MECHANICAL PROPERTY OF SOLITHANE 113

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INTRODUCTION

In studies of polymer failure under stress, a knowledge of the mechanical properties of the material becomes a necessary pre-requisite to the understanding of the mechanics of failure. This requirement is appreciated when one realizes that the occurrence of failure is related closely to the loading history, and therefore the mechanical behavior of the material. The more detailed aspects of the study involve the comparisons of behavior under various monotonic and sinusoidal cycling histories. It often assists in these studies to look at the ability of a viscoelastic material to convert the boundary forces into elastically stored or dissipated energies. The heat build in a specimen is synonomous with dissipation and could have significant thermal effects on the materials' failure. If these effects cannot be calculated one can investigate failure isothermally, using appropriate geometry and test conditions. The other extreme temperature condition worthy of consideration in testing is to produce a test environment in order that the temperature in the material increases adiabatically. This kind of information would also be useful when the dissipation is large and failure occurs over short intervals of time.

This paper is concerned with the mechanical properties of a test material known as Solithane 113 which is being used for studies in failure. Some physical properties have been calculated from linear viscoelastic theory using spring and dashpot model representations. The elements of the model are determined from the relaxation modulus for the material obtain in tests at small strains in tension. The energy responses of the material to a wide spectrum of load inputs are presented in a set of graphs. Other details include the complex modulus and complex compliance behavior in which their components have been plotted as functions of the input frequency. Various strain rate histories and their combinations have also been investigated and a special case is treated where two strain rates are combined at various ratios and the results are plotted for solithane material.

The next portion of the paper deals with estimating the specimen geometry which will enable isothermal tests to be carried out under sinusoidal strain inputs. Also the adiabatic rise of temperature in a specimen is considered for sinusoidal inputs using an incremental method of approximating the temperature increases.

SECTION 1. MECHANICAL PROPERTIES

The complex modulus and energy equations have been derived from one dimensional linear viscoelastic theory using a Wiechert model representation (Figure 1) for prescribed strain inputs and a Kelvin model representation (Figure 2) for prescribed stress inputs.[1,8] The relaxation modulus curve for the material is approximated by a Prony Series representation[1,2] given in the following equation

$$E_{rel}(t) = m_e + \sum_{i=1}^{n} m_i e^{-\frac{t}{\tau_i}} \tag{1}$$

which is shown plotted with the actual relaxation curve in Figure 3.

The stress response to unit step strain is then given by the equation

$$\sigma(t) = \left(m_e + \sum_{i=1}^{n} m_i e^{-\frac{t}{\tau_i}}\right) H(t) \tag{2}$$

With this equation the general equation for stress response to a strain input can now be generated in the Duhamel superposition integral[4] and is given by the equation

$$\sigma(t) = \epsilon(t)\, m_e + \sum_{i=1}^{n} m_i \int_0^t e^{-\left(\frac{t-\tau}{\tau_i}\right)} \dot{\epsilon}(\tau)\, d\tau \tag{3}$$

From these equations the complex modulus can be determined if we let the

input $\epsilon(t) = \epsilon_o e^{i\omega t}$ (4)

Then for long times

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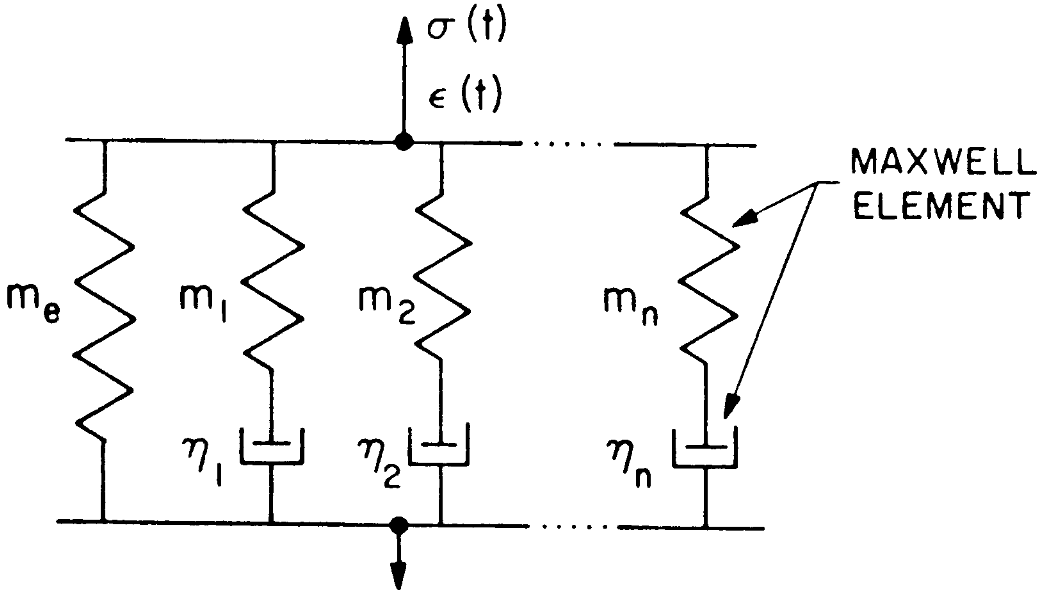

FIGURE 1 - WIECHERT MODEL

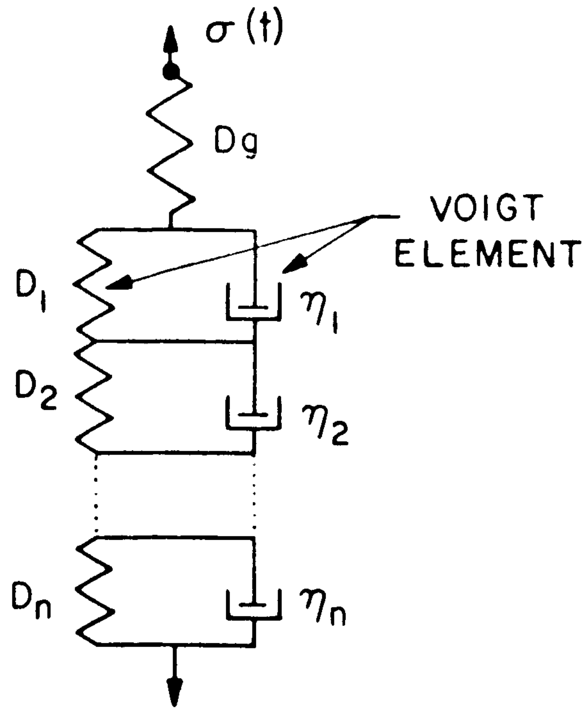

FIGURE 2 - KELVIN MODEL

$$\frac{\sigma(t)}{\epsilon_o e^{i\omega t}} \equiv E^*(\omega) = E' + i E'' \tag{5}$$

where $E^*(\omega)$ is the complex or dynamic modulus which can be represented explicitly by the following set of equations

$$E^*(\omega) = m_e + \sum_{i=1}^{n} m_i \frac{i\omega\tau_i}{1 + i\omega\tau_i} \tag{6}$$

$$E' = m_e + \sum_{i=1}^{n} m_i \frac{(\omega\tau_i)^2}{(1 + \omega\tau_i)^2} \tag{7}$$

$$E'' = \sum_{i=1}^{n} m_i \frac{\omega\tau_i}{1 + (\omega\tau_i)^2} \tag{8}$$

Equations 7 and 8 were calculated for Solithane 113 and the results are shown plotted as a function of frequency in Figure 4.

The dynamic compliance can be expressed by the following equation

$$D^*(\omega) = D' - i D'' \tag{9}$$

and is defined as the inverse of the dynamic modulus. It is shown plotted in a similar manner for D' and D'' in Figure 5. The following energy response equations for strain inputs were calculated for the model[3] in Figure 1.

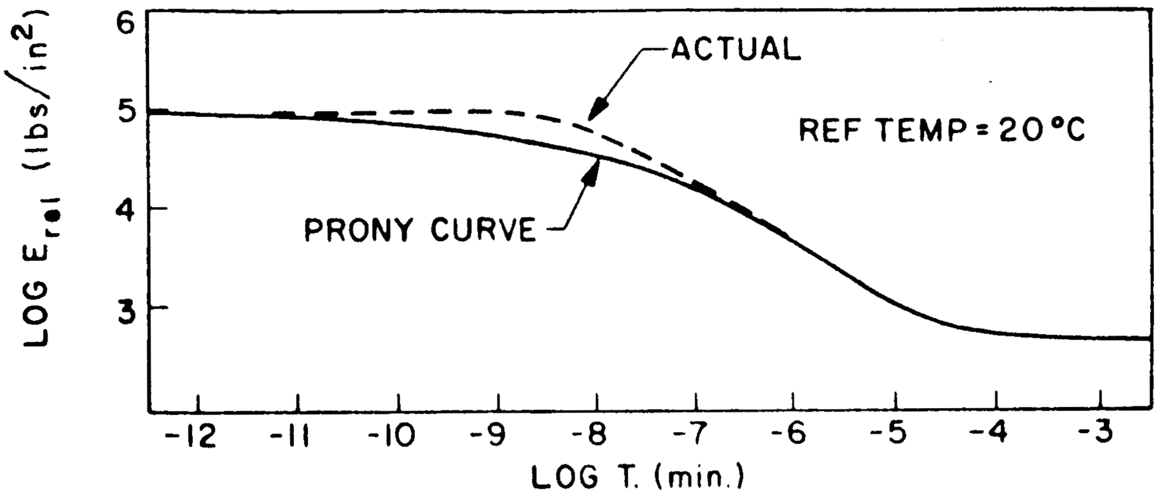

FIG. 3 RELAXATION MODULUS FOR SOLITHANE 113

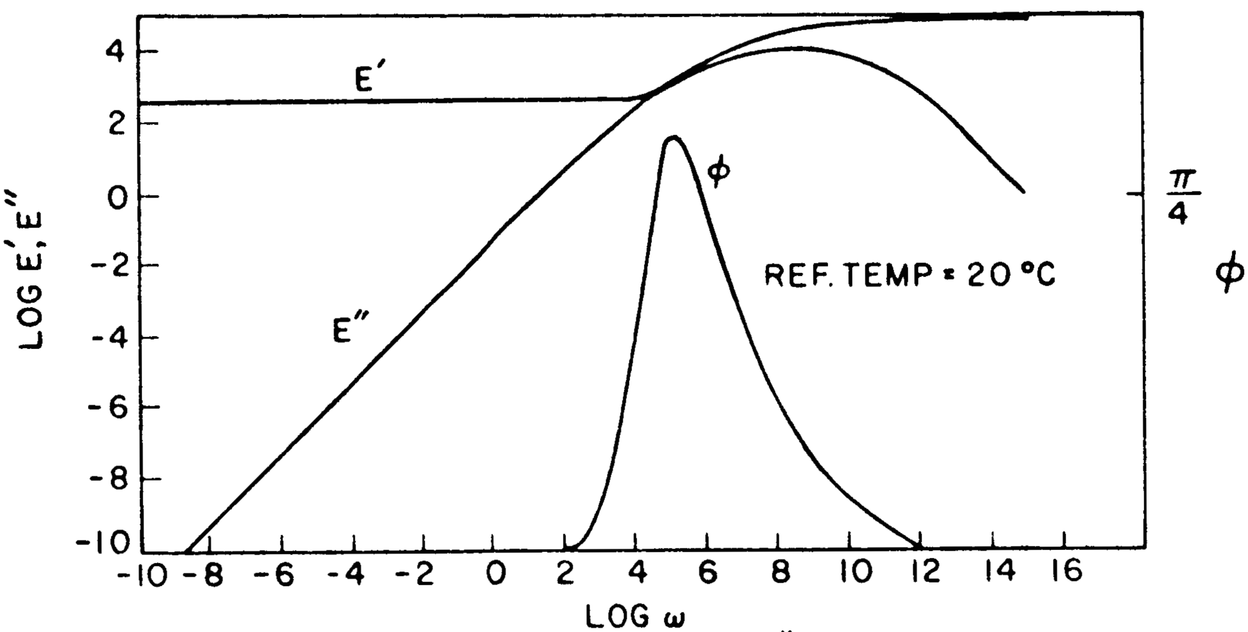

FIG. 4 COMPONENTS OF COMPLEX MODULUS; $E^* = E' + iE''$; $\tan\phi = E''/E'$

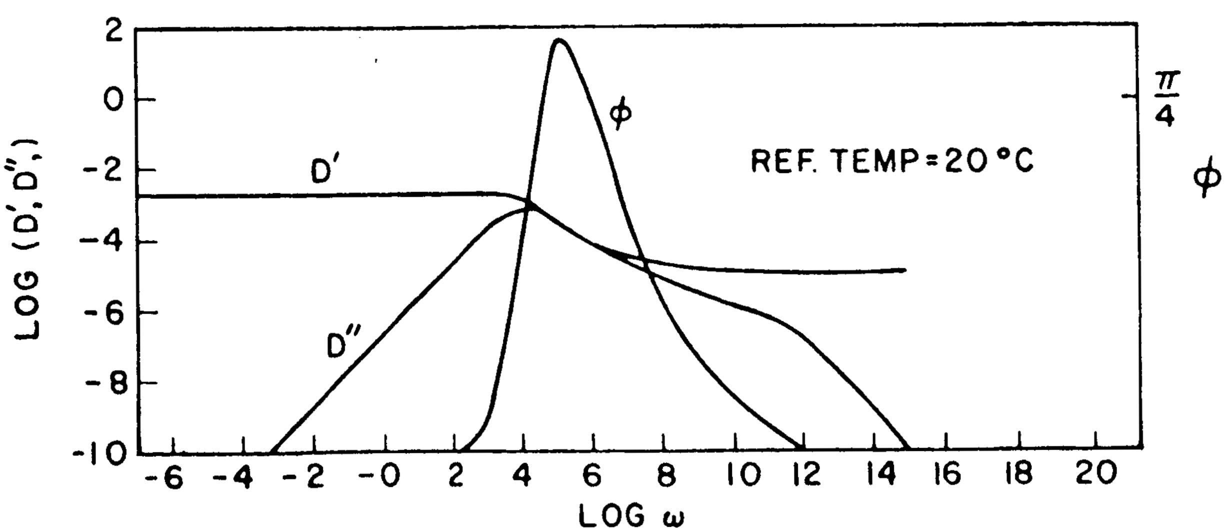

FIG. 5 COMPLEX COMPLIANCE; $D^*(\omega) = D' - iD''$ $\tan\phi = D''/D'$

The elastic energy for the i^{th} Maxwell element is given by

$$W_{el} = \int_0^t \sigma_i^s \, d\epsilon_i^s = \int_0^t \frac{\sigma_i}{m_i} \, d\sigma_i = \frac{\sigma_i^2}{2 m_i} \tag{10}$$

