NOTES ON THE RESTRICTED THREE BODY PROBLEM:

APPROXIMATE BEHAVIOR OF SOLUTIONS NEAR THE COLLINEAR LAGRANGIAN POINTS

C. C. Conley

Introduction

The purpose of these remarks is to describe in some detail the geometry of solutions of the restricted three body problem (as viewed in the rotating coordinate system) near those equilibrium points which are collinear with the two positive masses.

We deal only with the linearized equations, but make some qualitative observations which can be carried over without difficulty to the nonlinear equations for suitable values of the Jacobi Constant.

This report is intended to be the first in a series whose ultimate aims include an existence proof for the "periodic" solutions discovered numerically by M. Davidson [1]. Whether or not this can be accomplished remains to be seen, but it does seem clear that a thorough understanding of the behavior of orbits near the equilibrium point will be required. More will be said about this question in later reports.

From the work in this report we obtain the following qualitative picture of solutions of the linearized equations for values of the "Jacobi Constant" slightly above that of the equilibrium point.

The projections of orbits into the configuration space are constrained to lie in the region R between the two branches of a hyperbola symmetric with respect to the line, ℓ, joining the positive mass points, which line is contained in R.

We will generally restrict our attention to the portion of the phase space corresponding to a closed interval I of ℓ about the projection of the equilibrium point. Recalling that the value of the integral is fixed, we will see that this portion of the phase space is homeomorphic to $S^2 \times I$ (S^2 is the two-sphere) and so may be viewed as the space between 2-concentric spheres together with the bounding spheres.

If I is large enough we will see there there is exactly one closed orbit in this portion of the phase space. This corresponds to one of the family of periodic solutions which are known (by a theorem of Lyapounov) to exist in a neighborhood of the equilibrium point even for the nonlinear equations.

There are four "cylinders" in the phase space which abut on this periodic orbit and which are invariant under the flow. Two of these run to the outer bounding sphere and two to the inner. One of each of these two pair of cylinders corresponds to a family of solutions which is asymptotic to the periodic solution as the time goes to $+\infty$; the others to families asymptotic as time goes to $-\infty$. These cylinders act as separatrices. They separate those solutions which go from the inner to the outer sphere (or vice versa) from those that do not: in the language of the configuration space, they separate those solutions which make a transit of the region of the equilibrium from those which do not cross this region. (The existence of such cylinders for the restricted problem is apparent. From a theorem of J. Moser [2] it can be seen that they are described by real analytic functions near the equilibrium point.)

The projection of these cylinders into the configuration space covers the union of two infinite strips the boundaries of which are the enveloping lines of the solutions asymptotic to the periodic solution (figure 1). These four enveloping lines (which are tangent to the hyperbolas bounding R as well as to the periodic orbit) divide R into several regions and we will be able to determine the nature of solutions in these different regions. Further description will be easier to give later.

An amusing result is that exactly one solution from each of the four cylinders of solutions asymptotic to the periodic solution has a cusp (as viewed in the configuration space). A modification of this statement holds as well for the restricted three body problem. These four cusp points determine arcs on the hyperbolas bounding R, and any solution which cusps on these arcs is making a transit of the equilibrium region.

A statement which is perhaps a little more useful is that there are two unique solutions which are "best" for making a transit of the equilibrium region in that they take the least time. One of the (possible) difficulties in using orbits which correspond to the solutions of M. Davidson is the amount of time it is possible to spend in the region of the equilibrium.* It may be useful to have a simple criterion for decreasing

*The values of the Jacobi Constant considered here are small relative to the ones usually considered.

