NUMERICAL ANALYSIS OF LAPLACE'S EQUATION WITH NONLINEAR BOUNDARY CONDITIONS

A Thesis Submitted to

Case Institute of Technology

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

by

Edward Clarence Bittner

1965

Thesis Advisor: Professor Richard S. Varga

NUMERICAL ANALYSIS OF LAPLACE'S EQUATION WITH NONLINEAR BOUNDARY CONDITIONS

A Thesis Submitted to

Case Institute of Technology

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

by

Edward Clarence Bittner

1965

Thesis Advisor: Professor Richard S. Varga

ABSTRACT

32250

Existence of solutions of two-dimensional boundary-value problems of the type

$$\nabla^2 u = 0 \qquad \text{in } R,$$

$$u(P) = 0 \qquad \text{on } \Gamma_1,$$

$$\frac{\partial u}{\partial n} = \gamma(P,u) \qquad \text{on } \Gamma_2,$$

is discussed under conditions subsequently strengthened for uniqueness. In general, solutions are shown to lie between a maximum possible and a minimum possible, which extremes are obtainable as limits of sequences of solutions of certain linear problems. Under convexity assumptions on γ, the unique solution is shown to be obtainable as a maximum over solutions of an entire class of linear problems, and is also obtainable by Newton's method.

Finite difference approximations are shown by a Gerschgorin-type analysis to converge to solutions of the above, at essentially the rate $O(h^{1/3})$ as $h \to 0$, where h is the maximum mesh width. The order of convergence is tied to the smoothness of γ, Γ_1, and Γ_2, and more general results are actually obtained.

Author

The two-dimensional Laplace equation was used for simplicity and for conciseness of treatment. Generalizations are indicated in the direction of higher dimensions, more general elliptic operators, non-homogeneous, non-linear differential equations and conditions other than Dirichlet on Γ_1.

ACKNOWLEDGMENTS

I wish to express my appreciation to Professor Richard S. Varga for his advice, encouragement, and time.

I also wish to thank the Lewis Research Center of the National Aeronautics and Space Administration for support through its graduate study leave program.

Thanks are also due to my wife, Ann, who typed the early drafts.

The final draft was carefully executed with great patience by Miss Evelyn K. Spala and Mrs. June G. Masek of the Lewis Research Center.

LIST OF FIGURES

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

1.1 Object

Mixed boundary problems of the type

$$\nabla^2 u = 0 \qquad \text{in} \quad R, \tag{1.1.1}$$

$$u = 0 \qquad \text{on} \quad \bar{\Gamma}_1, \tag{1.1.2}$$

$$\frac{\partial u}{\partial n} = \gamma(u) \qquad \text{on} \quad \Gamma_2, \tag{1.1.3}$$

in two dimensions are investigated. R is a bounded two-dimensional region with boundary $\bar{\Gamma}_1 \cup \Gamma_2$. Existence of solutions is obtained under the mild assumption

$$\limsup_{|u| \to \infty} \frac{\gamma(u)}{u} \leq 0, \tag{1.1.4}$$

which, in effect, permits the replacement of γ by a bounded function γ_0, having the same values as γ within the a-priori estimates on u made possible by (1.1.4). Uniqueness is established under the additional, more restrictive hypothesis

$$\gamma_u \leq 0. \tag{1.1.5}$$

The problems of existence, uniqueness, and numerical approximations of solutions of Dirichlet problems for elliptic equations with solution-dependent source terms, e.g.

$$\nabla^2 u = f(u) \qquad (1.1.6)$$

have received a great deal of attention recently and a number of results have been obtained, for example, in [1, 4, 9, 12, 13, 14, 18, 20].

Problems like (1.1.1 - 1.1.3) have received some attention, for example, in [24] where they are treated from the point of view of the variational calculus and (1.1.3) is "weakly" satisfied, but I have not seen any work on the numerical approximation to their solutions with estimates of the error in the approximation.

Techniques similar to those applied to Dirichlet problems with a non-linear source term and to linear problems with mixed conditions can be applied to yield the desired results for problem (1.1.1 - 1.1.3). Thus the existence proof has a procedure somewhat parallel to that of [18] with results of [14] and [17] being called upon. Uniqueness is demonstrated in a manner analogous to that in [9] for (1.1.6). The proof that the solutions of (1.1.1 - 1.1.3) can be approximated arbitrarily closely by solutions of corresponding sets of difference equations is a Gerschgorin-type of analysis. The solutions being permitted discontinuities where Γ_1 intersects Γ_2, the error estimates must follow lines similar to those set down by [22] and [21] for Dirichlet problems with non-smooth boundaries. The result here is $O\left[h^{\alpha^2/(2+\alpha)}\right]$ convergence where

α is the Hölder coefficient in the smoothness hypotheses on γ, Γ_1, and Γ_2.

CHAPTER II contains the above-mentioned results on existence and uniqueness of solutions u of (1.1.1 - 1.1.3) and, under suitable further restrictions on γ, a proof of a maximum operation for u similar to that established in [13] for (1.1.6). CHAPTER III contains the method of finite difference approximations and results for them analogous to those of CHAPTER II. The error analyses are presented in CHAPTER IV, and some possible extensions of this paper in CHAPTER V. The results of CHAPTERS II through IV are then summarized in CHAPTER VI. Some numerical results are given in CHAPTER VII.

The remaining portion of this chapter is devoted to two problems from the transfer of heat which can be treated by proper modifications of the methods given in this paper. A special case of the first is used as a numerical example illustrating the results.

1.2 Motivation

The first example of a problem which can be put into a form suitable for analysis by the methods of this paper arises from requirements of rejection of waste heat into space by radiation. The heat transfer here is governed by the Stephan-Boltzmann law and the mathematical model yields

$$\nabla^2 T = 0 \qquad \text{in } R, \tag{1.2.1}$$

$$T = T_0 \qquad \text{on } \overline{\Gamma}_1, \tag{1.2.2}$$

$$\frac{\partial T}{\partial n} = -\frac{\epsilon\sigma}{K} T^4 \qquad \text{on} \quad \Gamma_2, \tag{1.2.3}$$

where $T(P)$ is the temperature at points P of $\overline{R}$, ϵ is emissivity, K is conductivity, σ is the Stefan-Boltzmann constant and $\partial T/\partial n$ is the derivative in the direction of the outer normal.

The second example arises from an attempt to keep the walls of a combustion chamber from melting. In many heat transfer problems arising in combustion, cryogenics or in those arising in combining both technologies, large temperature gradients are frequently encountered. The solution of the equations arising from the mathematical model assuming constant material properties no longer represents the temperature at points in the heat-carrying body.

Assume that part of the boundary is kept at a constant temperature while the transfer of heat at the remaining portion is governed by Newton's law. Let the conductivity $K > 0$ be a function of temperature. Then the equations for the temperature are

$$\operatorname{div}\left[K(T)\operatorname{grad} T\right] = 0 \qquad \text{in} \quad R, \tag{1.2.4}$$

$$T = T_c \qquad \text{on} \quad \overline{\Gamma}_1 \tag{1.2.5}$$

$$K(T)\frac{\partial T}{\partial n} = h(T_g - T) \qquad \text{on} \quad \Gamma_2, \tag{1.2.6}$$

where $h > 0$ is the heat transfer coefficient and, typically, T_g, the temperature of the gas, is such that $T_g \gg T_c$.

Under the transformation

$$u = \int_{T_c}^{T} K(T)\ dT \qquad (1.2.7)$$

which is 1 - 1, the boundary value problem becomes

$$\nabla^2 u = 0 \qquad \text{in} \quad R, \qquad (1.2.8)$$

$$u = 0 \qquad \text{on} \quad \bar{\Gamma}_1, \qquad (1.2.9)$$

$$\frac{\partial u}{\partial n} = h\left[T_g - T(u)\right] \qquad \text{on} \quad \Gamma_2. \qquad (1.2.10)$$

Thus (1.2.4 - 1.2.6) can also be placed (by (1.2.7)) into a form suitable for treatment by the methods of this paper.

The advantage of having a problem of this type in this latter form becomes apparent upon examining the system of finite difference equations. The system corresponding to (1.2.4 - 1.2.6) has most of its equations non-linear while that corresponding to (1.2.8 - 1.2.10) has most of them linear.

CHAPTER II

SOLUTIONS OF THE BOUNDARY VALUE PROBLEMS

FOR THE PARTIAL DIFFERENTIAL EQUATIONS

2.1 Existence

Consider

$$\nabla^2 u = 0 \quad \text{in} \quad R, \tag{2.1.1}$$

$$u(P) = 0 \quad \text{on} \quad \overline{\Gamma}_1, \tag{2.1.2}$$

$$\frac{\partial u}{\partial n} = \gamma(P,u) \quad \text{on} \quad \Gamma_2, \tag{2.1.3}$$

where R is a bounded, two-dimensional region with boundary $\overline{\Gamma}_1 \cup \Gamma_2$, consisting of a finite number of smooth arcs. Here $\overline{\Gamma}_1$ denotes the closure of Γ_1. As was done for Dirichlet problems with non-linear source terms by Levinson [14] and Parter [18], $\gamma(P,u)$ is to be replaceable by a bounded function γ_0 by requiring that

$$\limsup_{|u| \to \infty} \frac{\gamma(P,u)}{u} \leq 0. \tag{2.1.4}$$

Before proceeding to smoothness requirements on γ and on the boundary of R, it is necessary to make some definitions.

A point P of the boundary $\overline{\Gamma_1} \cup \overline{\Gamma_2}$ is said to have a barrier function there, if there exists a superharmonic function ω_P, continuous and single valued in $\overline{R}$, such that

(a) $\omega_P(P) = 0$,

(b) $\omega_P(Q) > 0$, for $Q \in \overline{R} - \{P\}$.

Γ_1 is said to be <u>smooth</u> if it can be covered by a finite number of circles in each of which, one of the co-ordinates can be expressed as a function of the other, the "arc parameter". These functions are required to have Hölder continuous second derivatives.

Some smoothness in the form of Hölder conditions on derivatives of γ with respect to arc parameters and with respect to the dependent variable is also required. Namely, let there be a number α such that $0 < \alpha < 1$, and five positive functions $K(M), K_0(M), K_1(M), K_2(M)$, and $K_3(M)$ such that for all P, P_1, and $P_2 \epsilon \Gamma_2$ and all u, u_1, and u_2 whose absolute value is bounded above by M,

$$|\gamma_P(P_1,u) - \gamma_P(P_2,u)| \leq K(M)|P_1 - P_2|^\alpha, \qquad (2.1.5)$$

$$|\gamma(P,u_1) - \gamma(P,u_2)| \leq K_0(M)|u_1 - u_2|, \qquad (2.1.6)$$

$$|\gamma_u(P_1,u) - \gamma_u(P_2,u)| \leq K_1(M)|P_1 - P_2|^\alpha, \qquad (2.1.7)$$

$$|\gamma_u(P,u_1) - \gamma_u(P,u_2) \leq K_2(M)|u_1 - u_2|, \qquad (2.1.8)$$

$$|\gamma_P(P,u_1) - \gamma_P(P,u_2)| \leq K_3(M)|u_1 - u_2|, \qquad (2.1.9)$$

where γ_P denotes the partial derivative with respect to arc parameter.

Some additional restrictions of ∂R are necessary to insure existence, uniqueness, and some degree of smoothness of solutions of some auxiliary problems of this section. Namely,

(i) A barrier function must exist at each point of $\bar{\Gamma}_1 \cup \Gamma_2$.

(ii) The tangents to Γ_1 and Γ_2 at the points of $\bar{\Gamma}_1 \cap \bar{\Gamma}_2$ must not meet at either of the two angles 0 or π.

(iii) Γ_2 must be smooth.

(iv) Γ_1 is smooth.

Although used in this section and in section 4.1, this last hypothesis can be dispensed with, under modifications to be indicated.

It will now be shown that the search for a solution of (2.1.1 - 2.1.3) may be carried out in a bounded portion of the (P,u) - space.

Condition (2.1.4) implies the existence of a positive function $\eta(u)$ which is decreasing for large u, such that,

$$\lim_{u\to\infty} \eta(u) = 0, \tag{2.1.10}$$

which satisfies, for large u,

$$\frac{\gamma(u)}{u} \leq \eta(u). \tag{2.1.11}$$

Furthermore, Levinson showed that η could be modified, without destroying these properties, so that $u\eta(u) + u$ is increasing for large u (see [14] lemma 3.1).

To derive the a priori estimates, φ is postulated as a solution of (2.1.3). The proof, that φ has a bound depending only on R and γ, depends on the existence of a positive function ζ for which upper and lower estimates are available, and the definition of a new function $\psi = \varphi/\zeta$ which is shown to be bounded, in two steps: The first is the proof that any positive maximum or negative minimum must occur on the boundary Γ_2. The second step is to show that these extremes have a bound depending on ζ.

By Miranda [17], there is a function ζ continuous in $\overline{R}$ which satisfies

$$\nabla^2\zeta = -1 \quad \text{in } R, \tag{2.1.12}$$

$$\zeta = 1 \quad \text{on } \overline{\Gamma}_1, \tag{2.1.13}$$

$$\frac{\partial\zeta}{\partial r} = 1 \quad \text{on } \Gamma_2. \tag{2.1.14}$$

Greenspan [11] and Batschelet [3] use this function to prove convergence of finite difference approximations to solutions of linear mixed problems under enough conditions that ζ will have four continuous derivatives. Here, we are only guaranteed that the first and second derivatives satisfy Hölder conditions in compact subsets of $\overline{R} - \overline{\Gamma}_1 \cap \overline{\Gamma}_2$.

Let φ be a solution of (2.1.1 - 2.1.3) and define ψ by

$$\varphi = \zeta\psi. \tag{2.1.15}$$

Assume that ψ has a positive maximum at an interior point P of R. Since at P

$$\nabla^2\varphi = 0,$$

i.e.

$$\psi\nabla^2\zeta + 2\left(\frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial y}\right) + \zeta\nabla^2\psi = 0,$$

we have

$$\begin{aligned}\nabla^2\psi &= -\frac{\psi\nabla^2\zeta}{\zeta}\\ &= \frac{\psi}{\zeta}\\ &> 0\end{aligned} \tag{2.1.16}$$

at this point. Thus ψ must take on any positive maximum on the boundary. Since ψ is zero on $\overline{\Gamma}_1$, this positive maximum is possible only on Γ_2. A contradiction is also reached for a postulated interior negative minimum.