Substituting Equation 3 with Equation 10 results in an energy equation for the Wiechert model representation and is given by

$$W_{el} = \frac{1}{2} m_e \epsilon^2(t) + \frac{1}{2} \sum_{i=1}^{n} m_i e^{-\frac{2t}{\tau_i}} \left(\int_0^t e^{\frac{\tau}{\tau_i}} \dot{\epsilon}(\tau) d\tau \right)^2 \tag{11}$$

Similarly, the energy dissipation rate for the i^{th} Maxwell element is given by

$$\dot{W}_v = \sigma_i^d \dot{\epsilon}_i^d = \frac{\sigma_i^2}{\eta_i} \tag{12}$$

now since τ_i the relaxation time $\equiv \frac{\eta_i}{m_i}$, then Equations 12 and 3 will combine to give the following equation for dissipation rate

$$\dot{W}_v = \sum_{i=1}^{n} \frac{m_i}{\tau_i} e^{-\frac{2t}{\tau_i}} \left(\int_0^t e^{\frac{\tau}{\tau_i}} \dot{\epsilon}(\tau) d\tau \right)^2 \tag{13}$$

from which the dissipation can be obtained by the equation

$$W_v = \int_0^t \dot{W}_v \, dt = \sum_{i=1}^{n} \frac{m_i}{\tau_i} \int_0^t e^{-\frac{2t}{\tau_i}} I^2 dt \qquad (14)$$

where I in Equation 14 is given by

$$I = \int_0^t e^{\frac{\tau}{\tau_i}} \dot{\epsilon}(\tau) \, d\tau$$

Similar equations for prescribed stresses can be established by referring to the Kelvin model representation shown in Figure 2. The strain response (or creep behavior) to input stresses is given by the equation

$$\epsilon(t) = \int_0^t D_{crp}(t-\tau) \, \dot{\sigma}(\tau) \, d\tau \qquad (15)$$

where $D_{crp}(t)$ in the above expression is defined as the creep compliance and can be fitted by a Prony series equation of the form

$$D_{crp}(t) = D_g + \sum_{i=1}^{n} D_i \left(1 - e^{-\frac{t}{\tau_i}}\right) \qquad (16)$$

The conversion of complex modulus data to creep compliance has been treated in other papers[9, 10] and is not therefore included in this discussion.

The above sets of equations have been used to determine the energy and stress responses for the following inputs:

$$\begin{aligned} &\text{i.} \quad \epsilon = \epsilon_0 H(t) \\ &\text{ii.} \quad \epsilon = Rt \qquad (17) \\ &\text{iii.} \quad \epsilon = \epsilon_0 \sin \omega t \end{aligned}$$

iv. $\sigma = \sigma_o H(t)$

v. $\sigma = Pt$ (17 contd.)

vi. $\sigma = \sigma_o \sin \omega t$

for $t > 0$.

Details of these equations for stress, strain, and energy responses are given in Appendix 1. These equations have been used to compute the properties of Solithane 113 (on an IBM 7090/7094 digital computer) and the results are shown plotted in Figures 6-11.

MULTI-STRAIN HISTORIES

In this example of multi-strain histories the behaviors of a material could be particularly useful when investigating various paths of strain history to failure.

In order to investigate these histories, a generalized input function is used which comprised a ramp strain, followed by a ramp superimposed with a sinusoidal strain, and can be represented by the following expression,

$$\epsilon(t) = R_1 t\, H(t_o - t) + (R_1 t_o + R_2(t - t_o) + \epsilon_o \sin \omega(t - t_o))\, H(t - t_o) \tag{18}$$

for $t > 0$.

The stress and energy results for the above input are given in Appendix 2. Various combinations can be obtained from Equation 18 and their portions of the solution given in Appendix 2 are easily distinguishable.

In studies of various strain histories to failure it is often desirable to obtain the unique values of a given type of strain history for a prescribed point in $\sigma - \epsilon$ space. An example is now given of determining two constant

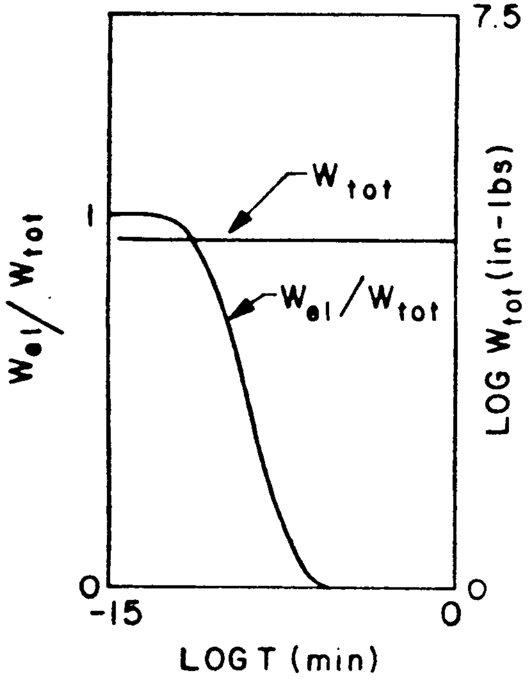

FIG. 6 ENERGY RESPONSE UNIT STEP STRAIN

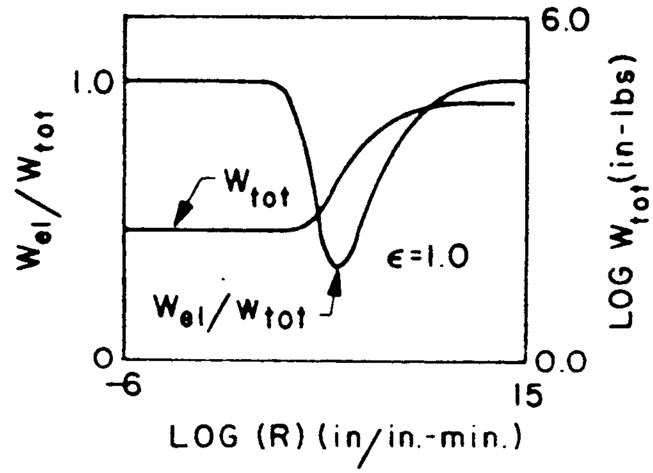

FIG. 7 ENERGY RESPONSE CONSTANT STRAIN RATE

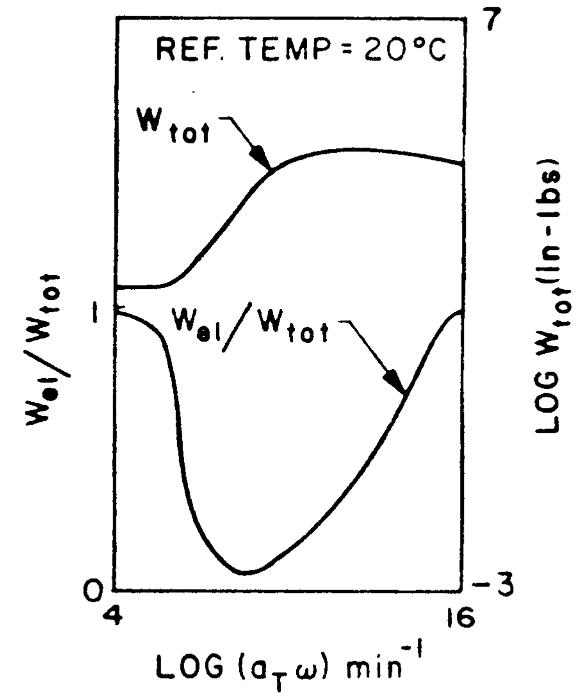

FIG. 8 ENERGY RESPONSE SINE STRAIN

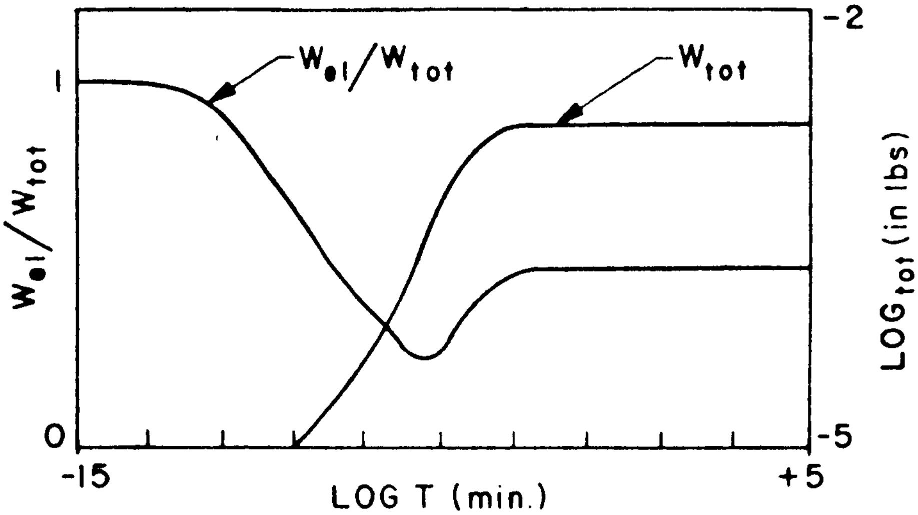

FIG. 9 ENERGY RESPONSE FOR A STEP STRESS

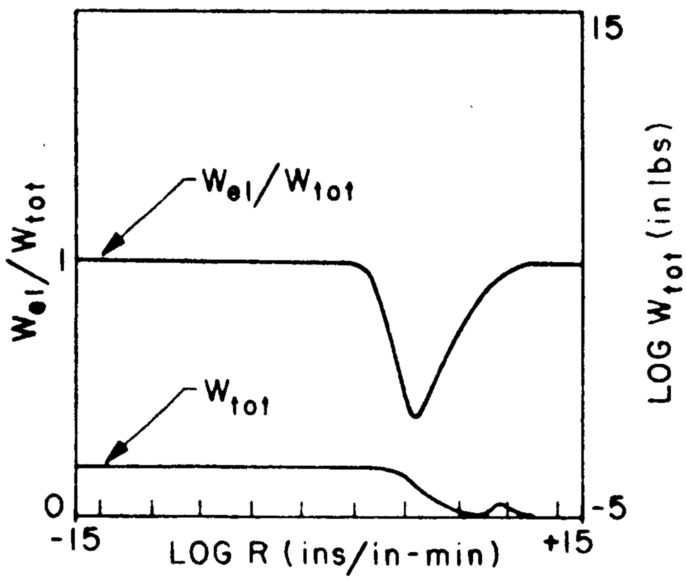

FIG. 10 ENERGY RESPONSE TO A CONSTANT STRESS RATE

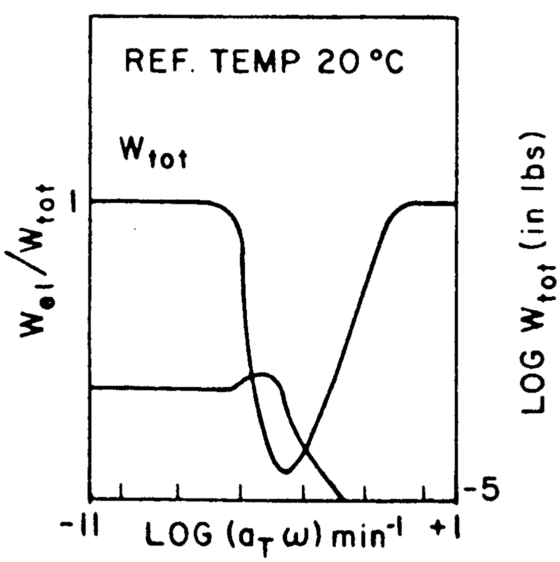

FIG. 11 ENERGY RESPONSES TO SINUSOIDAL STRESS

strain rates R_1 and R_2 in which a ratio R_1/R_2 is prescribed, the position of change known between R_1 and R_2 as a strain and the final point in stress-strain space known.