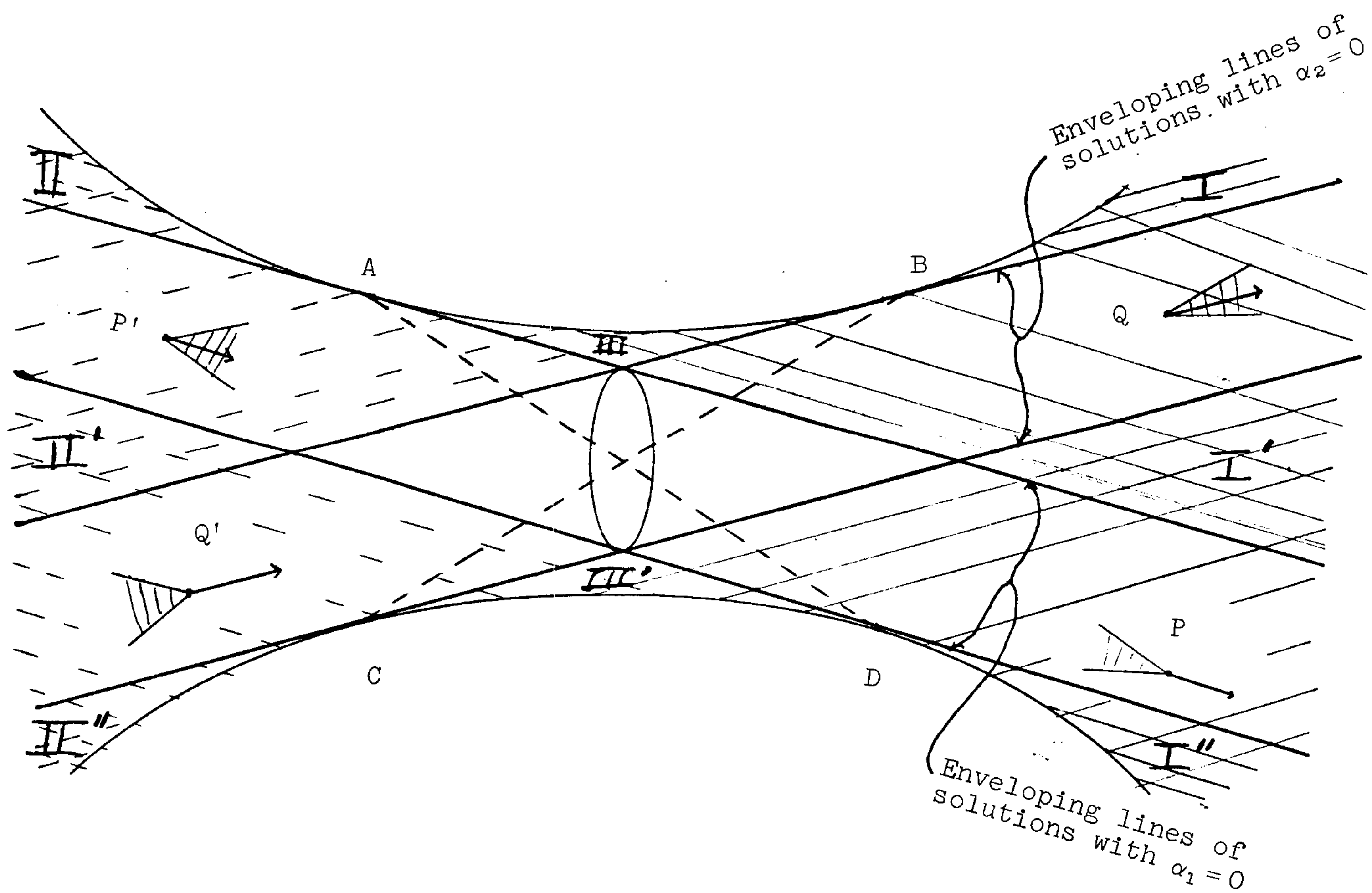

FIGURE 1. SOLUTIONS WITH VELOCITY VECTOR IN SHADED WEDGES GO ACROSS THE EQUILIBRIUM REGION

this time. An approximate means to determine the "best" orbit is given in statement eleven; a more accurate one could be derived using the result of J. Moser [2].

As stated above, these remarks have been collected primarily with a view to later applications. However, it is hoped they are of some value in themselves in gaining insight into the nature of solutions of the restricted three body problem.

1. The Equations

Without going through the arguments, we can state that the linearized equation near the equilibrium points in which we are presently interested form a hamiltonian system with Hamiltonian function:

$$H(x_1, x_2, y_1, y_2) = \frac{1}{2}\{(y_1 - \omega x_2)^2 + (y_2 + \omega x_1)^2 - a x_1^2 + b x_2^2\} \tag{1}$$

(ω, a, b are positive constants)

The equations are

$$\dot{x} = Hy \qquad \dot{y} = -Hx. \tag{2}$$

In these equations, ω is the frequency of rotation of the coordinate system; we assume ω is positive.

The constants a, b will be arbitrary positive constants in our discussion. In the case of the equilibrium point between the two positive masses of the restricted problem, $a = 2b$.* If the mass ratio is that of the Earth and Moon, then with $\omega = 1$, a is slightly larger than 8.

We introduce the following notation:

$$\hat{u} = (x_1, x_2, y_1, y_2) \qquad S = \begin{pmatrix} -a & 0 \\ 0 & b \end{pmatrix}; \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \tag{3}$$

---

*This statement is also true of the other two equilibria considered, however, the next is not.

$$\mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{3 cont.}$$

$$\Sigma = \begin{pmatrix} \omega^2 I + S & \omega J \\ -\omega J & I \end{pmatrix}$$

Our equations are then written as

$$\begin{aligned} \hat{H}(\hat{u}) &= \frac{1}{2}(\hat{u}, \Sigma \hat{u}) \\ \dot{\hat{u}} &= \mathcal{J} \hat{H}_{\hat{u}} = \mathcal{J} \Sigma \hat{u}. \end{aligned} \tag{4}$$

