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ON THE PROBLEM OF SUSTAINED TURBULENCE

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ABSTRACT

Some steady-state solutions are obtained for thermal and for shear-flow turbulence by expanding two-point nonlinear correlation equations in power series in the space variables. The correlation equations, which are for inhomogeneous turbulence, are constructed from the Navier-Stokes and energy equations. To make the problem determinate, the weak-turbulence approximation is used. Steady-state solutions are possible because of the presence of nonlinear production terms in the correlation equations. Because only the low-order terms are retained in the power series used in the expansions, the solutions are accurate only for small values of the space variables, and specific boundary conditions cannot be applied. The forms of the solutions show that critical values of parameters (similar to Rayleigh or Reynolds numbers) exist below which the turbulent fluctuations are zero. The main conclusion of the study is that the Navier-Stokes and energy equations (averaged for turbulent flow) can yield solutions in which the energy fed into a turbulent field by buoyancy or shear forces is equal to that dissipated by viscosity.

Author

INTRODUCTION

Most of the analyses based on statistical turbulence theory have been made for a decaying turbulence that is initially generated by external means, as by flow through a grid.[1-5] Sustained turbulence, such as

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that produced by shear or buoyancy forces, has generally been investigated by using a phenomenological approach,[4-7] or by using simplified equations in place of the equations for the real fluid.[8] Statistical methods have also been used for sustained turbulence,[4,5,9] but the work is generally limited to one-point correlation equations that by themselves do not lead to solutions, although they are of considerable schematic value.

The work done to date offers little evidence that the Navier-Stokes and energy equations either are, or are not, capable of yielding solutions for steady-state turbulence. Studies of the effect of a uniform velocity gradient[10] and of a uniform temperature gradient and body force[11] on an initially isotropic turbulent field indicate that in those cases, although energy can be fed into a turbulent field by shear or buoyancy forces, the energy fed in is less than that dissipated, and the turbulence decays with time. If it were not for the abundance of sustained turbulent flows in nature, one might, on the basis of the available solutions, be led to the conclusion that steady-state turbulence will not occur.

The present study is an attempt to provide evidence that the Navier-Stokes and energy equations in averaged form can yield solutions for steady-state turbulence. The work is based on generalized two-point correlation equations that are constructed from the Navier-Stokes and energy equations by methods similar to those used by von Kármán and Howarth for isotropic turbulence.[1] To make the problem determinate, the weak turbulence approximation (triple correlations neglected) is used. This approximation was also used by von Kármán and Howarth for the case of low

Reynolds number turbulence. Although the approximation might be considered somewhat restrictive, it appears to be the only reasonable basis of analysis, unless three- or four-point correlation equations are considered.[3] Moreover, since we are studying sustained turbulence, we are more interested in the production terms in the equations than in the transfer terms. (In the case of shear-flow turbulence the velocity gradient causes energy transfer between wave numbers even when triple correlations are neglected.[10])

Sustained turbulence is essentially a nonlinear phenomenon.[12] The nonlinear character of the two-point correlation equations is made evident when the mean temperatures or velocities are eliminated by introducing one point correlation equations into the two-point correlation equations. Plane heat transfer and shear layers are considered. The correlation equations are expanded in power series in the space variables to obtain algebriac expressions for the correlations. Because only the low-order terms are retained in the series, the solutions are accurate only for small values of the space variables.

The case of sustained thermal turbulence will be considered in the next section, after which sustained shear-flow turbulence will be taken up.

SUSTAINED THERMAL TURBULENCE

The term thermal turbulence as used here designates turbulence that is sustained by buoyance forces arising from temperature gradients and a body force. For steady-state turbulence, the energy fed into the turbulent field by buoyancy forces just balances that dissipated by viscous action.

Correlation Equations

Two-point correlation equations for homogeneous turbulence with a body force and a temperature gradient are constructed from the Navier-Stokes and energy equations in reference 11. Here only the modifications necessary for inhomogeneous turbulence will be considered. From reference 11 the correlation between velocity components at two points is given by

$$\frac{\partial}{\partial t}\overline{u_i u_j'} + \frac{\partial \overline{u_i u_k u_j'}}{\partial x_k} + \frac{\partial \overline{u_i u_k' u_j'}}{\partial x_k'} = -\frac{1}{\rho}\frac{\partial}{\partial x_i}\overline{p u_j'} - \frac{1}{\rho}\frac{\partial}{\partial x_j'}\overline{u_i p'} + \nu\frac{\partial^2 \overline{u_i u_j'}}{\partial x_k \partial x_k}$$

$$+ \nu\frac{\partial^2 \overline{u_i u_j'}}{\partial x_k' \partial x_k'} - \beta g_i \overline{\tau u_j'} - \beta g_j \overline{u_i \tau'} \qquad (1)$$