The solution for R_1 (or R_2) can be obtained by an iterative procedure used with the aid of a high speed computer. Details of this iterative method are outlined in Appendix 3. In Figure 12 the stress-strain curve for three ratios of R_1/R_2 are shown for a given point in $\sigma - \epsilon$ space using Solithane 113. The energies involved in reaching this stress-strain level by the three paths are shown in Figure 13. These results should be treated cautiously because considerable deviations could occur between these curves and experimental results due to the non-linearity of the material at finite strains.[11] The interpretation of these results (Figures 12 and 13) in relation to failure is discussed in another paper.[12]

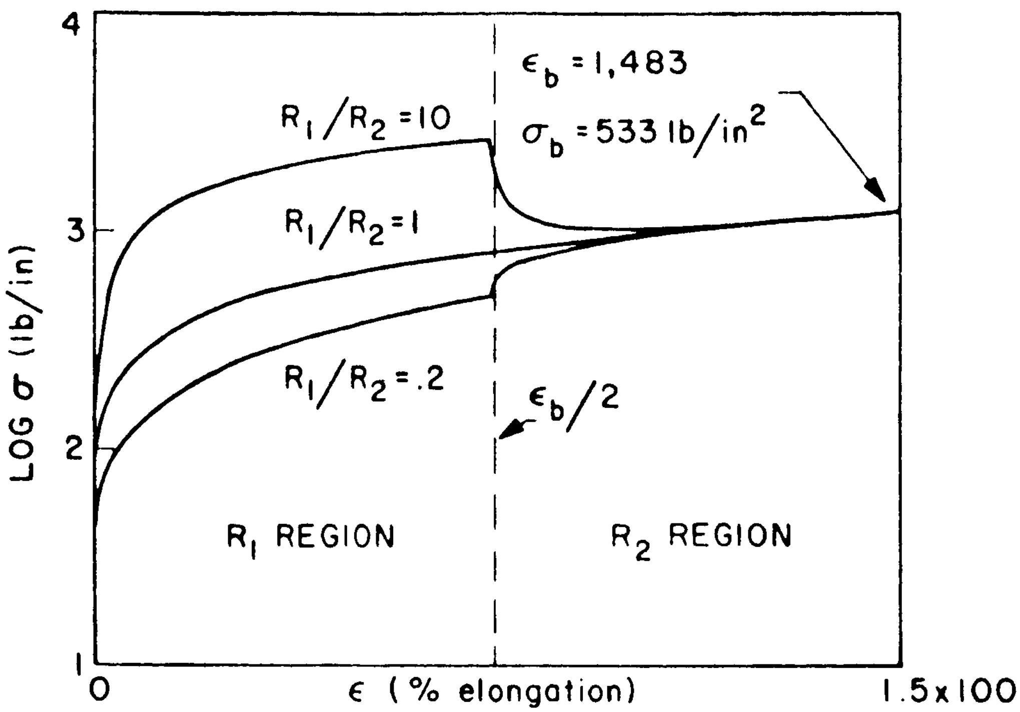

FIG.12 STRESS STRAIN RESPONSES TO DUAL CONSTANT STRAIN RATES

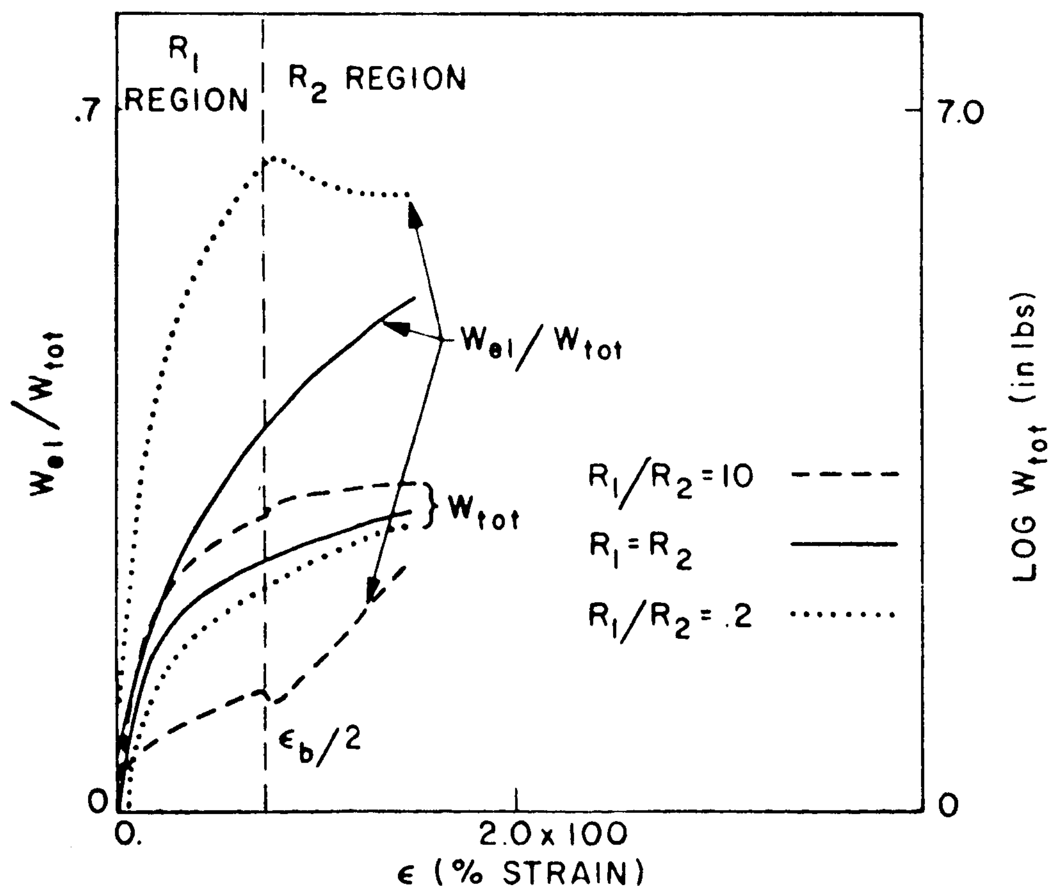

FIG.13 ENERGY RESPONSES VERSUS STRAIN FOR DUAL CONSTANT STRAIN RATE

SECTION 2. TEMPERATURE CONDITIONS

When dissipation is significant it becomes mandatory to calculate the thermal behavior of the material and details of this type of calculation are given in another paper,[8] indicating that the presence of sizeable temperature effects could cause much difficulty in failure studies. Considerable simplification might be achieved if a uniform and constant temperature state can be realized in testing for sinusoidal inputs. An approximate set of calculations have been made to estimate the upper limit of the input frequency which will give near uniform temperature distribution across the thickness of the specimen.

The other case considered is adiabatic increase in the temperature of a specimen for sinusoidal input. These calculations of temperature increase are based on the effect of an incremental temperature change on the properties of the material. Details of these cases of isothermal and adiabatic temperature conditions for sinusoidal inputs are given in the following

(a) Isothermal tests

If the test specimen is considered, for simplicity, to be an infinite sheet of uniform thickness with flat surfaces and that both surface temperatures of the specimen are taken to be constant at zero, then the heat flow could be assumed as one dimensional and expressed by the equation[5]

$$\frac{\partial^2 T}{\partial x^2} - \frac{1}{K_1}\frac{\partial T}{\partial t} = -\frac{W_v'}{K} \qquad (19)$$

where T = temperature

t = time

K_1 = thermal diffusivity

K = thermal conductivity

and W_V' = dissipation rate.

Assume that W_V' is constant with respect to time and distance x, that is

$$-\frac{1}{K_1}\frac{\partial T}{\partial t} = 0$$

then, for a thickness $-\ell < x < \ell$, the steady state solution for temperature distribution across the specimen satisfying these conditions can be expressed by the equation

$$T(x) = W_V'(\ell'^2 - x^2)/2K \tag{20}$$

when $x = 0$, an equation for half thickness ℓ' is given by

$$\ell = \sqrt{\frac{2TK}{W_V'}} \tag{21}$$

The dissipation rate W_V' can be obtained from the following equation

$$W_V' = \frac{W_V \text{ per cycle}}{\text{time of one cycle}} = \text{an average dissipation rate} \tag{22}$$

The ratios of thickness and $T(x = 0)$ versus ω_{a_T} have been computed from the normalized expression

$$\frac{\ell \epsilon_o}{\sqrt{KT}} = \sqrt{\frac{\epsilon_o^{\,2}}{W_V'}}$$

and plotted out as

$$\log \frac{\epsilon_o \ell}{\sqrt{KT}} \qquad \text{versus} \qquad \log \omega a_T$$

for Solithane 113 in Figure 14. The frequency ω can be determined as an upper limit when a thickness and temperature T at $x = 0$ are prescribed. It should be observed in applying Figure 14 that for prescribing T at $x = 0$ the variation in dissipation across the specimen must be sufficiently small that it can be considered uniform.

(b) <u>Adiabatic temperature increases for sinusoidal strain input.</u>

Let W_{V_1} be defined as the dissipation per cycle at temperature $T = T_1$ (see Appendix 1), then W_{V_n} is the dissipation per cycle at temperature $T = T_n$

$$T_n \Rightarrow \omega_n = \omega_o \, a_{T_n}$$

where a_{T_n} is the temperature-frequency shift factor given by the WLF equation.[7]

Let N = number of cycles at W_{V_n} dissipation necessary to cause a temperature change at ΔT_{n+1} given by the equation

$$\Delta T_{n+1} = T_{n+1} - T_n = \Delta T \text{ (constant)}$$

Let C be the heat capacity per unit volume,

then $$\frac{N}{C} W_{V_{n+1}} \leqslant \Delta T_{n+1} \leqslant \frac{N}{C} W_{V_n}$$

It was found that either side of the equality gave comparable results provided ΔT was made small.

Therefore in general

$$N_k \doteq \frac{\Delta T_o . C}{\overline{W}_V \left[\omega_o \; a_T \left(T_o + (K-1) \Delta T_o \right) \right.}$$

with the additional simultaneous relations

$$t_m = \frac{2\pi}{\omega_o} \sum_{k=1}^{m} N_K$$

$$T_m(t) = T_o + m T_o$$

Example of these calculations are given for Solithane 113 in Figure 15, the initial temperature $T_o = -20^\circ C$ and a frequency of one radian per second.

Note that due to the strong initial dissipation the temperature rises first rapidly and then, as the material heats up and dissipation increases, the rate of temperature decreases also.

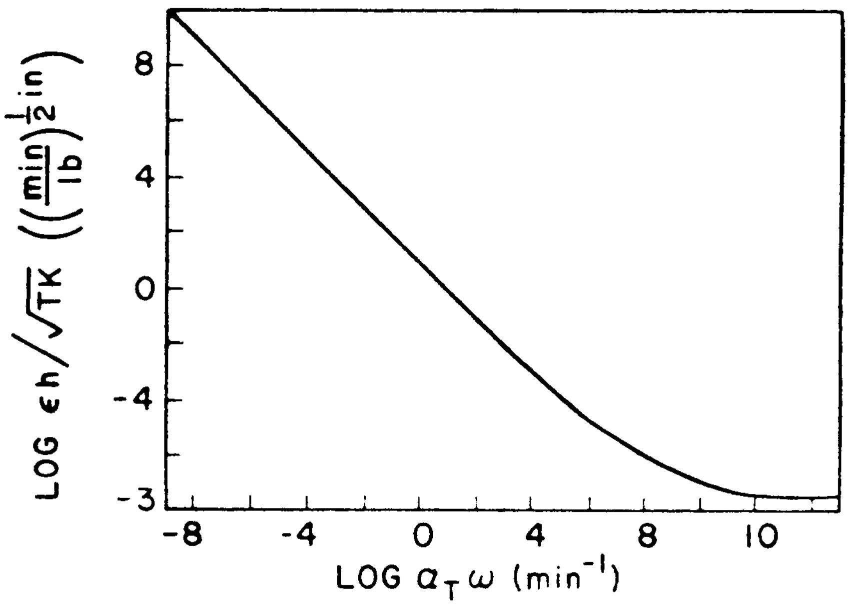

FIG. 14 TEMP.-THICKNESS VS FREQ. FOR SINE STRAIN

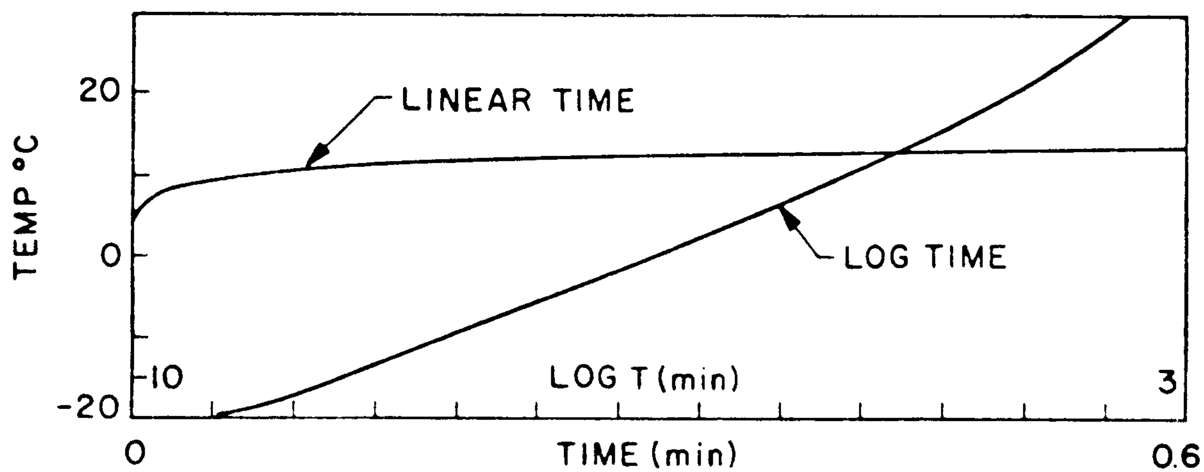

FIG. 15 ADIABATIC TEMP RISE $\epsilon = 0.10 \sin 60t$
$T_0 = -20°C$ $C = 1.8$ in-lb/in^3 °C

SUMMARY

A set of equations have been derived for stress, strain, and energy responses of linear viscoelastic systems for a wide range of inputs, these equations were then employed to compute the mechanical properties of the material Solithane 113. The temperature conditions for isothermal and adiabatic testing of the material were then calculated for sinusoidal strain inputs and graphs presented for measuring a specimen thickness or an adiabatic temperature increase.

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APPENDIX 1

VISCOELASTIC BEHAVIOR OF SOLITHANE 113

From experimental relaxation data for Solithane 113, a Prony series was fitted to the relaxation modulus curve, using 42 decades of time to collocate to points which will span the transition region from asymptotics E glassy to E rubbery shown in Figure 3. The following list of equations, expanded from Equations 3, 11, and 14, are used to compute the stress-strains, and the energy ratios in time.