Now to make the computations easier we introduce the non-canonical transformation

$$\begin{aligned} \hat{u} &= Au \\ A &= \begin{pmatrix} I & 0 \\ \omega J & I \end{pmatrix} \end{aligned} \tag{5}$$

The equations then transform to:

$$\begin{aligned} \dot{u} &= Bu \\ B &= A^{-1} \mathcal{J} \Sigma A = \begin{pmatrix} 0 & I \\ -S & -2\omega J \end{pmatrix} \end{aligned} \tag{6}$$

and the integral is given by

$$\begin{aligned} H(u) &= \hat{H}(A\hat{u}) = \frac{1}{2}(u, Eu) \\ E &= A^T \Sigma A = \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} \end{aligned} \tag{7}$$

If we now write $u = (x_1, x_2, z_1, z_2)$, the equations above give $\dot{x}_1 = z_2$. Thus if we consider projections of orbits in the x-plane, $z = (z_1, z_2)$ corresponds to the tangent vector.

In this notation we have for the integral:

$$H(u) = \frac{1}{2}(z_1^2 + z_2^2 - a\,x_1^2 + b\,x_2^2)$$

2. The Phase Space

We will be primarily interested in those orbits for which

$$H = h > 0, \tag{8}$$

and will describe the projections of these orbits in the x-plane.

Statement 1. a) For $H = h$, the projected orbits arc constrained to move in the region R given by

$$R: \qquad -a\,x_1^2 + b\,x_2^2 \leq h.$$

b) If $h \geq 0$ R is a connected region, otherwise it has two components.

c) If $h > 0$, the phase space is homeomorphic to $S^2 \times E^1$ (S^2 is the 2-sphere, E^1 the real line). We will be most interested in that part of the phase space for which $|x_1| \leq c > 0$. This region can be considered as the space between two concentric spheres including the boundaries.

Proof: Only part c) needs comment. To see this statement, consider the line $x_1 = c_1$. On this line we have

$$z_1^2 + z_2^2 + b\,x_2^2 = 2h + ac_1^2$$

So the corresponding points in the phase space form a 2-sphere. The rest follows.

3. Computations

Statement 2. a) The matrix B has one pair of real eigenvalues and one pair of imaginary eigenvalues. These we denote by

$$\pm\mu, \quad \pm i\nu \quad \text{where} \quad \mu, \nu > 0.$$

b) The corresponding eigenvectors can be chosen to be:

μ	$-\mu$	$i\nu$	$-i\nu$
$v_1 = \begin{pmatrix} 1 \\ \sigma \\ \mu \\ \mu\sigma \end{pmatrix}$	$v_2 = \begin{pmatrix} 1 \\ -\sigma \\ -\mu \\ \mu\sigma \end{pmatrix}$	$w_1 = \begin{pmatrix} 1 \\ i\tau \\ i\nu \\ -\nu\tau \end{pmatrix}$	$w_2 = \bar{w}_1 = \begin{pmatrix} 1 \\ -i\tau \\ -i\nu \\ -\nu\tau \end{pmatrix}$

where σ and τ are real, $\sigma > 0$; $\tau < 0$ (cf. e) of this Statement)

c) The general solution is of the form

$$u(t) = \alpha_1 e^{\mu t} v_1 + \alpha_2 e^{-\mu t} v_2 + 2\mathrm{Re}\,(\beta e^{i\nu t} w_1)$$

where α_1, α_2 are real, β is complex.

d) The value of the integral on the solution is

$$\frac{1}{2}(u(t),\ E\,u(t)) = \alpha_1 \alpha_2 e_1 + |\beta|^2 e_2$$

where

$$e_1 = (v_1,\ E\,v_2)$$

$$e_2 = (w_1,\ E\,w_2)$$

(Note: the inner product is the real one even when vectors are complex.)

e) The constants μ, ν, σ, τ, e_1, e_2 satisfy:

1) $a - 2\omega\sigma\mu = \mu^2$; in particular, $\mu > 0$

2) $-b\sigma + 2\omega\mu = \mu^2\sigma$

3) $a + 2\omega\tau\nu = -\nu^2$; in particular, $\tau < 0$

4) $-b\tau + 2\omega\nu = -\nu^2\tau$

5) $(v_1\ E\,v_1) = -a + b\sigma^2 + \mu^2 + \sigma^2\mu^2 = 0$

6) $(w_1\ E\,w_1) = -a - b\tau^2 - \nu^2 + \nu^2\tau^2 = 0$

7) $(v_1\ E\,w_1) = -a + ib\tau\sigma + i\mu\nu - \sigma\mu\tau\nu = 0$

8) $e_1 \equiv (v_1\ E\ v_2) = -a - b\sigma^2 - \mu^2 + \sigma^2\mu^2$

$$= -2(b\sigma^2 + \mu^2) < 0$$

9) $e_2 \equiv (w_1\,E\,w_2) = -a + b\tau^2 + \nu^2 + \nu^2\tau$

$$= 2(b\tau^2 + \nu^2) > 0$$

10) $2ab(\sigma^2 + \tau^2) = e_2(a + b\sigma^2)$

11) $\sigma\mu\tau\nu = a \quad ; \quad b\tau\sigma = -\mu\nu \qquad$ (from 7))