where the overbars indicate averaged values. The subscripts can take on the values 1, 2, or 3, and a repeated subscript in a term indicates a summation. The quantity u_i is an instantaneous velocity component, x_i and x_i' are space coordinates at the points P and P', t is the time, ρ is the density, ν is the kinematic viscosity, p is the instantaneous pressure, g_i is a component of the body force, τ is the fluctuating part of the instantaneous temperature, and β is the thermal expansion coefficient defined by $\beta \equiv -(1/\rho)(\partial\rho/\partial T)_p$. For inhomogeneous turbulence it is convenient to introduce the variables $r_k \equiv x_k' - x_k$ and $(x_k)_m \equiv 1/2\ (x_k + x_k')$ (see Fig. 1). Then Eq. (1) becomes

$$\frac{\partial}{\partial t}\overline{u_iu_j'} + \frac{1}{2}\frac{\partial}{\partial(x_k)_m}(\overline{u_iu_j'u_k'} + \overline{u_iu_ku_j'}) + \frac{\partial}{\partial r_k}(\overline{u_iu_j'u_k'} - \overline{u_iu_ku_j'})$$

$$= -\frac{1}{\rho}\left\{\frac{1}{2}\left[\frac{\partial}{\partial(x_i)_m}\overline{pu_j'} + \frac{\partial}{\partial(x_j)_m}\overline{pu_i'}(-\vec{r})\right] + \frac{\partial}{\partial r_j}\overline{pu_i'}(-\vec{r}) - \frac{\partial}{\partial r_i}\overline{pu_j'}\right\}$$

$$+ \frac{1}{2}\nu\frac{\partial^2\overline{u_iu_j'}}{\partial(x_k)_m\partial(x_k)_m} + 2\nu\frac{\partial^2\overline{u_iu_j'}}{\partial r_k\partial r_k} - \beta g_i\overline{\tau u_j'} - \beta g_j\overline{\tau u_i'}(-\vec{r}) \qquad (2)$$

where the following transformations were used:

$$\frac{\partial}{\partial x_k} = \frac{1}{2}\frac{\partial}{\partial(x_k)_m} - \frac{\partial}{\partial r_k}$$

$$\frac{\partial}{\partial x_k'} = \frac{1}{2}\frac{\partial}{\partial(x_k)_m} + \frac{\partial}{\partial r_k}$$

$$\frac{\partial^2}{\partial x_k\partial x_k} + \frac{\partial^2}{\partial x_k'\partial x_k'} = \frac{1}{2}\frac{\partial^2}{\partial(x_k)_m\partial(x_k)_m} + 2\frac{\partial^2}{\partial r_k\partial r_k}$$

Equation (2) reduces to the Kármán-Howarth equation[1] if the turbulence is homogeneous and body forces are absent.

In a similar way the following equations are obtained.

Pressure-velocity correlations:

$$\frac{1}{\rho}\left[\frac{1}{4}\frac{\partial^2\overline{pu_j'}}{\partial(x_k)_m\partial(x_k)_m} - \frac{\partial^2\overline{pu_j'}}{\partial(x_k)_m\partial r_k} + \frac{\partial^2\overline{pu_j'}}{\partial r_k\partial r_k}\right] = -\frac{1}{4}\frac{\partial^2\overline{u_lu_ku_j'}}{\partial(x_l)_m\partial(x_k)_m}$$

$$+\frac{1}{2}\frac{\partial^2\overline{u_lu_ku_j'}}{\partial(x_l)_m\partial r_k} + \frac{1}{2}\frac{\partial^2\overline{u_lu_ku_j'}}{\partial(x_k)_m\partial r_l} - \frac{\partial^2\overline{u_lu_ku_j'}}{\partial r_l\partial r_k} - \beta g_k\left[\frac{1}{2}\frac{\partial\overline{\tau u_j'}}{\partial(x_k)_m} - \frac{\partial\overline{\tau u_j'}}{\partial r_k}\right], \qquad (3)$$

Temperature-velocity correlations:

$$\frac{\partial\overline{\tau u_j'}}{\partial t} + \overline{u_k u_j'}\frac{\partial T}{\partial x_k} + \frac{1}{2}\frac{\partial}{\partial(x_k)_m}\left(\overline{\tau u_j' u_k'} + \overline{\tau u_k u_j'}\right) + \frac{\partial}{\partial r_k}\left(\overline{\tau u_j' u_k'} - \overline{\tau u_k u_j'}\right)$$