(i) Step Strain Input

$$\epsilon = \epsilon_o \; H(t)$$

(a) Stress Response

$$\sigma(t) = \epsilon_o m_e + \sum_{i=1}^{n} m_i \epsilon_o e^{-\frac{t}{\tau_i}}$$

(b) Total Elastic Energy

$$W_{el} = \frac{1}{2} m_e \epsilon_o^2 + \frac{1}{2} \sum_{i=1}^{n} m_i \epsilon_o^2 e^{-\frac{2t}{\tau_i}}$$

(c) Viscous Damping

$$W_v = \frac{1}{2} \sum_{i=1}^{n} m_i \epsilon_o^2 \left(1 - e^{-\frac{2t}{\tau_i}}\right)$$

(ii) Constant Strain Rate

$$\epsilon = R t \qquad t > 0$$

(a) Stress Response

$$\sigma(t) = R t m_e + \sum_{i=1}^{n} m_i R \tau_i \left(1 - e^{-\frac{t}{\tau_i}}\right)$$

(b) Total Elastic Energy

$$W_{el} = \frac{1}{2} m_e R^2 t^2 + \frac{1}{2} \sum_{i=1}^{n} m_i R^2 \tau_i^2 \left(1 - e^{-\frac{t}{\tau_i}}\right)^2$$

(c) Viscous Dissipation

$$W_V = \sum_{i=1}^{n} m_i R^2 \tau_i^2 \left(\frac{t}{\tau_i} - 2\left(1 - e^{-\frac{t}{\tau_i}}\right) + \frac{1}{2}\left(1 - e^{-\frac{2t}{\tau_i}}\right)\right)$$

(iii) Sinusoidal Strain Rate for Steady State Stress

$$\epsilon = \epsilon_o \sin \omega t \qquad t > 0$$

(a) Stress Response

$$\sigma(t) = m_e \epsilon_o \sin \omega t + \sum_{i=1}^{n} \frac{m_i \epsilon_o \omega^2 \tau_i^2}{1 + \omega^2 \tau_i^2} \left(\sin \omega t + \frac{1}{\omega \tau_i} \cos \omega t\right)$$

(b) Total Elastic Energy

$$W_{el} = \frac{1}{2} m_e \epsilon_o^2 + \sum_{i=1}^{n} m_i \epsilon_o^2 \left(1 + (\omega \tau_i)^2\right)^{-1} \sin^2(\omega t + \phi_c)$$

(c) Viscous Dissipation

$$W_V = \frac{1}{4}\sum_{i=1}^{n} m_i \omega \tau_i \epsilon_o^2 (1+\omega^2\tau_i^2)^{-1}(2\omega t - \sin 2(\omega t+\phi_c) + \sin 2\phi_c)$$

where

$$\sin 2\phi_c = \frac{2\omega\tau_i}{1+\omega^2\tau_i^2}$$

EQUATIONS FOR THE STRESS INPUTS PRESCRIBED

(i) <u>Step Stress Input</u>

$$\sigma = \sigma_o \; H(t)$$

(a) Strain Response

$$\epsilon(t) = \sigma_o D_g + \sum_{i=1}^{n} \sigma_o D_i (1 - e^{-\frac{t}{\tau_i}})$$

(b) Total Elastic Energy

$$W_{el} \begin{Bmatrix} \text{for } i^{th} \text{ element} \\ + \text{ glassy } '' \end{Bmatrix} = \frac{1}{2}\sigma_o^2 D_g + \int_0^t \sigma_i^s \, d\epsilon_i^s$$

$$= \frac{1}{2}\sigma_o^2 D_g + \int_0^t \frac{\epsilon_i}{D_i} \, d\epsilon_i$$

therefore W_{el} (for Kelvin Model)

$$W_{el} = \frac{1}{2}\sigma_o^2 D_g + \frac{1}{2}\sum_{i=1}^{n} \sigma_o^2 D_i (1 - e^{-\frac{t}{\tau_i}})^2$$

(c) Viscous Dissipation

W_V (for i^{th} element) $= \int_0^t \sigma_i^d \, d\epsilon_i^d$

$$= \int_0^t \frac{\tau_i}{D_i} \left(\frac{d\epsilon_i}{dt}\right)^2 dt$$

hence W_V (for Kelvin model)

$$W_V = \frac{1}{2} \sum_{i=1}^{n} D_i \sigma_0^2 \left(1 - e^{\frac{-2t}{\tau_i}}\right)$$

(ii) Constant Stress Rate

$$\sigma = Pt \qquad t > 0$$

and

$$D_{crp}(t) = D_g + \sum_{i=1}^{n} D_i \left(1 - e^{-\frac{t}{\tau_i}}\right)$$

(a) Strain Response

$$\epsilon(t) = \int_0^t D(t-\tau) \frac{\partial}{\partial \tau} \sigma(\tau) \, d\tau$$

$$= P\left(D_g t + \sum_{i=1}^{n} D_i \left(t - \tau_i \left(1 - e^{-\frac{t}{\tau_i}}\right)\right)\right)$$

(b) Total Elastic Energy

$$W_{el} = \frac{1}{2} (Pt)^2 D_g + \int_0^t \frac{\epsilon_i}{D_i} \, d\epsilon_i$$

Hence W_{el} (for Kelvin model)

$$W_{el} = \frac{1}{2}(Pt)^2 D_g + \frac{1}{2}\sum_{i=1}^{n} P^2 \tau_i^2 D_i \left(\frac{t^2}{\tau_i^2} - \frac{2t}{\tau_i}\left(1-e^{-\frac{t}{\tau_i}}\right) + 2\left(1-e^{-\frac{t}{\tau_i}}\right) - \left(1-e^{-\frac{2t}{\tau_i}}\right)\right)$$

(c) Viscous Dissipation

(for i^{th} element) $$W_V = \int_0^t \frac{\tau_i}{D_i}\left(\frac{d\epsilon}{dt}\right)^2 dt$$

Hence W_V (for Kelvin model)

$$W_V = \sum_{i=1}^{n} (P\tau_i)^2 D_i \left(\frac{t}{\tau_i} - 2\left(1-e^{-\frac{t}{\tau_i}}\right) + \frac{1}{2}\left(1-e^{-\frac{2t}{\tau_i}}\right)\right)$$

(iii) Sinusoidal Stress Input

$$\sigma = \sigma_0 \sin \omega t \qquad\qquad t > 0$$

(a) Strain Response

$$\epsilon(t) = \sigma_0 \int_0^t \left(D_g + \sum_{i=1}^{n} D_i \left(1 - e^{-\left(\frac{t-\tau}{\tau_i}\right)}\right)\right) \omega \cos \omega \tau \, d\tau$$

$$= D_g \sigma_0 \sin \omega t + \sum_{i=1}^{n} D_i \sigma_0 \sin \omega t - \frac{\omega \sigma_0 D_i \tau_i}{1+\omega^2 \tau_i^2}\left(\cos \omega t + \omega \tau_i \sin \omega t - e^{-\frac{t}{\tau_i}}\right)$$

(b) Total Elastic Energy

$$W_{el} = \frac{1}{2}\sigma_0^2(t)D_g + \int_0^t \frac{\epsilon_i^2}{2D_i}\,d\epsilon_i$$

Hence W_{el} (for Kelvin Model)

$$W_{el} = \frac{1}{2}D_g\sigma_0^2 \sin^2\omega t + \frac{1}{2}\sum_{i=1}^{n} D_i\sigma_0^2\left(\sin\omega t - \frac{\omega\tau_i}{1+\omega^2\tau_i^2}\left(\cos\omega t + \omega\tau_i\sin\omega t - e^{-\frac{t}{\tau_i}}\right)\right)^2$$

(c) Viscous Dissipation

$$W_V \text{ (for } i^{th} \text{ element)} = \int_0^t \frac{\tau_i}{D_i}\left(\frac{d\epsilon_i}{dt}\right)^2 dt$$

Hence W_V (for Kelvin model)

$$W_V = \sum_{i=1}^{n} D_i\sigma_0^2\frac{\omega^2\tau_i^2}{(1+\omega^2\tau_i^2)^2}\left[2e^{-\frac{t}{\tau_i}}\cos\omega t - 1 - \frac{1}{2}\cos 2\omega t - \frac{1}{2}e^{-\frac{2t}{\tau_i}} + \frac{1}{2}t\omega^2\tau_i + \frac{1}{2}\frac{t}{\tau_i} + \sin 2\omega t\left(\frac{1-\omega^2\tau_i^2}{4\omega\tau_i}\right)\right]$$

APPENDIX 2

DUAL STRAIN HISTORIES WITH TRANSIENT SOLUTION GIVEN FOR STRESS RESPONSE, ETC.

$$\epsilon(t) = R_1\, t\, H(t_0 - t) + \left(R_1 t_0 + R_2 (t - t_0) + \epsilon_0 \sin \omega (t - t_0)\right) H(t - t_0)$$

(a) Stress Response

$$\sigma(t) = m_e\, \epsilon(t) + \sum_{i=1}^{n} m_i\, e^{-\frac{t}{\tau_i}} \left(I_1 + I_2 + I_3\right)$$

where

$$I_1 = R_1 \tau_i \left(e^{\frac{t_0}{\tau_i}} - 1\right)$$

$$I_2 = R_2 \tau_i \left(e^{\frac{t}{\tau_i}} - e^{\frac{t_0}{\tau_i}}\right)$$

and

$$I_3 = \frac{\epsilon_0 \omega \tau_i^2}{1 + \omega^2 \tau_i^2} \left(e^{\frac{t}{\tau_i}} \left(\frac{\cos \omega (t - t_0)}{\tau_i} + \omega \sin \omega (t - t_0)\right) - \frac{e^{\frac{t_0}{\tau_i}}}{\tau_i}\right)$$

(b) Total Elastic Energy

$$W_{el} = \frac{1}{2} m_e\, \epsilon^2(t) + \frac{1}{2} \sum_{i=1}^{n} m_i\, e^{-\frac{2t}{\tau_i}} \left(I_1 + I_2 + I_3\right)^2$$

(c) Viscous Dissipation

$$W_v = \int_0^t e^{-\frac{2t}{\tau_i}}\, I^2\, dt$$

$$= \int_0^{t_0} e^{-\frac{2t}{\tau_i}} I_1^2\, dt + \int_{t_0}^{t} e^{-\frac{2t}{\tau_i}} \left(I_1(t_0) + I_2 + I_3\right)^2 dt.$$

Expanding and simplifying

$$
\begin{aligned}
W_v = \sum_{i=1}^{n} m_i \Bigg[& R_1^2 \tau_i^2 \left(\frac{t_0}{\tau_i} - 1 + e^{-\frac{t_0}{\tau_i}} - \frac{1}{2} e^{-\frac{2(t-t_0)}{\tau_i}} + e^{-\frac{(2t-t_0)}{\tau_i}} - \frac{1}{2} e^{-\frac{2t}{\tau_i}} \right) \\
& + R_2^2 \tau_i^2 \left(\frac{t-t_0}{\tau_i} + 2e^{-\frac{(t-t_0)}{\tau_i}} - \frac{e^{-\frac{2(t-t_0)}{\tau_i}}}{2} - \frac{3}{2} \right) \\
& + \left(\frac{\epsilon_0 \omega \tau_i}{1+\omega^2 \tau_i^2} \right)^2 \Bigg(2e^{-\frac{(t-t_0)}{\tau_i}} \cos \omega(t-t_0) + \frac{1}{2} \Bigg(\frac{t-t_0}{\tau_i} \left(1+\omega^2 \tau_i^2\right) \\
& \qquad + \sin 2\omega(t-t_0) \left(\frac{1-\omega^2 \tau_i^2}{2\omega \tau_i} \right) - \cos 2\omega(t-t_0) \\
& \qquad - e^{-\frac{2(t-t_0)}{\tau_i}} + 2 \Bigg) \Bigg) \\
& + R_1 R_2 \tau_i^2 \left(2e^{-\frac{t}{\tau_i}} + e^{-\frac{2(t-t_0)}{\tau_i}} + 1 - 2e^{-\frac{(t-t_0)}{\tau_i}} - e^{-\frac{(2t-t_0)}{\tau_i}} - e^{-\frac{t_0}{\tau_i}} \right) \\
& + 2 R_2 \tau_i \left(\frac{\epsilon_0 \omega \tau_i}{1+\omega^2 \tau_i^2} \right) \Bigg(\frac{\sin \omega(t-t_0)}{\omega \tau_i} + \cos \omega(t-t_0) \left(e^{-\frac{(t-t_0)}{\tau_i}} - 1 \right) \\
& \qquad - \frac{1}{2} e^{-\frac{2(t-t_0)}{\tau_i}} + e^{-\frac{(t-t_0)}{\tau_i}} - \frac{1}{2} \Bigg) \\
& + R_1 \tau_i \left(\frac{\epsilon_0 \omega \tau_i}{1+\omega^2 \tau_i^2} \right) \left(1 - e^{-\frac{t_0}{\tau_i}} \right) \left(e^{-\frac{2(t-t_0)}{\tau_i}} - 1 + 2e^{-\frac{(t-t_0)}{\tau_i}} \cos \omega(t-t_0) \right) \Bigg]
\end{aligned}
$$

APPENDIX 3

As explained in the test it is sometimes useful to find a strain history that will reach a given point in ϵ, σ space. Outlined below is a method for finding the related dual strains that will reach a given ϵ_b, σ_b point. In order to reach a unique solution we will take

$$R_1/R_2 = \text{constant} = K \quad \text{and} \quad R_1 T_1 = \epsilon_b/c \quad (c = \text{constant})$$

Thus:

$$\epsilon_b = \frac{\epsilon_b}{c} + R_2 (t_2 - t_1)$$

$$= \frac{\epsilon_b}{c} + t_2 \frac{R_1}{K} - \frac{R_1 t_1}{K}$$

$$= \frac{\epsilon_b}{c} - \frac{\epsilon_b}{cK} + t_2 \frac{R_1}{K}$$

Thus

$$R_1 t_2 = \epsilon_b \left(1 + \frac{1}{cK} - \frac{1}{c}\right) = K'$$

The problem is therefore reduced to finding the R_1 that will produce σ_b. Because of the elastic limit not all ϵ_b, σ_b points can be reached. In order for a solution to exist $\sigma_b/\epsilon_b \geqslant E_{rubbery}$.

Computing: to isolate R_1 to within one decade, run through R_1 values starting at R_1 very small (say 10^{-8}) and go to R_1 very large (say 10^{10}). Run through these R_1 values by powers of 10 calculating $\sigma(R_1)$ each time and comparing it with σ_b. Now σ is monotonically increasing in R_1 so if ϵ_b, σ_b is a good point there will be a decade for R_1 between 10^{-4} and 10^{10} for which σ will be less than σ_b to the left and greater than σ_b to the right. Once this decade is found, cut it in half and see if the σ for this R_1 is less than σ_b. If so, R_1 lies in the upper half. This is repeated until R_1 is satisfactorily resolved. It takes on the average 100 iterations to find an R_1 which yields a σ within 0.01% of σ_b. The ϵ_b level is automatically reached and once R_1 is found, $R_2 t_1$ and t_2 are also defined.