$$\mu^2\nu^2 = ab; \quad -\tau\sigma = \sqrt{\frac{a}{b}}$$

Proof of Statement 2: (Recall $B = \begin{pmatrix} 0 & I \\ -S & -2\omega J \end{pmatrix}$)

To prove parts a) and b) and equations 1) - 4) of e), we first observe that any eigenvector must have a non-zero first component which we can take to be 1. The form of B then forces the eigenvector to be $u = \{1, \rho, \lambda, \rho\lambda\}$ where λ is the eigenvalue. Now the last two equations in the system $Au = \lambda u$ require that

$$a - 2\omega\lambda\rho = \lambda^2$$

$$-b\rho + 2\omega\lambda = \lambda^2\rho$$

Elimination of ρ gives

$$\lambda^4 + (b-a + 4\omega^2)\lambda^2 - ab = 0$$

and parts a) and b) as well as the first two equations of part e) follow.

Part c) needs no comment.

Parts d) and e) follow from general considerations:

Lemma 1. Let v and w be eigenvectors of the matrix $\mathcal{J}\Sigma$ where Σ is symmetric and $\mathcal{J}$ is skew symmetric and orthogonal, and let the corresponding eigenvalues be λ and μ respectively.

Then either

$$(v, \Sigma w) = 0,$$

or

$$\lambda + \mu = 0$$

Proof: Since $\mathcal{J}$ is orthogonal,

$$(v, \Sigma w) = (\mathcal{J}v, \mathcal{J}\Sigma w) = \mu(\mathcal{J}v, w)$$

$$(\Sigma v, w) = (\mathcal{J}\Sigma v, \mathcal{J}w) = \lambda(v, \mathcal{J}w).$$

The result follows by the symmetry of Σ and skew symmetry of $\mathcal{J}$.

To apply this lemma to our problem we use the fact that, since $B = A^{-1}\mathcal{J}\Sigma A$, the vectors Av_i and Aw_i are eigenvectors of $\mathcal{J}\Sigma$. (The notation is that of §1.)

Part d) and equations 5) through 9) of e) now follow. The remaining equations and statements in e) are proved with a little algebra. The harder ones will be seen geometrically later so the computations are omitted.

Statement 3. If $u(t)$ is a solution such that $u(0) = (x_1, x_2, z_1, z_2)$, then the constants α, β (Statement 2, c) are given by:

$$e_1\alpha_1 = - ax_1 - b\sigma x_2 - \mu z_1 + \mu\sigma z_2 = (u, E v_2)$$

$$e_1\alpha_2 = - ax_1 + b\sigma x_2 + \mu z_1 + \mu\sigma z_2 = (u, E v_1)$$

$$e_2\beta = - ax_1 - ib\tau x_2 - i\nu z_1 - \nu\tau z_2 = (u, E w_2)$$

Proof: This follows on dotting the equation

$$u(0) = \alpha_1 v_1 + \alpha_2 v_2 + \beta w_1 + \bar{\beta} w_2$$

with

Ev_2, Ev_1, Ew_2 respectively, and using 5) - 9) of Statement 1).

Statement 4. (Recall that

$$H(u) = \frac{1}{2}\{z_1^2 + z_2^2 - ax_1^2 + bx_2^2\}$$
$$= \alpha_1\alpha_2 e_1 + |\beta|^2 e_2$$

where $e_1 < 0;\ e_2 > 0$).

Consider the projection in the x-plane of solutions in the integral surface

$$H(u) = h > 0$$

The solutions in the integral surface divide into classes as follows:

1) The (unique) periodic solution: $\alpha_1 = \alpha_2 = 0$

2) Solutions which are asymptotic to the periodic solution as $t \to \infty$ $(t \to -\infty)$:

$$\alpha_1 = 0 \qquad (\alpha_2 = 0)$$

3) Solutions whose x_1 component tends to $+\infty$ $(-\infty)$ as $t \to \pm\infty$:

$$\alpha_1, \alpha_2 > 0 \qquad (\alpha_1, \alpha_2 < 0).$$

These are solutions whose projected orbits in the x-space lie in a half space $x_1 > c$ or $x_1 < \tilde{c}$. They do not make a "transit" of the equilibrium region.