Let $M_1 > 0$ be such that for $u > M_1$, $\eta(u)$ is decreasing and

$$\eta(u) < \frac{1}{\max\limits_{\overline{\Gamma}_2} \zeta} \tag{2.1.18}$$

and

$$\frac{\gamma(p,u)}{u} \leq \eta(u). \tag{2.1.19}$$

Now suppose M_2, the positive maximum of ψ at $P \epsilon \Gamma_2$ is such that

$$M_2 > M_1. \tag{2.1.20}$$

Then at P, by (2.1.3), (2.1.14), and (2.1.15),

$$\begin{aligned} \frac{\partial\psi}{\partial n} &= \left[\frac{\gamma(P,\varphi)}{\varphi} - \frac{1}{\zeta}\frac{\partial\zeta}{\partial n}\right]\psi \\ &= \left[\frac{\gamma(P,\varphi)}{\varphi} - \frac{1}{\zeta}\right]M_2 \qquad (2.1.21) \\ &< \left[\frac{1}{\max\limits_{\overline{\Gamma}_2} \zeta} - \frac{1}{\zeta}\right]M_2 \\ &\leq 0, \qquad (2.1.22) \end{aligned}$$

which contradicts the maximality of ψ at P. Thus (2.1.20) is false, i.e.

$$M_2 \leq M_1. \tag{2.1.23}$$

Similarly a negative minimum for ψ on Γ_2 has smaller magnitude than M_1.

So, by (2.1.15), any solution φ of (2.1.1 - 2.1.3) has the a-priori bound

$$|\varphi| \le M_3 = \max_{\bar{R}} \zeta \cdot M_1 \qquad (2.1.24)$$

which depends only on R and η. This establishes the following theorem:

Theorem 2.1.1. - If φ is a solution of (2.1.1 - 2.1.3), then φ has the a priori estimate

$$|\varphi| \le \max_{R} \zeta \cdot M_1$$

where ζ is the solution of (2.1.12 - 2.1.14) and M_1 is so large that $\eta(u)$ is decreasing for $u > M_1$ and (2.1.18) is satisfied.

This paves the way for the replacement of γ by a bounded function γ_0. In fact, let $M_4 \ge M_3 + 1$ and

$$\gamma_0(P,u) = \begin{cases} \gamma(P,u) & |u| \le M_4 \qquad (2.1.25) \\ \left.\begin{array}{l}\gamma(P,M_4) \text{ if } \gamma(P,M_4) \le 0 \\ \max[0,\gamma(P,M_4) - u + M_4] \text{ otherwise}\end{array}\right\} & u > M_4 \qquad (2.1.26) \\ \left.\begin{array}{l}\gamma(P,-M_4) \text{ if } \gamma(P,-M_4) \ge 0 \\ \max[0,\gamma(P,-M_4) - u - M_4] \text{ otherwise}\end{array}\right\} & u < -M_4 \qquad (2.1.27) \end{cases}$$

The hypotheses satisfied by γ are also satisfied by γ_0. The easiest to verify are (2.1.5), (2.1.6). A little more work is required to verify (2.1.11), and this is now carried out.

For large u but where $|u| \leq M_4$, (2.1.11) is immediate for γ_0 from (2.1.25). Let $u > M_4$. Then since $u\eta(u) + u$ is increasing,

$$M_4\eta(M_4) + M_4 \leq u\eta(u) + u, \qquad (2.1.28)$$

so that

$$\frac{\gamma_0(P,u)}{u} = \frac{\gamma(P,M_4)}{u} - 1 + \frac{M_4}{u}$$

$$\leq \frac{M_4\eta(M_4)}{u} - 1 + \frac{M_4}{u}$$

by (2.1.19). Therefore by (2.1.28) it follows that

$$\frac{\gamma_0(P,u)}{u} \leq \eta(u) \qquad (2.1.29)$$

if $u > M_4$.

The argument is similar for $u < -M_4$. Thus γ_0 satisfies (2.1.11). By examining the several cases which arise it can be seen that the transitions at

$$u = \gamma(P,-M_4) - M_4, \qquad (\text{if } \gamma(P,-M_4) < 0),$$

$$u = -M_4,$$

$$u = M_4,$$

$$u = \gamma(P,M_4) + M_4 \qquad (\text{if } \gamma(P,M_4) > 0),$$

can be made smooth enough so that, besides (2.1.5) and (2.1.6), γ_0 satisfies the inequalities (2.1.7 - 2.1.9) without violating (2.1.11). Since γ_0 does not differ from γ for $u \leq M_3 < M_4$, φ is a solution of (2.1.1 - 2.1.3) if and only if it is a solution of this set of equations with γ replaced by γ_0.

The point of this latter result is that we may proceed to investigating existence of the solution of (2.1.1 - 2.1.3) with γ having one more (crucial) property: There is an $N > 0$ such that for all $P \epsilon \Gamma_2$ and $|u| < \infty$,

$$|\gamma(P,u)| \leq N. \qquad (2.1.30)$$

Let the distance between two sets be defined as usual:

$$d(A,B) = \inf_{\substack{P \in A \\ Q \in B}} |P - Q|.$$

In what follows, the various functions z, say, defined by solving certain boundary value problems, exist by the results of Miranda [17]. Here, if the conditions on Γ_2 are written as

$$\frac{\partial z}{\partial n} = \Phi(P), \qquad (2.1.32)$$

then the first derivative of Φ with respect to the arc parameter needs to satisfy a Hölder condition on compact subsets. $\tilde{\Gamma}_2$ of Γ_2, where $d(\tilde{\Gamma}_2, \bar{\Gamma}_1 \cap \bar{\Gamma}_2) > 0$, and Φ to be bounded on such sets. The results for z in R then are that z is continuous in $\bar{R}$ and its first and second partials satisfy Hölder conditions in compact subsets of $\bar{R} - \bar{\Gamma}_1 \cap \bar{\Gamma}_2$.

Let u_0 be defined for $N' \geq N$ by

$$\nabla^2 u_0 = 0 \quad \text{in } R, \qquad (2.1.33)$$

$$u_0 = 0 \quad \text{on } \bar{\Gamma}_1, \qquad (2.1.34)$$

$$\frac{\partial u_0}{\partial n} = N' \quad \text{on } \Gamma_2. \qquad (2.1.35)$$

Then $u_0 \geq 0$ in $\bar{R}$. Let

$$u_{OM} = \max_{P\epsilon\overline{\Gamma}_2} u_0(P) \tag{2.1.36}$$

and

$$k = K_0(u_{OM}). \tag{2.1.37}$$

Then

$$\gamma(u) - \gamma(w) - k(u - w) \geq 0, \tag{2.1.38}$$

for

$$-u_{OM} \leq u \leq w \leq u_{OM}. \tag{2.1.39}$$

Let $u_{m+1}(P)$ be defined in $\overline{R}$ by

$$\nabla^2 u_{m+1} = 0 \qquad \text{in } R, \tag{2.1.40}$$

$$u_{m+1} = 0 \qquad \text{on } \overline{\Gamma}_1, \tag{2.1.41}$$

$$\frac{\partial u_{m+1}}{\partial n} + k u_{m+1} = \gamma(u_m) + k u_m \qquad \text{on } \Gamma_2, m \geq 0. \tag{2.1.42}$$

Examine the case $m = 0$ on Γ_2:

$$\begin{aligned} \frac{\partial u_1}{\partial n} + k u_1 &= \gamma(u_0) + k u_0 \\ &\leq N' + k u_0 \\ &= \frac{\partial u_0}{\partial n} + k u_0. \end{aligned} \tag{2.1.43}$$

Therefore,

$$u_1 \leq u_0. \tag{2.1.44}$$

Also, on Γ_2

$$\begin{aligned} \frac{\partial u_1}{\partial n} &= -k(u_1 - u_0) + \gamma(u_0) \\ &\geq \gamma(u_0) \\ &\geq -N' \\ &= -\frac{\partial u_0}{\partial n}, \end{aligned} \tag{2.1.45}$$

which implies

$$u_1 \geq - u_0. \tag{2.1.46}$$

Having proved for $m = 1$ that

$$-u_0 \leq u_m \leq u_{m-1} \leq u_0 \tag{2.1.47}$$

proceed inductively to the case $m + 1$: On Γ_2

$$\frac{\partial}{\partial n}(u_{m+1} - u_m) + k(u_{m+1} - u_m) = \gamma(u_m) - \gamma(u_{m-1}) + k(u_m - u_{m-1})$$
$$\leq 0, \tag{2.1.48}$$

so that

$$u_{m+1} \leq u_m \tag{2.1.49}$$

and

$$\begin{aligned} \frac{\partial u_{m+1}}{\partial n} &= -k(u_{m+1} - u_m) + \gamma(u_m) \\ &\geq \gamma(u_m) \\ &\geq -N' \\ &= -\frac{\partial u_0}{\partial n} \end{aligned} \tag{2.1.50}$$

implying (2.1.47) for all m.

Thus, $\{u_m\}$ is a uniformly bounded, monotonic sequence of functions. They are, in addition, harmonic, so that by Harnack's second theorem (Petrovsky [19] p. 178) convergence is uniform in every compact subset of R and the limit is harmonic. Since $\overline{\Gamma}_1 \cup \Gamma_2$ is assumed to have a barrier function at each point, the limit

$$u = \lim_{m\to\infty} u_m \tag{2.1.51}$$

is continuous in $R \cup \overline{\Gamma}_1$ and takes the value 0 on $\overline{\Gamma}_1$. (Petrovsky [19] p. 184). Using the results of Miranda, it is possible to say more.

Let $\delta > 0$ and $\overline{R}_\delta$ be the compact subset of $\overline{R} - \overline{\Gamma}_1 \cap \overline{\Gamma}_2$ such that $d(\overline{R}_\delta, \overline{\Gamma}_1 \cap \overline{\Gamma}_2) = \delta$:

$$\overline{R}_\delta = \{P \epsilon \overline{R} \mid d(P, \overline{\Gamma}_1 \cap \overline{\Gamma}_2) \geq \delta\}. \tag{2.1.52}$$

Recalling (2.1.42), let

$$\Phi_{m+1}(P) = \gamma[P, u_m(P)] + ku_m(P) \tag{2.1.53}$$

for $m \geq 0$. Then by (2.1.5), (2.1.7 - 2.1.9), the sequence of first derivatives of the Φ_{m+1} with respect to arc parameter satisfies a Hölder condition uniformly in $\overline{R}_\delta \cap \Gamma_2$. By Theorem 6,I of Miranda [17] the second partial derivatives satisfy uniform Hölder conditions in $\overline{R}_\delta$ and, therefore, the sequence of second partial derivatives of u_m form an equicontinuous set. There is, then, a subsequence of second derivatives which converges uniformly, so that the boundary conditions on Γ_2 are satisfied by the limit u. Furthermore, the first and second derivatives of u are Hölder continuous on compact subsets of $\overline{R} - \overline{\Gamma}_1 \cap \overline{\Gamma}_2$.

These last results can be summarized as:

<u>Theorem 2.1.2.</u> - If R is a bounded region of the xy-plane with boundary $\overline{\Gamma}_1 \cup \Gamma_2$, satisfying the smoothness conditions (i) through (iv) on page 7, and $\gamma(P,u)$ is a function defined for all $P \epsilon \Gamma_2$ and all finite u, satisfying the smoothness condi-

tions (2.1.5) through (2.1.9) on page 7 and the fundamental condition (2.1.4) on page 6, then there exist solutions of (2.1.1 - 2.1.3) continuous in $\overline{R}$ and having Hölder continuous first and second derivatives in compact subsets of $R - \overline{\Gamma}_1 \cap \overline{\Gamma}_2$.

The bases of the existence theorem 6,I of Miranda are some Schauder-like estimates on spaces of functions which, with some of its derivatives, are permitted various orders of discontinuities as points approach $\overline{\Gamma}_1 \cap \overline{\Gamma}_2$. If it is desired to relax the smoothness of Γ_1 (to dispense with (iv) on page 8), Miranda has other estimates which become more nearly like the usual Schauder estimates (see Courant and Hilbert [9] p.331 ff.). The above arguments are modified, chiefly in the definition of $\overline{R}_\delta$, namely,

$$\overline{R}_\delta = \{P \epsilon \overline{R} \mid d(P, \overline{\Gamma}_1) \geq \delta\}. \qquad (2.1.54)$$

This gives rise to a modification of Theorem 2.1.2 where the compact subsets in the conclusion are those of $\overline{R} - \overline{\Gamma}_1$.

The proof of this theorem could also call upon results of Agmon, Douglis, and Nirenberg [2], especially on Theorem 12.2, but more care would have been necessary in defining $\overline{R}_\delta$ and φ_m, since the boundary operators need to be smooth on the entire boundary.

The solution u will be seen in what follows to be a maximum possible solution to (2.1.1 - 2.1.3). In a similar fashion a minimum possible solution v can also be obtained.