$$= -\frac{1}{\rho}\left[\frac{1}{2}\frac{\partial}{\partial(x_j)_m}\overline{\tau p'} + \frac{\partial}{\partial r_j}\overline{\tau p'}\right] + \nu\left[\frac{1}{4}\frac{\partial^2\overline{\tau u_j'}}{\partial(x_k)_m\partial(x_k)_m} + \frac{\partial^2\overline{\tau u_j'}}{\partial(x_k)_m\partial r_k} + \frac{\partial^2\overline{\tau u_j'}}{\partial r_k\partial r_k}\right]$$

$$+ \alpha\left[\frac{1}{4}\frac{\partial^2\overline{\tau u_j'}}{\partial(x_k)_m\partial(x_k)_m} - \frac{\partial^2\overline{\tau u_j'}}{\partial(x_k)_m\partial r_k} + \frac{\partial^2\overline{\tau u_j'}}{\partial r_k\partial r_k}\right] - \beta g_j\overline{\tau\tau'}, \tag{4}$$

Temperature-temperature correlations:

$$\frac{\partial}{\partial t}\overline{\tau\tau'} + \overline{u_k\tau'}\frac{\partial T}{\partial x_k} + \overline{\tau u_k'}\frac{\partial T'}{\partial x_k'} + \frac{1}{2}\frac{\partial}{\partial(x_k)_m}\left(\overline{\tau\tau' u_k'} + \overline{u_k\tau\tau'}\right)$$

$$+ \frac{\partial}{\partial r_k}\left(\overline{\tau\tau' u_k'} - \overline{u_k\tau\tau'}\right) = \alpha\left[\frac{1}{2}\frac{\partial^2\overline{\tau\tau'}}{\partial(x_k)_m\partial(x_k)_m} + 2\frac{\partial^2\overline{\tau\tau'}}{\partial r_k\partial r_k}\right], \tag{5}$$

Temperature-pressure correlations:

$$\frac{1}{\rho}\left[\frac{1}{4}\frac{\partial^2\overline{\tau p'}}{\partial(x_k)_m\partial(x_k)_m} + \frac{\partial^2\overline{\tau p'}}{\partial(x_k)_m\partial r_k} + \frac{\partial^2\overline{\tau p'}}{\partial r_k\partial r_k}\right] = -\frac{1}{4}\frac{\partial^2\overline{\tau u_l' u_k'}}{\partial(x_l)_m\partial(x_k)_m} - \frac{1}{2}\frac{\partial^2\overline{\tau u_l' u_k'}}{\partial(x_l)_m\partial r_k}$$

$$-\frac{1}{2}\frac{\partial^2\overline{\tau u_l' u_k'}}{\partial(x_k)_m\partial r_l} - \frac{\partial^2\overline{\tau u_l' u_k'}}{\partial r_l\partial r_k} - \beta g_k\left[\frac{1}{2}\frac{\partial\overline{\tau\tau'}}{\partial(x_k)_m} + \frac{\partial\overline{\tau\tau'}}{\partial r_k}\right] \tag{6}$$

where T is the mean temperature and α is the thermal diffusivity. These equations reduce to Eqs. (7), (9), (15), and (19) of reference 11 if the turbulence is homogeneous. It is also of interest to compare them with Eqs. (5), (7), and (8) of reference 10.

Assume now that the only nonzero component of $\vec{g}$ is in the negative vertical direction, and let

$$g \equiv -g_3 \tag{7}$$

Also, let the temperature gradient be in the vertical direction so that it is given by $\partial T/\partial x_3$. The turbulence can then be homogeneous in horizontal planes, and vertical axes will be axes of symmetry. Let

$$r_1^2 + r_2^2 = \xi^2 \tag{8}$$

If the turbulence is weak enough for triple correlations to be neglected and if we let $i = j = 3$ in Eqs. (2) to (6),

$$\frac{\partial}{\partial t}\overline{u_3u_3'} = -\frac{1}{\rho}\left\{\frac{1}{2}\left[\frac{\partial}{\partial(x_3)_m}\overline{pu_3'} + \frac{\partial}{\partial(x_3)_m}\overline{pu_3'}(-\vec{r})\right] + \frac{\partial}{\partial r_3}\overline{pu_3'}(-\vec{r}) - \frac{\partial}{\partial r_3}\overline{pu_3'}\right\}$$
$$+\frac{1}{2}\nu\frac{\partial^2\overline{u_3u_3'}}{\partial(x_3)_m^2} + 2\nu\left(\frac{\partial^2\overline{u_3u_3'}}{\partial r_3^2} + \frac{\partial^2\overline{u_3u_3'}}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial\overline{u_3u_3'}}{\partial\xi}\right) + \beta g\left[\overline{\tau u_3'} + \overline{\tau u_3'}(-\vec{r})\right] \tag{9}$$

