MULTIPLE SCATTERING BY LARGE PARTICLES

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ABSTRACT

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The Neumann solution to the scalar equation of transfer in a homogeneous layer of optical thickness $\tau^* \leq 1$ is obtained numerically for sample phase functions with large forward and backward peaks. The results are presented graphically and are compared with the intensities and albedos computed by several approximate methods.

Author

I. Introduction

The problem of multiple scattering by particles whose dimensions are comparable to or larger than the wavelength of the radiation scattered is made difficult by the extreme asymmetry of the individual particle scattering diagrams. Such particles, which are predominant in terrestrial clouds and haze, scatter radiation primarily in the forward direction. Methods of solution for the corresponding equation of radiative transfer are known in principle (Chandrasekhar 1960), but very few exact numerical results have been obtained. The present paper gives such exact results. The purpose is two-fold: to show the effect that a large asymmetry in the scattering diagram has on the angular distribution of diffuse light in a plane scattering layer, and to evaluate some of the approximate methods employed in radiative transfer problems.

The calculations are intended to be illustrative rather than to form an exhaustive critique of existing methods, and are confined to layers of optical thickness $\tau^* \leq 1$ and to situations with azimuthal symmetry and conservative scattering.

II. Problem

We shall consider an idealized problem: monochromatic radiation is scattered by plane-parallel, homogeneous, non-absorbing layer of optical thickness τ^*. The angle between a given direction and the direction of increasing optical depth τ will be designated θ. The scattering per unit volume is characterized by a scalar phase function (scattering indicatrix) $\Phi(\cos\alpha)$, where α is the scattering angle and $\Phi(\cos\alpha)$ is asymmetric about $\alpha=\pi/2$. Two alternative azimuth-independent sources of the radiation will be considered: radiation confined to the cone $\theta=\theta_0$ incident on the top of a layer containing no internal sources (the resulting intensity is the average over azimuth of the intensity resulting from irradiation by a parallel beam); or a uniform distribution of sources within the layer, and no radiation incident from outside. The specific intensity of diffuse radiation (of total radiation in the case of internal sources) in the layer is then governed by the equation of transfer in the form

$$\mu \frac{dI}{d\tau} = -I(\tau,\mu) + J(\tau,\mu) + J_1(\tau,\mu) \quad , \tag{1}$$

$$J(\tau,\mu) = \frac{1}{2}\int_{-1}^{1} d\mu' \, F(\mu,\mu') \, I(\tau,\mu') \quad ; \tag{2}$$

where $\mu = \cos\theta$, J_1 is the source function for once-scattered (eq. 3a) or unscattered (eq. 3b) light:

$$J_1(\tau,\mu) = e^{-\tau/\mu_0} F(\mu,\mu_0)/4\pi\mu_0 \quad , \text{(external source)} \qquad (3a)$$

$$J_1(\tau,\mu) \equiv 1 \quad ; \qquad \text{(internal source)} \qquad (3b)$$

and

$$F(\mu,\mu') = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, \Phi(\mu,\phi:\mu',0) \, . \qquad (4)$$

Exact solutions to the above problem have been obtained in only a few isolated cases (Chu et al. 1963; Romanova 1963; van de Hulst and Davis 1961). An exception to this statement occurs for $\Phi(\cos\alpha) = 1 + k\cos\alpha$, but even in this case results have been tabulated only for $\tau = \infty$ (see Harris 1961 and Sobolev 1956).

III Method

As is well known, the equation of transfer may be rewritten as an integral equation for the source function J, in which the integral operator Λ involves an integration over both angle and optical depth:

$$J(\tau,\mu) = \Lambda\{J\} + J_1 \quad , \qquad (5)$$

$$\Lambda\{\ldots\} = \frac{1}{2}\int_{-1}^{1} d\mu' F(\mu,\mu') \int_0^{\tau^*} d\tau' \ldots k(\tau-\tau',\mu') \quad , \qquad (6)$$

$$k(\tau,\mu) = \begin{cases} \frac{1}{|\mu|} e^{-\tau/\mu} \, , & \tau/\mu > 0 \\ 0 \, . & \tau/\mu < 0 \end{cases} \qquad (7)$$

A solution to this equation (the Neumann solution) is then an infinite series (Busbridge 1960), each term of which involves a successive application of the Λ operator to J_1:

$$J(\tau,\mu) = \sum_{n=1}^{\infty} \Lambda^{n-1}\{J_1\} \equiv \sum_{n=1}^{\infty} J_n \quad . \tag{8}$$

Physically, this series is nothing but an expansion of the source function in successive orders of scattering (e.g. van de Hulst 1948). The diffuse intensity I may then be written as

$$I\ (\tau,\mu) = \sum_{n=1}^{\infty} I_n\ (\tau,\mu), \tag{9}$$

where

$$I_n(\tau,\mu) = \int_0^{\tau^*} d\tau' \ J_n(\tau',\mu)\ k(\tau-\tau',\mu)\ . \tag{10}$$

The calculations, which involved a double numerical integration in order to obtain J_n from J_{n-1} in accordance with equation (8), were performed on an IBM 7094 Model 1; the time for one iteration with the Λ operator varied from .2 minute to about .7 minute, depending on the numer of points used for the integrations, which depended in turn on the asymmetry of the phase function. Simpson's rule was used. If a sufficient numer of terms n_o are computed, $J_{n_o} \approx \eta J_{n_o-1}$, where η is the maximum eigenvalue of Λ and is independent of τ and of angle (Leonard and Mullikin 1964). The remainder of the series (8) may then be replaced by a geometric series. By varying the number of points used for the integrations and the value of n_o, it was found that the computed results are accurate to about 1 per cent.

The particular phase function used for most of the calculations was that first introduced into astrophysics by Henyey and Greenstein (1941)

$$\Phi_{HG}(\cos\alpha;g) = \frac{1-g^2}{(1+g^2-2g\cos\alpha)^{3/2}} \tag{11}$$

which gives a sharp forward peak in the scattering using only one parameter. This parameter, g, which may be called the asymmetry factor, is the average over the unit sphere of the cosine of the scattering angle, weighted by the phase function (see Irvine 1965). For some of the computations a sum of two such phase functions was used

$$\Phi(\cos\alpha) = b\,\Phi_{HG}(g_1) + (1-b)\,\Phi_{HG}(g_2) \quad . \tag{12}$$

This allowed the introduction of peaks in both the forward and backward directions of scattering.

Figure 1 shows the phase functions which were used for numerical computations. Their significance is explained in Table 1.

IV. Comparison with Approximate Methods

The present method is "exact" in the sense that it provides a numerical approximation to an exact solution of equation (1), and this approximation can be made arbitrarily close to the exact solution if sufficient computer time is used. In contrast, we shall call those methods "approximate" which are not based on an exact solution of our idealized problem. Such methods fall into two categories: those that take into account the asymmetry of the phase function, but treat the multiple scattering problem only approximately; and those that utilize exact solutions of the equation of transfer, but use a simplified phase function. The first can say little about the intensity I as a function of angle; they may, however, provide a good approximation to the total flux reflected or transmitted by a layer. Illustrations are discussed in § a) below. The second have frequently been used in an attempt to obtain at least qualitative information about the angular distribution of radiation. Examples are given in §b).

It must be stressed that our idealized problem differs in several respects from even the simplest physically realizable situations (cf. van de Hulst and Irvine 1962). To comment on just two points: first of all, we have here neglected polarization. Apart from the loss of information which results, errors are introduced into the resultant intensity even if the unscattered light is initially unpolarized. For Rayleigh scattering, these errors are negligible for very thin layers (single scattering dominant) and are of the order of 10 per cent for a semi-infinite atmosphere (Chandrasekhar 1960). The polarization due to single scattering will be less for spherical drops or for randomly oriented irregular particles. Hence, the error due to neglect of polarization should be less than in the Rayleigh case, except for a situation with aligned, asymmetric particles (such as might occur in the presence of a magnetic field). Secondly, the phase functions, such as eq. (12), used for the exact calculations correspond only approximately to those of real particle distributions (see Figure 1). We are interested only in the qualitative nature of the radiation field for large particle multiple scattering, however, and not in the details of specific situations; consequently, the omission of rainbows and related phenomena is not important.

In Figures 2 - 9 the full curves give the Neumann solution corresponding to the phase function that labels the curve (see Table 1), while the dashed curves are various approximations. In Figures 4 - 9, θ represents the angle to the outward normal at the surface considered.

a) Albedo of a layer

The total reflectivity or albedo A of a layer is of vital importance to computations of atmospheric heat balance. Figure 2 shows the albedo of a layer of optical thickness τ^* for normal incidence and two choices of the phase function. As is to be expected, the albedo is much larger for isotropic scattering than for forward-directed scattering. The approximation (dashed curves) shown is the familiar two-stream theory, in the formulation of Chu and Churchill (1955). This approximation produces very good results for normal incidence and thin layers. For larger angles of incidence the two-stream theory breaks down, but comparable accuracy could perhaps be obtained by using a six-stream theory such as that of Chu and Churchill (1955).