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MECHANICAL PROPERTY OF SOLITHANE 113

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MECHANICAL PROPERTY OF SOLITHANE 113

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JULY 1965

INTRODUCTION

In studies of polymer failure under stress, a knowledge of the mechanical properties of the material becomes a necessary pre-requisite to the understanding of the mechanics of failure. This requirement is appreciated when one realizes that the occurrence of failure is related closely to the loading history, and therefore the mechanical behavior of the material. The more detailed aspects of the study involve the comparisons of behavior under various monotonic and sinusoidal cycling histories. It often assists in these studies to look at the ability of a viscoelastic material to convert the boundary forces into elastically stored or dissipated energies. The heat build in a specimen is synonomous with dissipation and could have significant thermal effects on the materials' failure. If these effects cannot be calculated one can investigate failure isothermally, using appropriate geometry and test conditions. The other extreme temperature condition worthy of consideration in testing is to produce a test environment in order that the temperature in the material increases adiabatically. This kind of information would also be useful when the dissipation is large and failure occurs over short intervals of time.

This paper is concerned with the mechanical properties of a test material known as Solithane 113 which is being used for studies in failure. Some physical properties have been calculated from linear viscoelastic theory using spring and dashpot model representations. The elements of the model are determined from the relaxation modulus for the material obtain in tests at small strains in tension. The energy responses of the material to a wide spectrum of load inputs are presented in a set of graphs. Other details include the complex modulus and complex compliance behavior in which their components have been plotted as functions of the input frequency. Various strain rate histories and their combinations have also been investigated and a special case is treated where two strain rates are combined at various ratios and the results are plotted for solithane material.

The next portion of the paper deals with estimating the specimen geometry which will enable isothermal tests to be carried out under sinusoidal strain inputs. Also the adiabatic rise of temperature in a specimen is considered for sinusoidal inputs using an incremental method of approximating the temperature increases.

SECTION 1. MECHANICAL PROPERTIES

The complex modulus and energy equations have been derived from one dimensional linear viscoelastic theory using a Wiechert model representation (Figure 1) for prescribed strain inputs and a Kelvin model representation (Figure 2) for prescribed stress inputs.[1,8] The relaxation modulus curve for the material is approximated by a Prony Series representation[1,2] given in the following equation

$$E_{rel}(t) = m_e + \sum_{i=1}^{n} m_i e^{-\frac{t}{\tau_i}} \tag{1}$$

which is shown plotted with the actual relaxation curve in Figure 3.

The stress response to unit step strain is then given by the equation

$$\sigma(t) = \left(m_e + \sum_{i=1}^{n} m_i e^{-\frac{t}{\tau_i}}\right) H(t) \tag{2}$$

With this equation the general equation for stress response to a strain input can now be generated in the Duhamel superposition integral[4] and is given by the equation

$$\sigma(t) = \epsilon(t)\, m_e + \sum_{i=1}^{n} m_i \int_0^t e^{-\left(\frac{t-\tau}{\tau_i}\right)} \dot{\epsilon}(\tau)\, d\tau \tag{3}$$

From these equations the complex modulus can be determined if we let the input $\epsilon(t) = \epsilon_o e^{i\omega t}$ (4)

Then for long times

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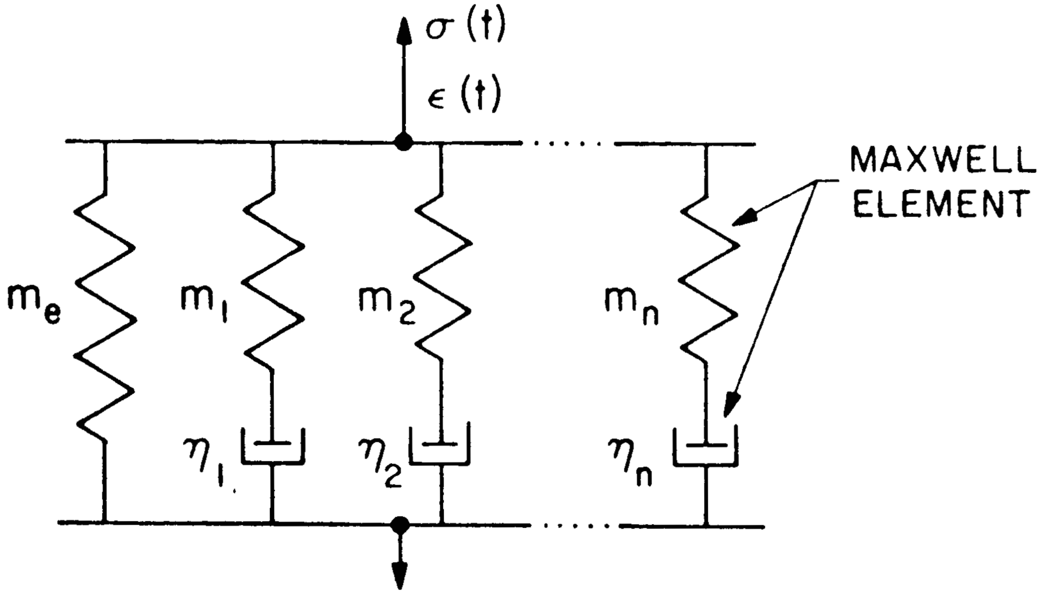

FIGURE 1 - WIECHERT MODEL

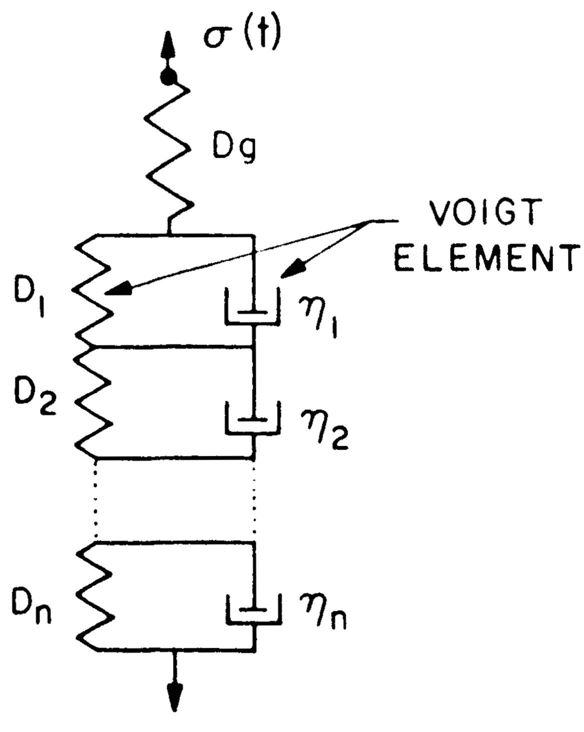

FIGURE 2 - KELVIN MODEL

$$\frac{\sigma(t)}{\epsilon_o e^{i\omega t}} \equiv E^*(\omega) = E' + i E'' \tag{5}$$

where E*(ω) is the complex or dynamic modulus which can be represented explicitly by the following set of equations

$$E^*(\omega) = m_e + \sum_{i=1}^{n} m_i \frac{i\omega\tau_i}{1 + i\omega\tau_i} \tag{6}$$

$$E' = m_e + \sum_{i=1}^{n} m_i \frac{(\omega\tau_i)^2}{(1 + \omega\tau_i)^2} \tag{7}$$

$$E'' = \sum_{i=1}^{n} m_i \frac{\omega\tau_i}{1 + (\omega\tau_i)^2} \tag{8}$$

Equations 7 and 8 were calculated for Solithane 113 and the results are shown plotted as a function of frequency in Figure 4.

The dynamic compliance can be expressed by the following equation

$$D^*(\omega) = D' - i D'' \tag{9}$$

and is defined as the inverse of the dynamic modulus. It is shown plotted in a similar manner for D' and D'' in Figure 5. The following energy response equations for strain inputs were calculated for the model[3] in Figure 1.

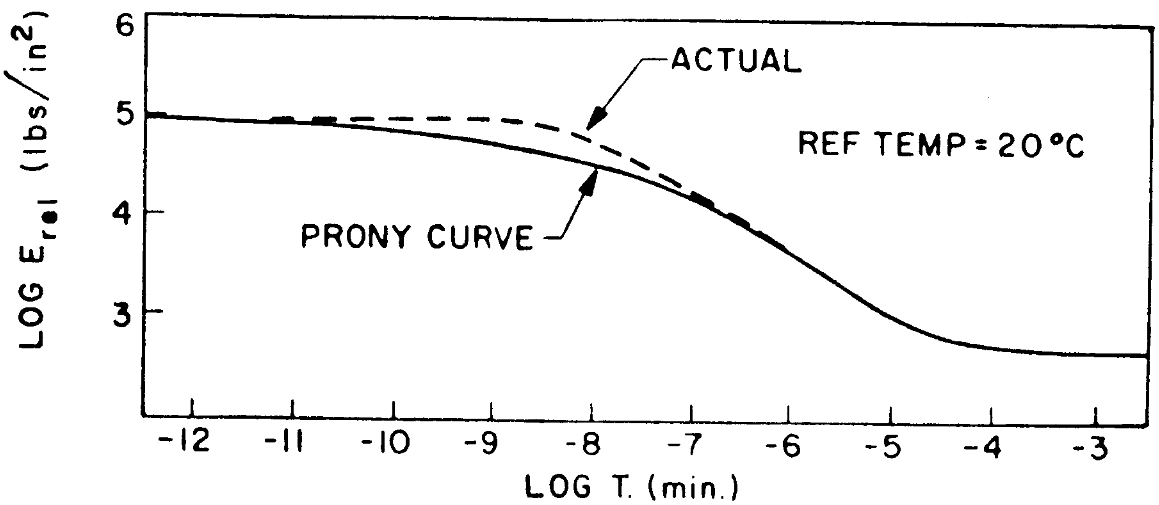

FIG. 3 RELAXATION MODULUS FOR SOLITHANE 113

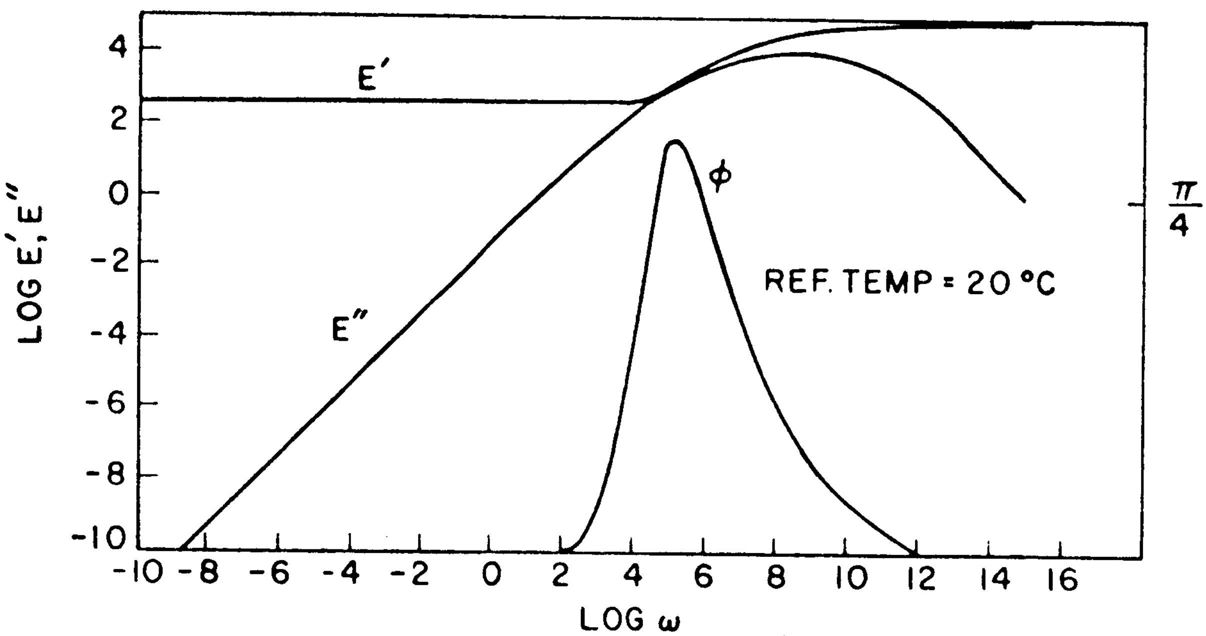

FIG. 4 COMPONENTS OF COMPLEX MODULUS; $E^* = E' + iE''$; $\tan \phi = E''/E'$

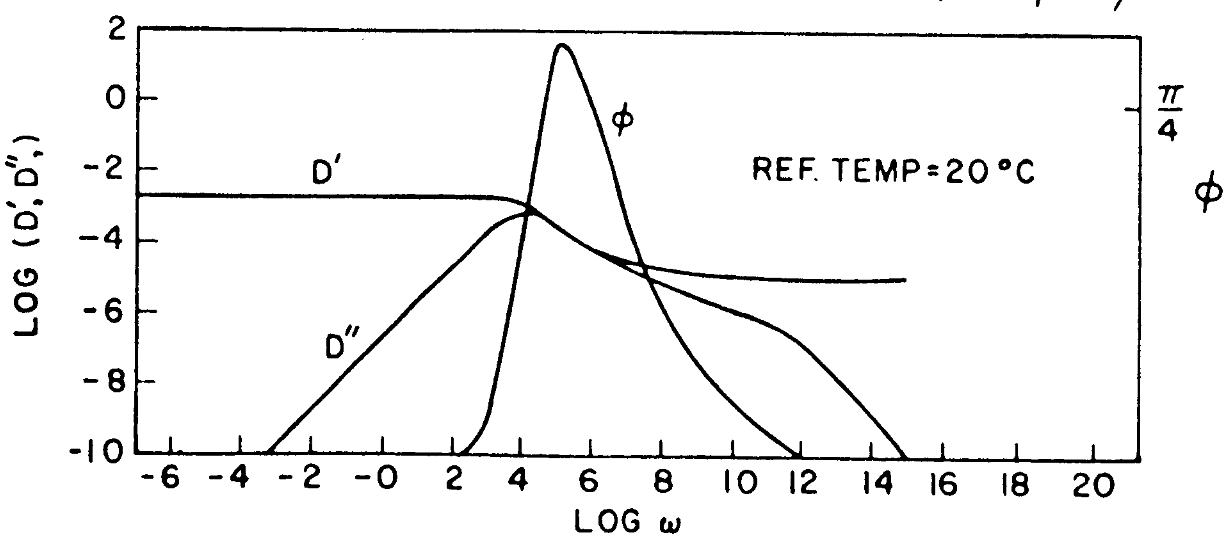

FIG. 5 COMPLEX COMPLIANCE; $D^*(\omega) = D' - iD''$ $\tan \phi = D''/D'$

The elastic energy for the i^{th} Maxwell element is given by

$$
\begin{aligned}
W_{el} &= \int_0^t \sigma_i^s \, d\epsilon_i^s \\
&= \int_0^t \frac{\sigma_i}{m_i} \, d\sigma_i \qquad (10) \\
&= \frac{\sigma_i^2}{2 m_i}
\end{aligned}
$$

