions which do not dominate each other but each of which dominates all solutions not of the set. In this case one will find that any solution started with arbitrary boundary conditions at x_o can be expressed for all $x>x_1$ as a linear combination of M other arbitrary solutions. Of course, if M = N it simply means that no solution dominates any other and all solutions can be calculated numerically by standard methods.

Now that we have a good working definition of dominance we can label the solutions in order of their dominance u_1, u_{2a}, u_{2b}, u_3, etc. The additional subscripts, a, b, etc. will be used to distinguish

linearly independent solutions of a set with nearly equal dominance. It should be noted here that unless two independent solutions have exactly equal growth rates (for instance e^{x+ix} and e^{x-ix}) it is a matter of choice and convenience whether they be treated as having nearly equal dominance or not. If they are treated as nearly equal, the interval over which they are integrated must be short enough that one does not dominate the other. If they are treated separately the "intervals of separation" (to be defined) must be long enough to ensure dominance of one over the other.

The "intervals of separation" are extra intervals both before and after the interval in which the solutions are required. The integrations must be carried out over these intervals as well in order to separate out the dominant solutions in a pure (within roundoff error) form. Specifically we must require that u_i dominate all u_j $(j>i)$ in the intervals $(x_{i-1}<x<x_i)$ and $(x_{m-i}<x<x_m)$. Then if the original equation is integrable throughout $(x_o<x<x_m)$, u_1 can be obtained throughout $(x_1<x<x_m)$, u_N can be obtained throughout $(x_{N-1}<x<x_{m-N})$ and all other solutions u_K can be obtained throughout $(x_K<x<x_{m-K})$.

3. Method for Nondegenerate Case

Assume that there are no solutions with nearly equal dominance (the degenerate case will be treated later). Then if the integration of (1) is begun at x_o for two sets of arbitrary boundary conditions, the ratio of the two solutions will eventually converge to a constant at some x, which will be labeled x_1. Either solution after this point will be the pure solution u_1.

Define the function

$$v(x) = \frac{u(x)}{u_1(x)} \tag{3}$$

in terms of which one obtains for $u^{(n)}$.

$$u^{(n)} = u_1{}^{(n)} v + \sum_{r=0}^{n-1} \frac{n!}{(n-r)!r!} u_1{}^{(r)} v^{(n-r)} \tag{4}$$

Substitute this in (1), and drop the first term since u_1 is a solution of (1):

$$\sum_{n=1}^{N} \sum_{r=0}^{n-1} \frac{n!\, a_n}{(n-r)!r!} u_1{}^{(r)} v^{(n-r)} = 0 \tag{5}$$

Set r = n-s-1, exchange the order of summation, and divide by u_1

$$\sum_{s=0}^{N-1} b_{s+1} v^{(s+1)} = 0 \qquad b_s = \sum_{n=s}^{N} \frac{n!\, a_n}{s!(n-s)!} \frac{u_1{}^{(n-s)}}{u_1} \tag{6}$$

This is clearly a well defined linear equation of order N-1 in v'. If this is solved with arbitrary boundary conditions at x_1 and the result is integrated to obtain v, by definition it must be of the form

$$v(x) = \sum_{i=2}^{N} A_i \frac{u_i(x)}{u_1(x)} + A_1 \tag{7}$$

Multiplication by u_1 would recover u(x), which for any finite A_1 would of course be dominated by u_1. Therefore in integrating v' the constant

of integration must be chosen so as to make A_1 vanish. This is done by setting $v(x_m) = 0$ and then integrating v' from this point in the direction of smaller x. Since u_1 dominates all other u_i in the interval $x_{m-1}<x<x_m$, any linear combination of the remaining u_i will dominate u_1 in that interval in the reverse direction. Thus for all $x<x_{m-1}$ the contribution of u_1 to $u(x) = u_1 v = \sum_{i=2}^{N} A_i u_i + A_1 u_1$ will be below the level of roundoff error. It is not possible to say at this point that the u(x) thus obtained is a pure u_2, but if the calculation of v' is repeated with different boundary conditions at x_1, and the value of x at which the ratio v'_a/v'_b approaches a constant is called x_2, then u(x) will be a pure u_2 solution in the interval $x_2<x<x_{m-1}$.