Diffusion theory has frequently been applied to problems of radiative or neutron transport (Glasstone and Edlund 1952). In its standard form this method is useful if the distribution of sources in the layer is reasonably homogeneous, and if the point considered is not too near the boundary of the layer (say $2 \leq \tau \leq \tau^* - 2$). These conditions are not fulfilled for reflection and transmission of uni-directional radiation by a plane layer of large particles. To study the albedo of terrestrial clouds, Fritz (1954) proposed a modified diffusion theory in which only light that has been scattered by at least 60° away from the direction of the incident beam contributes to the source term in the diffusion equation. This approximation works very well for normal incidence and thin layers (Fig. 2).

The albedo of a layer of unit optical thickness for various values of the angle of incidence is shown in Figure 3 for two choices of phase function. The dashed curve is taken from Fritz (1954). Fritz's main result, the sharp increase in the albedo of a cloud for large angles of incidence, is confirmed by the exact calculations. The difference between this curve and the exact results at large μ_0 may be due in part to the use of slightly different phase functions; it probably also reflects the loss in accuracy which Fritz anticipated for large zenith angles.

b) Intensity

Let us now consider the variation with angle of the intensity emanating from a plane scattering layer. We shall compare with the present calculations two approaches that have been used in the past: (i) exact solutions to the equations of transfer obtained for only slightly elongated phase functions; and (ii) approximate methods based on the use of exact expression for first order large particle scattering.

(i) Several authors (e.g. Horak 1950, and Harris 1961) have hoped that in certain situations the diffuse intensity produced by large particle multiple scattering would not differ qualitatively from that obtained with a phase function consisting of a three-, two-, or even one-term expansion in Legendre polynomials.

Let us test this idea for a thin layer. To eliminate any preferred direction resulting from the initial conditions, consider a scattering layer with a uniform distribution of internal sources (eq. [3b]; this model has been used to compute a first order approximation to the diffuse light in the Galaxy by Horak 1952 and van de Hulst and Davis 1961). Although the intensity $I(\theta)$ emitted by such a layer differs little among the cases B, C, and D of Table 1 (Horak 1952, and unpublished calculations by the author), considerable differences develop for more elongated phase functions (Fig. 4). The relative difference between the intensities corresponding to the two phase functions in Figure 4 decreases as τ^* increases, but the process is very slow. That this difference is not primarily a result of low-order scattering can be shown by an examination of the eigenfunctions, I_{n_0}, which differ even more than the total diffuse intensities. In other words, even after the photon has been scattered many times, it still knows that the phase function of the layer is asymmetric (essentially because it can "see" the boundaries).

For the more asymmetric initial conditions (eq. [3a]) typical of planetary problems, the use of a two- or even a three-term Legendre expansion of the phase function gives results for the diffuse reflection which may differ by a factor of 2 or 3 from those for large particle scattering (Fig. 5), while the diffuse transmission is qualitatively different. A limited Legendre expansion can be made more elongated if negative scattering is allowed at certain angles (Churchill et al. 1961). This procedure does not bring much improvement in the diffuse transmission, however, and is entirely inappropriate for the diffuse reflection. In Figures 6 and 7 curves F and G were obtained for two- and three-term Legendre expansions for which the first (and first and second) moments were chosen to equal those of the function E, for which the diffuse reflection and transmission are also shown.

The hope has sometimes been expressed that one could approximate an elongated phase function, such as that for cloud droplets, by isotropic scattering if the radiation scattered in the forward peak were considered as unscattered. This corresponds to using in the transfer equation a scattering coefficient that is some fraction of the true value (close to 0.5, since for large particles half of the scattered light is diffracted and is confined to the forward peak). The large errors inherent in such an approach are clear from Fig. 1, which shows that typical phase functions for haze and clouds are sharply varying functions of angle over almost their entire range. The diffuse intensity calculated in this approximation ($\phi = 1$, $\tau^* = \tau^*/2$) is labeled δ and is compared in Figures 6 and 7b to that for the phase function E.

One might hope that the scattering for a simple phase function such as C or D would correspond more closely to that for large particle scattering in the case of a thick layer ($\tau^* \gg 1$). Indeed, within the depths of a thick, conservatively scattering layer illuminated from outside the intensity is independent of phase function (Sobolev 1956). Significant departures from this independence occur for the reflected (Romanova 1963) and the transmitted (Piotrowski 1961) light, however.

(ii) For layers of optical thickness $\tau^* \lesssim 1$ the Neumann solution converges rapidly and it is natural to consider approximate solutions based on exact first order scattering. The first order scattering (I_1) by a plane homogeneous layer can be found very easily for an arbitrary phase function from equations (3) and (10). One may then approximate the higher order scattering appropriate to the exact phase function by the corresponding scattering for simpler phase functions. We shall consider three possible approximations of this type -- exact first order scattering plus higher order scattering computed for :

α. isotropic scattering and a value of τ^* equal to that used in the exact calculation.

β. isotropic scattering and a value of τ^* equal to half of that used in the exact calculation (i.e., assuming that half the scattered light is confined to such small angles that it may be treated as unscattered).

γ. a phase function $1 + 3g \cos \alpha$, where g is chosen to equal the asymmetry factor of the exact phase function. We shall call this the Sobolev (1956) approximation.

Figures 6 - 9 compare the diffuse reflection and transmission computed from the Neumann solution for three choices of elongated phase function with the intensities computed from the approximations just described. We note the following points for normal incidence and transmitted light:

1. All three approximations give similar and reasonable good results for the forward peak in the transmitted light; this is due,of course, to the large contribution of single scattering. The Sobolev approximation is slightly better in this region than approximation (α), which is in turn slightly better than (β).

2. For angles $\theta \geq 45°$ the Sobolev approximation may err by a factor of 2 or 3, becoming worse for more forward-directed Φ. In this region approximation (β) can five fairly good results, even for the most asymmetric phase function tried (H in Table 1). Approximation (α)

seems to consistently over-estimate I in this region, and not to be as safe an estimate as (β).

3. In the intermediate range ($20° \leq \theta \leq 45°$) approximations (β) and (γ) seem to bracket the true intensity fairly well, while (α) may be closer to the exact value.

Likewise we observe for the reflected light that

1. Approximation (α) is considerably worse than (β) or (γ), giving an overestimate of up to a factor 3. It does not improve rapidly for smaller τ.

2. If there is not a large backward peak in Φ, the Sobolev approximation may give good values for both $\theta \leq 10°$ and $\theta \leq 87°$. In the intermediate range, however, and if there is large back-scattering, the agreement with the exact results may be only qualitative.

3. All three approximations overestimate the back scattering, except when ϕ has a sharp backward peak (Fig. 9b). In the latter case approximation (β) most closely fits the true curve for angles close to the directly backscattered light.

A further comment concerning the Sobolev approximation may be made. The X and Y functions for diffuse reflection and transmission with a phase function $\Phi = 1 + 3g \cos \alpha$ are not tabulated for finite layers. Consequently, unless a considerable computational program is undertaken, the Sobolev approximation can be used only by making additional approximations in order to obtain the intensity corresponding to such a phase function (Sobolev 1956). This process introduces further errors (Atroshenko et al. 1962).

V. Conclusion

The previous examples show that for optically thin layers of large particles the albedo may be obtained with reasonable accuracy through the use of methods which take into account the asymmetry of the scattering but describe multiple scattering only crudely (two-stream theory, diffusion-type theories). Such methods, however, can say little about the diffuse intensity as a function of angle. Approximate methods based on the use of exact first order scattering can give good results for the intensity diffusely transmitted by a thin layer; the reflected intensity so obtained may be only qualitatively correct.