Briefly, (omitting the details as they are parallel to those for u) define v_0 by

$$\nabla^2 v_0 = 0 \quad \text{in } R, \tag{2.1.55}$$

$$v_0 = 0 \quad \text{on } \bar{\Gamma}_1, \tag{2.1.56}$$

$$\frac{\partial v_0}{\partial n} = -N' \quad \text{on } \Gamma_2. \tag{2.1.57}$$

Define k analogously to (2.1.37) and $\{v_m\}$ by (2.1.40 - 2.1.42). Then

$$v = \lim_{m\to\infty} v_m . \tag{2.1.58}$$

Let Z be any solution of (2.1.1 - 2.1.3). On Γ_2

$$\frac{\partial Z}{\partial n} = \gamma(Z) \tag{2.1.59}$$

$$\leq N' \tag{2.1.60}$$

Letting $w = u_0 - Z$, by (2.1.60), since $\partial u_0/\partial n = N'$,

$$\frac{\partial w}{\partial n} \geq 0 \tag{2.1.61}$$

on Γ_2, so that

$$Z \leq u_0 \quad \text{in } R. \tag{2.1.62}$$

To show that Z is bounded above by all iterates u_m another inequality implied by (2.1.6), (2.1.37), and (2.1.39) is needed, namely, if $-u_{OM} \leq u \leq w \leq u_{OM}$ then

$$\gamma(w) - \gamma(u) - k(u - w) \geq 0. \tag{2.1.63}$$

Let $w = u_1 - Z$. By (2.1.43), on Γ_2

$$\begin{aligned} \frac{\partial u_1}{\partial n} &= \gamma(u_0) + k(u_0 - u_1) \\ &= \gamma(u_0) + k(u_0 - Z) + k(Z - u_1) \\ &= \gamma(u_0) + k(u_0 - Z) - kw. \end{aligned} \tag{2.1.64}$$

Subtract (2.1.59) from (2.1.64) and obtain

$$\frac{\partial w}{\partial n} + kw = \gamma(u_0) - \gamma(Z) - k(Z - u_0)$$
$$\geq 0 \qquad (2.1.65)$$

by (2.1.62) and (2.1.63). Therefore

$$Z \leq u_1. \qquad (2.1.66)$$

Replacing u_0 by u_m and u_1 by u_{m+1} completes the induction. That is, $m \geq 0$ implies

$$Z \leq u_m \qquad (2.1.67)$$

for any solution Z of (2.1.1 - 2.1.3). The result is

$$Z \leq u. \qquad (2.1.68)$$

Carrying out a similar argument with v obtain finally that

$$v \leq Z \leq u, \qquad (2.1.69)$$

for any Z solving (2.1.1 - 2.1.3).

These final results may be summarized as

Theorem 2.1.3. - Under the hypotheses of Theorem 2.1.2, if the solution u is defined as in its proof, namely, by (2.1.51), and the solution v by (2.1.58), then $v \leq u$ and if z is any solution of (2.1.1 - 2.1.3), then

$$v \leq z \leq u.$$

This result is like that obtained in Courant and Hilbert [9], pp. 369-372, for Dirichlet problems for

$$\nabla^2 u = f(P,u) \qquad \text{in } R,$$

and later by Parter [18] under more general conditions on f

and the boundary of R, still however, for Dirichlet problems for the non-linear Poisson equation directly above.

2.2 Uniqueness.

In addition to conditions (2.1.4 - 2.1.9), imposed to guarantee the existence of solutions to problem (2.1.1 - 2.1.3), further restrictions must be placed on γ to guarantee the uniqueness of a solution. For an example of a boundary value problem satisfying all conditions of section 2.1 but possessing infinitely many solutions, consider the two-point boundary-value problem,

$$\frac{d^2u}{dx^2} = 0, \qquad 0 < x < 1, \tag{2.2.1}$$

$$u = 0, \qquad x = 0, \tag{2.2.2}$$

$$\frac{du}{dx} = \gamma(u), \qquad x = 1, \tag{2.2.3}$$

where

$$\gamma(u) = \begin{cases} -1, & u < -1 \\ u, & -1 \leq u \leq 1 \\ 1, & u > 1. \end{cases} \tag{2.2.4}$$

Here one may take $K(M) \equiv 1$, any α, $0 < \alpha < 1$, $K_0(M) \equiv 1$ and $\eta(u) = 1/u$. The solution ζ of (2.1.12), (2.1.13), (2.1.14), where $\overline{\Gamma}_1$ is $\{0\}$ is

$$\zeta(x) = 1 + 2x - \frac{1}{2}x^2, \tag{2.2.5}$$

so that

$$\max_{\overline{R}} \zeta = 2.5, \tag{2.2.6}$$

and so

$$M_1 = 2.5,$$

and the a priori bound (2.1.24) for solutions of (2.2.1), (2.2.2), (2.2.3) reads

$$|\varphi| \leq M_3 = 6.25. \tag{2.2.7}$$

But there are many solutions of this problem, some of which are given by

$$u = ax \tag{2.2.8}$$

for any $|a| \leq 1$. The constant N of (2.1.30) may be taken to be $N = 1$. The function u_0 of (2.1.33), (2.1.34), (2.1.35) is

$$u_0(x) = x, \tag{2.2.9}$$

so that $u_{0M} = 1$, $k = 1$, and therefore for all $m \geq 0$,

$$u_m(x) = x. \tag{2.2.10}$$

Therefore, the limit is

$$u(x) = x, \tag{2.2.11}$$

and similarly

$$v(x) = -x. \tag{2.2.12}$$

So if z is any solution of (2.2.1), (2.2.2.), (2.2.3),

$$-x \leq z(x) \leq x, \qquad 0 \leq x \leq 1. \tag{2.2.13}$$

In order to obtain uniqueness it is further required that γ satisfies

$$\gamma_u(P,u) \leq 0. \tag{2.2.14}$$

Other similar conditions also can be used to deduce uniqueness. For example, see the papers of Ablow and Perry [1], Kalaba [13], and Pohožaev [20].

First, consider the linear problem

$$\nabla^2 w = 0 \qquad \text{in } R, \tag{2.2.15}$$

$$w = 0 \qquad \text{on } \overline{\Gamma}_1, \tag{2.2.16}$$

$$\frac{\partial w}{\partial n} = -c(P)w \qquad \text{on } \Gamma_2, \tag{2.2.17}$$

where $c \geq 0$. That any postulated positive maximum or negative minimum for w must be assumed on Γ_2 follows from (2.2.15) and (2.2.16). If $c > 0$, then either of these postulates contradicts (2.2.17) at the point $P^* \in \Gamma_2$ where the postulated maximum (minimum) occurs. If at P^*, $c = 0$, then consider ψ defined by

$$w = \zeta\psi \tag{2.2.18}$$

where ζ is defined by (2.1.12 - 2.1.14) so that as before, (sec. 2.1) if ψ has a positive maximum (negative minimum) it occurs on Γ_2. But by (2.2.18), (2.1.14), (2.2.17),

$$\frac{\partial\psi}{\partial n} = -\frac{\psi}{\zeta} \qquad \text{at } P^* \tag{2.2.19}$$

$$\frac{\partial\psi}{\partial n} = -\frac{c+1}{\zeta}\psi \qquad P \neq P^*. \tag{2.2.20}$$

For either postulate, $\partial\psi/\partial n$ will have the wrong sign.

Therefore, the following lemma is obtained:

<u>Lemma 2.2.1.</u> - The linear problem (2.2.15 - 2.2.17) with $c(P) \geq 0$, has only the one solution $w \equiv 0$.

Next, suppose the non-linear problem (2.1.1 - 2.1.3) has two solutions u and v and define

$$w = u - v, \tag{2.2.21}$$

which satisfies

$$\nabla^2 w = 0 \qquad \text{in} \quad R, \tag{2.2.22}$$

$$w(P) = 0 \qquad \text{on} \quad \overline{\Gamma}_1, \tag{2.2.23}$$

$$\frac{\partial w}{\partial n} = \gamma(u) - \gamma(v) \qquad \text{on} \quad \Gamma_2. \tag{2.2.24}$$

By the mean-value theorem, (2.2.24) becomes

$$\frac{\partial w}{\partial n} = -c(P)w \qquad \text{on} \quad \Gamma_2, \tag{2.2.25}$$

where

$$c(P) = -\gamma_u(P,\varphi) \geq 0, \tag{2.2.26}$$

and φ is some number between $u(P)$ and $v(P)$. By Lemma 2.2.1, problem (2.2.22), (2.2.23), (2.2.25) has only the solution $w \equiv 0$, i.e.

$$u \equiv v. \tag{2.2.27}$$

It is possible to get uniqueness even if γ is not differentiable, by replacing (2.2.14) by

$$\text{if} \quad u < v \quad \text{then} \quad \gamma(v) \leq \gamma(u). \tag{2.2.28}$$

With this and (2.1.6), if $u(P) \neq v(P)$, the boundary condition (2.2.25) still holds with

$$c(P) = \theta K_0(M) \geq 0 \tag{2.2.29}$$

for some γ such that $0 \leq \theta \leq 1$.

These results can be stated as

Theorem 2.2.1. - Under the conditions of Theorem 2.1.2, if $\gamma(P,u)$ is non-increasing as a function of u, in particular if $\gamma_u \leq 0$, then there is only one solution of (2.1.1 - 2.1.3).

The proof is similar in spirit to that of Courant and Hilbert [9], pp. 320 - 324.

2.3 Newton's Method and the Maximum Operation.

By strengthening the hypotheses on γ, results may be obtained which parallel those of Kalaba [13] where solutions of Dirichlet problems for quasi-linear differential equations, for example

$$\nabla^2 u = f(P,u) \qquad \text{in } R, \tag{2.3.1}$$

were obtained by Newton's method as a maximum of solutions of a class of linear equations.

For problem (2.1.1 - 2.1.3), the conditions of Theorem (2.1.2) are to be supplemented by $\gamma_u \leq 0$ and

$$\gamma(u) = \max_v [\gamma(v) + (u - v)\gamma'(v)], \tag{2.3.2}$$

where the prime indicates the partial derivative with respect to u, i.e., γ is convex in u. That γ satisfies the usual inequality for convexity can be seen by adding the following two inequalities resulting from (2.3.2): Let $w = (1/2)(u + v)$. Then

$$\gamma(u) \geq \gamma\left[\frac{1}{2}(u + v)\right] + (u - w)\gamma'(w),$$

and

$$\gamma(v) \geq \gamma\left[\frac{1}{2}(u + v)\right] + (v - w)\gamma'(w).$$

A sufficient conditions for convexity is that γ be twice continuously differentiable in u and $\gamma_{uu} \geq 0$.

The property of u being a maximum is altered to that of being a minimum if γ is concave, i.e., if

$$\gamma(u) = \min_{v}\left[\gamma(v) + (u - v)\gamma'(v)\right]. \qquad (2.3.3)$$

For the concave case, however, it is not necessary to separately require (2.1.4) since it is then a consequence of (2.1.14) and (2.3.3).

Newton's method applied to the problem of (2.1.1 - 2.1.3) is carried out as follows. Let u_0 be any initial approximation, good or poor, to u, the unique solution of (2.1.1 - 2.1.3) guaranteed by sections 2.1 and 2.2. Consider the sequence of approximations $\{u_m\}$ defined by the linear problems, $m \geq 1$

$$\nabla^2 u_m = 0 \quad \text{in } R, \qquad (2.3.4)$$

$$u_m = 0 \quad \text{on } \bar{\Gamma}_1, \qquad (2.3.5)$$

$$\frac{\partial u_m}{\partial n} = \gamma(u_{m-1}) + \gamma'(u_{m-1})(u_m - u_{m-1}) \quad \text{on } \Gamma_2. \qquad (2.3.6)$$

To compare u with u_m let $w = u - u_m$ and note on Γ_2 that

$$\frac{\partial w}{\partial n} = \gamma(u) - \left[\gamma(u_{m-1}) + \gamma'(u_{m-1})(u_m - u_{m-1})\right]$$

$$\geq \gamma(u^{m-1}) + \gamma'(u_{m-1})(u - u_{m-1}) - \left[\gamma(u_{m-1}) + \gamma'(u_{m-1})(u_m - u_{m-1})\right]$$

by (2.3.2). That is

$$\nabla^2 w = 0 \quad \text{in } R, \qquad (2.3.7)$$

$$w = 0 \quad \text{on } \bar{\Gamma}_1, \qquad (2.3.8)$$

$$\frac{\partial w}{\partial n} - \gamma'(u_{m-1})w \geq 0 \qquad \text{on } \Gamma_2. \qquad (2.3.9)$$

Therefore, because of (2.2.14), $w \geq 0$, i.e., for all $m \geq 1$

$$u_m \leq u \qquad \text{in } \bar{R} \qquad (2.3.10)$$

independently of the choice of u_0.

To compare u_{m+1} with u_m, $m \geq 1$ note that on Γ_2

$$\begin{aligned} \frac{\partial u_m}{\partial n} &= \gamma(u_{m-1}) + (u_m - u_{m-1})\gamma'(u_{m-1}) \\ &\leq \gamma(u_m) \qquad (2.3.11) \end{aligned}$$

by (2.3.2) and

$$\frac{\partial u_{m+1}}{\partial n} = \gamma(u_m) + (u_{m+1} - u_m)\gamma'(u_m). \qquad (2.3.12)$$

Subtracting (2.3.11) from (2.3.12) obtain for $w = u_{m+1} - u_m$ that

$$\frac{\partial w}{\partial n} - \gamma'(u_m)w \geq 0 \qquad \text{on } \Gamma_2$$

whereas in R, (2.3.7) holds and on Γ_1, (2.3.8) holds, so that again $w \geq 0$. That is

$$u_{m+1} \geq u_m \qquad \text{in } \bar{R}. \qquad (2.3.13)$$

With this and (2.3.10), u_m is a uniformly bounded, monotonic sequence of harmonic functions and, as in section 2.1, its limit solves (2.1.1 - 2.1.3). By Theorem 2.2.1 this is the same function defined by the limit of the sequence defined in section 2.1.

Since this problem is equivalent to (2.3.4 - 2.3.6) with $u_0 = u$ (so that $u_1 = u$), (2.3.10) implies that

$$u = \max_{u_0} \left\{ u_1 \,\middle|\, (2.3.4 - 2.3.6) \text{ with } m = 1 \right\}. \qquad (2.3.14)$$

If γ were instead concave, a similar result with "max" replaced by "min" in (2.3.14) would hold for u. Here, of course, $\{u_m\}$ is a non-increasing sequence.

Summarizing,

Theorem 2.3.1. - Under the conditions of Theorem 2.2.1, namely smoothness and

$$\gamma_u \leq 0,$$

if γ is convex, then the solution of (2.1.1 - 2.1.3) may be obtained by Newton's method and the maximum operation (2.3.14) is valid.