Substituting Equation 3 with Equation 10 results in an energy equation for the Wiechert model representation and is given by

$$W_{el} = \frac{1}{2} m_e \epsilon^2(t) + \frac{1}{2} \sum_{i=1}^{n} m_i e^{-\frac{2t}{\tau_i}} \left(\int_0^t e^{\frac{\tau}{\tau_i}} \dot{\epsilon}(\tau) d\tau \right)^2 \qquad (11)$$

Similarly, the energy dissipation rate for the i^{th} Maxwell element is given by

$$\dot{W}_V = \sigma_i^d \dot{\epsilon}_i^d = \frac{\sigma_i^2}{\eta_i} \qquad (12)$$

now since τ_i the relaxation time $\equiv \frac{\eta_i}{m_i}$, then Equations 12 and 3 will combine to give the following equation for dissipation rate

$$\dot{W}_V = \sum_{i=1}^{n} \frac{m_i}{\tau_i} e^{-\frac{2t}{\tau_i}} \left(\int_0^t e^{\frac{\tau}{\tau_i}} \dot{\epsilon}(\tau) d\tau \right)^2 \qquad (13)$$

from which the dissipation can be obtained by the equation

$$W_V = \int_0^t \dot{W}_V \, dt = \sum_{i=1}^{n} \frac{m_i}{\tau_i} \int_0^t e^{-\frac{2t}{\tau_i}} I^2 dt \tag{14}$$

where I in Equation 14 is given by

$$I = \int_0^t e^{\frac{\tau}{\tau_i}} \dot{\epsilon}(\tau) d\tau$$

Similar equations for prescribed stresses can be established by referring to the Kelvin model representation shown in Figure 2. The strain response (or creep behavior) to input stresses is given by the equation

$$\epsilon(t) = \int_0^t D_{crp}(t-\tau) \dot{\sigma}(\tau) d\tau \tag{15}$$

where $D_{crp}(t)$ in the above expression is defined as the creep compliance and can be fitted by a Prony series equation of the form

$$D_{crp}(t) = D_g + \sum_{i=1}^{n} D_i \left(1 - e^{-\frac{t}{\tau_i}}\right) \tag{16}$$

The conversion of complex modulus data to creep compliance has been treated in other papers[9, 10] and is not therefore included in this discussion.

The above sets of equations have been used to determine the energy and stress responses for the following inputs:

$$\begin{aligned} &\text{i.} \quad \epsilon = \epsilon_o H(t) \\ &\text{ii.} \quad \epsilon = Rt \\ &\text{iii.} \quad \epsilon = \epsilon_o \sin \omega t \end{aligned} \tag{17}$$

$$\text{iv.} \quad \sigma = \sigma_o H(t)$$

$$\text{v.} \quad \sigma = Pt \qquad (17\ \text{contd.})$$

$$\text{vi.} \quad \sigma = \sigma_o \sin \omega t$$

for $t > 0$.

Details of these equations for stress, strain, and energy responses are given in Appendix 1. These equations have been used to compute the properties of Solithane 113 (on an IBM 7090/7094 digital computer) and the results are shown plotted in Figures 6-11.

MULTI-STRAIN HISTORIES

In this example of multi-strain histories the behaviors of a material could be particularly useful when investigating various paths of strain history to failure.

In order to investigate these histories, a generalized input function is used which comprised a ramp strain, followed by a ramp superimposed with a sinusoidal strain, and can be represented by the following expression,

$$\epsilon(t) = R_1 t\, H(t_o - t) + (R_1 t_o + R_2 (t - t_o) + \epsilon_o \sin \omega(t - t_o))\, H(t - t_o) \qquad (18)$$

for $t > 0$.

The stress and energy results for the above input are given in Appendix 2. Various combinations can be obtained from Equation 18 and their portions of the solution given in Appendix 2 are easily distinguishable.

In studies of various strain histories to failure it is often desirable to obtain the unique values of a given type of strain history for a prescribed point in $\sigma - \epsilon$ space. An example is now given of determining two constant

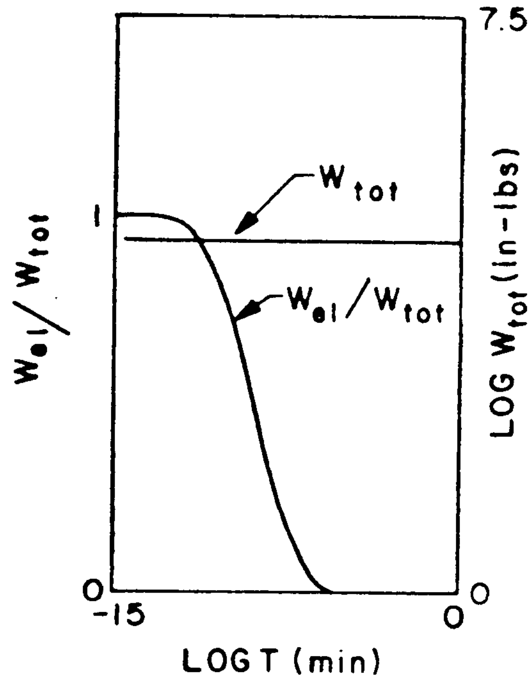

FIG.6 ENERGY RESPONSE UNIT STEP STRAIN

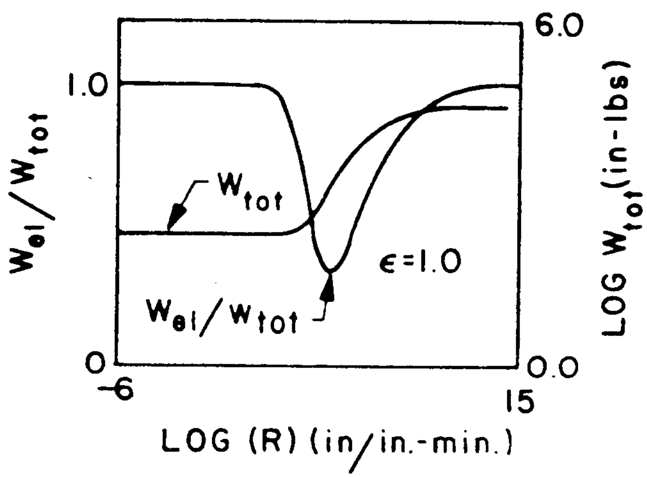

FIG.7 ENERGY RESPONSE CONSTANT STRAIN RATE

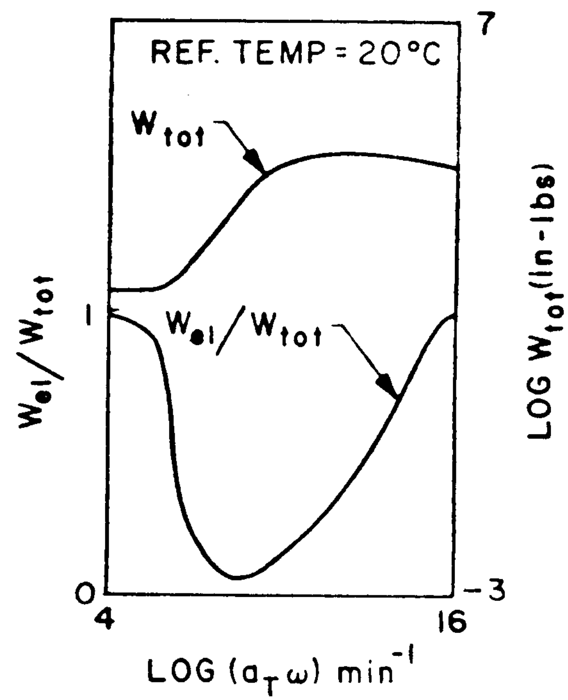

FIG.8 ENERGY RESPONSE SINE STRAIN

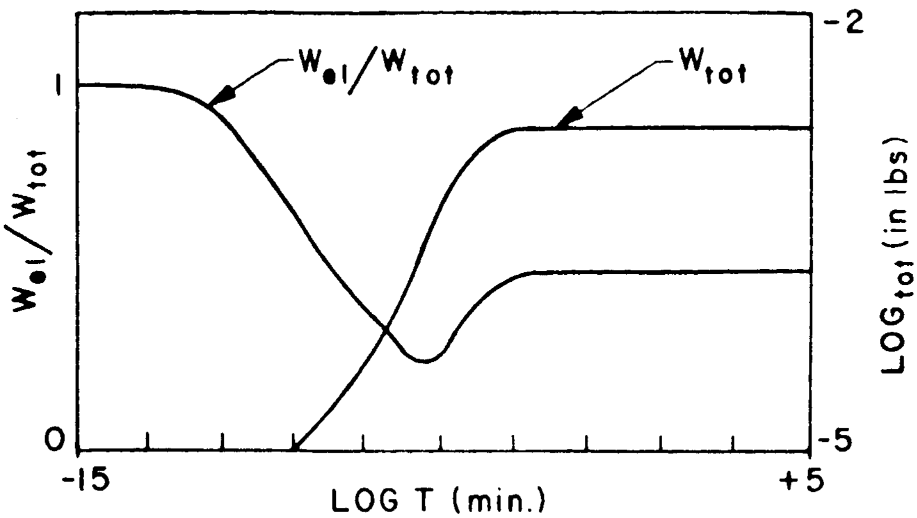

FIG. 9 ENERGY RESPONSE FOR A STEP STRESS

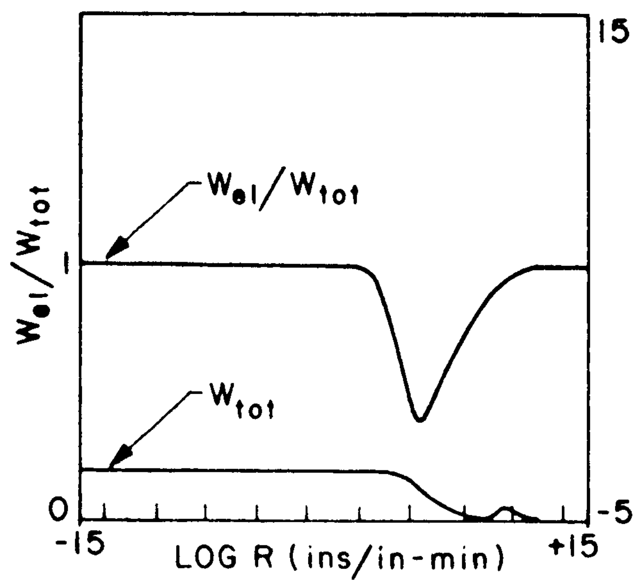

FIG. 10 ENERGY RESPONSE TO A CONSTANT STRESS RATE

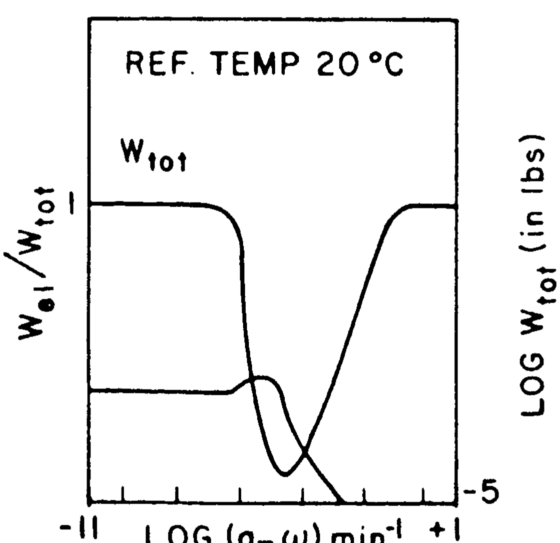

FIG. 11 ENERGY RESPONSES TO SINUSOIDAL STRESS

strain rates R_1 and R_2 in which a ratio R_1/R_2 is prescribed, the position of change known between R_1 and R_2 as a strain and the final point in stress-strain space known.