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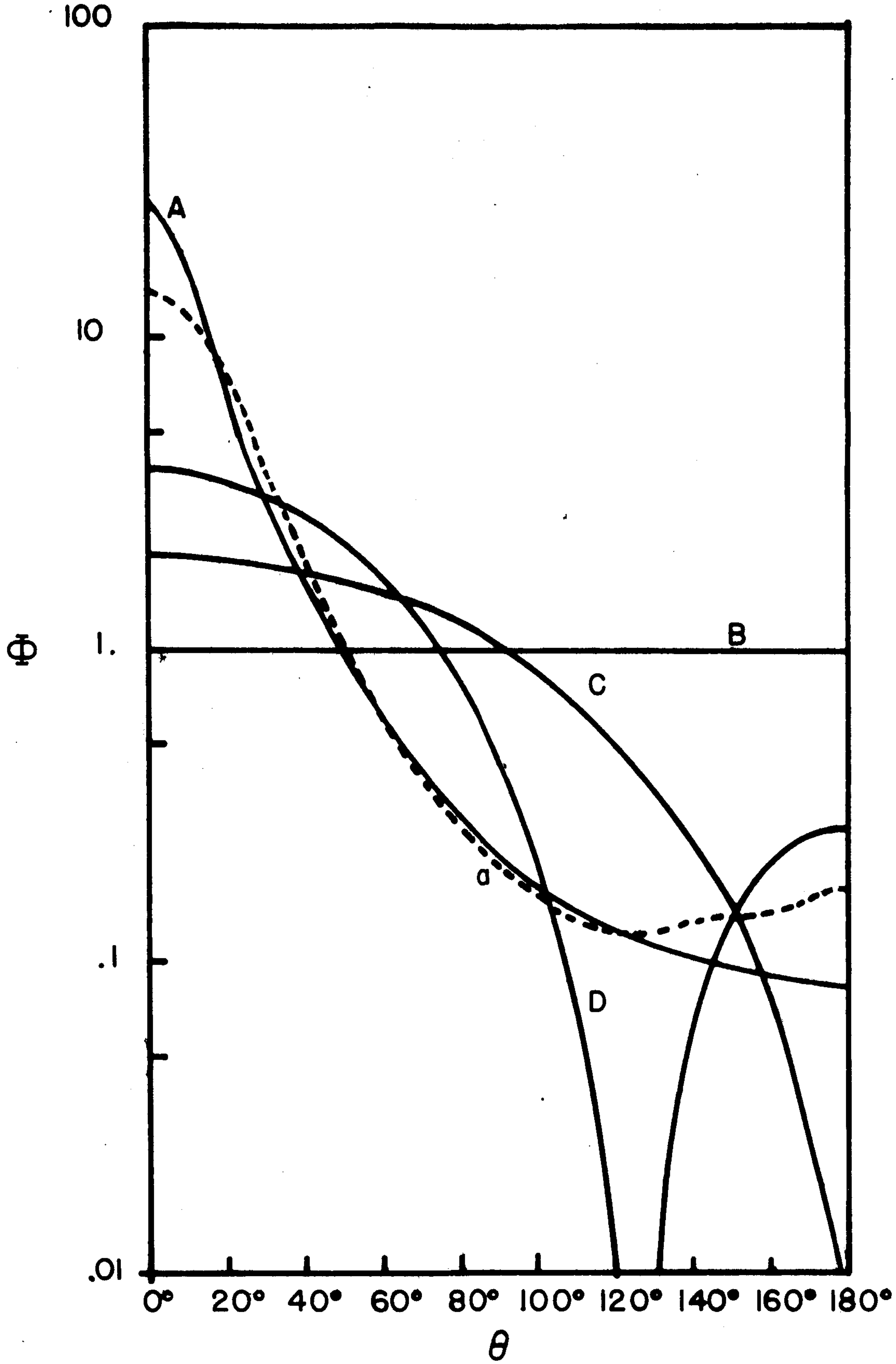
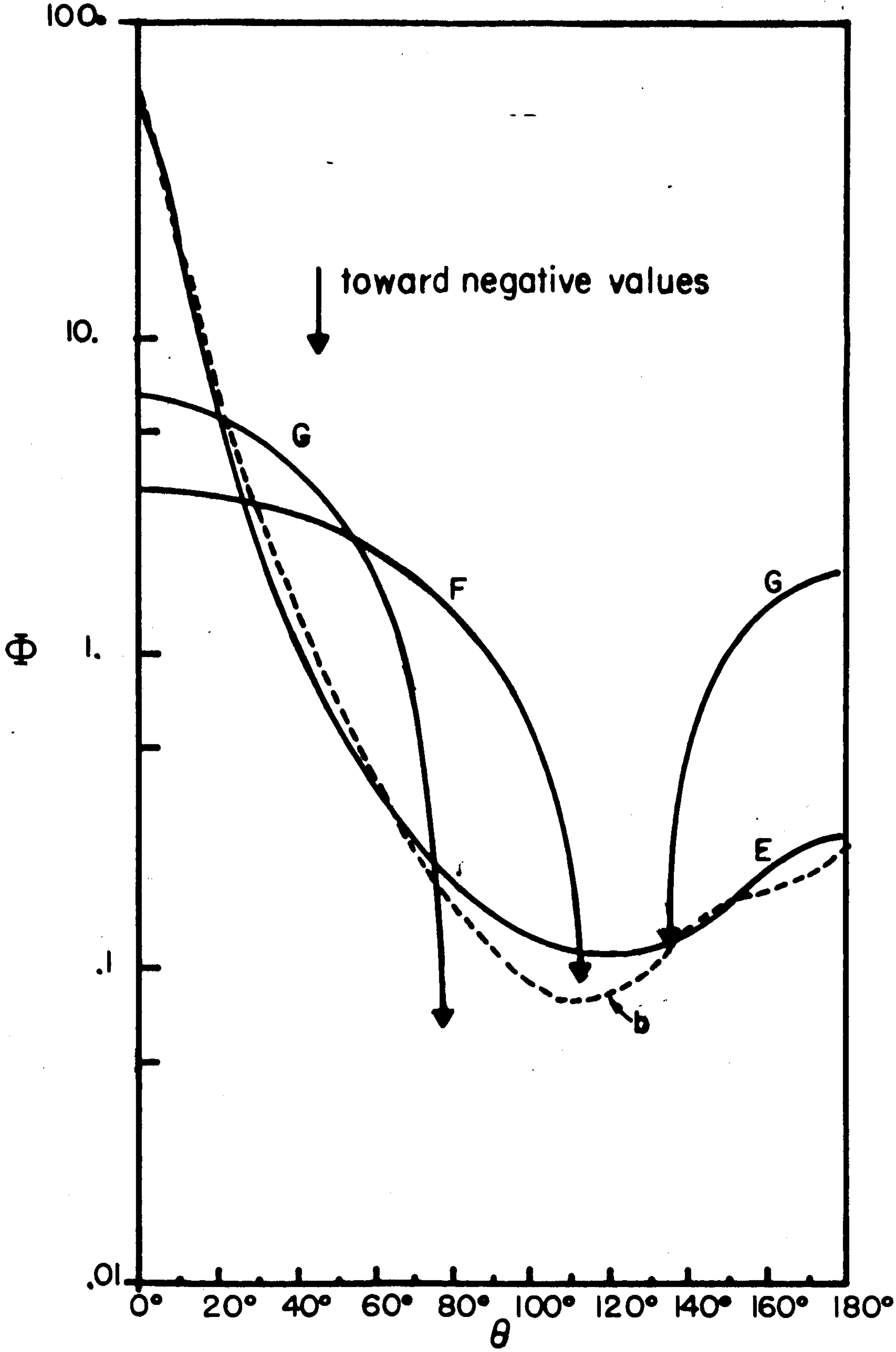
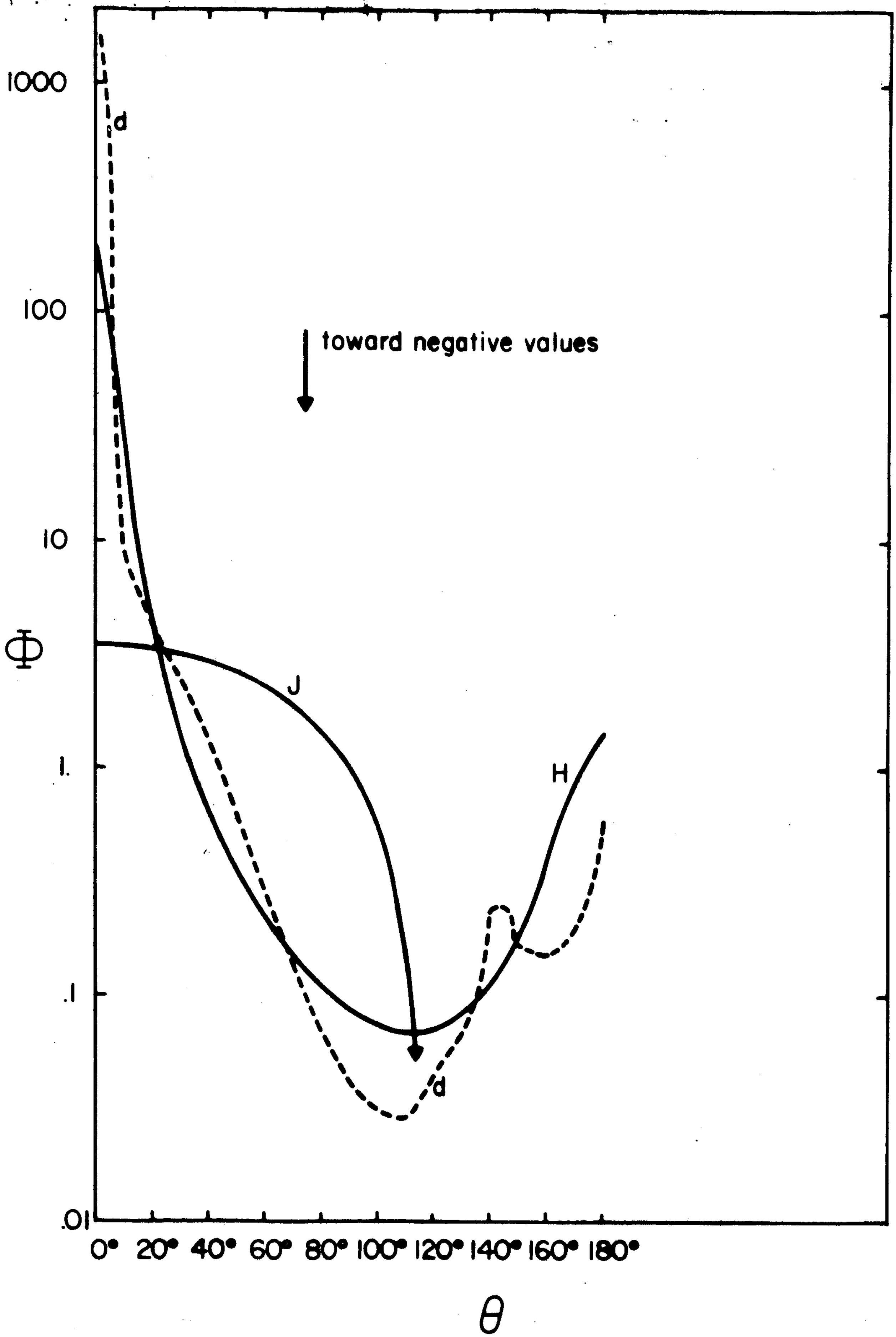