Strictly speaking x_{m-1} is to be determined as the largest value of x for which one could restart the integration of (1) with arbitrary boundary conditions and still recover a pure u_1 solution before reaching x_m. In practice it is much more easily determined to a good approximation by the formula:

$$|v(x_{m-1})| > \frac{|v(x_t)|}{\epsilon} \tag{8}$$

where

$$|u_1(x_t)| < \frac{|u_1(x_m)|}{10}\left[1 - \left|1 + \frac{u_1(x_s)v'(x_s)}{u_1'(x_s)v(x_s)}\right|\right] \qquad x_s \cong x_t$$

This is just a restatement of the condition for dominance, complicated by the fact that $v(x_m) = 0$, so that $v(x_t)$ must be used instead. This x_t must be just far enough from x_m so that u_1 has decreased by about 10 unless v is not growing locally by a comparable rate.

This completes the detailed description of the extraction of the first subdominant solution and the determination of its interval of validity. Of course this interval may be extended in either direction, if necessary, by starting the entire calculation at a smaller x_o and/or extending it to a larger x_m.

If the next solution, $u_3(x)$, is desired it is only necessary to repeat the entire procedure on the equation for v' that was just used on the equation for u. Define the function

$$w(x) = \frac{v'(x)}{v_2'(x)} \tag{9}$$

Substitute this into the equation for v' to obtain the following equation for w':

$$\sum_{s=0}^{N-2} c_{s+1} w^{(S+1)} = 0 \qquad c_s = \sum_{n=s}^{N-1} \frac{n! b_{n+1}}{s!(n-s)!} \frac{v_2^{(n-s-1)}}{v_2'} \tag{10}$$

This equation of order N-2 in w' is integrated from x_2 using two sets of arbitrary boundary conditions. The point at which the ratio of the two solutions becomes constant is x_3 and the calculation of w' is carried from there on to x_m. Although v_2 is known only up to x_{m-1}, v_2' is known all the way to x_m, so the $c_s(x)$ are defined up to x_m. Setting $w(x_m) = 0$, the integration of w' is done from x_m to x_3, and as before the value of x_{m-2} is determined from the approximation:

$$|w(x_{m-2})| > \frac{|w(x_t)|}{\epsilon} \tag{11}$$

where

$$|v_2'(x_t)| < \frac{|v_2'(x_m)|}{10}\left[1 - \left|1 + \frac{v_2'(x_s)w'(x_s)}{v_2''(x_s)w(x_s)}\right|\right], x_s \cong x_t$$

Then in the interval $x_3<x<x_{m-2}$ the w(x) just calculated is a pure w_3 solution and $v_3' = w_3 v_2'$ by (9). Set $v_3(x_m) = 0$ and integrate v_3' from there x_3 to obtain v_3, and then multiply this by u_1 to obtain the desired $u_3(x)$ in the interval $x_3<x<x_{m-2}$. Since v_3 is growing faster than w_3 (in the negative x direction), any errors in v_3 due to an inaccurate w_3 in the interval $x_{m-2}<x<x_m$ should be washed out by the time the integration of v_3' reaches x_{m-2}.

Note that x_{m-2} may be either greater or less than x_{m-1} depending on the relative dominance of the solutions u_1, u_2 and u_3.

The extension of this process to successive solutions is obvious and will not be pursued further, except to note that one need not pursue it farther than $u_{N/2}$, since reversing the direction of integration reverses the order of dominance of allthe solutions.

4. Method for Degenerate Case

Suppose that u_1 is a double solution; that is, the ratio of two arbitrary solutions u_{1a} and u_{1b} does not approach a constant after a reasonable distance, but any other arbitrary solution does approach a linear combination of the first two, $u = Au_{1a} + Bu_{1b}$. Then the value of x beyond which any solution can be expressed as a linear combination of u_{1a} and u_{1b} will be designated x_1. Define

$$v(x) = \frac{u(x)}{u_{1a}(x)} \tag{12}$$

as before, and obtain the resulting equation for v'. If this equation were solved for v', the result would be formally:

$$v' = \sum_{i=2} A_i \left(\frac{u_i}{u_{1a}}\right)' + A_{1b} \left(\frac{u_{1b}}{u_{1a}}\right)' \tag{13}$$

The term with A_{1b} must be eliminated (as the term with A_{1a} has been) so (13) is divided by $(u_{1b}/u_{1a})'$ and differentiated, which gives a formal expression for w' where

$$w(x) = \frac{v'(x)}{(u_{1b}/u_{1a})'} \tag{14}$$

The equation for w' is easily obtained from the equation for v', just exactly as the equation for v' was obtained from the equation for u (see Eq. 1, 3 and 6), since $\left(\frac{u_{1b}}{u_{1a}}\right)'$ is a solution of the equation for v'.

If u_1 had been a triple solution, it would only be a matter of tedious calculation to repeat the process again, defining $y(x) = w'(x)/[(u_{1c}/u_{1a})'/(u_{1b}/u_{1a})']'$ and obtaining the appropriate equation for y'.