$$\frac{\partial}{\partial t}\overline{\tau u_3'} = -\overline{u_3u_3'}\frac{\partial T}{\partial x_3} - \frac{1}{\rho}\left[\frac{1}{2}\frac{\partial}{\partial(x_3)_m}\overline{\tau p'} + \frac{\partial}{\partial r_3}\overline{\tau p'}\right] + \frac{1}{4}(\nu+\alpha)\frac{\partial^2\overline{\tau u_3'}}{\partial(x_3)_m^2}$$
$$+(\nu-\alpha)\frac{\partial^2\overline{\tau u_3'}}{\partial(x_3)_m\partial r_3} + (\alpha+\nu)\left(\frac{\partial^2\overline{\tau u_3'}}{\partial r_3^2} + \frac{\partial^2\overline{\tau u_3'}}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial\overline{\tau u_3'}}{\partial\xi}\right) + \beta g\overline{\tau\tau'} \tag{10}$$

$$\frac{\partial}{\partial t}\overline{\tau\tau'} = -\overline{\tau u_3'}(-\vec{r})\frac{\partial T}{\partial x_3} - \overline{\tau u_3'}\frac{\partial T'}{\partial x_3'} + \alpha\left[\frac{1}{2}\frac{\partial^2\overline{\tau\tau}}{\partial(x_3)_m^2}\right.$$
$$\left. + 2\frac{\partial^2\overline{\tau\tau'}}{\partial r_3^2} + 2\frac{\partial^2\overline{\tau\tau'}}{\partial\xi^2} + \frac{2}{\xi}\frac{\partial\overline{\tau\tau'}}{\partial\xi}\right] \tag{11}$$

$$\frac{1}{\rho}\left[\frac{1}{4}\frac{\partial^2\overline{pu_3'}}{\partial(x_3)_m^2} - \frac{\partial^2\overline{pu_3'}}{\partial(x_3)_m\partial r_3} + \frac{\partial^2\overline{pu_3'}}{\partial r_3^2} + \frac{\partial^2\overline{pu_3'}}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial\overline{pu_3'}}{\partial\xi}\right] = \beta g\left[\frac{1}{2}\frac{\partial\overline{\tau u_3'}}{\partial(x_3)_m} - \frac{\partial\overline{\tau u_3'}}{\partial r_3}\right] \tag{12}$$

$$\frac{1}{\rho}\left[\frac{1}{4}\frac{\partial^2\overline{\tau p'}}{\partial(x_3)_m^2} + \frac{\partial^2\overline{\tau p'}}{\partial(x_3)_m\partial r_3} + \frac{\partial^2\overline{\tau p'}}{\partial r_3^2} + \frac{\partial^2\overline{\tau p'}}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial\overline{\tau p'}}{\partial\xi}\right] = \beta g\left[\frac{1}{2}\frac{\partial\overline{\tau\tau'}}{\partial(x_3)_m} + \frac{\partial\overline{\tau\tau'}}{\partial r_3}\right] \tag{13}$$

In order to eliminate the mean temperature gradient from the preceding two-point correlation equations and to emphasize their nonlinear character, we obtain a one-point correlation equation by sbustituting Eq. (5) into Eq. (11) of reference 11 and by averaging. This gives

$$\frac{\partial T}{\partial t} + \frac{\partial \overline{\tau u_k}}{\partial x_k} = \alpha \frac{\partial^2 T}{\partial x_k \partial x_k} \tag{14}$$

In the remainder of the section we will be concerned only with the steady-state case. Also, since the correlations change only in the x_3 direction, Eq. (14) becomes

$$\frac{\partial}{\partial x_3}\left(\overline{\tau u_3} - \alpha \frac{\partial T}{\partial x_3}\right) = 0$$

or

$$\overline{\tau u_3} - \alpha \frac{\partial T}{\partial x_3} = \frac{\alpha}{k} q_3 \tag{15}$$

where q_3 is the heat transfer per unit area and is independent of position. The quantity k is the thermal conductivity. The temperature gradient at point P is then given by

$$-\frac{\partial T}{\partial x_3} = \frac{q_3}{k} - \frac{\overline{\tau u_3}}{\alpha} \tag{16}$$

and that at point P' is given by

$$-\frac{\partial T'}{\partial x_3'} = \frac{q_3}{k} - \frac{\overline{\tau' u_3'}}{\alpha} \tag{17}$$