Fig. 1 a, b, c.--Phase functions used for numerical computations (full curves) and illustrative naturally occurring phase functions (dashed curves). See Table 1.



100.
10.
1.
.1
.01
Φ
toward negative values
G
F
G
E
b
0°
20°
40°
60°
80°
100°
120°
140°
160°
180°
θ

Fig. 1 b.



1000
100
10
1.
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toward negative values
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H
d
0° 20° 40° 60° 80° 100° 120° 140° 160° 180°
θ

Fig. 1 c.

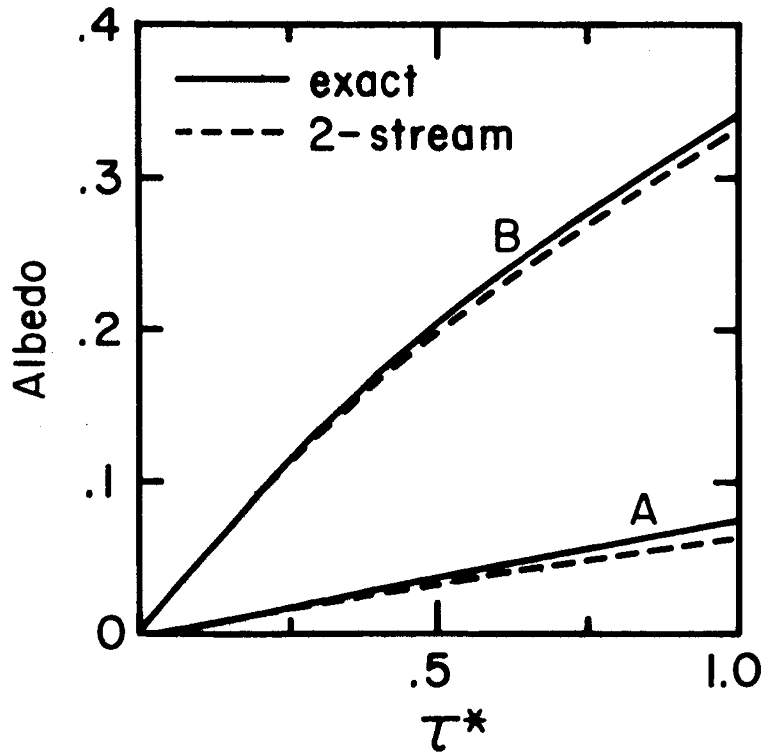


Fig. 2.--Albedo of a plane-parallel layer of optical thickness τ^* illuminated normally as computed from the exact and from the two-stream theory. Both isotropic (B) and forward-directed (A; see Table 1) scattering shown. Results of Fritz (1954) lie on exact curve A.

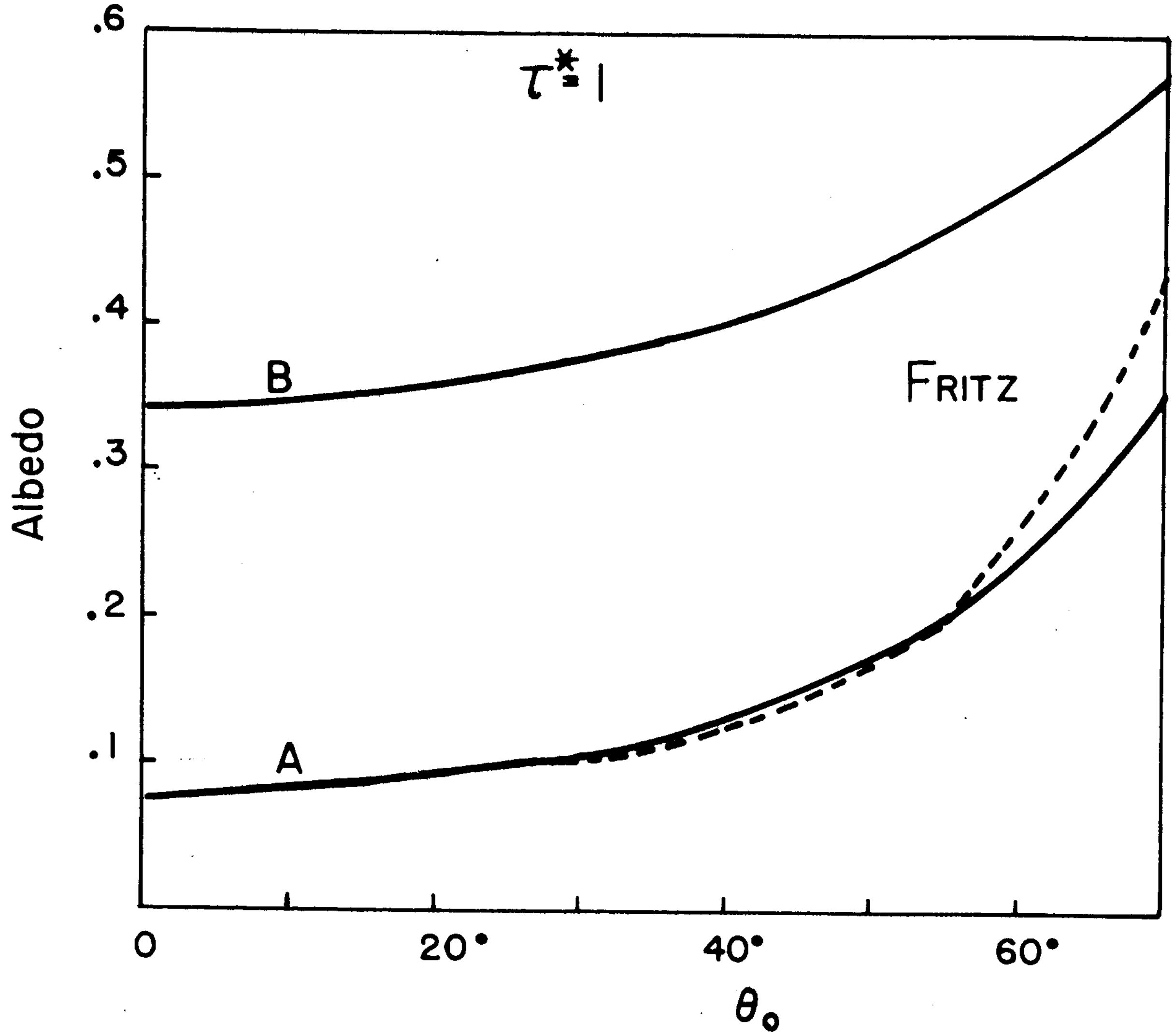


Fig. 3.--Albedo of a plane-parallel layer vs. angle of incidence for isotropic (B) and forward-directed (A; see Table 1) scattering. Dashed curve is approximation of Fritz (1954) for a cloud of water droplets.