CHAPTER III

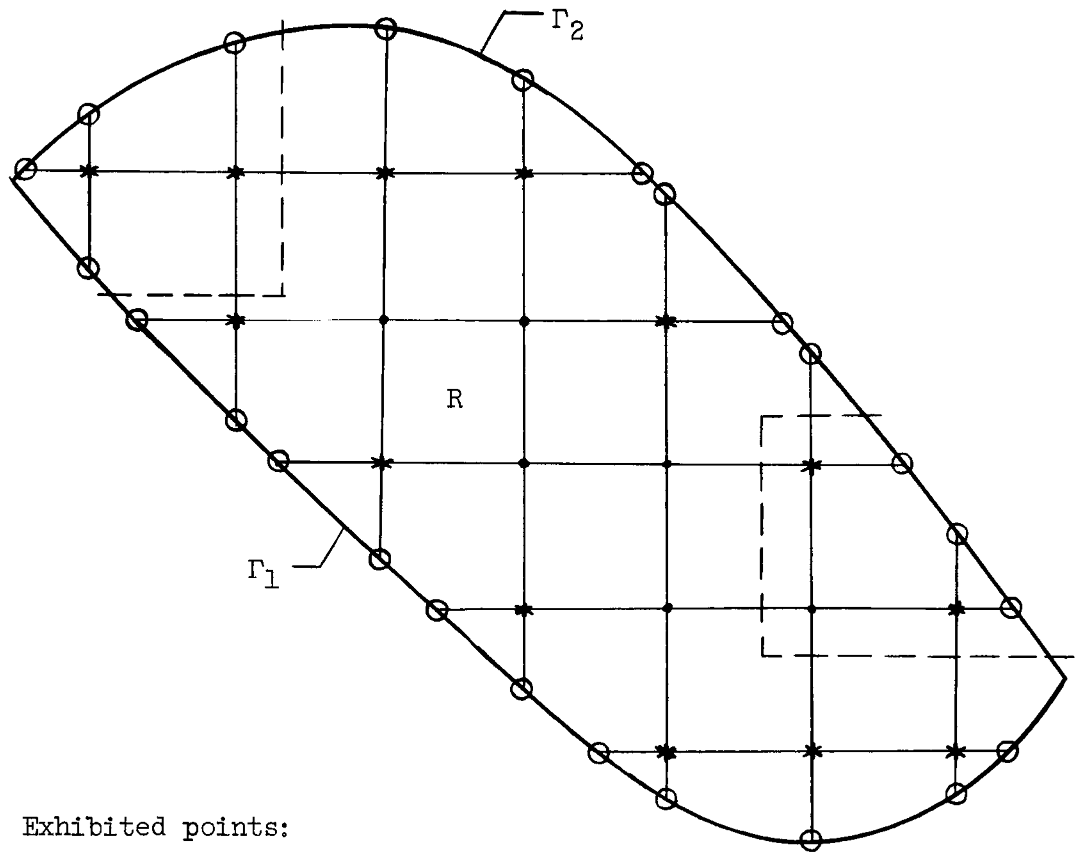
APPROXIMATE SOLUTIONS

3.1 Derivation of the Difference Equations.

In this section, difference approximations are made which lead to systems of algebraic equations whose solution is shown in Chapter IV, to differ from the solution of the original boundary value problem by an amount tending to zero with h at the rate $h^{1/3}$. Higher order approximations are discussed in Chapter V.

In what follows it will be convenient to consult figure 1 where $\overline{R}_h$ is displayed. Here Γ_h has its points circled, Γ_h^* has its points with asterisks superimposed, and R_h has its points merely emphasized. Dashed lines on this figure indicate sections which are singled out for further treatment in figures 2 and 3.

To derive the difference equations, a square net of mesh width h is superimposed on the bounded region R and a system of equations is obtained, one equation at each node of the net. These equations differ, depending upon the relative position in the net and three types must be distinguished according



Exhibited points:

- ○ Γ_h
- ✱ Γ_h^*
- • R_h

Figure 1. - $\overline{R}_h$.

as the node lies in R, near $\bar{\Gamma}_1$ or on Γ_2.

All intersections of $\bar{\Gamma}_1 \cup \Gamma_2$ with grid lines will be considered along with intersections, interior to R, of horizontal grid lines with vertical grid lines, as nodes of the net.

The totality of all nodes is denoted by $\bar{R}_h$. Each point of $\bar{R}_h$ will have its corresponding difference equations (degenerating on $\bar{\Gamma}_1$ to a mere statement of the functional value at that point). The notation used here is like that of various papers of Bramble and Hubbard (see, e.g., [7]). Let $(\Gamma \equiv \bar{\Gamma}_1 \cup \Gamma_2)$

$$\Gamma_h = \Gamma \cap \bar{R}_h \tag{3.1.1}$$

$$\Gamma_h^* = \left\{ P \,\middle|\, \begin{matrix} P\epsilon\bar{R}_h - \Gamma_h \text{ but } P \text{ has} \\ \text{a neighbor } Q\epsilon\Gamma_h \end{matrix} \right\} \tag{3.1.2}$$

$$R_h = \bar{R}_h - (\Gamma_h \cup \Gamma_h^*). \tag{3.1.3}$$

The sets Γ_h and Γ_h^* are further decomposed into

$$\Gamma_{h1} = \Gamma_h \cap \bar{\Gamma}_1, \tag{3.1.4}$$

and

$$\Gamma_{h2} = \Gamma_h \cap \Gamma_2. \tag{3.1.5}$$

Γ_{h1}^* and Γ_{h2}^* are defined analogously except here the representation

$$\Gamma_h^* = \Gamma_{h1}^* \cup \Gamma_{h2}^* \tag{3.1.6}$$

would not necessarily be a disjoint union unless it is required that those points P which have neighbors in both

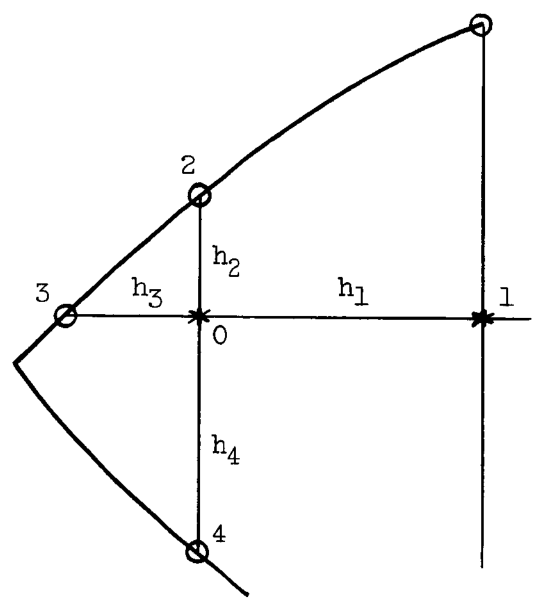


Figure 2. - Points used in interior approximations.

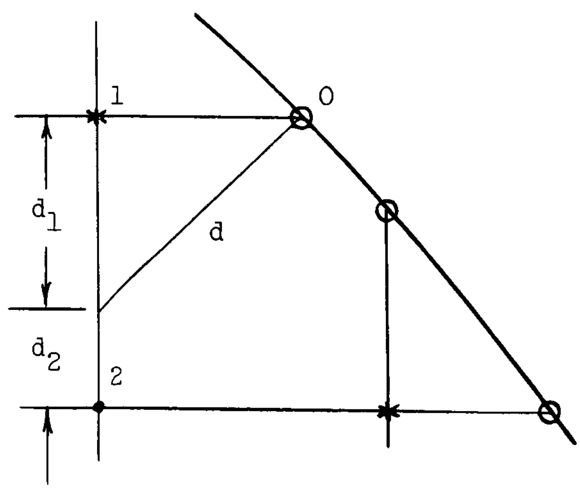


Figure 3. - Points used in boundary approximations.

Γ_{h1} and Γ_{h2} be placed in Γ^*_{h1}, thus making

$$\Gamma^*_{h1} \cap \Gamma^*_{h2} = \emptyset. \qquad (3.1.7)$$

Figure 2, with $h_i \leq h$, exhibits 5 points singled out by dashed lines in figure 1, and labelled 0, 1, 2, 3, and 4 in this new figure. Here, for example, 0 corresponds to the point $P_0 = (x_0, y_0)$, 1 to the point $P_1 = (x_0 + h_1, y_0)$ etc. Let $P_0 \in R_h \cup \Gamma^*_h$. Then either through Taylor's series (see Forsythe and Wasow [10] pp. 179 ff.) or through an integration method (Varga [26] pp. 181 ff.) one obtains at the point P_0 the equation

$$\left(\frac{2}{h_1h_3} + \frac{2}{h_2h_4}\right) u_0 - \frac{2u_1}{h_1(h_1 + h_3)} - \frac{2u_2}{h_2(h_2 + h_4)} - \frac{2u_3}{h_3(h_1 + h_3)} - \frac{2u_4}{h_4(h_2 + h_4)} = 0, \qquad (3.1.8)$$

corresponding to (2.1.1). For $P_0 \in \Gamma^*_{h1}$, one or more values (say u_4) are known to be zero and (3.1.8) reads

$$\left(\frac{2}{h_1h_3} + \frac{2}{h_2h_4}\right) u_0 - \frac{2u_1}{h_1(h_1 + h_3)} - \frac{2u_2}{h_2(h_2 + h_4)} - \frac{2u_3}{h_3(h_3 + h_1)} = 0. \qquad (3.1.9)$$

For $P_0 \in \Gamma_{h1}$ the equation is merely

$$u_0 = 0. \qquad (3.1.10)$$

For $P_0 \in R_h$ the equation is the usual

$$\frac{(4u_0 - u_1 - u_2 - u_3 - u_4)}{h^2} = 0 \qquad (3.1.11)$$

leaving only points in Γ_{h2} to consider.

Following Greenspan [11], either the boundary normal extended inwardly from P_0 meets a grid line obliquely at another node labelled P_1, or between two nodes. In the first case, the outwardly directed normal derivative is replaced simply by

$$\frac{u_0 - u_1}{d},$$

where $d = O(h)$ as $h \to 0$ is the distance between P_0 and P_1. In the second case, consider figure 3. The normal derivative at P_0 is replaced by

$$\frac{u_0}{d} - \frac{d_1}{d(d_1 + d_2)} u_2 - \frac{d_2}{d(d_1 + d_2)} u_1 .$$

The equations at these points are thus in one of two forms,

$$\frac{u_0 - u_1}{d} = \gamma(P_0, u_0) \tag{3.1.12}$$

or

$$\frac{u_0}{d} - \frac{d_1 u_2}{d(d_1 + d_2)} - \frac{d_2 u_1}{d(d_1 + d_2)} = \gamma(P_0, u_0) . \tag{3.1.13}$$

3.2. Solution of the Difference Equations.

The system of equations obtained by combining equations of type (3.1.8 - 3.1.13) written at all points of $\bar{R}_h$ where they apply may be considered in the form

$$A\underline{u} = \underline{d}(\underline{u}) \tag{3.2.1}$$

where the matrix A is such that $\det A \neq 0$ and in fact, A is an M-matrix so that

$$A^{-1} \geq 0. \qquad (3.2.2)$$

This follows from a lemma due to Varga (see Roudebush [22] lemma 2, p. 11) since $a_{ij} \leq 0$ for $i \neq j$, $a_{ii} > 0$, and A is reducible but has normal reduced form (Varga [26], p. 46) with irreducibly diagonally dominant submatrices along its diagonal.

Let N be defined by (2.1.30) and consider $\underline{u}_0$ defined by

$$A\underline{u}_0 = \underline{b} \qquad (3.2.3)$$

where b_i is the same as the i^{th} component of $\underline{d}(\underline{u})$ in (3.2.1) unless $P_i \epsilon \Gamma_{h2}$. At these points, (3.1.12) (analogously (3.1.13)) is replaced by

$$\frac{u_{00} - u_{01}}{d} = N + 1. \qquad (3.2.4)$$

Then $\underline{u}_0 \geq 0$. Let

$$k = K_0(\|\underline{u}_0\|), \qquad (3.2.5)$$

where $\|\underline{u}_0\|$ denotes

$$\max_i |u_{0,i}|.$$

Then

$$\gamma(u) - \gamma(w) - k(u - w) \geq 0 \qquad (3.2.6)$$

for

$$- \|\underline{u}_0\| \leq u \leq w \leq \|\underline{u}_0\|. \qquad (3.2.6a)$$

Define $\underline{u}_{m+1}$, $m \geq 0$ by

$$A\underline{u}_{m+1} = \underline{b}(\underline{u}_m) \qquad (3.2.7)$$

where the i^{th} component of $\underline{b}(\underline{u}_m)$ is the same as that of $\underline{d}(\underline{u})$ of (3.2.1) unless $P_i \epsilon \Gamma_{h2}$. Here

$$\underline{b}(\underline{u}_m)_i = \underline{d}(\underline{u}_m)_i + k(u_{m,i} - u_{m+1,i})$$
$$= \gamma(u_{mi}) + k(u_{m,i} - u_{m+1,i}). \qquad (3.2.8)$$

This system of equations can also be written as

$$\tilde{A}\underline{u}_{m+1} = \underline{d}(\underline{u}_m) + \kappa(\underline{u}_m) \qquad (3.2.9)$$

where

$$\kappa_i = 0, P_i \notin \Gamma_{h2}$$
$$\kappa_i = ku_{mi}, P_i \in \Gamma_{h2}, \qquad (3.2.10)$$

and $\tilde{A}$ differs from A only in the corresponding rows by having k added to the (already positive) diagonal element in those rows, so that

$$0 \leq \tilde{A}^{-1} \leq A^{-1}. \qquad (3.2.11)$$

Let $P_0 \in \Gamma_{h2}$. Then for $m = 0$, (3.2.9) reads

$$\frac{u_{1,0} - u_{1,1}}{d} + ku_{1,0} = \gamma(u_{0,0}) + ku_{0,0}$$
$$\leq N + ku_{0,0} + 1$$
$$\stackrel{\rightarrow}{=} \frac{u_{0,0} - u_{0,1}}{d} + ku_{0,0}. \qquad (3.2.12)$$

A like inequality holds if the equation at P_0 were of the form (3.1.13). At other points of $\overline{R}_h$,

$$\sum a_{ij} u_{m+1,j} = \sum a_{ij} u_{m,j} = 0. \qquad (3.2.13)$$

If (3.2.12) and (3.2.13) are combined, one obtains

$$\tilde{A}\underline{u}_1 \leq \tilde{A}\underline{u}_0, \tag{3.2.14}$$

from which follows

$$\underline{u}_1 \leq \underline{u}_0. \tag{3.2.15}$$

If (3.2.9) is again examined in the light of (3.2.15), one obtains

$$\begin{aligned} \frac{u_{1,0} - u_{1,1}}{d} &= \gamma(u_{0,0}) + k(u_{0,0} - u_{1,0}) \\ &\geq \gamma(u_{0,0}) \\ &\geq -N - 1 \\ &= -\frac{u_{0,0} - u_{0,1}}{d}, \end{aligned} \tag{3.2.16}$$

by (2.1.30). Therefore,

$$A\underline{u}_1 \geq -A\underline{u}_0, \tag{3.2.17}$$

so that

$$-\underline{u}_0 \leq \underline{u}_1. \tag{3.2.18}$$

If (3.2.15) and (3.2.18) is applied to (3.2.6) one obtains

$$\tilde{A}(\underline{u}_2 - \underline{u}_1) \leq 0 \tag{3.2.19}$$

or

$$\underline{u}_2 \leq \underline{u}_1. \tag{3.2.20}$$

This and (2.1.30) implies (as (3.2.17) followed) that

$$A\underline{u}_2 \geq -A\underline{u}_0 \tag{3.2.21}$$

or

$$-\underline{u}_0 \leq \underline{u}_2. \tag{3.2.22}$$

Repeating the steps leading from (3.2.15) and (3.2.18) to

(3.2.20) and (3.2.22) with the subscripts 1 and 2 replaced by m and $m + 1$, $m \geq 1$, completes the induction, i.e.

$$-\underline{u}_0 \leq \underline{u}_{m+1} \leq \underline{u}_m \leq \cdots \leq \underline{u}_2 \leq \underline{u}_1 \leq \underline{u}_0. \qquad (3.2.23)$$

Therefore, the components of $\{\underline{u}_m\}$ form a bounded monotonic sequence of real numbers and so have limits. Since only a finite number of arithmetic operations are involved in each equation of (3.2.7), the limit vector

$$\underline{u} = \lim_{m\to\infty} \underline{u}_m \qquad (3.2.24)$$

satisfies (3.2.1) by virtue of (3.2.8).