Returning to the double solution case, integrate the equation for w' from x_1 to x_m. The value of x beyond which any arbitrary solution is a linear combination of M other arbitrary solutions is called x_2, and u_2 consists of M independent solutions. Each independent w' is integrated

from x_m to x_2 starting with $w(x_m) = 0$, giving w_{2a}, w_{2b},... which are accurate for $x_2<x<x_{m-2}$, where x_{m-2} is determined by the analog of (11) with $(u_{1b}/u_{1a})'$ replacing v_2'. Equation (14) then gives v_{2a}', v_{ab}',... which are integrated from x_m to x_2 with $v_2(x_m) = 0$. If $v_2(x)$ grows more rapidly (in the negative x direction) then $w_2(x)$, the resultant v_{2a}, v_{2b}, etc. will be accurate in the interval $x_2<x<x_{m-2}$. If v_2 grows more slowly than w_2, a new x_{m-2} will need to be determined by a formula analogous to (8). Of course, u_{2a}, u_{2b}, etc. are determined now from (12).

The continuation of this to obtain u_3, etc., should be obvious.

5. System of First Order Equations

The method is best described in terms of a single N^{th} order equation, and so this was developed first, but in most physical problems the equations occur rather as a system of N first order equations which may be very difficult to reduce to a single N^{th} order equation. Thus the method will now be described for this case.

Suppose we have the following linear system of N first order equations:

$$u_n'(x) = \sum_{i=1}^{N} a_{ni}(x)\, u_i(x) \qquad n=1, 2, \ldots N \qquad (15)$$

As before, if any solution of this equation starting with arbitrary boundary conditions at x_o is a unique linear combination of M other arbitrary solutions u_{n1a}, u_{n1b},... for all $x>x_1$, then these M independent solutions dominate all other solutions and can be calculated exactly (within roundoff error) in the interval $x_1<x<x_m$, where x_m is the maximum value of x to which the solution of (15) is carried.

Define

$$v_n(x) = \frac{u_n(x)}{u_{nla}(x)} \tag{16}$$

Substitute this in (15) to obtain the equations for v_n.

$$v_n' = \sum_{i=1}^{N} a_{ni} \frac{u_{ila}}{u_{nla}} [v_i - v_n] \tag{17}$$

Rewriting this equation in terms of the differences $\Delta v_n = v_n - v_N$ gives a system of one lower order

$$\Delta v_n' = \sum_{i=1}^{N-1} b_{ni} \Delta v_i \qquad n=1,2,\ldots N-1$$

$$b_{ni} = \left(\frac{a_{ni}}{u_{nla}} - \frac{a_{Ni}}{u_{Nla}}\right) u_{ila} - \delta_{in} \left(\sum_{s=1}^{N} a_{ns} \frac{u_{sla}}{u_{nla}}\right) \tag{18}$$

If u_{n1} were a single solution (i.e. dominated all others) one would merely solve (18) with arbitrary boundary conditions at x_1, determining the lower boundary of validity x_2 as before. Equation (17) then gives v_n' in terms of Δv_n as follows:

$$v_n' = \sum_{i=1}^{N} a_{ni} \frac{u_{ila}}{u_{nla}} [\Delta v_i - \Delta v_n] \tag{19}$$

This is integrated to obtain v_n, starting at x_m with the boundary condition $v_n(x_m) = 0$. The upper limit of validity, x_{m-1}, of the v_n thus determined is the largest value of x at which one could restart the integration of (15) with arbitrary boundary conditions and still recover

a pure u_{n1} solution before reaching x_m. In practice it is more easily determined by a generalization of the approximation given in (8).

$$|v_n(x_{m-1})| > \frac{|v_n(x_t)|}{\epsilon} \qquad n = 1, 2, \ldots N \tag{20}$$

where

$$|u_{n1}(x_t)| < \frac{|u_{n1}(x_m)|}{10} \left[1 - \left|1 + \frac{u_{n1}(x_s)v_n'(x_s)}{u_{n1}'(x_s)v_n(x_s)}\right|\right] \qquad x_s \cong x_t$$

If u_{n1} is a double solution, then the formal solution for Δv_n is

$$\Delta v_n = \sum_{i=1} A_i \left(\frac{u_{ni}}{u_{n1a}} - \frac{u_{Ni}}{u_{N1a}}\right) + A_{1b} \left(\frac{u_{n1b}}{u_{n1a}} - \frac{u_{N1b}}{u_{N1a}}\right) \tag{21}$$

which may be divided by the factor multiplying A_{1b} and differenced again in order to eliminate A_{1b}. Specifically, define

$$w_n = \frac{\Delta v_n}{F_n} \quad , \qquad F_n = \frac{u_{n1b}}{u_{n1a}} - \frac{u_{N1b}}{u_{N1a}} \tag{22}$$