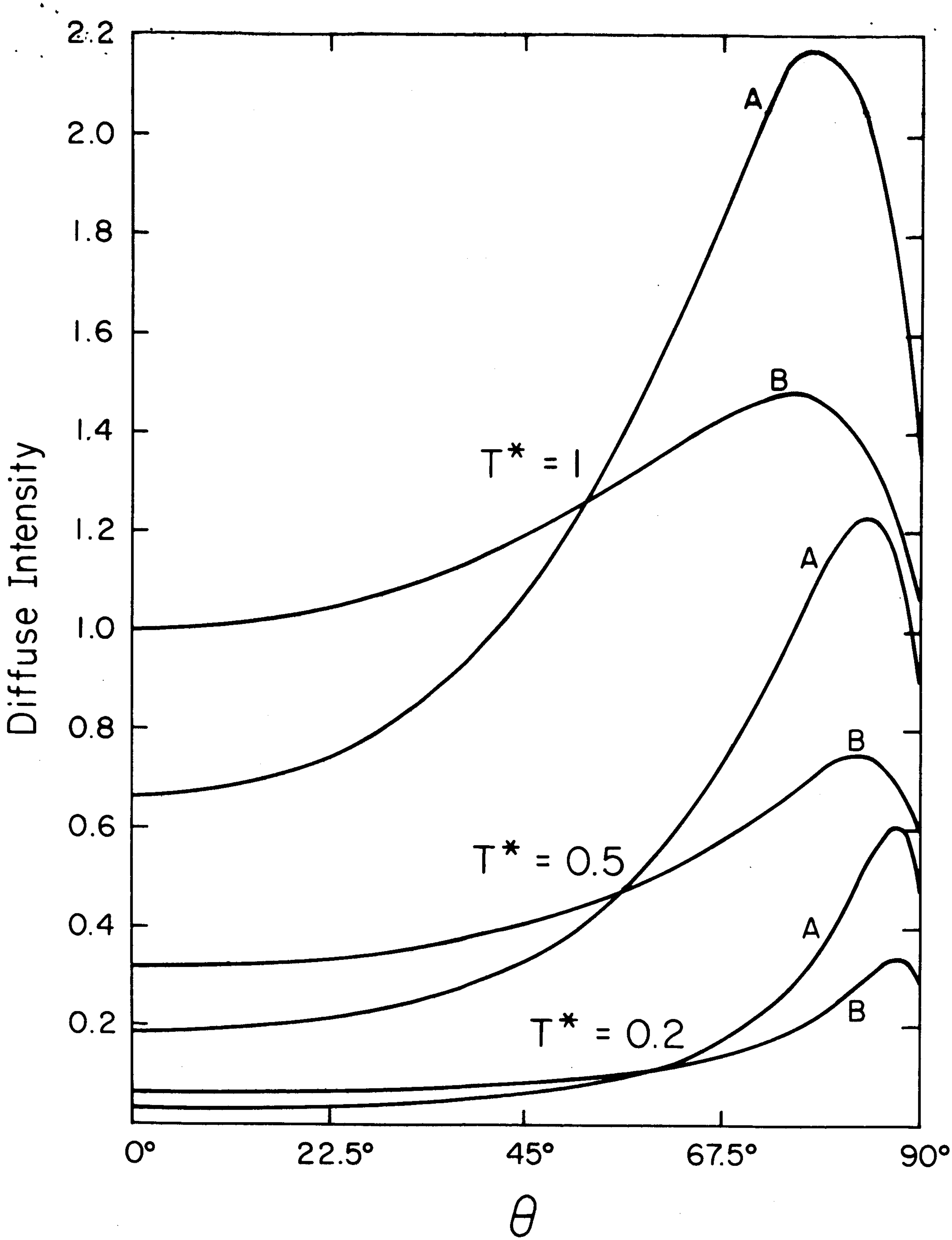


Fig. 4.--Diffuse intensity emitted by a plane layer of optical thickness τ^* containing a uniform distribution of internal sources for isotropic (B) and forward-directed (A; see Table 1) scattering.

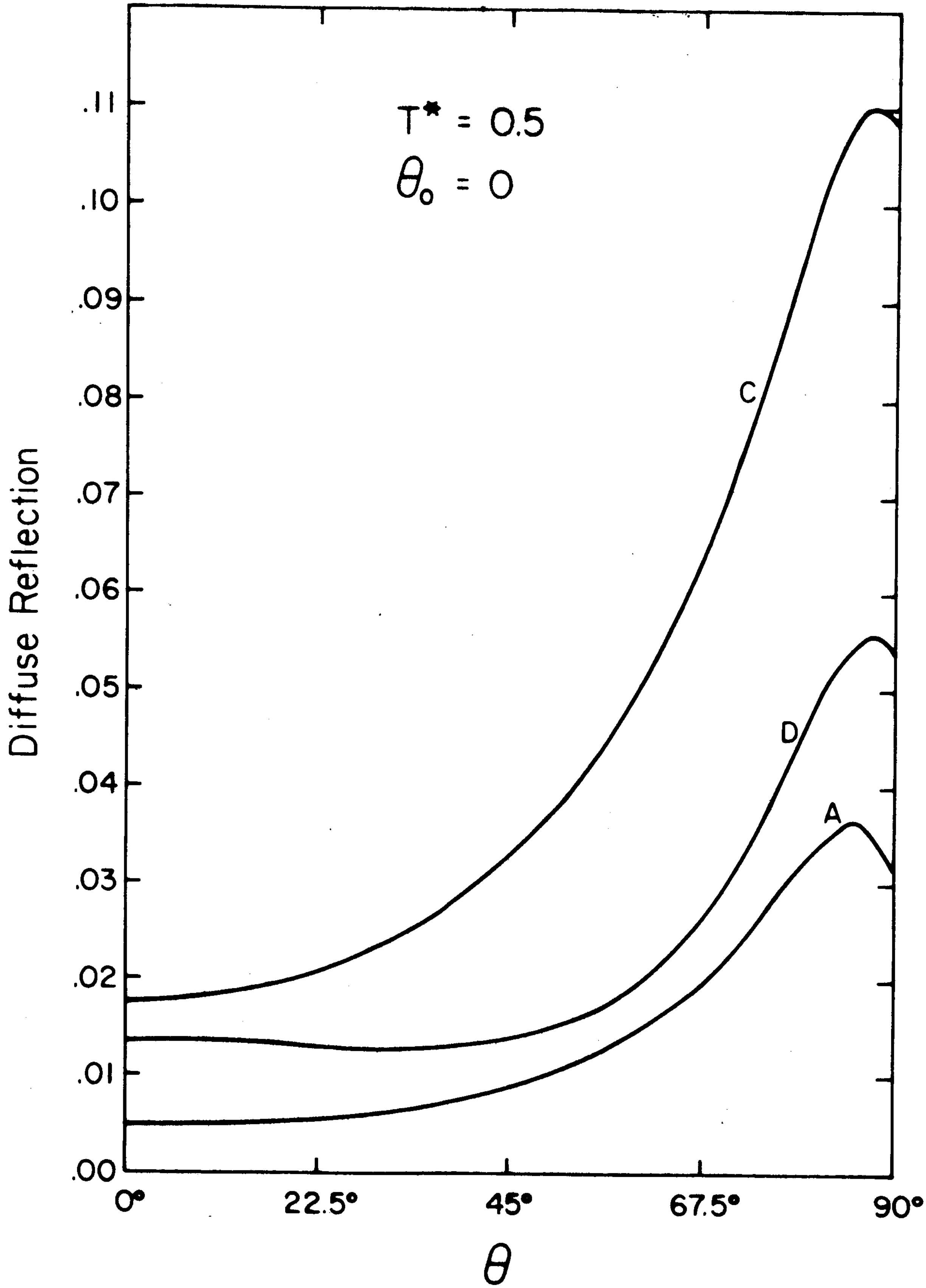


Fig. 5.--Diffuse reflection from a plane-parallel layer of optical thickness 0.5 illuminated normally for three choices of phase function (see Table 1).