Similarly defining $\underline{v}_0$ by

$$A\underline{v}_0 = \tilde{\underline{b}} \qquad (3.2.25)$$

where instead of (3.2.4)

$$\frac{v_{00} - v_{0,1}}{d} = -N - 1 \qquad (3.2.26)$$

is satisfied, and $\underline{v}_m$ by (3.2.7) for $m \geq 1$, obtain a monotonically increasing sequence whose limit

$$\underline{v} = \lim_{m\to\infty} \underline{v}_m \qquad (3.2.27)$$

also satisfies (3.2.1).

It is now shown that any solution of (3.2.1) is bounded from above by $\underline{u}$ and from below by $\underline{v}$. To this end, let $\underline{z}$ be any solution of (3.2.1). Again comparisons of difference expressions are carried out only on Γ_{h2}, the only place they might differ. Subtracting the latter of

$$\frac{z_0 - z_1}{d} = \gamma(z_0) \tag{3.2.28}$$

$$\leq N + 1 \tag{3.2.29}$$

from (3.2.4) obtain for $\underline{w} = \underline{u}_0 - \underline{z}$

$$\frac{w_0 - w_1}{d} \geq 0. \tag{3.2.30}$$

Therefore

$$\underline{z} \leq \underline{u}_0. \tag{3.2.31}$$

One more inequality beside (3.2.6) is implied by (3.2.5), (3.2.6a) and (2.1.6), namely, if $-\|u_0\| \leq u \leq w \leq \|u_0\|$, then

$$\gamma(w) - \gamma(u) - k(u - w) \geq 0. \tag{3.2.32}$$

Now let $\underline{w} = \underline{u}_1 - \underline{z}$. By (3.2.8)

$$\frac{u_{10} - u_{11}}{d} = \gamma(u_{00}) + k(u_{00} - u_{10})$$

$$= \gamma(u_{00}) + k(u_{00} - z_0) + k(z_0 - u_{10})$$

$$= \gamma(u_{00}) + k(u_{00} - z_0) - kw_0. \tag{3.2.33}$$

Subtracting (3.2.28) from this one obtains

$$\frac{w_0 - w_1}{d} + kw_0 = \gamma(u_{00}) - \gamma(z_0) - k(z_0 - u_{00})$$

$$\geq 0 \tag{3.2.34}$$

by (3.2.31) and (3.2.32). But this implies

$$\tilde{A}\underline{w} \geq 0, \tag{3.2.35}$$

i.e.

$$\underline{z} \leq \underline{u}_1. \tag{3.2.36}$$

Induction is completed in the same way as before for (3.2.23) and the conclusion is that

$$\underline{z} \leq \underline{u}. \tag{3.2.37}$$

Similarly,

$$\underline{v} \leq \underline{z}. \tag{3.2.38}$$

That is, any solution of (3.2.1) is in the "interval" [v,u].

Therefore, the following theorem is established:

Theorem 3.2.1. - Under the conditions of Theorem 2.1.3, the system of non-linear difference equations, obtained by the classical five-point scheme at interior grid points and the elementary approximations (3.1.12) or (3.1.13) to the boundary equation, possesses solutions. Moreover, there exists a maximal and a minimal solution, u, v, respectively, such that any solution $\underline{z}$ of (3.2.1) satisfies

$$\underline{v} \leq \underline{z} \leq \underline{u}.$$

3.3 Uniqueness of the Solution of the Difference Equations.

Add the additional hypothesis that γ is non-increasing in the dependent variable, i.e., if

$$w < z \qquad \text{then} \qquad \gamma(z) \leq \gamma(w), \tag{3.3.1}$$

(2.2.14) being a special case. Suppose that $\underline{u}$ and $\underline{v}$ are two different solutions of (3.2.1) defined by (3.2.24) and (3.2.27) respectively. Define w by

$$\underline{w} = \underline{u} - \underline{v}. \tag{3.3.2}$$

Then at all points of $\overline{R}_h$

$$\underline{w} \geq 0 \tag{3.3.3}$$

where, for at least one point, $w_i > 0$ by assumption. In addition if this were not true for some point of Γ_{h2} a contradiction is immediate since in that case,

$$\underline{A}\underline{w} = 0. \tag{3.3.4}$$

Thus it may be assumed that strict inequality holds for at least one point on Γ_{h2}. For such points P_0

$$\begin{aligned} \frac{w_0 - w_1}{d} &= \gamma(u_0) - \gamma(v_0) \\ &\leq 0 \end{aligned} \tag{3.3.5}$$

by (3.3.1). So that

$$\underline{A}\underline{w} \leq 0 \tag{3.3.6}$$

This with (3.3.3) implies $\underline{w} = 0$. That is,

$$\underline{u} = \underline{v}$$

and since any solution $\underline{z}$ of (3.2.1) must lie between them, uniqueness is proven.

Thus there is the result analogous to that of the continuous case:

Theorem 3.3.1. - If the conditions for the existence of solutions of (3.2.1) are supplemented by

$$\gamma_u \leq 0,$$

then there is only one solution of the non-linear difference equations.

3.4 The Practical Solution of the Difference Equations.

Although section 3.2 covers existence by a proof which is constructive, if h is very small as compared with the diameter of R, the algorithm calls for the repeated direct solution of large systems of linear equations. For one-dimensional problems and, if h is not too small, for two-dimensional problems as well, the method is practical for use on large computers to obtain the solution of (3.2.1).

Another possibility also requiring repeated direct solutions of linear systems is Newton's method set up for (3.2.1) analogously to the way it was done in section 2.3 for the continuous problem, again under the assumption that $\gamma_u \leq 0$. The advantage of this method is its convergence properties, namely

$$\|\underline{u} - \underline{u}_{m+1}\| \leq K\|\underline{u} - \underline{u}_m\|^2 \tag{3.4.1}$$

whereas the first method's convergence is only linear. A disadvantage of this latter method is the fact that the matrix of coefficients of the linear system of equations is different for each iteration. Therefore recourse cannot be made to "factoring methods" (see Varga [25]) to reduce the number of arithmetic operations per iteration as can be where the matrix is independent of m.

A third possibility is suggested by the "simplified Newton's method" of Collatz [8]. Here (examining the continuous problem, the discrete case being entirely analogous) the factor $\gamma'(u_{m-1})$ appearing in (2.3.6) is replaced by $\gamma'(u_0)$ and not changed from iteration to iteration as it is in the ordinary Newton's method. This third method appears "intermediate" to the other two in that the advantages of the first in the invariance of the iteration matrix are retained and, while the convergence properties are not those of the ordinary Newton's method, they are surely better than those of the first method if $\underline{u}_0$ is already a reasonable approximation to $\underline{u}$.

Requiring solutions with a fair amount of detail to two-dimensional problems and with a moderate amount of detail to three-dimensional problems is becoming increasingly common in technological applications. In view of this the previously mentioned methods of obtaining such solutions for (3.2.1) no longer become practical. Instead it is necessary to make use of some sort of "relaxation technique" for these large order non-linear systems.

There are some results available on the convergence of various relaxation methods for systems of non-linear algebraic (or transcendental, depending on γ) equations. See, for example, Lieberstein [15], and Bers [5] (where here the systems arise from a difference method for Dirichlet's problem for Poisson's equation with non-linear source term) and Schechter [23].

It is shown here with the aid of Schechter's results, how (3.2.1) may be solved by a (non-linear) "Gauss-Seidel" type relaxation and that even "over-relaxation" is possible. In fact, the numerical results of section 7.1 were obtained by using the one-line successive over-relaxation iterative method where only one equation per mesh line, corresponding to the mesh point on the boundary Γ_2, was non-linear. Thus the tri-diagonal matrix, which couples unknowns along the line, is partially factored into a product of upper and lower triangular matrices. The factoring per iteration can then be completed by solving exactly this single non-linear equation in a single variable.

To apply the results of Schechter's paper directly, it becomes necessary to take advantage of the smoothness of Γ_2 to transform R into another region R^t whose corresponding boundary Γ_2^t is a subset of the x_1-axis, $x_2 = 0$. Upon making the transformation, the discretization is carried out by means of a square grid whose elements are parallel to the axes, so that the difference equations on Γ_{2h} are all of the form (3.1.12). Having done this, it is possible, after rejecting the identity difference equations on Γ_{1h}, to make A a symmetric matrix, which is the point of the above transformation and a requirement for the application of Schechter's work.

It will be discussed in Chapter V, section 5.4 how symmetry of A (truncated) may be obtained without insisting upon such a transformation.

However symmetry is obtained, i.e., whether by the above transformation, or by the use of other boundary approximations, let

$$G(u) = \frac{1}{2}\, \underline{u}^T(A)\underline{u} - \frac{1}{h} \sum_{P_i \in \Gamma_{2h}} \int_0^{u_i} \gamma(P_i, v)\,dv\,, \qquad (3.4.2)$$

where here it has been assumed that Γ_2 is a mesh line and has been divided exactly into subintervals of length h. If the points of $\overline{\Gamma}_1 \cap \overline{\Gamma}_2$ turn out not to be nodes of the grid, the above summation is modified accordingly (being a trapezoidal approximation of a boundary integral).

A is positive definite. If B is defined by

$$b_{ij} = \frac{\partial^2 G}{\partial u_i\, \partial u_j} = a_{ij} - \frac{\delta_{ijk}}{h}\, \gamma_v(P_k, v)\Big]_{v=u_k}, \qquad (3.4.3)$$

since $\gamma_u < 0$, B is positive definite along with A. Define

$$\underline{r}(\underline{u}) = \text{grad } G(\underline{u}). \qquad (3.4.4)$$

Then (3.2.1) is equivalent to

$$\underline{r}(\underline{u}) = \underline{0}\,. \qquad (3.4.5)$$

It can be seen that for $\underline{u} \in \mathcal{R}_n$, the real n-dimensional vector space,

$$G(u) \geq -\frac{1}{h^2} M_4 N \times l(\Gamma_2) \qquad (3.4.6)$$

and $\mathcal{R}_n$ is a "solvent set". That is, [23], having chosen arbitrarily $\underline{u} \in \mathcal{R}_n$, u_i may be altered, determining a new $\underline{u}' \in \mathcal{R}_n$ such that $r_i(\underline{u}') = 0$. Here $\lambda(A)$ denotes eigenvalue of A and $l(\Gamma_2)$, the length of Γ_2. M_4 was defined prior to (2.1.25) and N by (2.1.30).

The iterative procedure, analogous to the Gauss-Seidel process for systems of linear equations, proven convergent in Schechter [23], is given as follows:

Let $\underline{u}^{(m)} \in \mathcal{R}_n$. Define $\underline{u}^{(m+1)}$ by

$$r_i\left(u_1^{(m)} \ldots u_{i-1}^{(m)}, u_i^{(m+1)}, u_{i+1}^{(m)} \ldots u_n^{(m)}\right) = 0$$

$$u_k^{(m+1)} = u_k^{(m)}, \; k \neq i, \tag{3.4.7}$$

That is, the i^{th} equation of (3.4.6) is regarded as a function of the i^{th} variable and is solved for u_i with guessed values (previous iterate) for all other $u_j (j \neq i)$ in that equation. Schechter actually proves that any "free steering" method of changing i converges, but as a special case, the non-linear Gauss-Seidel method, consisting of taking each equation of (3.4.6) in turn, does converge.

Schechter also proves a theorem on convergence of a type of over-relaxation, but the satisfaction of the conditions of his theorem depends very strongly upon the γ at hand and involves some greater restrictions.

The results of this section may be summarized in,

Theorem 3.4.1. - If the matrix A of coefficients of the system of finite difference equations (3.2.1) is symmetric and if γ_u is non-positive and satisfies the conditions of the existence theorem 2.1.2, then the solution of the non-linear system of equations may be determined by a non-linear Gauss-Seidel iterative method.

CHAPTER IV

CONVERGENCE OF SOLUTIONS OF THE DISCRETE PROBLEMS TO THE CONTINUOUS

That the solutions of the boundary value problems (2.1.1 - 2.1.3) may be approximated arbitrarily closely by the solutions of the non-linear system of equations (3.2.1) is proven first under the restriction $\gamma_u \leq 0$. It is then shown that the elements u_m of the sequence generated by (2.1.33 - 2.1.35), (2.1.40 - 2.1.42) can be approximated as closely as desired by the corresponding elements $\underline{u}_m$ of the sequence generated by (3.2.3), (3.2.7). Finally convergence of $\underline{u}$ of (3.2.24) to u of (2.1.51) as $h \to 0$ is shown under assumption of uniqueness of the solution.