Substitution of Eqs. (16) and (17) into the two-point correlation Eqs. (9), (10), and (11) gives, for the steady-state case,

$$0 = -\frac{1}{\rho}\left\{\frac{1}{2}\left[\frac{\partial}{\partial(x_3)_m}\overline{pu_3'} + \frac{\partial}{\partial(x_3)_m}\overline{pu_3'(-\vec{r})}\right] + \frac{\partial}{\partial r_3}\overline{pu_3'(-\vec{r})} - \frac{\partial}{\partial r_3}\overline{pu_3'}\right\}$$

$$+\frac{1}{2}\nu\frac{\partial^2\overline{u_3u_3'}}{\partial(x_3)_m^2} + 2\nu\left(\frac{\partial^2\overline{u_3u_3'}}{\partial r_3^2} + \frac{\partial^2\overline{u_3u_3'}}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial\overline{u_3u_3'}}{\partial\xi}\right) + \beta g\left[\overline{\tau u_3'} + \overline{\tau u_3'(-\vec{r})}\right] \qquad (18)$$

$$0 = \overline{u_3'u_3'}\left(\frac{q_3}{k} - \frac{\overline{\tau u_3}}{\alpha}\right) - \frac{1}{\rho}\left[\frac{1}{2}\frac{\partial}{\partial(x_3)_m}\overline{\tau p'} + \frac{\partial}{\partial r_3}\overline{\tau p'}\right] + \frac{1}{4}(\nu+\alpha)\frac{\partial^2\overline{\tau u_3'}}{\partial(x_3)_m^2}$$

$$+(\nu-\alpha)\frac{\partial^2\overline{\tau u_3'}}{\partial(x_3)_m\partial r_3} + (\alpha+\nu)\left(\frac{\partial^2\overline{\tau u_3'}}{\partial r_3^2} + \frac{\partial^2\overline{\tau u_3'}}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial\overline{\tau u_3'}}{\partial\xi}\right) + \beta g\overline{\tau\tau'} \qquad (19)$$

$$0 = \overline{\tau u_3'(-\vec{r})}\left(\frac{q_3}{k} - \frac{\overline{\tau u_3}}{\alpha}\right) + \overline{\tau u_3'}\left(\frac{q_3}{k} - \frac{\overline{\tau' u_3'}}{\alpha}\right)$$

$$+\alpha\left[\frac{1}{2}\frac{\partial^2\overline{\tau\tau'}}{\partial(x_3)_m^2} + 2\frac{\partial^2\overline{\tau\tau'}}{\partial r_3^2} + 2\frac{\partial^2\overline{\tau\tau'}}{\partial\xi^2} + \frac{2}{\xi}\frac{\partial\overline{\tau\tau'}}{\partial\xi}\right] \qquad (20)$$

Eqs. (18, (19), and (20) together with Eqs. (12) and (13) form a determinate set. (Note that $\overline{\tau u_3}$ and $\overline{\tau' u_3'}$ are special cases of $\overline{\tau u_3'}$.) Inspection of the equations shows that one possible solution is given when the correlations are all zero. In that case no turbulence will occur and the heat transfer will be entirely by conduction. We would expect that if the fluid is heated from below (positive q_3) there will, in addition, by a nonzero solution, inasmuch as experiment indicates that turbulence can be set up for that case. Also, the presence of the nonlinear terms $\overline{u_3u_3'}\ \overline{\tau u_3}$, $\overline{\tau u_3'(-\vec{r})}\ \overline{\tau u_3}$, and $\overline{\tau u_3'}\ \overline{\tau' u_3'}$ in Eqs. (19) and (20) leads us to suspect that nonzero solutions exist. If those terms were not present, the no-turbulence solution would in general, be the only pertinent solution; the equations in that case would be linear and homogeneous. In the next section the possibility of turbulent solutions of the steady-state correlation equations will be investigated by expanding them in power series. Before doing that, however, it is convenient to convert them to dimensionless form by introducing the following dimensionless variables:

$$\overline{u_3u_3'}^* = \frac{\lambda^2\overline{u_3u_3'}}{\nu^2},\ \overline{\tau u_3'}^* = \frac{\beta\lambda^4 g\overline{\tau u_3'}}{\nu^3},\ \overline{\tau\tau'}^* = \frac{\beta^2\lambda^6 g^2\overline{\tau\tau'}}{\nu^4},\ \overline{pu_3'}^* = \frac{\lambda^3\overline{pu_3'}}{\rho\nu^3},$$

$$\overline{\tau p'}^* = \frac{\beta\lambda^5 g\overline{\tau p'}}{\rho\nu^4},\ (x_3)_m^* = \frac{(x_3)_m}{\lambda},\ r_i^* = \frac{r_i}{\lambda},\ N_t = \frac{q_3\beta\lambda^4 g}{\alpha k\nu},\ Pr = \frac{\nu}{\alpha},$$

where λ is a typical microscale of the turbulence. The microscale is used as a length because it is defined in terms of the shape of the correlation curve for small values of the space variables, and the solutions to be obtained are accurate only for small values of those variables. The quantity N_t is a determining parameter for the thermal turbulence and is somewhat similar to a Rayleigh number. Eqs. (18) to (20) and Eqs. (12) and (13) become, in dimensionless form,