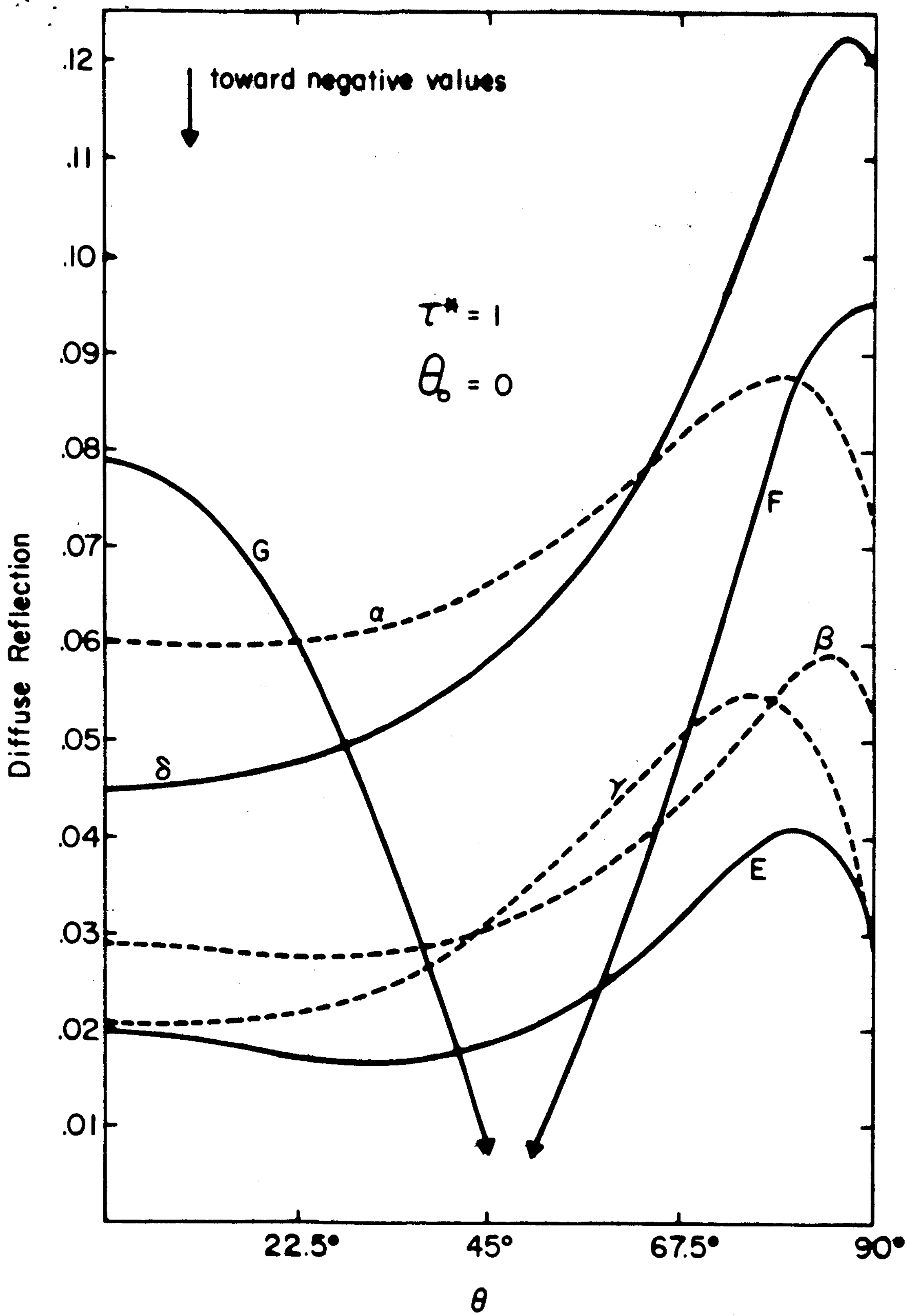


Fig. 6.--Diffuse reflection from a plane-parallel layer of unit optical thickness for normal incidence. Exact calculations for phase functions E, F, G (see Table 1) and for curve δ (isotropic scattering, $\tau^* = 0.5$). Curves α,β,γ represent approximations to curve E described in § IV. b. ii.

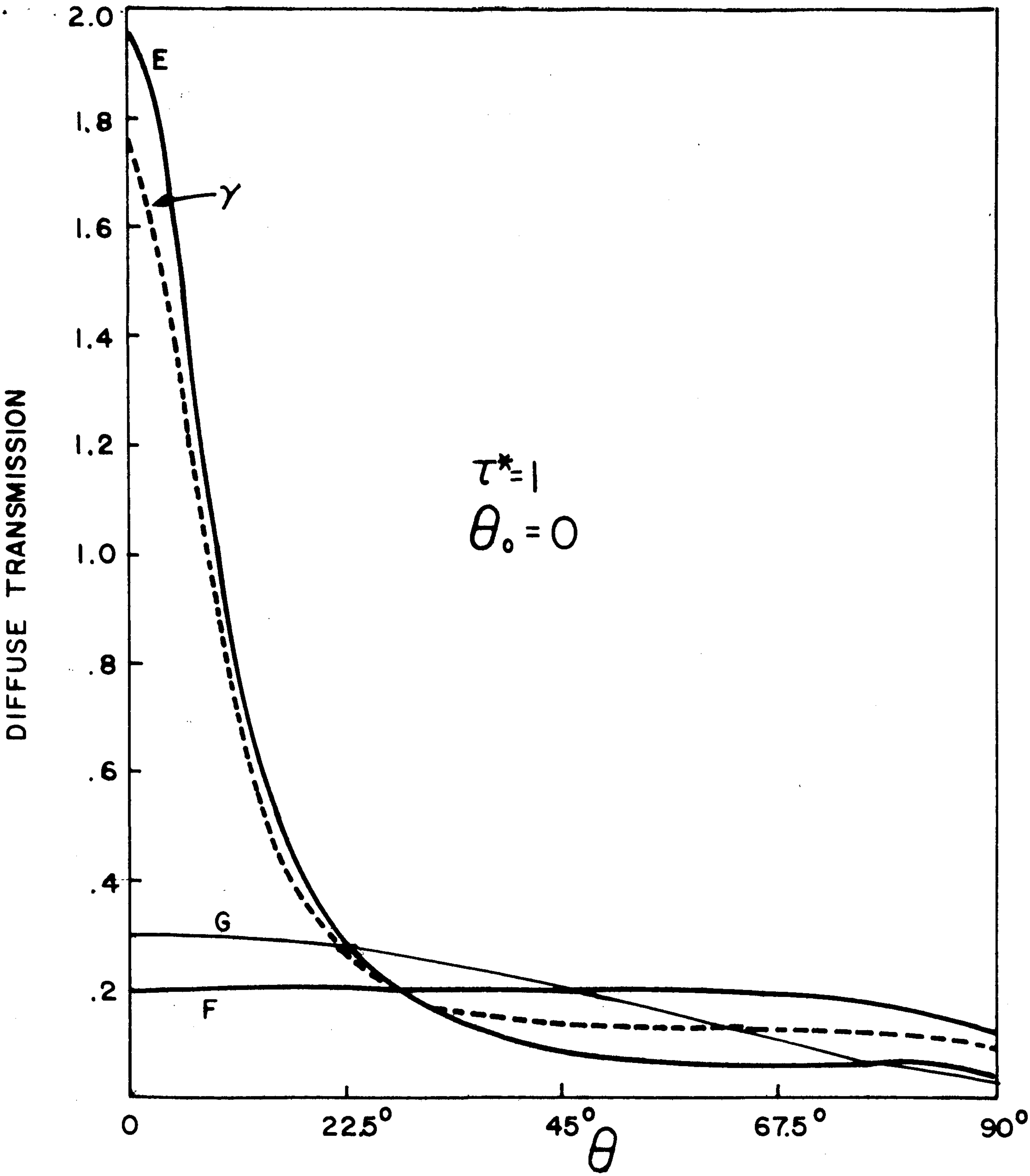


Fig. 7 a, b.--Diffuse transmission by a plane-parallel layer. Curves labeled as in Figure 6. Note omission of forward peak in (b).