Actually, to show convergence as h goes to zero, the approximating system is modified by rejecting equations written at points in a circular neighborhood of the corners where the "Dirichlet boundary" and the "Neumann boundary" intersect. The radius, k, is taken larger than h but goes to zero with h. This approach is similar to the one used by Roudebush [22] and Rosenbloom [21] where equations were rejected near all of a non-smooth boundary (assumed merely to have strong barrier function) to show (see [22], p. 115) that the discrete approximation approaches the solution, assumed Hölder continuous with exponent α, of a Dirichlet problem for a uniformly elliptic differential equation, at the rate $h^{\alpha^2/2(1+\alpha)}$ which is $h^{1/4}$ if u satisfies a Lipshitz condition.

This modification imposes no hardship on the results of the other chapters.

4.1 Convergence When $\gamma_u \leq 0$.

It is now assumed that $\gamma_u \leq 0$, that all the smoothness hypotheses (2.14 - 2.1.9) on γ are satisfied and that Γ_1 and Γ_2 are both smooth, the condition on Γ_1 being later relaxed to get slightly weaker results.

We choose h and k so that

$$0 < h < k, \tag{4.1.1}$$

both tending to zero in what follows. A modified grid is now defined where some of the grid points near the corners $\overline{\Gamma}_1 \cap \overline{\Gamma}_2$ are dropped from consideration, by setting

$$\left.\begin{aligned} R_k^0 &= \left\{P \in \overline{R}_h \,\middle|\, d(P,\overline{\Gamma}_1 \cap \overline{\Gamma}_2) \geq k\right\} \\ \Gamma_k^0 &= \left\{P \in \overline{R}_h - R_k^0 \,\middle|\, \begin{array}{l}\text{there is a } Q \in R_k^0 \\ \text{such that } |Q - P| \leq h\end{array}\right\} \\ \overline{R}_{h,k} &= R_k^0 \cup \Gamma_k^0. \end{aligned}\right\} \tag{4.1.2}$$

For these definitions it is helpful to examine figure 4 which is an illustration relative to figure 1 of $\overline{R}_{h,k}$ where k has been taken equal to 2h as an example. The sets $R_{h,k}$, $\Gamma_{h,k}$, $\Gamma^*_{h,k}$, $\Gamma_{h,k,1}$, etc. have the same relation to $\overline{R}_{h,k}$ that R_h, Γ_h, Γ^*_h, $\Gamma_{h,1}$, etc. respectively, have to $\overline{R}_h$, adding the points of Γ^0_k to the set $\Gamma_{h,k,1}$.

A new finite difference boundary problem is defined, in $\overline{R}_{h,k}$ by setting

$$\underline{U}_k(P) = 0 \qquad \text{if} \quad P \in \Gamma^0_k \tag{4.1.3}$$

retaining the original equations at all other points of $\overline{R}_{h,k}$. Its solution $\underline{U}_k$ is guaranteed unique by Theorem 3.3.1.

The difference function $\underline{Z}$ is now formed at points P_i of $\overline{R}_{h,k}$:

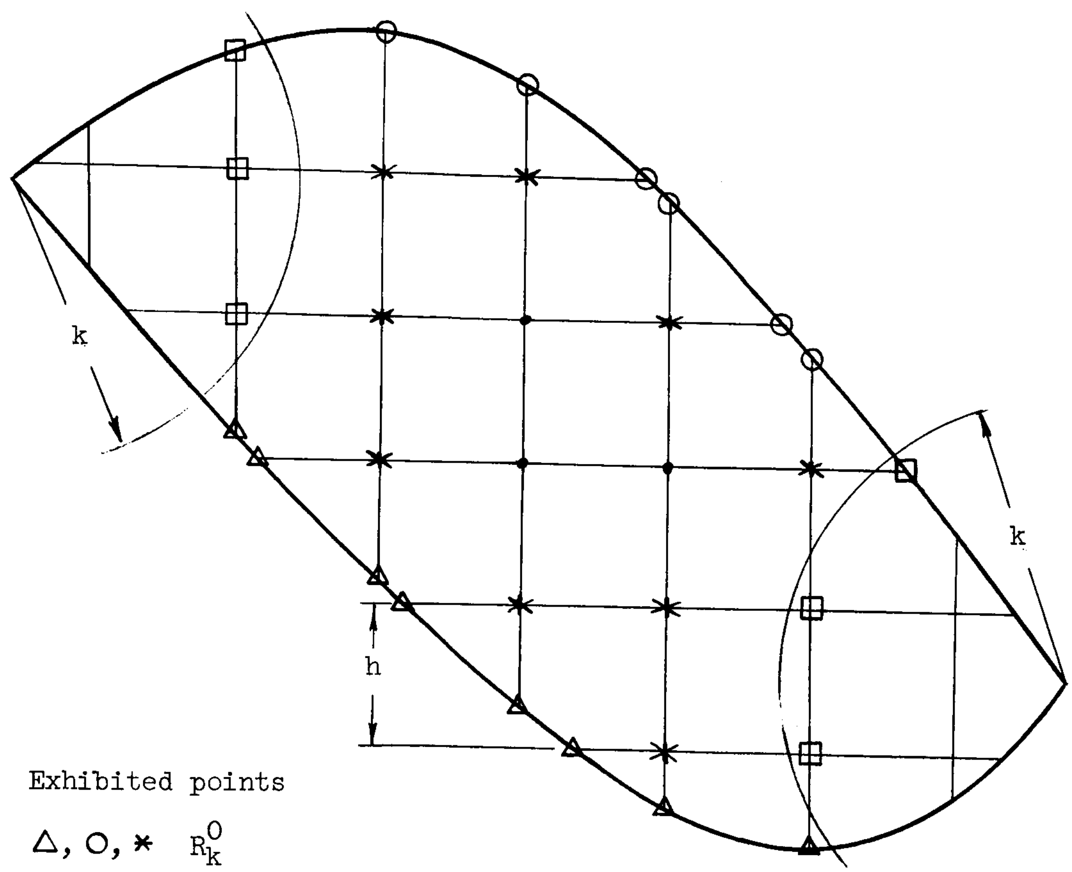
$$\underline{Z} = \underline{U}_k - \underline{u} \tag{4.1.4}$$

where $u_i = u(P_i)$ and u is the unique solution of (2.1.1 - 2.1.3) guaranteed by the results of CHAPTER II. To estimate the error in $\underline{U}_k$, the part of the matrix operator A of (3.2.1) which still remains will be applied to $\underline{Z}$. For later purposes call the "truncated" matrix A_k. Thus from (3.2.1)

$$\begin{aligned} A_k\underline{Z} &= A_k\underline{U}_k - A_k\underline{u} \\ &= \underline{d}_k(\underline{U}_k) - A_k\underline{u}. \end{aligned} \tag{4.1.5}$$

Let $D^2 v(P_i)$ denote any second partial derivative of v taken at the point P_i. Then results of Miranda [17] show that there is a constant M_2 such that

$$|D^2u(P_1) - D^2u(P_2)| \le \frac{M_2}{k^2} |P_1 - P_2|^\alpha. \tag{4.1.6}$$



Exhibited points

△, ○, ✱	R_k^0
□	Γ_k^0
✱	$\Gamma_{h,k}^*$
•	$R_{h,k}$
□, △	$\Gamma_{h,k,1}$
○	$\Gamma_{h,k,2}$

Figure 4. - $\overline{R}_{h,k}$

To examine the error $\underline{Z}$ at a point $P_0 \in R_{h,k} \cup \Gamma^*_{h,k}$ by considering the "formal error" $A_k\underline{Z}$, figure 2 should be consulted. A Taylor's series procedure similar to that of Roudebush [22] is used:

$$u_1 = u_0 + h_1 u_{x0} + \frac{1}{2} h_1^2 \tilde{u}_{xx1} \tag{4.1.7}$$

$$u_2 = u_0 + h_2 u_{y0} + \frac{1}{2} h_2^2 \tilde{u}_{yy2} \tag{4.1.8}$$

$$u_3 = u_0 - h_3 u_{x0} + \frac{1}{2} h_3^2 \tilde{u}_{xx3} \tag{4.1.9}$$

$$u_4 = u_0 - h_4 u_{y0} + \frac{1}{2} h_4^2 \tilde{u}_{yy4} \tag{4.1.10}$$

where, for example, u_{x0} is $\partial u/\partial x$ evaluated at the point P_0 and $\tilde{u}_{xx1}$ is $\partial^2 u/\partial x^2$ at some point between P_0 and P_1. Subtracting u_0 from both sides of (4.1.7) and (4.1.9), dividing (4.1.7) by h_1 and (4.1.9) by h_3, and adding one obtains

$$\frac{u_1 - u_0}{h_1} + \frac{u_3 - u_0}{h_3} = \frac{h_1}{2} \tilde{u}_{xx1} + \frac{h_3}{2} \tilde{u}_{xx3} . \tag{4.1.11}$$

Multiplying by $2/(h_1 + h_3)$ and subtracting u_{xx0} one obtains

$$\frac{2}{h_1 + h_3} \left(\frac{u_1 - u_0}{h_1} + \frac{u_3 - u_0}{h_3} \right) - u_{xx0} = \frac{h_1}{h_1 + h_3} (\tilde{u}_{xx1} - u_{xx0})$$

$$+ \frac{h_3}{h_1 + h_3} (\tilde{u}_{xx3} - u_{xx0}) \tag{4.1.12}$$

Analogously, operating on (4.1.8) and (4.1.10) one obtains

$$\frac{2}{h_2 + h_4}\left(\frac{u_2 - u_0}{h_2} + \frac{u_4 - u_0}{h_4}\right) - u_{yy0} = \frac{h_2}{h_2 + h_4}(\tilde{u}_{yy2} - u_{yy0})$$

$$+ \frac{h_4}{h_2 + h_4}(\tilde{u}_{yy4} - u_{yy0}) \qquad (4.1.13)$$

Adding (4.1.12) and (4.1.13) one obtains by (4.1.6)

$$\left|(A_k\underline{u})(P_0) + \nabla^2 u(P_0)\right| \le \frac{M_2}{k^2}\left[\frac{h_1}{h_1 + h_3}\left|\tilde{P}_1 - P_0\right|^\alpha + \frac{h_3}{h_1 + h_3}\left|\tilde{P}_3 - P_0\right|^\alpha\right.$$

$$\left. + \frac{h_2}{h_2 + h_4}\left|\tilde{P}_2 - P_0\right|^\alpha + \frac{h_4}{h_2 + h_4}\left|\tilde{P}_4 - P_0\right|^\alpha\right]$$

$$\le \frac{2M_2}{k^2} h^\alpha. \qquad (4.1.14)$$

To examine the formal error at points of $\Gamma_{h,k,2}$, figure 3 should be consulted.

Let u_* be the value of u where the normal intersects the grid line between the points labelled 1 and 2. Then

$$u_0 = u_* + d \left.\frac{\partial u}{\partial n}\right]_0 + \frac{d^2}{2}\frac{\partial^2 u}{\partial n^2}(\eta_0) \qquad (4.1.15)$$

as before. Also

$$u_2 = u_* - d_2 \left.\frac{\partial u}{\partial y}\right]_* + \frac{d_2^2}{2}\frac{\partial^2 u}{\partial y^2}(\eta_2) \qquad (4.1.16)$$

$$u_1 = u_* + d_1 \left.\frac{\partial u}{\partial y}\right]_* + \frac{d_1^2}{2}\frac{\partial^2 u}{\partial y^2}(\eta_1). \qquad (4.1.17)$$

Eliminating $(\partial u_*/\partial y)$ between these equations yields

$$\left| u_* - \frac{d_1u_2 + d_2u_1}{d_1 + d_2} \right| \leq \frac{d_1d_2}{2(d_1 + d_2)} \left[d_2 \left| \frac{\partial^2 u}{\partial y^2} (\eta_2) \right| + d_1 \left| \frac{\partial^2 u}{\partial y^2} (\eta_1) \right| \right]$$

$$\leq \frac{d_1d_2}{2} \frac{\tilde{M}_2}{k^2}, \qquad (4.1.18)$$

so that

$$\left| \frac{u_*}{d} - \frac{d_1u_2 + d_2u_1}{d(d_1 + d_2)} \right| \leq \frac{d_1d_2}{2d} \frac{\tilde{M}_2}{k^2}$$

$$\leq \frac{d_2}{2} \frac{M_2}{k^2} . \qquad (4.1.19)$$

By (4.1.15),

$$\frac{u_*}{d} = \frac{u_0}{d} - \frac{\partial u}{\partial n}\bigg]_0 + O\left(\frac{d}{k^2}\right), \qquad (4.1.20)$$

where $d \leq \sqrt{2}\, h$. Combining (4.1.19) and (4.1.20) one obtains for $P_0 \in \Gamma_{h,k,2}$,

$$(A_k\underline{u})(P_0) = \gamma(u_0) + O\left(\frac{h}{k^2}\right). \qquad (4.1.21)$$

Therefore

$$(A_k\underline{Z})(P_0) = \gamma[\underline{U}_k(P_0)] - \gamma(u_0) + O\left(\frac{h}{k^2}\right)$$

$$= \gamma'(\tilde{u})[\underline{U}_k(P_0) - u_0] + O\left(\frac{h}{k^2}\right), \qquad (4.1.23)$$

i.e.

$$\left| (\tilde{A}_k\underline{Z})(P_0) \right| = O\left(\frac{h}{k^2}\right), \qquad (4.1.24)$$

where $\tilde{A}_k$ here has a slightly different meaning than it has in section 3.2 in that the diagonal elements affected have positive numbers added which are not constant. Nevertheless, the inequality (3.2.11) still holds, and this is the key to the error estimate in the non-linear case where $\gamma_u \leq 0$. A similar result holds for the simpler boundary equation (3.1.12) wherever it may be applied. Here (4.1.17), with the subscript "*" replaced by "0" leads to a result like (4.1.24).