$$0 = -\frac{1}{2}\frac{\partial}{\partial(x_3)_m^*}\overline{pu_3'}^* - \frac{1}{2}\frac{\partial}{\partial(x_3)_m^*}\overline{pu_3'}^*(-\vec{r}^*) - \frac{\partial}{\partial r_3^*}\overline{pu_3'}^*(-\vec{r}^*) + \frac{\partial}{\partial r_3^*}\overline{pu_3'}^*$$
$$+\frac{1}{2}\frac{\partial^2\overline{u_3u_3'}^*}{\partial(x_3)_m^{*2}} + 2\frac{\partial^2\overline{u_3u_3'}^*}{\partial r_3^{*2}} + 2\frac{\partial^2\overline{u_3u_3'}^*}{\partial\xi^{*2}} + \frac{2}{\xi^*}\frac{\partial\overline{u_3u_3'}^*}{\partial\xi^*} + \overline{\tau u_3'}^* + \overline{\tau u_3'}^*(-\vec{r}^*) \qquad (21)$$

$$0 = \overline{u_3u_3'}^* \left(\frac{N_t}{Pr} - Pr\ \overline{\tau u_3}^*\right) - \frac{1}{2}\frac{\partial}{\partial(x_3)_m^*}\overline{\tau p'}^* - \frac{\partial}{\partial r_3^*}\overline{\tau p'}^* + \frac{1}{4}\left(1 + \frac{1}{Pr}\right)\frac{\partial^2\overline{\tau u_3'}^*}{\partial(x_3)_m^{*2}}$$

$$+ \left(1 - \frac{1}{Pr}\right)\frac{\partial^2\overline{\tau u_3'}^*}{\partial(x_3)_m^*\partial r_3^*} + \left(\frac{1}{Pr} + 1\right)\left(\frac{\partial^2\overline{\tau u_3'}^*}{\partial r_3^{*2}} + \frac{\partial^2\overline{\tau u_3'}^*}{\partial \xi^{*2}} + \frac{1}{\xi^*}\frac{\partial\overline{\tau u_3'}^*}{\partial\xi^*}\right) + \overline{\tau\tau'}^* \tag{22}$$

$$0 = \overline{\tau u_3'}^*(-\vec{r}^*)\left(\frac{N_t}{Pr} - Pr\ \overline{\tau u_3}^*\right) + \overline{\tau u_3'}^*\left(\frac{N_t}{Pr} - Pr\ \overline{\tau' u_3'}^*\right) + \frac{1}{Pr}\left[\frac{1}{2}\frac{\partial^2\overline{\tau\tau'}^*}{\partial(x_3)_m^{*2}}\right.$$

$$\left. + 2\frac{\partial^2\overline{\tau\tau'}^*}{\partial r_3^{*2}} + 2\frac{\partial^2\overline{\tau\tau'}^*}{\partial\xi^{*2}} + \frac{2}{\xi^*}\frac{\partial\overline{\tau\tau'}^*}{\partial\xi^*}\right] \tag{23}$$

$$\frac{1}{4}\frac{\partial^2\overline{pu_3'}^*}{\partial(x_3)_m^{*2}} - \frac{\partial^2\overline{pu_3'}^*}{\partial(x_3)_m^*\partial r_3^*} + \frac{\partial^2\overline{pu_3'}^*}{\partial r_3^{*2}} + \frac{\partial^2\overline{pu_3'}^*}{\partial\xi^{*2}} + \frac{1}{\xi^*}\frac{\partial\overline{pu_3'}^*}{\partial\xi^*} = \frac{1}{2}\frac{\partial\overline{\tau u_3'}^*}{\partial(x_3)_m^*} - \frac{\partial\overline{\tau u_3'}^*}{\partial r_3^*} \tag{24}$$

$$\frac{1}{4}\frac{\partial^2\overline{\tau p'}^*}{\partial(x_3)_m^{*2}} + \frac{\partial^2\overline{\tau p'}^*}{\partial(x_3)_m^*\partial r_3^*} + \frac{\partial^2\overline{\tau p'}^*}{\partial r_3^{*2}} + \frac{\partial^2\overline{\tau p'}^*}{\partial\xi^{*2}} + \frac{1}{\xi^*}\frac{\partial\overline{\tau p'}^*}{\partial\xi^*} = \frac{1}{2}\frac{\partial\overline{\tau\tau'}^*}{\partial(x_3)_m^*} + \frac{\partial\overline{\tau\tau'}^*}{\partial r_3^*} \tag{25}$$

Series Expansions

The plan now is to expand each correlation in Eqs. (21) to (25) in a power series in ξ^*, to substitute the series into the partial differential equations, and to equate the coefficient of each power of ξ^* to zero. This gives a set of partial differential equations that does not contain ξ^*. Each dependent variable in those equations is then expanded in a series in r_3^* to obtain ordinary differential equations that do not contain r_3^*. Finally, expansion in $(x_3)_m^*$ eliminates that variable, and we end up with a set of algebraic equations that can be solved simultaneously to obtain values for the correlations.