4) Solutions whose x_1 component goes from $-\infty$ to $+\infty$ $(+\infty$ to $-\infty)$ as t goes from $-\infty$ to $+\infty$:

$$\alpha_1 > 0, \quad \alpha_2 < 0 \qquad (\alpha_1 < 0; \; \alpha_2 > 0).$$

These are the solutions which do cross the equilibrium region.

Proof: By inspection of the corresponding general solution.

We are particularly interested in the solutions of class 4) which, in the case of the equilibrium between the two positive mass points, can be interpreted as solutions going from the earth side of the equilibrium to the moon side (or vice versa). Clearly the most "efficient" (least time expenditure) such orbit is that for which $\beta = 0$ since the "β-portion" of a solution contributes only useless oscillation — we will come back to this point later.

Interpretation for restricted Problem:

Solution 1) corresponds of course to the periodic solution about the equilibrium point of the restricted problem whose existence is guaranteed by a theorem of Lyapounov.

The solutions of 2) correspond to the four families which are asymptotic to the periodic solution as described in the introduction. Since the argument of β is free and can vary on a "circle," these four families are easily seen to be "cylinders" which abut on the periodic solution. One can now check that two of these cylinders "go to $+\infty$" and two "to $-\infty$" (as $t \to +\infty$), i.e., the region of the earth (say) or the moon (resp.) Again, one easily checks that one of each of these pairs is asymptotic to the periodic solution as t goes to $+\infty$, the other as t goes to $-\infty$.

The solutions of 3) are those which enter the region of the equilibrium only to return whence they came while those of 4) make the transit.

While we have considered only the linearized equations, simple considerations ensure the same qualitative picture for the equations of the restricted problem.

Statement 5. If $x_1^2 > \dfrac{2h(a - \mu^2)}{a\mu^2} \equiv c^2$

then a) $x_1 z_1 \geq 0 \Rightarrow \alpha_1 x_1 > 0$

b) $x_1 z_1 \leq 0 \Rightarrow \alpha_2 x_1 > 0$

Interpretation: If a solution crosses the line $x_1 = c_1$ going away from the origin, then if $c_1 > c$, the x_1 component of this solution must tend to $+\infty$. If a solution crosses the line coming toward the origin, it's x_1 component goes to $+\infty$ as $t \to -\infty$. Corresponding statements hold if $x_1 = c_1 < -c$.

In particular, a solution of class 2) or 4) $(\alpha_1\alpha_2 \leq 0)$ can cross the line $x_1 = c$, only once and must do so with $z_1 \neq 0$. We will make use of this remark later.

Also, we can see that a solution crosses both of the lines $x_1 = \pm c_1$ if and only if $\alpha_1\alpha_2 < 0$. This comment allows us to give a precise geometric meaning to the statement that "a solution makes a transit of the equilibrium region." A similar definition works for the restricted problem for the same reason.

Proof of Statement 5.

a) We have (Statement 3)

$$e_1\alpha_1 = -ax_1 - b\sigma x_2 - \mu z_1 + \mu\sigma z_2,$$

where $e_1 < 0$ (Statement 2), e), 9)

Thus

$$\operatorname{sgn} x_1\alpha_1 = \operatorname{sgn}(ax_1^2 + b\sigma x_1x_2 + \mu z_1x_1 - \mu\sigma x_1z_2)$$

Since $x_1z_1 \geq 0$, we need only show that

$$ax_1^2 > |b\sigma x_1x_2 - \mu\sigma x_1z_2|$$

We estimate (Schwarz)

$$|b\sigma x_1x_2 - \mu\sigma x_1z_2| \leq |x_1|(b\sigma^2 + \mu^2\sigma^2)^{\frac{1}{2}}(bx_2^2 + z_2^2)^{\frac{1}{2}}$$

Using Statement 2, e) 1) and the energy integral we have

$$a - \mu^2 = b\sigma^2 + \mu^2\sigma^2$$

$$bx_2^2 + z_2^2 \leq 2h + a\,x_1^2$$

so that

$$|b\sigma x_1x_2 - \mu\sigma x_1x_2| \leq |x_1|(a - \mu^2)^{\frac{1}{2}}(h + a\,x_1^2)^{\frac{1}{2}}$$
$$= |x_1|(a^2x_1^2 + 2ha - 2h\mu^2 - a\mu^2x_1^2)^{\frac{1}{2}}$$

This last quantity is less than ax_1^2 provided $2ha - 2h\mu^2 - a\mu^2x_1^2 < 0$ which is the hypothesis. A similar proof holds for part b).