The solution for R_1 (or R_2) can be obtained by an iterative procedure used with the aid of a high speed computer. Details of this iterative method are outlined in Appendix 3. In Figure 12 the stress-strain curve for three ratios of R_1/R_2 are shown for a given point in σ - ϵ space using Solithane 113. The energies involved in reaching this stress-strain level by the three paths are shown in Figure 13. These results should be treated cautiously because considerable deviations could occur between these curves and experimental results due to the non-linearity of the material at finite strains.[11] The interpretation of these results (Figures 12 and 13) in relation to failure is discussed in another paper.[12]

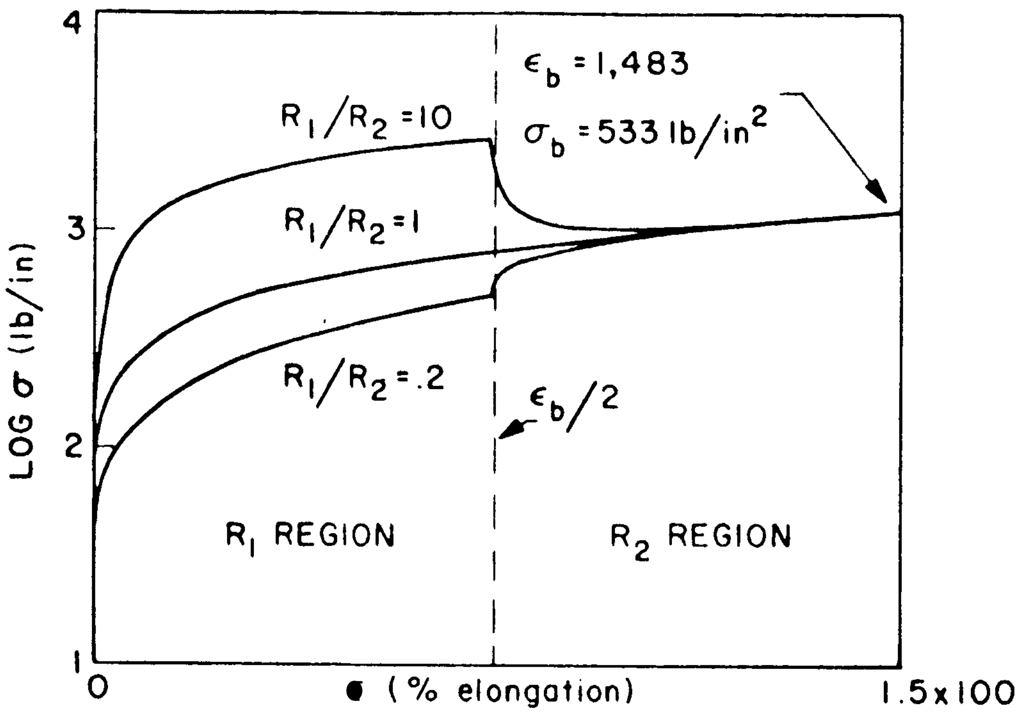

FIG.12 STRESS STRAIN RESPONSES TO DUAL CONSTANT STRAIN RATES

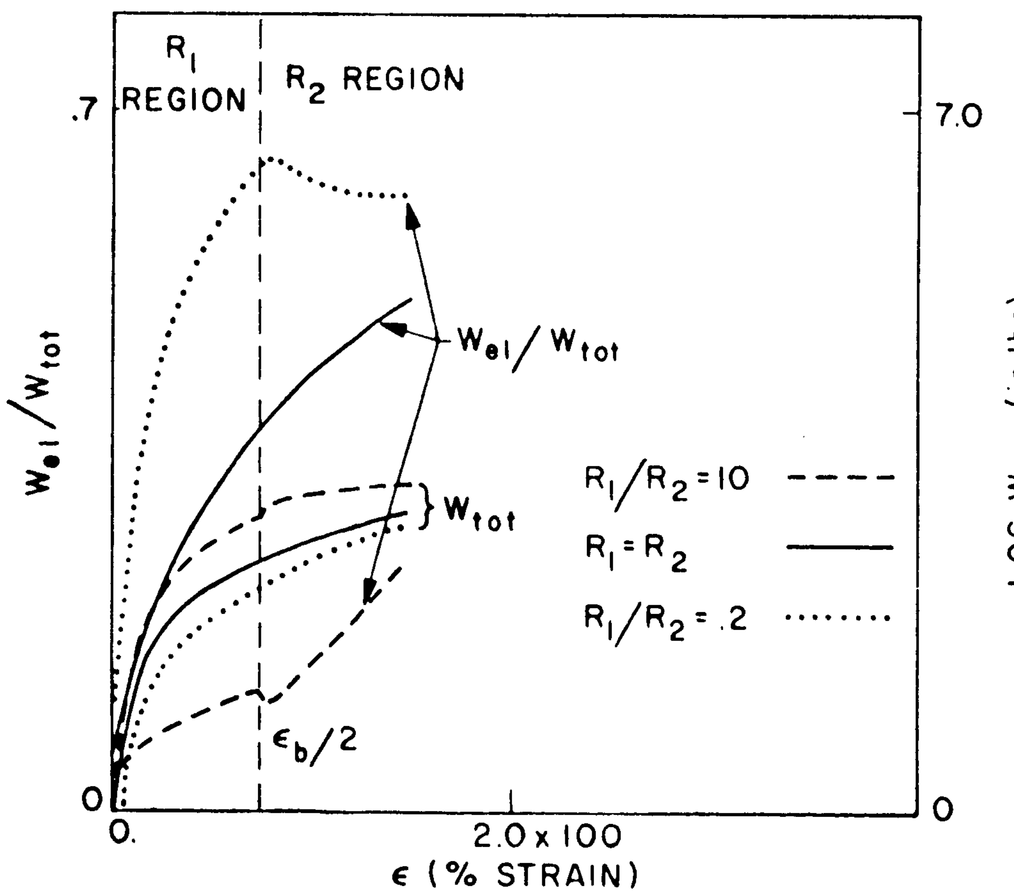

FIG.13 ENERGY RESPONSES VERSUS STRAIN FOR DUAL CONSTANT STRAIN RATE

SECTION 2. TEMPERATURE CONDITIONS

When dissipation is significant it becomes mandatory to calculate the thermal behavior of the material and details of this type of calculation are given in another paper,[8] indicating that the presence of sizeable temperature effects could cause much difficulty in failure studies. Considerable simplification might be achieved if a uniform and constant temperature state can be realized in testing for sinusoidal inputs. An approximate set of calculations have been made to estimate the upper limit of the input frequency which will give near uniform temperature distribution across the thickness of the specimen.

The other case considered is adiabatic increase in the temperature of a specimen for sinusoidal input These calculations of temperature increase are based on the effect of an incremental temperature change on the properties of the material. Details of these cases of isothermal and adiabatic temperature conditions for sinusoidal inputs are given in the following

(a) <u>Isothermal tests</u>

If the test specimen is considered, for simplicity, to be an infinite sheet of uniform thickness with flat surfaces and that both surface temperatures of the specimen are taken to be constant at zero, then the heat flow could be assumed as one dimensional and expressed by the equation[5]

$$\frac{\partial^2 T}{\partial X^2} - \frac{1}{K_1}\frac{\partial T}{\partial t} = -\frac{W_v'}{K} \tag{19}$$

where T = temperature

t = time

K_1 = thermal diffusivity

K = thermal conductivity

and W_V' = dissipation rate.

Assume that W_V' is constant with respect to time and distance x, that is

$$-\frac{1}{K_1}\frac{\partial T}{\partial t} = 0$$

then, for a thickness $-\ell < x < \ell$, the steady state solution for temperature distribution across the specimen satisfying these conditions can be expressed by the equation

$$T(x) = W_V'(\ell^2 - x^2)/2K \tag{20}$$

when $x = 0$, an equation for half thickness ℓ is given by

$$\ell = \sqrt{\frac{2TK}{W_V'}} \tag{21}$$

The dissipation rate W_V' can be obtained from the following equation

$$W_V' = \frac{W_V \text{ per cycle}}{\text{time of one cycle}} = \text{an average dissipation rate} \tag{22}$$

The ratios of thickness and $T(x = 0)$ versus ω_{a_T} have been computed from the normalized expression

$$\frac{\ell \epsilon_o}{\sqrt{KT}} = \sqrt{\frac{\epsilon_o^2}{W_V'}}$$

and plotted out as

$$\log \frac{\epsilon_o \ell}{\sqrt{KT}} \qquad \text{versus} \qquad \log \omega a_T$$

for Solithane 113 in Figure 14. The frequency ω can be determined as an upper limit when a thickness and temperature T at $x = 0$ are prescribed. It should be observed in applying Figure 14 that for prescribing T at $x = 0$ the variation in dissipation across the specimen must be sufficiently small that it can be considered uniform.

(b) Adiabatic temperature increases for sinusoidal strain input.

Let W_{V_1} be defined as the dissipation per cycle at temperature $T = T_1$ (see Appendix 1), then W_{V_n} is the dissipation per cycle at temperature $T = T_n$

$$T_n \Rightarrow \omega_n = \omega_o \, a_{T_n}$$

where a_{T_n} is the temperature-frequency shift factor given by the WLF equation.[7]

Let N = number of cycles at W_{V_n} dissipation necessary to cause a temperature change at ΔT_{n+1} given by the equation

$$\Delta T_{n+1} = T_{n+1} - T_n = \Delta T \text{ (constant)}$$

Let C be the heat capacity per unit volume,

then $$\frac{N}{C} W_{V_{n+1}} \leqslant \Delta T_{n+1} \leqslant \frac{N}{C} W_{V_n}$$

It was found that either side of the equality gave comparable results provided ΔT was made small.

Therefore in general

$$N_k \doteq \frac{\Delta T_o . C}{\bar{W}_V [\omega_o \; a_T(T_o + (K-1)\Delta T_o)}$$

with the additional simultaneous relations

$$t_m = \frac{2\pi}{\omega_o} \sum_{K=1}^{m} N_K$$

$$T_m(t) = T_o + m T_o$$

Example of these calculations are given for Solithane 113 in Figure 15, the initial temperature $T_o = -20^oC$ and a frequency of one radian per second.

Note that due to the strong initial dissipation the temperature rises first rapidly and then, as the material heats up and dissipation increases, the rate of temperature decreases also.

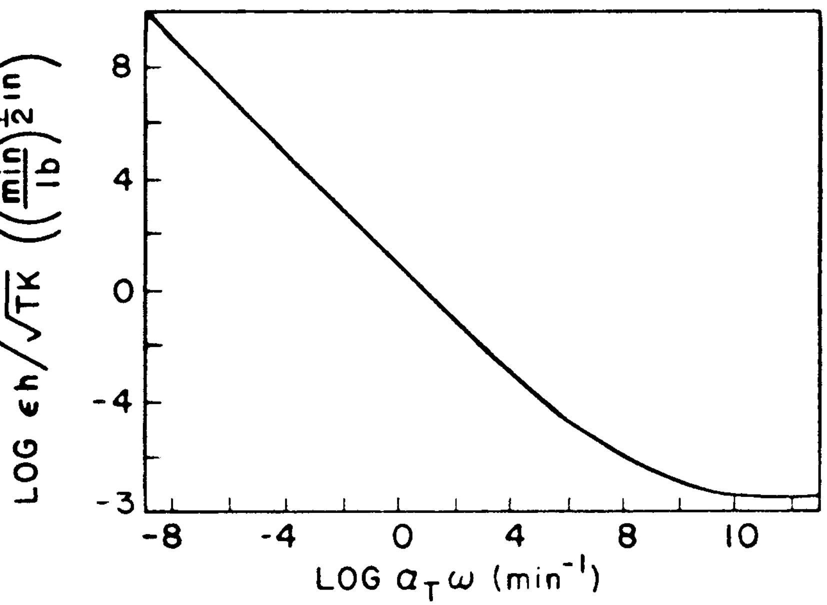

FIG. 14 TEMP.-THICKNESS VS FREQ. FOR SINE STRAIN

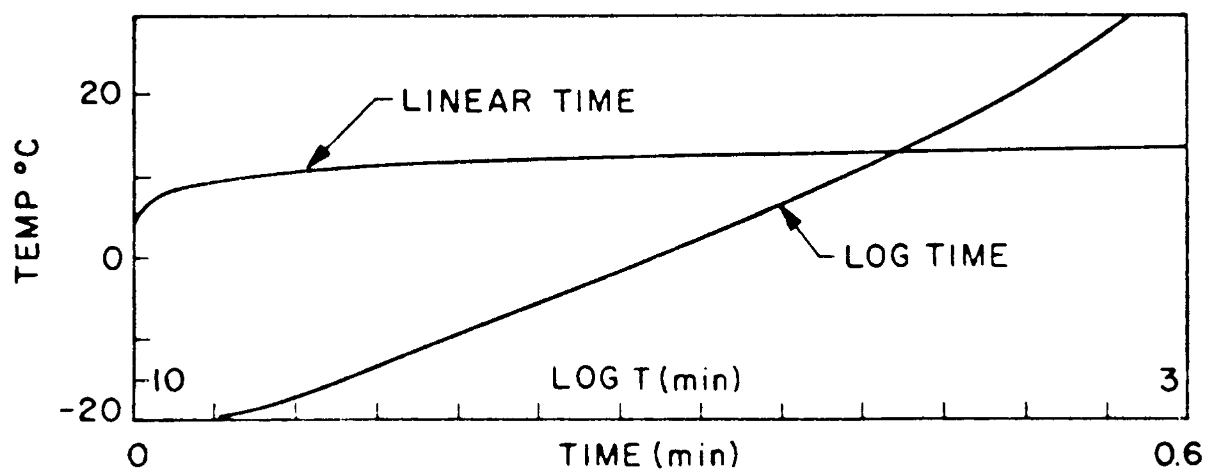

FIG. 15 ADIABATIC TEMP RISE $\epsilon = 0.10 \sin 60t$
$T_0 = -20°C$ $C = 1.8$ in-lb/in^3 °C

SUMMARY

A set of equations have been derived for stress, strain, and energy responses of linear viscoelastic systems for a wide range of inputs, these equations were then employed to compute the mechanical properties of the material Solithane 113. The temperature conditions for isothermal and adiabatic testing of the material were then calculated for sinusoidal strain inputs and graphs presented for measuring a specimen thickness or an adiabatic temperature increase.

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APPENDIX 1

VISCOELASTIC BEHAVIOR OF SOLITHANE 113

From experimental relaxation data for Solithane 113, a Prony series was fitted to the relaxation modulus curve, using 42 decades of time to collocate to points which will span the transition region from asymptotics E glassy to E rubbery shown in Figure 3. The following list of equations, expanded from Equations 3, 11, and 14, are used to compute the stress-strains, and the energy ratios in time.