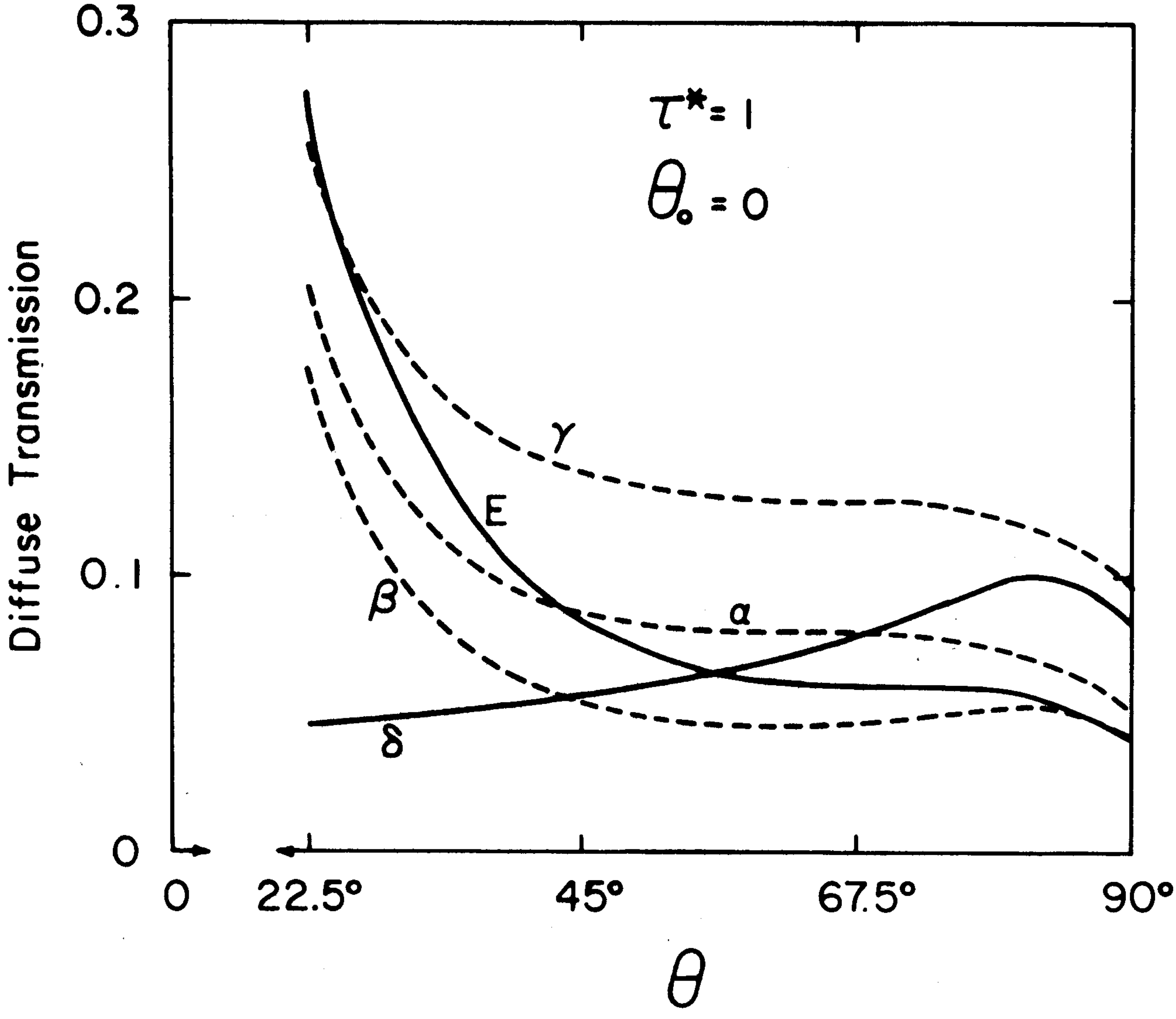

0.3
0.2
0.1
0
Diffuse Transmission
$\tau^* = 1$
$\theta_0 = 0$
γ
E
α
β
δ
0
22.5°
45°
67.5°
90°
θ

Fig. 7 b.

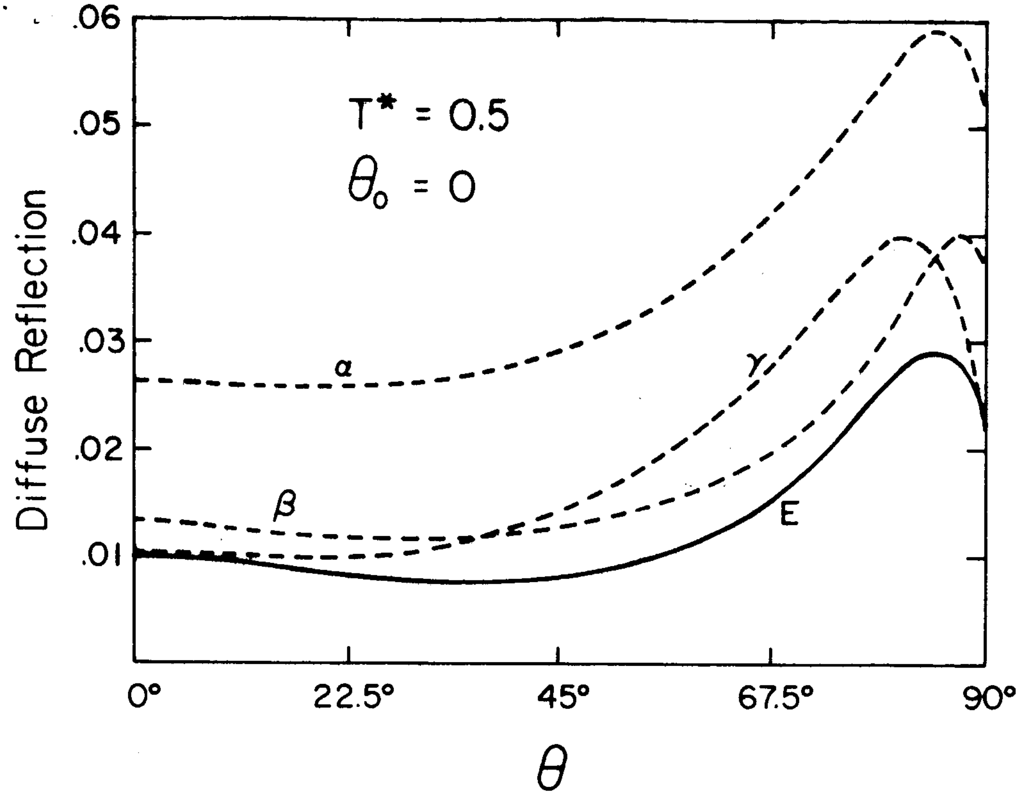


Fig. 8 b.

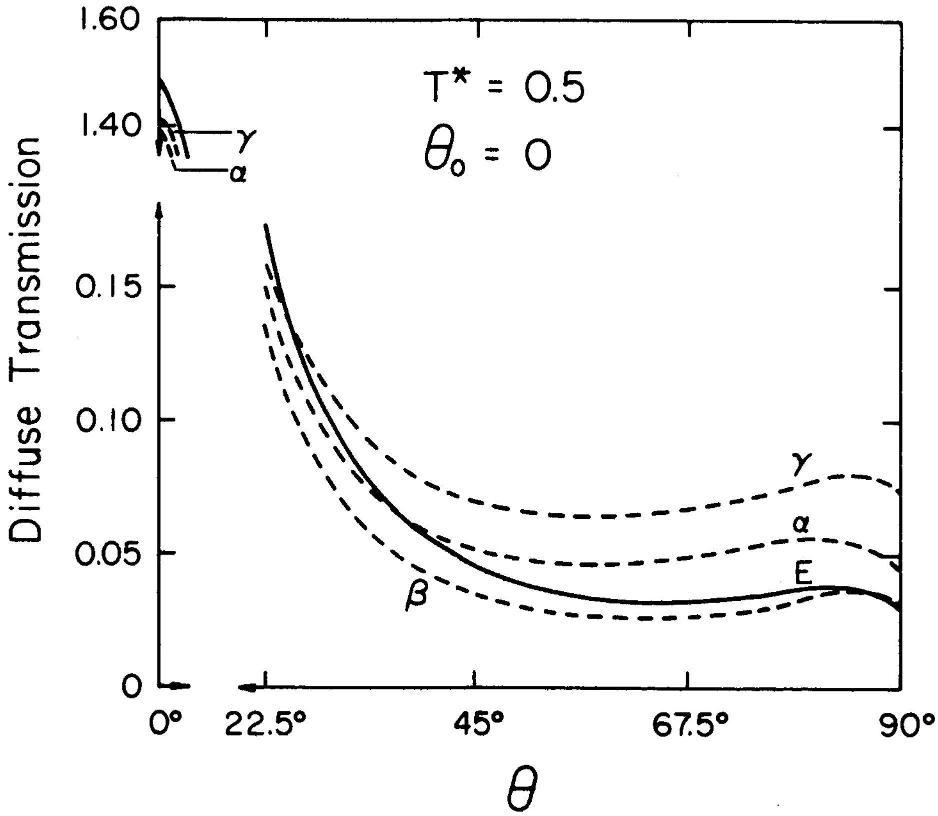


Fig. 8 a,b.--Diffuse transmission and reflection by a plane-parallel layer of optical thickness one-half for normal incidence. Curves labeled as in Figure 6. Note change of scale in ordinate and break in abscissa of (a).