A statement stronger than the above can be proved if we place a restriction on the constants a and ω: Namely

Statement 6. Recall the equations are given by

$$\dot{x}_1 = z_1 \; ; \qquad \dot{z}_1 = -2\omega z_2 + ax_1$$
$$\dot{x}_2 = z_2 \qquad \dot{z}_2 = 2\omega z_1 - bx_2,$$

If

$$x_1 = c_1 \geq \sqrt{\left(\frac{8\omega^2h}{a^2 - 4\omega^2a}\right)}$$

Then $z_1 \geq 0$ implies the corresponding solution never returns to the line $x_1 = c_1$. (Also $x_1(t) \to \infty$) Furthermore if $z_1 = 0$, c_1 is an absolute minimum for x_1. A similar statement holds if
$x_1 = c_1 < -\sqrt{\frac{8\omega^2 h}{a^2 - 4\omega^2 a}}$.

Proof: The proof consist of showing that $\dot{z}_1 > 0$ under the above circumstances. We have:

$$|z_2| \leq \sqrt{2h + ax_1^2}$$

so that

$$4\omega^2 z_2^2 \leq 4\omega^2 (2h + ax_1^2) < a^2 x_1^2$$

The last inequality being the hypothesis. The result now follows.

This statement has no force unless

$$a^2 - 4\omega^2 a > 0$$

which situation does however hold for the equilibrium point of the restricted problem between the two positive masses. ($a \geq 8$; $\omega = 1$).

Geometrically, we see from Statement 6 that the points where the x_1 component of a solution can have a maximum must lie to the left of the line $x_1 = \sqrt{\frac{8\omega^2 h}{a^2 - 4\omega^2 a}}$. Such a restriction is valid only when $a^2 - 4\omega^2 a > 0$ as can easily be seen. This remark will be useful in a later report.

Statement 7

The projection of the periodic solution in the x-plane is an ellipse with minor axis of length $2\sqrt{\frac{h}{e_2}}$ in the direction of the x_1-axis and major axis of length $-2\tau\sqrt{\frac{h}{e_2}}$ in the direction of the x_2-axis.

Proof: (Assume β is real.) The projection is given by

$$x_1(t) = 2\,\mathrm{Re}(\beta e^{i\nu t}) = 2\beta \cos \nu t$$

$$x_2(t) = -2\tau\,\mathrm{Im}(\beta e^{i\nu t}) = -2\tau\beta \sin \nu t$$

Also the energy integral gives

$$\beta^2 e_2 = h$$

That $|\tau| > 1$ follows from Statement 2, e), 6):

$$0 < \tau^2 = \frac{\nu^2 + a}{\nu^2 - b} > 1.$$

The result follows.

Statement 8. (Recall the solutions with $\alpha_1\alpha_2 = 0$ are those asymptotic to the periodic solution.)

a) The envelopes of projections in x-space of orbits with $\alpha_1 = 0$ are the straight lines

$$x_2 = -\sigma x_1 \pm (a - \sigma^2 b)\sqrt{\frac{2h}{ab}}$$

$$= -\sigma x_1 \pm 2\sqrt{\frac{h}{e_2}}\cdot(\sigma^2 + \tau^2)^{\frac{1}{2}}.$$

The corresponding envelopes for $\alpha_2 = 0$ are:

$$x_2 = \sigma x_1 \pm (a - \sigma^2 b)\sqrt{\frac{2h}{ab}} .$$

b) All four of these lines are tangent to the boundaries of R (i.e., of the region of x-space wherein solutions must move — see Statement 1.)

c) The points of tangency lie on the lines

$$x_1 = \pm\sigma\sqrt{\frac{2bh(a - b\sigma^2)}{a}} = \mp\frac{1}{\tau}\sqrt{2h(a-b\sigma^2)}$$

(See figure 1)

Proof of Statement 8

a) If $\alpha_1 = 0$ we have (Statement 2)

$$x_1 = \alpha_2 e^{-\mu t} + 2\,\mathrm{Re}(\beta e^{i\nu t})$$

$$x_2 = -\sigma \alpha_2 e^{-\mu t} - 2\tau\,\mathrm{Im}(\beta e^{i\nu t})$$

$$= -\sigma x_1 + 2\sigma\,\mathrm{Re}\,(\beta e^{i\nu t}) - 2\tau\,\mathrm{Im}(\beta e^{-i\nu t})$$

The extreme values of x_2 for fixed x_1 are obtained by varing $\arg\beta$. These are computed to be

$$x_2 = -\sigma x_1 \pm 2|\beta|(\sigma^2 + \tau^2)^{\frac{1}{2}},$$

Finally, we have $|\beta| = \sqrt{\frac{h}{e_2}}$ from the energy integral which gives one of the alternate expressions in a). Observe that the extreme values are achieved.

b) We could prove b) by computation; however, the following geometric argument carries over to the corresponding statement (that "envelopes of solutions asymptotic to the periodic solution touch the boundaries of R") for the restricted problem:

We first observe that we can obtain a space homeomorphic to the phase space as follows: First deform R to an infinite strip (i.e., squeeze the boundaries down to straight lines). Noting that at each point of R (except the boundaries) there is a "circle" of possible velocities (i.e., $z_1^2 + z_2^2 = \text{const} > 0$) we cross the infinite strip with a circle to obtain a "pipe," i.e., the space between two coaxial cylinders.