Then since F_n is a solution of the equation for Δv_n, if we define $\Delta w_n = w_n - w_{N-1}$, the equation for Δw_n is:

$$\Delta w_n' = \sum_{i=1}^{N-2} c_{ni}\, \Delta w_i \qquad n = 1, 2, \ldots N-2$$

$$c_{ni} = \left(\frac{b_{ni}}{F_n} - \frac{b_{N-1\,i}}{F_{N-1}}\right) F_i - \delta_{in} \left(\sum_{s=1}^{N-1} b_{ns} \frac{F_s}{F_n}\right) \tag{23}$$

Solve this equation for arbitrary boundary conditions at x_1, determining the lower boundary of validity x_2 as before. The w_n' are obtained from these Δw_n by the formula:

$$w_n' = \sum_{i=1}^{N-1} b_{ni} \frac{F_i}{F_n} (\Delta w_i - \Delta w_n) \qquad (24)$$

These w_n' are integrated from x_m down to x_2, starting with the boundary condition $w_n(x_m) = 0$, and determining the upper limit of validity, x_{m-1}, by a formula analogous to (20). Using these w_n in (22), and the resulting Δv_n in (19) gives v_n' which is likewise integrated from x_m to x_2 starting with the boundary condition $v_n(x_m) = 0$. As pointed out before, if this v_{n2} grows faster than w_n, then x_{m-1} is still the upper limit of validity for v_{n2}; if not, then x_{m-1} must be lowered appropriately. Finally u_{n2} is calculated from (16) and is accurate in the range $x_2 < x < x_{m-1}$.

The modifications necessary to find the solutions subdominant to three or more equally dominant solutions are a tedious but straightforward extension of the above.

6. Summary

A method has been developed which makes possible the numerical calculation of all the solutions of almost any N^{th} order system of linear differential equations, even when some of the solutions grow so rapidly as to completely dominate a standard numerical initial value calculation. This is the situation that arises physically whenever the inclusion of a physical effect (such as viscosity, thermal conductivity, resistivity, etc.) in the equations has only a small effect on the desired solution but raises the order of the equations. Invariably, then, it also

introduces into the equations other solutions which vary rapidly relative to the original solutions and therefore cannot be eliminated by proper choice of boundary conditions in any calculation subject to roundoff error.

The method described makes it possible to calculate numerically all the solutions of the equations in those cases where the following conditions are fulfilled.

1) The solutions can be clearly "ordered" as to their dominance, in the sense described in Section 2.

2) No solution which is treated as dominant over any other may have a zero.

3) It must be possible to solve the equations over intervals both before and after the interval where accurate solutions are required. These intervals are longer for the more sub-dominant solutions.

The approach consists of isolating these dominant solutions one at a time in order of their dominance, expressing the general solution as a product of the dominant solution and a new function. The amount of the dominant solution that appears in the final solution then is controlled by an additive constant in the new function. This additive constant is determined at the other end of the interval of integration in such a way that the dominant solution is effectively eliminated. Treating the equation for this new function (which is one order lower) in the same way as the original equation eliminates the next dominant solution, and so on. The method for eliminating several solutions none of which dominates any of the others is also described.

Appendix

For purposes of illustration, the foregoing method is here applied to the problem of calculating the four solutions of the equation

$$-\varepsilon^2[(x^2+2\varepsilon^2)u'''' - 2xu'''] + x^2u'' - 2xu'+2u = 0 \qquad \text{(A-1)}$$

where ε is a small parameter. Clearly when $\varepsilon = 0$, this becomes a second order equation with solutions $u = x$ and $u = x^2$. These are still solutions for $\varepsilon \neq 0$ and in addition one also obtains the two solutions $u = e^{\pm x/\varepsilon}$. Qualitatively this is just what happens in many physical situations when the inclusion of a small additional effect raises the order of the equations.

The specific value $\varepsilon = 0.1$ was chosen and (A-1) was integrated over the range $0 \leq x \leq 10$ by Runge-Kutta integration using a step size of .00125. The boundary values at $x = 0$ included equal amounts of all four solutions. Referring to the first column of Table 1, one can see that the first solution had dominated the other three before $x = 2$. The first pair of numbers in each column gives two significant figures and the exponent of the quantity and the next pair gives one significant figure and the exponent of the fractional error (obtained by comparison with the analytic solution). Only 8 significant figures were printed out so errors below 10^{-8} are given as 1-8.