In the present analysis we will consider only equations obtained by setting the coefficients of ξ^{*0}, r_3^{*0}, r_3^{*1}, $(x_3)_m^{*0}$, and $(x_3)_m^{*1}$ equal to zero. Thus the solution obtained will be accurate only for small values of those variables and will approach an exact solution only as ξ^*, r_3^*, and $(x_3)_m^*$ approach zero. We will not be able to accurately apply boundary conditions that state, for instance, that a correlation is zero at given points in ξ^*, r_3^*, $(x_3)_m^*$ space (e.g., at walls or at ∞), since, in general, ξ^*, etc. will not be small at the points the boundary conditions are applied. In lieu of boundary conditions we will introduce microscales that depend only on the shapes of the correlation curves near their origins. As defined by Taylor, a microscale is the distance at which the inscribed parabola at the origin of a correlation curve goes to zero.[2,4] For the low Reynolds numbers considered here (final period for decaying turbulence), the microscale of the turbulence differs but slightly from the macroscale.[3] The microscales used here are, in some cases, slight generalizations of the usual concept inasmuch as we include microscales associated with $(x_3)_m$ as well as with ξ and r_3. We will also consider microscales for the case where the slope of the correlation curve at its origin is not zero. In the latter case a third degree rather than a second degree curve is inscribed in the correlation curve at its origin.

The various correlations will, in general, have different microscales. For the sake of definiteness, they will here be arbitrarily taken as equal in most cases and will be designated by λ. (The λ used here is twice the usual microscale.) In order to obtain the actual relation between the microscales for specific boundary conditions (which we are not specifying here), it would be necessary to consider higher-order expansions of the correlation equations. The present analysis, however, should be adequate for determining whether or not reasonable solutions of the correlation equations exist for steady-state turbulence. Let

$$\left.\begin{aligned}
\overline{u_3u_3'}^* &= (\overline{u_3u_3'})_0 + (\overline{u_3u_3'})_2\,\xi^{*2}\ \ldots\\
\overline{pu_3'}^* &= (\overline{pu_3'})_0 + (\overline{pu_3'})_2\,\xi^{*2}\ \ldots\\
\overline{\tau u_3'}^* &= (\overline{\tau u_3'})_0 + (\overline{\tau u_3'})_2\,\xi^{*2}\ \ldots\\
\overline{\tau\tau'}^* &= (\overline{\tau\tau'})_0 + (\overline{\tau\tau'})_2\,\xi^{*2}\ \ldots\\
\overline{\tau p'}^* &= (\overline{\tau p'})_0 + (\overline{\tau p'})_2\,\xi^{*2}\ \ldots
\end{aligned}\right\} \tag{26}$$

where the barred quantities in parenthesis are independent of ξ^*. Then $(\overline{u_3u_3'})_0$, for instance, is the value of $(\overline{u_3u_3'}^*)$ for $\xi^* = 0$, and $(\overline{u_3u_3'})_2$ is $(1/2)\partial^2\overline{u_3u_3'}/\partial\xi^{*2}$ evaluated at $\xi^* = 0$. The odd powers of ξ^* are omitted in these expressions because of symmetry. By using a Taylor series we can write $\overline{\tau u_3}^*$ and $\overline{\tau' u_3}^*$ in Eqs. (22) and (23) as (Fig. 1)

$$\overline{\tau u_3'}^* = (\overline{\tau u_3'})_{00} - \frac{\partial}{\partial(x_3)_m^*}(\overline{\tau u_3'})_{00}\frac{r_3^*}{2} + \frac{\partial^2}{\partial(x_3)_m^{*2}}(\overline{\tau u_3'})_{00}\frac{r_3^{*2}}{8}\ \ldots \tag{27}$$

$$\overline{\tau' u_3'}^* = (\overline{\tau u_3'})_{00} + \frac{\partial}{\partial(x_3)_m^*}(\overline{\tau u_3'})_{00}\frac{r_3^*}{2} + \frac{\partial^2}{\partial(x_3)_m^{*2}}(\overline{\tau u_3'})_{00}\frac{r_3^{*2}}{8}\ \ldots \tag{28}$$

where $(\overline{\tau u_3'})_{00}^*$ is the value of $\overline{\tau u_3'}^*$ at $(x_3)_m^*$ for $\xi^* = r_3^* = 0$. Substituting Eqs. (26) to (28) in Eqs. (21) to (25) and setting the coefficient of ξ^{*0} equal to zero in each equation give