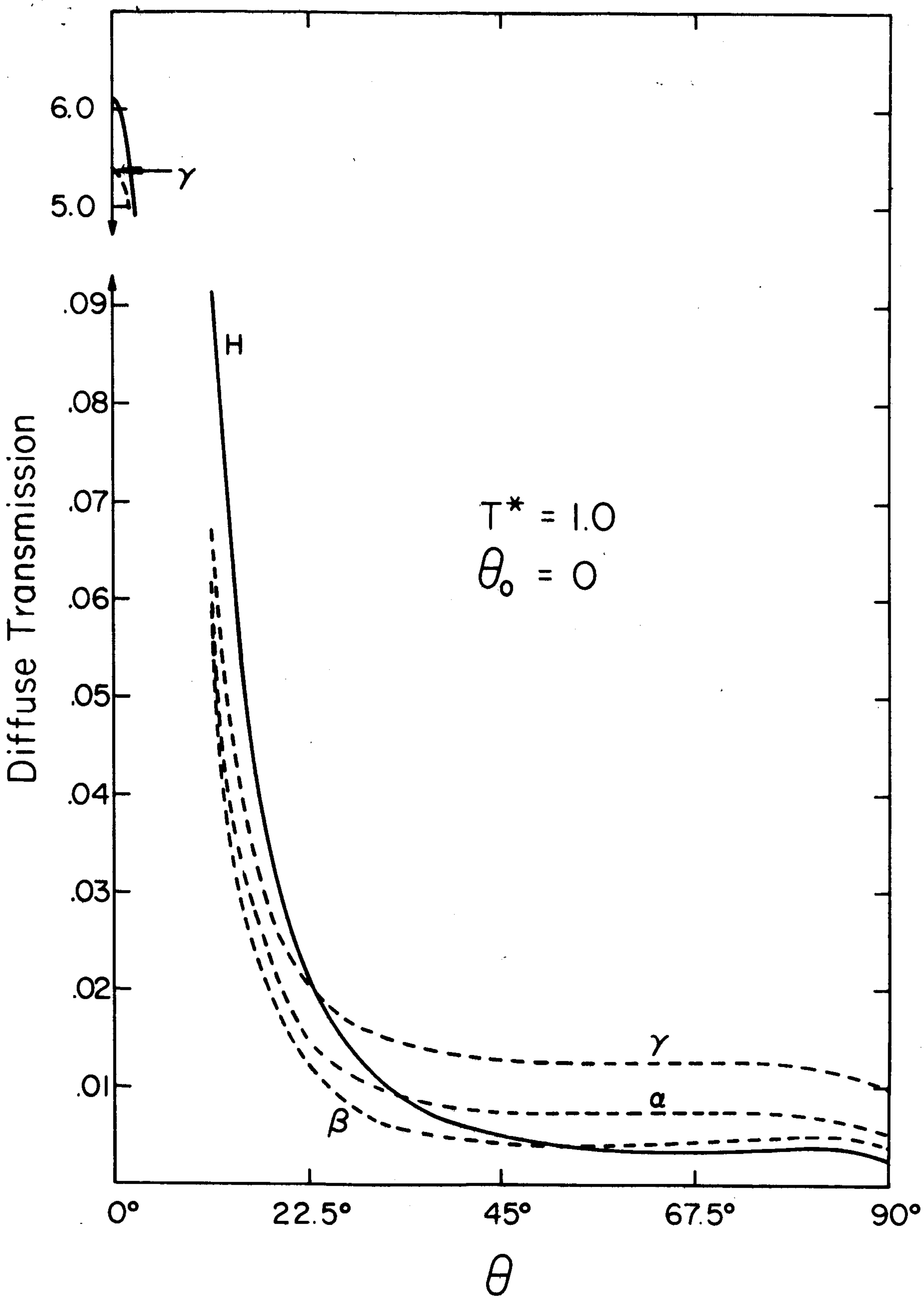
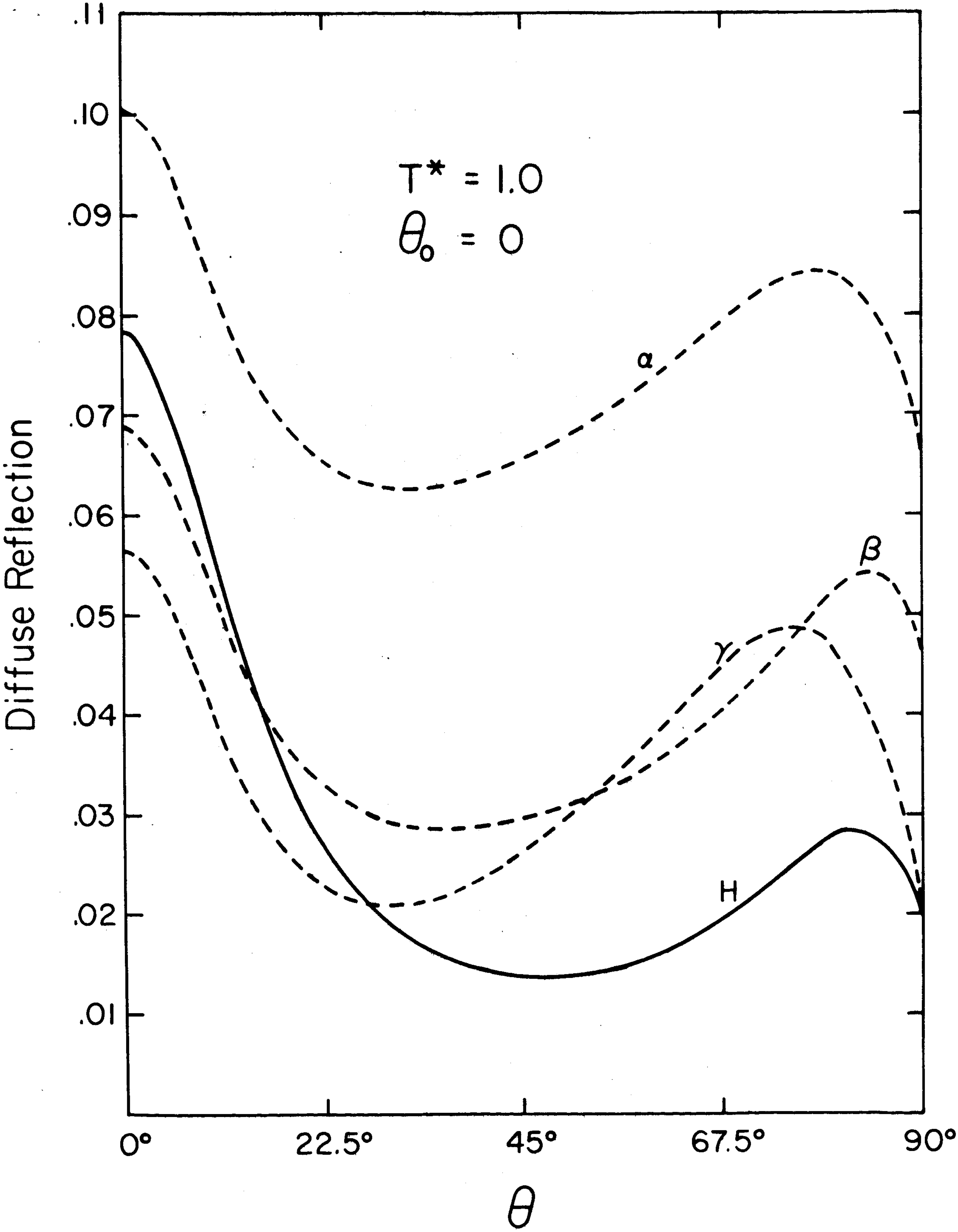


Fig. 9 a,b.--Diffuse transmission and reflection by a plane-parallel layer of unit opitcal depth for normal incidence. Exact calculations for phase function H (see Table 1). Curves α,β,γ represent approximations described in § IV. b. ii. Note change of scale in ordinate of (a).



.11
.10
.09
.08
.07
.06
.05
.04
.03
.02
.01
Diffuse Reflection
T* = 1.0
θo = 0
α
β
γ
H
0°
22.5°
45°
67.5°
90°
θ

Fig. 9 b.

TABLE 1.

PHASE FUNCTIONS ILLUSTRATED IN FIGURE 1_{a}, b, c

Label	Form* of Φ	Comments
A	eq.(11): $g = 0.75$	same first moment as for cloud droplets (van de Hulst and Irvine 1962) and interstellar grains (?;van de Hulst 1954)
B	eq.(11): $g = 0$	isotropic scattering
C	$1 + P_1$	most elongated, non-negative Φ for 2 Legendre polynomials
D	$1 + 3^{1/2}P_1 + P_2$	most elongated, non-negative Φ for 3 Legendre polynomials
E	eq.(12): $g_1 = 0.824$ $g_2 = -0.55$, $b = 0.9724$	similar to b
F	$1 + 2.3583P_1$	same first moment as E
G	$1 + 2.3583P_1 + 3.1445P_2$	same first and second moments as E
H	eq.(12): $g_1 = 0.9$, $g_2 = -0.75$, $b = 0.95$	elongated in forward direction, backward peak similar to d
J	$1 + 2.4525 P_1$	same first moment as H
a	Mie theory	continental haze at 5.3 μ
b	Mie theory	maritime haze at 0.7 μ
d	Mie theory	cumulus cloud at 0.7 μ

* P_1 are the Legendre polynomials; curves a, b, d (Deirmendjian 1963) were obtained by integrating Mie theory results over naturally occurring size distributions.