Beginning at $x = 2$ the differential equation for v' was determined from Eq. (6) and solved (using Runge-Kutta with step size .0025) for two

sets of boundary values to determine v_{2a}' and v_{2b}'. The u_2 were determined from the formula

$$u_2(x) = u_1(x)\int_x^{10} v_2'(x')dx' \qquad (A\text{-}2)$$

which comes from (3) and the condition that $u_2(x_m) = 0$. One does not know a priori what linear combinations of x and x^2 correspond to u_{2a} and u_{2b} but these are easily determined near the center of the interval. The linear combinations thus determined were used as the analytic solutions in calculating the indicated errors.

Beginning at $x = 4$, where the u_2 are accurately known, y_3' is determined from the differential equation

$$y_3'' = \left[\frac{2x}{x^2+.02} - \frac{4u_1'}{u_1} + \frac{v_{2a}''}{v_{2a}'} - 2\left(\frac{v_{2b}''' v_{2a}' - v_{2a}''' v_{2b}'}{v_{2b}'' v_{2a}' - v_{2a}'' v_{2b}'}\right)\right] y_3' \qquad (A\text{-}3)$$

by a Runge-Kutta integration with step size of .005. It was necessary to double the step size for each successive integration because the Runge-Kutta method requires that the coefficient functions be known at the middle as well as the ends of each interval.

Finally the last three columns of the table were determined by the formulae:

$$w_3'(x) = \left(\frac{v_{2b}'}{v_{2a}'}\right)' \int_x^{10} y_3'(x')dx' \qquad (A\text{-}4)$$

$$v_3'(x) = v_{2a}'(x)\int_x^{10} w_3'(x')dx' \qquad (A\text{-}5)$$

$$u_3(x) = u_1(x) \int_x^{10} v_3'(x')dx' \qquad \text{(A-6)}$$

The fractional error in the u_3, namely about 10^{-6}, is larger than the error in the other solutions primarily because of the larger interval size in the final integrations, but certainly also as a result of the cumulative errors from going through six successive integrations.

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x	u_1		V_{2a}'	u_{2a}		V_{2b}'	u_{2b}		y_3'	w_3'	v_3'	u_3	
0	20-1	9-1											
.5	15+1	5-3											
1.0	22+3	9-5											
1.5	33+5	1-6											
2.0	49+7	1-8	75-0	34+8	2-1	37-0	15+8	5-1					
2.5	72+9	1-8	55-2	42+9	8-4	23-2	18+8	2-3					
3.0	11+12	1-8	51-4	58+8	3-7	25-4	29+8	7-6					
3.5	16+14	1-8	46-6	76+8	2-8	25-6	42+8	3-8					
4.0	24+16	1-8	39-8	97+8	1-8	23-8	58+8	2-8	10-1	83-4	28-11	32+5	1-6
4.5	35+18	1-8	33-10	12+9	1-8	21-10	76+8	1-8	65-4	45-6	13-15	21+3	1-6
5.0	52+20	1-8	27-12	15+9	1-8	18-12	97+8	1-8	43-6	25-8	58-20	14+1	1-6
5.5	77+22	1-8	22-14	17+9	1-8	15-14	12+9	1-8	28-8	14-10	26-24	97-2	1-6
6.0	11+25	1-8	17-16	20+9	1-8	12-16	15+9	1-8	19-10	79-13	12-28	65-4	1-6
6.5	17+27	1-8	14-18	24+9	1-8	99-19	17+9	1-8	13-12	46-15	54-33	44-6	1-6
7.0	25+29	1-8	11-20	27+9	1-8	79-19	20+8	1-8	83-15	27-17	25-37	30-8	1-6
7.5	37+31	1-8	81-23	31+9	1-8	62-23	24+9	1-8	56-17	16-19	11-41	20-10	1-6
8.0	55+33	1-8	62-25	35+9	2-8	48-25	27+9	2-8	37-19	94-22	51-46	13-12	1-6
8.5	82+35	1-8	47-27	39+9	4-7	37-27	31+9	4-7	25-21	56-24	23-50	90-15	1-5
9.0	12+38	1-8	35-29	44+9	5-5	28-29	35+9	5-5	17-23	34-26	11-54	61-17	1-3
9.5	18+40	1-8	26-31	48+9	7-3	21-31	39+9	7-3	11-25	20-28	46-59	38-19	7-2
10.0	27+42	1-8	19-33	0		16-33	0		74-28	0	0	0	

Table 1 - Results for sample problem.