The length along the cylinder corresponds to the x_1 coordinate. For each fixed x_1 there corresponds an annulus of points; the radial variable in this annulus corresponds to x_2, while the angular variable corresponds to the direction of the velocity vector $z = (z_1, z_2)$. The inner and outer boundaries of the annulus correspond to boundary points of R. These boundaries should be identified to (different) points since $z_1^2 + z_2^2$ is zero on the boundary of R; however, we neglect this point for the moment.

Now consider the "cylinder" of solutions with $\alpha_1 = 0$ say. For fixed x_1, the corresponding points on the cylinder make a closed loop in the pipe.

Now if $x_1 > c$ (Statement 5), and $\alpha_1 = 0$, then $z_1 < 0$. Thus the corresponding "circle" does not go around the hole in the pipe. On the other hand, the periodic orbit does encircle the hole since the velocity vector on this orbit goes through all angles. Since the cylinder abuts on this periodic orbit, some section of it must enclose the hole. It follows that this cylinder must cross one of the bounding cylinders of the pipe.

This implies that some orbit with $\alpha_1 = 0$ must touch the boundary of R and so the envelopes of these orbits must cut this boundary. However, they cannot go out of the region R, and therefore are tangent to the boundary.

Part c) (and alternate expression in part a))

From parts a) and b) it follows that, for example, the equations

$$a\,x_1^2 - b\,x_2^2 + 2h = 0$$

$$x_2 = -\sigma x_1 + 2\sqrt{\frac{h}{e_2}}\,(\sigma^2 + \tau^2)^{\frac{1}{2}}$$

have a unique solution for x_1.

This means the following quadratic equation has double roots:

$$(a - b\sigma)x_1^2 + 4b\sigma\sqrt{\frac{h}{e_2}}\,(\sigma^2 + \tau^2)^{\frac{1}{2}} - 4b\frac{h}{e_2}\,(\sigma^2 + \tau^2) + 2h = 0.$$

The condition for a double root is:

$$4b^2\sigma^2\frac{h}{e_2}(\sigma^2 + \tau^2) = (a - b\sigma^2)\left\{2h - \frac{4bh}{e_2}(\sigma^2 + \tau^2)\right\}$$

which reduces to the equation:

$$(\sigma^2 + \tau^2) = \frac{e_2(a - b\sigma^2)}{2ab}$$

This equation (which is Statement 2, e), 10)) could of course be verified algebraically from the other equations of e); the algebra is left out since the geometric proof suffices.

The remaining computations are now easily completed and similar arguments complete the proof of Statement 8.

The following statement enables us to give a fairly clear picture of the approximate location of those orbits which make a transit of the region near the equilibrium point $(\alpha_1\alpha_2 < 0)$. This picture carries over to the restricted problem with little difficulty and suggests a "possible" means of giving an existence proof for the periodic orbits of M. Davidson. (However, the present author has not been able to carry out any proof as yet.)

Before giving this statement, we state a lemma. In the lemma, $\cos^{-1}(\gamma)$ denotes that angle between 0 and π whose cosine is γ: (provided $|\gamma| < 1$)

Lemma:

$$\alpha \cos\theta + \beta \sin\theta \geq \gamma \Longleftrightarrow |\chi - \theta| \leq \cos^{-1} \frac{\gamma}{(\alpha^2 + \beta^2)^{\frac{1}{2}}}$$

where

$$\cos \chi \sim \alpha; \quad \sin \chi \sim \beta$$

the equality signs hold simultaneously. If $\gamma^2 > \alpha^2 + \beta^2$ the inequality never holds.

Statement 9

Let $z_1 = \rho \cos\theta$; $z_2 = \rho \sin\theta$.

Let $x = (x_1, x_2)$ denote any point in R.

If $\gamma_1 = -\dfrac{ax_1 + b\sigma x_2}{\mu\rho}$; $\quad \gamma_2 = \dfrac{ax_1 - b\sigma x_2}{\mu\rho}$

$$\cos \chi_1 \sim 1 \qquad \cos \chi_2 \sim 1$$

$$\sin \chi_1 \sim -\sigma \qquad \sin \chi_2 \sim \sigma$$

a) Then for $|\gamma_2| \leq 1$, we have:

$$\alpha_i \geq 0 \Longleftrightarrow |\theta - \chi_i| \leq \cos^{-1} \frac{\gamma_i}{(1 + \sigma^2)^{\frac{1}{2}}} \quad .$$

b) It follows (Statement 8) that $|\gamma_i| \leq (1 + \sigma^2)^{\frac{1}{2}}$ only in the strip between the lines enveloping the orbits with $\alpha_i = 0$ and that $|\gamma_i| = (1+\sigma^2)^{\frac{1}{2}}$ on the boundary of these strips.