(i) Step Strain Input

$$\epsilon = \epsilon_o \; H(t)$$

(a) Stress Response

$$\sigma(t) = \epsilon_o m_e + \sum_{i=1}^{n} m_i \epsilon_o e^{-\frac{t}{\tau_i}}$$

(b) Total Elastic Energy

$$W_{el} = \frac{1}{2} m_e \epsilon_o^{\,2} + \frac{1}{2} \sum_{i=1}^{n} m_i \epsilon_o^{\,2} e^{-\frac{2t}{\tau_i}}$$

(c) Viscous Damping

$$W_v = \frac{1}{2} \sum_{i=1}^{n} m_i \epsilon_o^{\,2} \left(1 - e^{-\frac{2t}{\tau_i}}\right)$$

(ii) Constant Strain Rate

$$\epsilon = Rt \qquad t > 0$$

(a) Stress Response

$$\sigma(t) = Rt\,m_e + \sum_{i=1}^{n} m_i R \tau_i \left(1 - e^{-\frac{t}{\tau_i}}\right)$$

(b) Total Elastic Energy

$$W_{el} = \frac{1}{2} m_e R^2 t^2 + \frac{1}{2} \sum_{i=1}^{n} m_i R^2 \tau_i^2 \left(1 - e^{-\frac{t}{\tau_i}}\right)^2$$

(c) Viscous Dissipation

$$W_V = \sum_{i=1}^{n} m_i R^2 \tau_i^2 \left(\frac{t}{\tau_i} - 2\left(1 - e^{-\frac{t}{\tau_i}}\right) + \frac{1}{2}\left(1 - e^{-\frac{2t}{\tau_i}}\right)\right)$$

(iii) Sinusoidal Strain Rate for Steady State Stress

$$\epsilon = \epsilon_o \sin \omega t \qquad t > 0$$

(a) Stress Response

$$\sigma(t) = m_e \epsilon_o \sin \omega t + \sum_{i=1}^{n} \frac{m_i \epsilon_o \omega^2 \tau_i^2}{1 + \omega^2 \tau_i^2} \left(\sin \omega t + \frac{1}{\omega \tau_i} \cos \omega t\right)$$

(b) Total Elastic Energy

$$W_{el} = \frac{1}{2} m_e \epsilon_o^2 + \sum_{i=1}^{n} m_i \epsilon_o^2 \left(1 + (\omega \tau_i)^2\right)^{-1} \sin^2(\omega t + \phi_c)$$

(c) Viscous Dissipation

$$W_V = \frac{1}{4}\sum_{i=1}^{n} m_i \omega \tau_i \epsilon_o^2 (1+\omega^2\tau_i^2)^{-1}(2\omega t - \sin 2(\omega t+\phi_c) + \sin 2\phi_c)$$

where

$$\sin 2\phi_c = \frac{2\omega\tau_i}{1+\omega^2\tau_i^2}$$

EQUATIONS FOR THE STRESS INPUTS PRESCRIBED

(i) <u>Step Stress Input</u>

$$\sigma = \sigma_o \; H(t)$$

(a) Strain Response

$$\epsilon(t) = \sigma_o D_g + \sum_{i=1}^{n} \sigma_o D_i (1 - e^{-\frac{t}{\tau_i}})$$

(b) Total Elastic Energy

$$W_{el} \left\{\begin{matrix}\text{for } i^{th} \text{ element} \\ \text{+ glassy } \text{"}\end{matrix}\right\} = \frac{1}{2}\sigma_o^2 D_g + \int_0^t \sigma_i^s \, d\epsilon_i^s$$

$$= \frac{1}{2}\sigma_o^2 D_g + \int_0^t \frac{\epsilon_i}{D_i} \, d\epsilon_i$$

therefore W_{el} (for Kelvin Model)

$$W_{el} = \frac{1}{2}\sigma_o^2 D_g + \frac{1}{2}\sum_{i=1}^{n} \sigma_o^2 D_i (1 - e^{-\frac{t}{\tau_i}})^2$$

(c) Viscous Dissipation

W_V (for i^{th} element) $= \int_0^t \sigma_i^d \, d\epsilon_i^d$

$$= \int_0^t \frac{\tau_i}{D_i} \left(\frac{d\epsilon_i}{dt}\right)^2 dt$$

hence W_V (for Kelvin model)

$$W_V = \frac{1}{2} \sum_{i=1}^{n} D_i \sigma_o^2 \left(1 - e^{\frac{-2t}{\tau_i}}\right)$$

(ii) Constant Stress Rate

$$\sigma = P t \qquad t > 0$$

and

$$D_{crp}(t) = D_g + \sum_{i=1}^{n} D_i \left(1 - e^{-\frac{t}{\tau_i}}\right)$$

(a) Strain Response

$$\epsilon(t) = \int_0^t D(t-\tau) \frac{\partial}{\partial \tau} \sigma(\tau) \, d\tau$$

$$= P\left(D_g t + \sum_{i=1}^{n} D_i \left(t - \tau_i \left(1 - e^{-\frac{t}{\tau_i}}\right)\right)\right)$$

(b) Total Elastic Energy

$$W_{el} = \frac{1}{2} (Pt)^2 D_g + \int_0^t \frac{\epsilon_i}{D_i} \, d\epsilon_i$$

Hence W_{el} (for Kelvin model)

$$W_{el} = \frac{1}{2}(Pt)^2 D_g + \frac{1}{2}\sum_{i=1}^{n} P^2 \tau_i^2 D_i \left(\frac{t^2}{\tau_i^2} - \frac{2t}{\tau_i}\left(1-e^{-\frac{t}{\tau_i}}\right) + 2\left(1-e^{-\frac{t}{\tau_i}}\right) - \left(1-e^{-\frac{2t}{\tau_i}}\right)\right)$$

(c) Viscous Dissipation

(for i^{th} element) $$W_V = \int_0^t \frac{\tau_i}{D_i}\left(\frac{d\epsilon}{dt}\right)^2 dt$$

Hence W_V (for Kelvin model)

$$W_V = \sum_{i=1}^{n} (P\tau_i)^2 D_i \left(\frac{t}{\tau_i} - 2\left(1-e^{-\frac{t}{\tau_i}}\right) + \frac{1}{2}\left(1-e^{-\frac{2t}{\tau_i}}\right)\right)$$

(iii) <u>Sinusoidal Stress Input</u>

$$\sigma = \sigma_0 \sin \omega t \qquad t > 0$$

(a) Strain Response

$$\epsilon(t) = \sigma_0 \int_0^t \left(D_g + \sum_{i=1}^{n} D_i\left(1-e^{-\left(\frac{t-\tau}{\tau_i}\right)}\right)\right)\omega \cos \omega\tau \, d\tau$$

$$= D_g \sigma_0 \sin \omega t + \sum_{i=1}^{n} D_i \sigma_0 \sin \omega t - \frac{\omega \sigma_0 D_i \tau_i}{1+\omega^2\tau_i^2}\left(\cos\omega t + \omega\tau_i \sin \omega t - e^{-\frac{t}{\tau_i}}\right)$$

(b) Total Elastic Energy

$$W_{el} = \frac{1}{2}\,\sigma_o^{\,2}(t) D_g + \int_0^t \frac{\epsilon_i^{\,2}}{2D_i}\, d\epsilon_i$$

Hence W_{el} (for Kelvin Model)

$$W_{el} = \frac{1}{2} D_g \sigma_o^{\,2} \sin^2\omega t + \frac{1}{2}\sum_{i=1}^{n} D_i \sigma_o^{\,2}\Big(\sin\omega t - \frac{\omega\tau_i}{1+\omega^2\tau_i^{\,2}}\Big(\cos\omega t + \omega\tau_i \sin\omega t - e^{-\frac{t}{\tau_i}}\Big)\Big)^2$$

(c) Viscous Dissipation

$$W_V \text{ (for } i^{th} \text{ element)} = \int_0^t \frac{\tau_i}{D_i}\left(\frac{d\epsilon_i}{dt}\right)^2 dt$$

Hence W_V (for Kelvin model)

$$W_V = \sum_{i=1}^{n} D_i \sigma_o^{\,2} \frac{\omega^2\tau_i^{\,2}}{(1+\omega^2\tau_i^{\,2})^2}\left[\begin{array}{l} 2e^{-\frac{t}{\tau_i}}\cos\omega t - 1 - \frac{1}{2}\cos 2\omega t \\ -\frac{1}{2}e^{-\frac{2t}{\tau_i}} + \frac{1}{2}t\omega^2\tau_i + \frac{1}{2}\frac{t}{\tau_i} \\ + \sin 2\omega t\left(\frac{1-\omega^2\tau_i^{\,2}}{4\omega\tau_i}\right) \end{array}\right]$$

APPENDIX 2

DUAL STRAIN HISTORIES WITH TRANSIENT SOLUTION GIVEN FOR STRESS RESPONSE, ETC.

$$\epsilon(t) = R_1 t\, H(t_o - t) + \left(R_1 t_o + R_2 (t - t_o) + \epsilon_o \sin \omega(t - t_o)\right) H(t - t_o)$$

(a) Stress Response

$$\sigma(t) = m_e\, \epsilon(t) + \sum_{i=1}^{n} m_i\, e^{-\frac{t}{\tau_i}} \left(I_1 + I_2 + I_3\right)$$

where

$$I_1 = R_1 \tau_i \left(e^{\frac{t_o}{\tau_i}} - 1\right)$$

$$I_2 = R_2 \tau_i \left(e^{\frac{t}{\tau_i}} - e^{\frac{t_o}{\tau_i}}\right)$$

and

$$I_3 = \frac{\epsilon_o \omega \tau_i^2}{1 + \omega^2 \tau_i^2} \left(e^{\frac{t}{\tau_i}} \left(\frac{\cos \omega(t - t_o)}{\tau_i} + \omega \sin \omega(t - t_o)\right) - \frac{e^{\frac{t_o}{\tau_i}}}{\tau_i}\right)$$

(b) Total Elastic Energy

$$W_{el} = \frac{1}{2} m_e\, \epsilon^2(t) + \frac{1}{2} \sum_{i=1}^{n} m_i\, e^{-\frac{2t}{\tau_i}} \left(I_1 + I_2 + I_3\right)^2$$

(c) Viscous Dissipation

$$W_v = \int_0^t e^{-\frac{2t}{\tau_i}} I^2\, dt$$

$$= \int_0^{t_o} e^{-\frac{2t}{\tau_i}} I_1^2\, dt + \int_{t_o}^{t} e^{-\frac{2t}{\tau_i}} \left(I_1(t_o) + I_2 + I_3\right)^2 dt.$$

Expanding and simplifying

$$
\begin{aligned}
W_V = \sum_{i=1}^{n} m_i \Bigg[& R_1^2 \tau_i^2 \left(\frac{t_0}{\tau_i} - 1 + e^{-\frac{t_0}{\tau_i}} - \frac{1}{2} e^{-\frac{2(t-t_0)}{\tau_i}} + e^{-\frac{(2t-t_0)}{\tau_i}} - \frac{1}{2} e^{-\frac{2t}{\tau_i}} \right) \\
& + R_2^2 \tau_i^2 \left(\frac{t-t_0}{\tau_i} + 2 e^{-\frac{(t-t_0)}{\tau_i}} - \frac{e^{-\frac{2(t-t_0)}{\tau_i}}}{2} - \frac{3}{2} \right) \\
& + \left(\frac{\epsilon_0 \omega \tau_i}{1+\omega^2 \tau_i^2} \right)^2 \Bigg(2 e^{-\frac{(t-t_0)}{\tau_i}} \cos \omega (t-t_0) + \frac{1}{2} \Bigg(\frac{t-t_0}{\tau_i} \left(1+\omega^2 \tau_i^2 \right) \\
& \qquad + \sin 2\omega (t-t_0) \left(\frac{1-\omega^2 \tau_i^2}{2\omega \tau_i} \right) - \cos 2\omega (t-t_0) \\
& \qquad - e^{-\frac{2(t-t_0)}{\tau_i}} + 2 \Bigg) \Bigg) \\
& + R_1 R_2 \tau_i^2 \left(2 e^{-\frac{t}{\tau_i}} + e^{-\frac{2(t-t_0)}{\tau_i}} + 1 - 2 e^{-\frac{(t-t_0)}{\tau_i}} - e^{-\frac{(2t-t_0)}{\tau_i}} - e^{-\frac{t_0}{\tau_i}} \right) \\
& + 2 R_2 \tau_i \left(\frac{\epsilon_0 \omega \tau_i}{1+\omega^2 \tau_i^2} \right) \Bigg(\frac{\sin \omega (t-t_0)}{\omega \tau_i} + \cos \omega (t-t_0) \left(e^{-\frac{(t-t_0)}{\tau_i}} - 1 \right) \\
& \qquad - \frac{1}{2} e^{-\frac{2(t-t_0)}{\tau_i}} + e^{-\frac{(t-t_0)}{\tau_i}} - \frac{1}{2} \Bigg) \\
& + R_1 \tau_i \left(\frac{\epsilon_0 \omega \tau_i}{1+\omega^2 \tau_i^2} \right) \left(1 - e^{-\frac{t_0}{\tau_i}} \right) \left(e^{-\frac{2(t-t_0)}{\tau_i}} - 1 + 2 e^{-\frac{(t-t_0)}{\tau_i}} \cos \omega (t-t_0) \right) \Bigg]
\end{aligned}
$$

APPENDIX 3

As explained in the test it is sometimes useful to find a strain history that will reach a given point in ϵ, σ space. Outlined below is a method for finding the related dual strains that will reach a given ϵ_b, σ_b point. In order to reach a unique solution we will take

$$R_1/R_2 = \text{constant} = K \quad \text{and} \quad R_1 T_1 = \epsilon_b/c \quad (c = \text{constant})$$

Thus:

$$\epsilon_b = \frac{\epsilon_b}{c} + R_2 (t_2 - t_1)$$

$$= \frac{\epsilon_b}{c} + t_2 \frac{R_1}{K} - \frac{R_1 t_1}{K}$$

$$= \frac{\epsilon_b}{c} - \frac{\epsilon_b}{cK} + t_2 \frac{R_1}{K}$$

Thus

$$R_1 t_2 = \epsilon_b \left(1 + \frac{1}{cK} - \frac{1}{c} \right) = K'$$

The problem is therefore reduced to finding the R_1 that will produce σ_b. Because of the elastic limit not all ϵ_b, σ_b points can be reached. In order for a solution to exist $\sigma_b/\epsilon_b \geqslant E_{rubbery}$.

Computing: to isolate R_1 to within one decade, run through R_1 values starting at R_1 very small (say 10^{-8}) and go to R_1 very large (say 10^{10}). Run through these R_1 values by powers of 10 calculating $\sigma(R_1)$ each time and comparing it with σ_b. Now σ is monotonically increasing in R_1 so if ϵ_b, σ_b is a good point there will be a decade for R_1 between 10^{-4} and 10^{10} for which σ will be less than σ_b to the left and greater than σ_b to the right. Once this decade is found, cut it in half and see if the σ for this R_1 is less than σ_b. If so, R_1 lies in the upper half. This is repeated until R_1 is satisfactorily resolved. It takes on the average 100 iterations to find an R_1 which yields a σ within 0.01% of σ_b. The ϵ_b level is automatically reached and once R_1 is found, $R_2 t_1$ and t_2 are also defined.