For $P_i \in \Gamma_k^0$ it is already true that

$$|\underline{U}_k(P_i) - u(P_i)| = O(k^\alpha). \tag{4.1.25}$$

For $P_i \in \Gamma_{h,k,1} - \Gamma_k^0$

$$\underline{U}_k(P_i) = u(P_i) = 0. \tag{4.1.26}$$

Compiling (4.1.14), (4.1.24), and (4.1.25) yields

$$\tilde{A}_k \underline{Z} = O(k^\alpha) + O\left(\frac{h^\alpha}{k^2}\right) \tag{4.1.27}$$

as $h \to 0$. Letting $k = h^\beta$ and maximizing the order of convergence in (4.1.27) yields finally $\beta = \alpha/(2 + \alpha)$ and thus there is a constant $M > 0$ such that

$$|\tilde{A}_k \underline{Z}| \leq M h^{\alpha^2/(2+\alpha)} \underline{e} \tag{4.1.28}$$

where $\underline{e}$ is a vector, all of whose components are 1,

$$\underline{e} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \tag{4.1.29}$$

and $|\underline{x}|$ is defined by

$$|\underline{x}|_i = |x_i|. \tag{4.1.30}$$

Since both A_k^{-1} and $\tilde{A}_k^{-1}$ are positive matrices, (4.1.28) and

(3.2.11) imply

$$|\underline{Z}| \leq Mh^{\alpha^2/(2+\alpha)} \tilde{A}_k^{-1} \underline{e}$$
$$\leq Mh^{\alpha^2/(2+\alpha)} A_k^{-1} \underline{e}. \qquad (4.1.31)$$

To complete the proof of convergence, it must be shown that the components of $A_k^{-1} \underline{e}$ are bounded independently of h.

Let ζ be the unique solution of (2.1.8 - 2.1.10), guaranteed to have the same smoothness as u as a special case of Miranda's theorem.

The vector $\underline{\zeta}$ is defined by

$$\zeta_i = \zeta(P_i). \qquad (4.1.32)$$

For $P_0 \in R_{h,k}$, applying (2.1.12), one obtains, in the manner that (4.1.14) was found, that

$$(A_k \underline{\zeta})(P_0) = 1 + 0\left[h^{\alpha^2/(2+\alpha)}\right], \qquad (4.1.33)$$

so that, for all sufficiently small h, one has

$$(A_k \underline{\zeta})(P_0) \geq \frac{1}{2}. \qquad (4.1.34)$$

Also, for $P_0 \in \Gamma_{h,k,2}$, applying (2.1.14), one obtains, in the manner that (4.1.24) was found, that

$$(A_k \underline{\zeta})(P_0) = 1 + 0\left[h^{\alpha/(2+\alpha)}\right] \qquad (4.1.35)$$

so that for all sufficiently small h, one again has

$$(A_k \underline{\zeta})(P_0) \geq \frac{1}{2}. \qquad (4.1.36)$$

This is surely true for $P_0 \in \Gamma_{h,k,1}$ since here, A_k is the identity. Therefore,

$$A_k^{-1}\underline{e} \le 2\zeta, \tag{4.1.37}$$

and by (4.1.31),

$$|\underline{Z}| \le 2Mh^{\alpha^2/(2+\alpha)}\underline{\zeta}. \tag{4.1.38}$$

Now, since ζ is a bounded function, we have thus proved:

Theorem 4.1.1. - Under the hypotheses of the uniqueness theorem, Theorem 2.2.1, if $k = h^{\alpha/(2+\alpha)}$, $\underline{U}_k$ converges to the solution u of (2.1.1 - 2.1.3) at the rate

$$|\underline{U}_k - u| = O\left[h^{\alpha}/(\alpha+2)\right].$$

In particular, if γ_u and the second derivative with respect to arc parameter of the representations of Γ_1 and Γ_2 satisfy Lipschitz conditions ($\alpha = 1$) with respect to arc parameter, then

$$|\underline{U}_k - u| = O(h^{1/3}).$$

4.2 Convergence of the Discrete Iterate to the Corresponding Continuous Iterate.

Let u_0 and v_0 be respectively defined by (2.1.33 - 2.1.35) and (2.1.55 - 2.1.57) with $N' = N + 1$. The sequences $\{u_n\}$ and $\{v_n\}$ are then defined by (2.1.40 - 2.1.42), starting with u_0 and v_0 respectively. Let $\underline{U}_0$ and $\underline{V}_0$ be the solutions of (3.2.3) and (3.2.25) with R_h replaced by $R_{h,k}$ which implies the replacement of A, $\underline{b}$ and $\tilde{\underline{b}}$ by A_k, $\underline{b}_k$, and $\tilde{\underline{b}}_k$, respectively. The sequences $\{\underline{U}_n\}$ and $\{\underline{V}_n\}$ are then defined by (3.2.7), again with A, $\underline{b}(\underline{U}_m)$, and $\underline{b}(\underline{V}_m)$ replaced by A_k, $\underline{b}_k(\underline{U}_m)$, and $\underline{b}_k(\underline{V}_m)$, respectively.

The results of section (4.1) already show that u_0 and v_0 may be approximated arbitrarily closely by $\underline{U}_0$ and $\underline{V}_0$. It remains to show that this is also true for u_n and v_n for any n.

Assume that u_{n-1} and v_{n-1} may be approximated by $\underline{U}_{n-1}$, and $\underline{V}_{n-1}$, respectively, as closely as desired. It will be proven that u_n can then be approximated arbitrarily closely by $\underline{U}_n$, the argument for v_n being completely analogous. Thus, by hypothesis, for arbitrary $\epsilon > 0$ there is an $h_0 < 0$ such that for all h satisfying $0 < h \leq h_0$

$$|\underline{U}_{n-1} - u_{n-1}| < \epsilon, \qquad (4.2.1)$$

where the subscript, k, appearing in the previous section, has been dropped for simplicity and will not appear modifying functions or sets. Let $\underline{W}_n$ be defined by

$$A\underline{W}_n = \underline{b}(u_{n-1}, \underline{W}_n) \qquad (4.2.3)$$

where the i^{th} component of $\underline{b}$ is zero unless $P_i \in \Gamma_{h2}$. Here

$$\underline{b}(u_{n-1}, \underline{W}_n)\Big]_i = \gamma(P_i, u_{n-1}) + k\Big[\underline{W}_n(P_i) - u_{n-1}(P_i)\Big]. \qquad (4.2.4)$$

But by the results of section 4.1, there is an $h_1 > 0$ such that for all h satisfying $0 < h \leq h_1$

$$|\underline{W}_n - u_n| < \epsilon. \qquad (4.2.5)$$

Let

$$\underline{\Omega} = \underline{U}_n - \underline{W}_n. \qquad (4.2.6)$$

Then

$$A_k\underline{\Omega} = A_k\underline{U} - A_k\underline{W}_n$$
$$= \underline{b}(\underline{U}_n) - \underline{b}(u_n,\underline{W}_n)$$
$$= \underline{\tilde{b}}(u_n,\underline{U}_n,\underline{W}_n). \quad (4.2.7)$$

where $\tilde{b}_i = 0$ unless $P_i \in \Gamma_{h2}$. Here

$$\underline{\tilde{b}}(u_n,\underline{U}_n,\underline{W}_n) = \gamma(P_i,\underline{U}_{n-1}) + k[\underline{U}_n(P_i) - \underline{U}_{n-1}(P_i)] - \gamma(P_i,u_{n-1})$$
$$- k[\underline{W}_n(P_i) - u_{n-1}(P_i)]. \quad (4.2.8)$$

Thus at these points,

$$|(A_k\underline{\Omega})(P_i) - k\Omega_i| \leq |\gamma(P_i,\underline{U}_{n-1}) - \gamma(P_i,u_{n-1})|$$
$$+ k|\underline{U}_{n-1}(P_i) - u_{n-1}(P_i)|$$
$$\leq 2k|\underline{U}_{n-1}(P_i) - u_{n-1}(P_i)|. \quad (4.2.9)$$

Therefore if h is such that $0 < h \leq \min(h_0,h_1)$, then

$$|\tilde{A}_k\underline{\Omega}| \leq 2k\,\epsilon\,\underline{e} \quad (4.2.10)$$

so that, as in section 4.1,

$$|\Omega_i| \leq 4k\,\epsilon\,\zeta(P_i). \quad (4.2.11)$$

Recalling that ζ is a bounded function, the following theorem holds:

<u>Theorem 4.2.1</u>. - Without the assumption $\gamma_u \leq 0$, the iterates u_n for the continuous problem (2.1.1 - 2.1.3) and the iterates $\underline{U}_{k,n}$ for the discrete problem $A_k\underline{U}_k = \underline{d}_k(\underline{U}_k)$ are considered. If $k = h^{\alpha/(2+\alpha)}$, then

$$|\underline{U}_{k,n} - u_n| = O\left[h^{\alpha^2/(2+\alpha)}\right],$$

for $n \geq 0$. Similarly, this is true for the iterates $\underline{V}_{k,n}$ and v_n.

4.3 Convergence of the Discrete Limit Function to the Continuous Limit Function

Under the assumption that the solution to (2.1.1 - 2.1.3) is unique (but not necessarily assuming that $\gamma_u \leq 0$) it is now shown that, if $\underline{U}_k$ is defined by $A_k\underline{U}_k = \underline{d}_k(\underline{U}_k)$ where $k = h^{\alpha/(2+\alpha)}$ and u by (2.1.51), u can be approximated as closely as desired by $\underline{U}_k$ by letting h go to zero.

If v is defined by (2.1.58), then uniqueness of u implies that

$$u = v \tag{4.3.1}$$

Although not done in the previous sections (e.g., 4.2, 3.2) it now becomes necessary to subscript $\underline{U}$ and $\underline{U}_n$ with k. That is, in CHAPTER III, where they were first discussed, $\underline{U}_n$ and $\underline{U}$ were defined for a particular h. Let $\epsilon > 0$. Then from the results of CHAPTER II prior to Theorem 2.1.2, there is an N_1 such that for all $P \in \overline{R}_{h,k}$ and $n \geq N$,

$$u \leq u_n < u + \frac{\epsilon}{2}. \tag{4.3.2}$$

By section 4.2 there is an $h_1 > 0$ such that for all h with $0 < h \leq h_1$ and all $P_i \in \overline{R}_{h,k}$

$$|\underline{U}_{n,k} - u_n| < \frac{\epsilon}{2}. \tag{4.3.3}$$

Therefore, for these n and h

$$\underline{U}_k \leq \underline{U}_{n,k} \leq u_n + \frac{\epsilon}{2} \leq u + \epsilon. \tag{4.3.4}$$

Similarly there are N_2, h_2 such that $n \geq N_2$, $0 < h \leq h_2$ implies

$$v - \epsilon \leq v_n - \frac{\epsilon}{2} \leq \underline{V}_{n,k} \leq \underline{V}_k. \quad (4.3.5)$$

Equation (4.3.4) implies

$$\underline{U}_k - u \leq \epsilon. \quad (4.3.6)$$

Equations (4.3.5) and (4.3.1) imply

$$u - \underline{U}_k = v - \underline{U}_k \leq v - \underline{V}_k \leq \epsilon. \quad (4.3.7)$$

But (4.3.6) and (4.3.7) are the results desired. That is, if h is such that $0 < h \leq \min(h_1, h_2)$, then

$$|\underline{U}_k - u| < \epsilon \quad (4.3.8)$$

at all points of $\overline{R}_{h,k}$.

The result is

<u>Theorem 4.3.1.</u> - If there is only one solution, u, of (2.1.1 - 2.1.3) then $\underline{U}_k$ converges to u as $h \to 0$ if $k = h^{\alpha/(2+\alpha)}$, i.e., for points of $\overline{R}_{h,k}$,

$$|\underline{U}_k - u| = 0\left[h^{\alpha^2/(2+\alpha)}\right].$$

CHAPTER V

GENERALIZATIONS

The results in this thesis thus far are stated for the Laplace equation in two dimensions for mixed boundary conditions which are in part Dirichlet. Extensions to more general elliptic equations, with non-linear source terms, extensions to higher dimensions, and extensions to more general assumptions on the boundaries and boundary conditions can be carried out in some spots very simply in others with some difficulty. Although the numerical analysis was carried out on the basis of the classical five-point difference approximation to the partial differential equation on square networks, it can be extended in the various directions taken in recent years by workers (for example Bramble and Hubbard [7] and Roudebush [22]) in this field. The purpose of this chapter is to indicate the directions of generalization.

5.1 Higher Dimensions.

Although the discussion in this thesis was carried out assuming everywhere that the equations were in two independent variables, almost all arguments could have been carried out for differential equations in n-dimensional space. One reason for not writing it all for n variables is the greater length of the expressions which would result in the numerical treatment starting in CHAPTER III, as, for example, for (3.1.9) and the correspondingly larger increase in detail in the arguments for the error estimates, as, for example, the arguments leading from (4.1.7) to (4.1.14). Most of the references are either already in n-dimensions, have statements indicating the extension of their results to n-dimensions, or can be extended to n-dimensions.

As an example, consider Miranda's work [17] which is basic to this paper. The results here are stated in m-dimensional space and carried out in great detail under the assumption $m \geq 3$, with statements about where arguments have to be modified for the case $m = 2$. Two reasons for this are the difference in form of Green's functions appearing in them, and the necessity of considering the hypergeometric function for $m \geq 3$, the case $m = 2$ being much simpler.

Probably the greatest difficulty would arise in the derivation and treatment in the error estimates of the difference equations on Γ_2 where the non-linear boundary conditions apply, if they are to be set up analogously with the two-dimensional treatment here.

5.2 Partial Differential Equations.

Although Laplace's equation was treated throughout this paper, immediate extensions can be made to more general homogeneous equations and to their non-homogeneous counterparts. In fact the paper upon which this thesis rests most heavily, Miranda [17], treats partial differential equations of the form

$$\sum_{i,k=1}^{m} a_{ik}(x) \frac{\partial^2 u}{\partial x_i \, \partial x_k} + \sum_{i=1}^{m} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u = f(x) \tag{5.2.1}$$

where (a_{ik}) is positive definite in all of R and satisfies conditions similar to those of uniform ellipticity. The coefficients in addition satisfy certain Hölder conditions in domains not containing $\bar{\Gamma}_1 \cap \bar{\Gamma}_2$ permitting certain orders of approach to infinity as the distance of these domains from $\bar{\Gamma}_1 \cap \bar{\Gamma}_2$ approach zero.

Again, the greatest difficulty appears to be in the numerical analysis and will require some relaxing of the class of difference approximations. This aspect is treated in section 5.4.