Proof of Statement 9

From Statement 2, we have

$$|e_1|\alpha_1 = a\,x_1 + b\sigma x_2 + \mu z_1 - \mu\sigma z_2$$

$$|e_1|\alpha_2 = ax_1 - b\sigma x_2 - \mu z_1 - \mu\sigma z_2$$

Replacing z_1 by $\rho\cos\theta$ and z_2 by $\rho\sin\theta$ we set

$$\alpha_1 \geq 0 \Longleftrightarrow \cos\theta + \sigma\sin\theta \geq -\frac{(ax_1 + b\sigma x_2)}{\mu\rho}$$

and

$$\alpha_2 \geq 0 \Longleftrightarrow \cos\theta + \sigma\sin\theta \geq \frac{(ax_1 - b\sigma x_2)}{\mu\rho}$$

An application of the lemma completes the proof.

Statement 10 (consequence of 9)

From 9, it follows that orbits with $\alpha_2 = 0$ cut the line $\gamma_2 = 0$ in a direction orthogonal to the enveloping lines of these orbits $(i = 1, 2)$. Thus the lines $\mu_2 = 0$ must pass through the points of tangency of the enveloping lines with the boundary of R.

We further observe that to the "right" of the line $\gamma_i = 0$, χ_i is acute, while to the left of the line $\gamma_i = 0$, χ_i is obtuse. The results implied by figure 1 are easy consequences. In particular, we see for example that any orbits in the regions I, I', I''; II, II', II'' have $\alpha_1\alpha_2 > 0$ while those in the regions III, III' have $\alpha_1\alpha_2 < 0$. The situation in the strips is not as simple, but is fairly clear.

Figure 1.

1) The (two) solid dark lines through the points A and D are the enveloping lines of solutions with $\alpha_1 = 0$. The corresponding lines through B and C are the enveloping lines of solutions with $\alpha_2 = 0$. Any solution with $\alpha_1 = 0$ or $\alpha_2 = 0$ must lie in the corresponding strip bounded by these lines.

2) At P, the shaded wedge indicates the directions at P for which the corresponding solution has $\alpha_1 \leq 0$. At P' the shaded wedge indicates $\alpha_1 \geq 0$. Similarly at $\underline{Q}$ the wedge indicates $\alpha_2 < 0$, at Q', $\alpha_2 > 0$. On the dotted line $\overline{AD}$ the wedge has angle π corresponding to $\gamma_1 = 0$. $\overline{CB}$ has a similar meaning with regard to the strip for α_2.

3) The solid lines parallel to the strips indicate the regions where the corresponding $\alpha_i > 0$ for all possible angles. The dotted lines similarly indicate where $\alpha_i < 0$.

4) Thus we can see that in regions I, I', I'', both of α_1, α_2 are positive, while in the regions II, II', II'', α_1 and α_2 are negative. Finally in regions III, $\alpha_1 > 0$; $\alpha_2 < 0$ while in III', $\alpha_1 < 0$; $\alpha_2 > 0$.

5) In the strips we must determine the sign of α from the direction of the velocity vector: e.g., at P, any solution whose velocity vector lies in the shaded wedge has $\alpha_2 > 0$, $\alpha_1 < 0$, etc.

Thus we have a geometric criterion for determining whether or not a solution will make a transit of the equilibrium region. Note in particular that such a solution must stay inside one or the other of the strips away from the equilibrium, and that as it crosses the equilibrium region it changes strips. Solutions going from right to left are "on the bottom"; those from left to right on top.

We conclude with a remark which may have some "engineering" value:

Statement 11. The (two) solutions for which $|\beta| = 0$ are hyperbolas; these solutions correspond to those orbits which cross the region of the equilibrium point the fastest.

(Corresponding solutions for the restricted problem exist and are well approximated by these — in the equilibrium region — for energies slightly larger than that of the equilibrium.)

The equation for these orbits are

$$-a x_1 = \nu \tau z_2$$

$$-b \tau x_2 = \nu z_1$$

or

$$-\sigma x_1^2 + x_2^2 = 2h\nu^2(b\tau^2 + \nu^2)^{-1}b^{-1}$$

$$= \frac{2h\nu^2}{e_2 b}$$

Proof: Statement 8 plus some algebra.

(Note that the left hand side is determined from geometrical considerations alone, while the right hand side follows by letting $x_1 = 0$ and using the energy equation.)

This completes the present collection of statements.

BIBLIOGRAPHY

1. Davidson, M. (to be published).

2. Moser, J., On the Generalization of a Theorem of A. Liapounoff, Comm. Pure Appl. Math., Vol. XI, 1958, pp. 257-271.