5.3 Boundary Conditions.

Miranda indicates some ways to weaken the smoothness required of φ (and thus for the Φ_m of section 2.1) in the boundary conditions (corresponding to (5.2.1))

$$-\sum_{k=1}^{m} a_{mk}(x) \frac{\partial u}{\partial x_k} + \beta(x)u = \varphi(x) \tag{5.3.1}$$

on Γ_2. Pursuing this would lead to convergence (in the sense of CHAPTER IV) of a lower order.

Stampacchia [24] discusses various weak ways of satisfying the boundary conditions and discusses non-linearities in both partial differential equation and boundary conditions from the point of view of calculus of variations.

With regard to Γ_1 there are also some directions of generalization:

(i) The type of boundary condition may be other than Dirichlet. For example

$$\frac{\partial u}{\partial n} = \alpha(u_0 - u), \ \alpha > 0, \tag{5.3.2}$$

is a possibility and does not lead to too much difficulty in the numerical analysis as monotone properties are still present and boundedness of the inverse independently of h still holds. This is an especially useful generalization in applications to heat transfer.

(ii) $\Gamma_1 = \emptyset$. In other words, the non-linear Neumann type of conditions is permitted all around. Here some additional restrictions need to be made on γ for existence as well as uniqueness.

(iii) Relaxation of smoothness of Γ_1: Here the results of this paper need to be combined with those of Roudebush [22] and results obtained are the essentially $O(h^{1/4})$ convergence obtained there.

5.4 Difference Equations.

Here, a generalization can quickly be made since it will be recalled that (3.1.11) was not even analyzed in CHAPTER IV. That is, the mesh might as well have been assumed not necessarily square, but rectangular, to begin with, gaining the advantage of possibly "fitting" the domain R a little better by R_h. Convergence is then analyzed on the basis of

$$h = \max h_i . \qquad (5.4.1)$$

Other mesh patterns have been considered in the past with varying degrees of success. It might be advantageous to consider them here since an arbitrary triangulation of R could possibly "fit" it better and is amenable to differencing based upon the integration technique. (See Varga [26], p. 184).

It would certainly be advantageous to consider the integration technique since it takes boundary conditions with normal derivatives without difficulty and produces automatically a symmetric matrix. This is the ideal situation for the application of Schechter's iterative method's. Some error analysis needs to be carried out yet for this type of boundary condition treatment.

CHAPTER VI

SUMMARY AND CONCLUSIONS

This thesis is concerned with solving Laplace's equation with mixed boundary conditions, part Dirichlet and part Neumann, in bounded two-dimensional regions. On that part of the boundary where the normal derivative enters, it is a non-linear function γ of the solution itself. Such boundary value problems have solutions under some, not too restrictive hypotheses on the smoothness of γ, including the condition

$$\limsup_{|u| \to \infty} \frac{\gamma(P,u)}{u} \leq 0, \qquad (6.1.1)$$

and on the smoothness of the boundary.

A constructive existence proof is carried out giving two solutions, (possibly equal) the largest possible and the smallest possible, as limits of two sequences of functions satisfying two sequences of linear boundary value problems related to the original. When

$$\gamma_u \leq 0, \qquad (6.1.2)$$

there is only one solution and thus the two limits are equal.

The intention of this research was to prove that the application of finite differences could be made to elliptic equations with non-linear boundary conditions, with a background of mathematical certainty that the functions so obtained are near those which are the solutions of these boundary value problems. This is carried out rigorously in the fourth chapter, using the most elementary differencing method based upon the classical five point scheme, as modified in the third chapter to fit problems of this type. Although the exponent is not large, suggesting the need for large numbers of points to get moderate detail in the solutions, numerical results suggest that with more care, better convergence of the solutions of the difference equations to that of the original problem is obtained in practice.

CHAPTER VII

NUMERICAL EXPERIMENTS

7.1 Radiation of Heat to Space.

The following example arises from the necessity of rejecting waste heat into space. It represents a section of a rectangular radiating fin.

It is required to find approximations to the temperature field, considered essentially to be two-dimensional and satisfying the following equations

$$\nabla^2 T = 0 \quad \text{in} \quad R: \quad 0 < x, y < 1, \tag{7.1.1}$$

$$\frac{\partial T}{\partial n} = 0 \quad \text{on} \quad \Gamma_0: \quad \begin{matrix} x = 0, & 0 < y < 1, \\ y = 0, & 0 < x < 1, \end{matrix} \tag{7.1.2}$$

$$T = 1 \quad \text{on} \quad \Gamma_1: \quad x = 1, \; 0 \leq y \leq 1, \tag{7.1.3}$$

$$\frac{\partial T}{\partial n} = \begin{Bmatrix} -T^4, & T \geq 0 \\ 0, & T < 0 \end{Bmatrix} \text{on} \quad \Gamma_2: \; y = 1, \; 0 < x < 1, \tag{7.1.4}$$

in what is really only the first quadrant portion of a square two units on a side, but considered as above because of the symmetry of the original problem.

Define

$$\zeta_1(x,y) = 2 - x^2 + \frac{1}{2} y^2, \tag{7.1.5}$$

which satisfies

$$\nabla^2 \zeta_1 = -1 \quad \text{in} \quad R, \tag{7.1.6}$$

$$\frac{\partial \zeta_1}{\partial n} = 0 \quad \text{on} \quad \Gamma_0, \tag{7.1.7}$$

$$\zeta_1 = 1 + \frac{1}{2} y^2 \quad \text{on} \quad \Gamma_1, \tag{7.1.8}$$

$$\frac{\partial \zeta_1}{\partial n} = 1 \quad \text{on} \quad \Gamma_2 . \tag{7.1.9}$$

Therefore, if ζ is to be defined by (2.1.12 - 2.1.14) on the entire 4-quadrant configuration, (7.1.6 - 7.1.9) implies that

$$\zeta(x,y) \leq \zeta_1(x,y) \tag{7.1.10}$$

for $(x,y) \in R$.

To place this in a form where the a priori estimate of section 2.1 may be obtained, it is necessary to define T_1 by

$$T = T_1 + 1, \tag{7.1.11}$$

and

$$\gamma(T_1) = \begin{cases} -(T_1 + 1)^4 & T_1 \geq -1 \\ 0 & T_1 < -1. \end{cases} \tag{7.1.12}$$

Problem (7.1.1 - 7.1.4) is then rewritten as

$$\nabla^2 T_1 = 0 \quad \text{in} \quad R, \tag{7.1.13}$$

$$\frac{\partial T_1}{\partial n} = 0 \quad \text{on} \quad \Gamma_0, \tag{7.1.14}$$

$$T_1 = 0 \quad \text{on} \quad \Gamma_1, \tag{7.1.15}$$

$$\frac{\partial T_1}{\partial n} = \gamma(T_1) \text{ on } \Gamma_2. \tag{7.1.16}$$

Then for $-1 \leq T_1 < 0$,

$$\frac{\gamma(T_1)}{T_1} = (T_1 + 1)^4 \left(- \frac{1}{T_1}\right)$$

$$\leq \frac{1}{|T_1|} .$$

Therefore, if we take

$$\eta(u) = \begin{cases} \dfrac{1}{|u|} & |u| \leq 1 \\ \dfrac{1}{|u|^m} & |u| > 1, \end{cases} \tag{7.1.17}$$

where $m > 1$ is free to be chosen, all conditions of section 2.1 are met.

Since

$$\max_{\bar{\Gamma}_2} \zeta_1 = 2.5, \tag{7.1.18}$$

it is possible to take M_1 any number such that

$$M_1 > 1, \tag{7.1.19}$$

simply by making m large enough in (7.1.17). Thus

$$M_3 = 2.5, \tag{7.1.20}$$

i.e.,

$$|T_1| \leq 2.5, \tag{7.1.21}$$

so that

$$|T| \leq 3.5. \tag{7.1.22}$$

This is a coarse estimate in this case, which is subject to some improvement, perhaps, if ζ were used instead of ζ_1. However, applying the strong maximum principles (see Bers, John, Schechter [5], p. 151) shows that

$$|T| \leq 1 \text{ in } \bar{R}. \tag{7.1.23}$$

The boundary value problem was replaced by a system of finite difference equations following the methods of section 3.1. A square mesh of mesh width h was used and was so situated in the square that the edges were parallel to the grid lines and of distance h from the resulting Γ_h^*.

Equations near the corners were not rejected as required for the convergence proof of CHAPTER IV but left in for ease of programming. The numerical results presented show that, at least in this case, convergence as $h \to 0$ is still possible.

h	Error	Apparent order
1/4	0.01211	----
1/8	.00797	1.52
1/16	.00468	1.70
1/32	.00264	1.77

The third column of this table is obtained by replacing (by assumption) the inequality implied in (4.1.39) by equality and the exponent by an unknown, α_a the "apparent order", namely we assume

$$\|error\| \equiv \epsilon = Mh^{\alpha_a}$$

and solve for α_a, based on the second column.

As already announced in section 3.4 this example was solved by a line-by-line successive over-relaxation scheme (S.O.R.). A series of experiments on some rectangular problems indicate that, as in the linear case where optimum values are obtainable rigorously, there is a value for ω, the S.O.R. parameter, which is optimum. This is taken in the sense that any other value leads to more iterations before the error in an aribtrary initial approximation is reduced to an acceptable level.

In conclusion, this numerical example has strengthened this thesis in indicating, that the predicted convergence of approximate solutions to the solution of the continuous, does indeed take place in practice. However, it serves to point up its weaknesses and thus indicate further directions for research. First, it should be possible to prove that one need not reject equations near the corners and an approach using "smoothing" operators might apply here as it did in Parter [18]. Secondly, something should be possible in analytically predicting a value which is optimum for the relaxation parameter.

REFERENCES

1. Ablow, C. M.; and Perry, C. L.: Iterative Solutions of the Dirichlet Problem for $\Delta u = u^2$. J. S. I. A. M. 7, 4, 1959, pp. 459-467.

2. Agmon, S.; Douglis, A.; and Nirenberg, L.: Estimates Near the Boundary for Solutions of Elliptic Partial Differential Equations Satisfying General Boundary Conditions 1. Comm. Pure App. Math., XII, 1959, pp. 623-727.

3. Batschelet, Eduard: Über Die Numerische Auflösung von Randwertproblemen bei Elliptischen Partiellen Differentialgleichungen. ZAMP III, 3, 1952, pp. 165-193.

4. Bers, Lipman: On Mildly Nonlinear Partial Difference Equations of Elliptic Type. J. Res. N. B. S. 51, 5, 1953, pp. 229-236.

5. Bers, L.; John, F.; and Schechter, M.: Partial Differential Equations. Interscience, 1964.

6. Bramble, J. H.; and Hubbard, B. E.: On a Finite Difference Analogue of an Elliptic Boundary Problem which is Neither Diagonally Dominant Nor of Non-Negative Type. J. Math. and Phys. XLIII, no. 2, 1964, pp. 117-132.

7. Bramble, J. H.; and Hubbard, B. E.: A Theorem on Error Estimation for Finite Difference Analogues of the Dirichlet Problem for Elliptic Equations. Contr. Diff. Eq. II, 1964, pp. 319-340.

8. Collatz, L.: Das Vereinfachte Newtonsche Verfahren bei Nichtlinearen Randwertaufgaben. Arch. Math. 5, 1954, pp. 233-240.

9. Courant, R.; and Hilbert, D.: Methods of Mathematical Physics. Vol. II. Interscience, 1962.

10. Forsythe, George E.; and Wasow, Wolfgang, R.: Finite Difference Methods for Partial Differential Equations. Wiley & Sons, New York, 1960.

11. Greenspan, Donald: On the Numerical Solutions of Problems Allowing Mixed Boundary Conditions. Math. Res. Ctr., U. Wisc., Techn. Summary Rep. 372, 1962.

12. Greenspan, Donald; and Parter, Seymour: Mildly Nonlinear Elliptic Partial Differential Equations and their Numerical Solution II. Math. Res. Ctr., U. Wisc., Tech. Summary Rep. 474, 1964.

13. Kalaba, Robert: On Nonlinear Differential Equations, the Maximum Operation and Monotone Convergence. J. Math. and Mech. 8 4, 1959, pp. 519-574.

14. Levinson, Norman: Dirichlet Problem for $\Delta u = f(P,u)$. J. Math. and Mech. 12 4, 1963, pp. 567-575.

15. Lieberstein, H. M.: Overrelaxation for Non-linear Elliptic Partial Differential Problems. Math. Res. Center, U. of Wisc., Tech. Summary Rep. no. 80, 1959.

16. Lieberstein, H. M.: A Numerical Test Case for the Non-linear Overrelaxation Algorithm. Math. Res. Center, U. Wisc., Tech. Summ. Rep. no. 122, 1960.

17. Miranda, Carlo: Sul Problema Misto per le Equazioni Lineari Ellitiche. Ann. Mat. Pura Appl., ser. 4 39, 1955, pp. 279-393.

18. Parter, Seymour V.: Mildly Nonlinear Partial Differential Equations and their Numerical Solution I. Math Res. Ctr., U. Wisc., Tech. Summary Rep. 470, 1964.

19. Petrovsky, I. G.: Lectures on Partial Differential Equations. Interscience, 1954.

20. Pohožaev, S. I.: The Dirichlet Problem for the Equation $\Delta u = u^2$. Soviet Math., 1, 1960, pp. 1143-1146.

21. Rosenbloom, P.: The Difference Equation Method for Solving the Dirichlet Problem. In: Construction and Applications of Conformal Maps. Appl. Math. Ser. no. 18, N.B.S., 1952, pp. 231-237.

22. Roudebush, William H.: Analysis of Discretization Errors for Differential Equations with Discontinuous Coefficients. Ph. D. Thesis, Case Institute of Technology, 1963.

23. Schechter, Samuel: Iteration Methods for Nonlinear Problems. Trans. A.M.S. 104, no. 1, 1962, pp. 179-189.

24. Stampacchia, Guido: Problemi al Contorno Misti per Equazioni del Calcolo Delle Variazioni. Ann. Mat. Pura Appl., ser. 4, 40, 1955, pp. 193-209.

25. Varga, Richard S.: Factorization and Normalized Iteration Methods. In: Boundary Problems in Differential Equations. U. of Wis. Press, 1959.

26. Varga, Richard S.: Matrix Iterative Analysis. Prentice Hall, 1962.