$$0 = -\frac{1}{2}\frac{\partial}{\partial(x_3)^*_m}(\overline{pu_3'})_0 - \frac{1}{2}\frac{\partial}{\partial(x_3)^*_m}(\overline{pu_3'})_0(-r_3^*) - \frac{\partial}{\partial r_3^*}(\overline{pu_3'})_0(-r_3^*)$$

$$+ \frac{\partial}{\partial r_3^*}(\overline{pu_3'})_0 + \frac{1}{2}\frac{\partial^2}{\partial(x_3)^{*2}_m}(\overline{u_3u_3'})_0 + 2\frac{\partial^2}{\partial r_3^{*2}}(\overline{u_3u_3'})_0$$

$$+ 8(\overline{u_3u_3'})_2 + (\overline{\tau u_3'})_0 + (\overline{\tau u_3'})_0(-r_3^*) \tag{29}$$

$$0 = (\overline{u_3u_3'})_0\left\{\frac{N_t}{Pr} - Pr\left[(\overline{\tau u_3'})_{00} - \frac{\partial}{\partial(x_3)^*_m}(\overline{\tau u_3'})_{00}\frac{r_3^*}{2} + \frac{\partial^2}{\partial(x_3)^{*2}_m}(\overline{\tau u_3'})_{00}\frac{r_3^{*2}}{8}\right]\right\}$$

$$-\frac{1}{2}\frac{\partial}{\partial(x_3)^*_m}(\overline{\tau p'})_0 - \frac{\partial}{\partial r_3^*}(\overline{\tau p'})_0 + \frac{1}{4}\left(1+\frac{1}{Pr}\right)\frac{\partial^2}{\partial(x_3)^{*2}_m}(\overline{\tau u_3'})_0$$

$$+\left(1-\frac{1}{Pr}\right)\frac{\partial^2}{\partial(x_3)^*_m\partial r_3^*}(\overline{\tau u_3'})_0$$

$$+\left(\frac{1}{Pr}+1\right)\left[\frac{\partial^2}{\partial r_3^{*2}}(\overline{\tau u_3'})_0 + 4(\overline{\tau u_3'})_2\right] + (\overline{\tau\tau'})_0 \tag{30}$$

$$0 = (\overline{\tau u_3'})_0(-r_3^*)\left\{\frac{N_t}{Pr} - Pr\left[(\overline{\tau u_3'})_{00} - \frac{\partial}{\partial(x_3)^*_m}(\overline{\tau u_3'})_{00}\frac{r_3^*}{2} + \frac{\partial^2}{\partial(x_3)^{*2}_m}(\overline{\tau u_3'})_{00}\frac{r_3^{*2}}{8}\right]\right\}$$

$$+ (\overline{\tau u_3'})_0\left\{\frac{N_t}{Pr} - Pr\left[(\overline{\tau u_3'})_{00} + \frac{\partial}{\partial(x_3)^*_m}(\overline{\tau u_3'})_{00}\frac{r_3^*}{2}\right.\right.$$

$$\left.\left. + \frac{\partial^2}{\partial(x_3)^{*2}_m}(\overline{\tau u_3'})_{00}\frac{r_3^{*2}}{8}\right]\right\}$$

$$+\frac{1}{Pr}\left[\frac{1}{2}\frac{\partial^2}{\partial(x_3)^{*2}_m}(\overline{\tau\tau'})_0 + 2\frac{\partial^2}{\partial r_3^{*2}}(\overline{\tau\tau'})_0 + 8(\overline{\tau\tau'})_2\right] \tag{31}$$

$$\frac{1}{4}\frac{\partial^2}{\partial(x_3)_m^{*2}}(\overline{pu_3'})_0 - \frac{\partial^2}{\partial(x_3)_m^*\partial r_3^*}(\overline{pu_3'})_0 + \frac{\partial^2}{\partial r_3^{*2}}(\overline{pu_3'})_0 + 4(\overline{pu_3'})_2$$

$$= \frac{1}{2}\frac{\partial}{\partial(x_3)_m^*}(\overline{\tau u_3'})_0 - \frac{\partial}{\partial r_3^*}(\overline{\tau u_3'})_0 \qquad (32)$$

$$\frac{1}{4}\frac{\partial^2}{\partial(x_3^*)^2}(\overline{\tau p'})_0 + \frac{\partial^2}{\partial(x_3)_m^*\partial r_3^*}(\overline{\tau p'})_0 + \frac{\partial^2}{\partial r_3^{*2}}(\overline{\tau p'})_0 + 4(\overline{\tau p'})_2$$

$$= \frac{1}{2}\frac{\partial}{\partial(x_3)_m^*}(\overline{\tau\tau'})_0 + \frac{\partial}{\partial r_3^*}(\overline{\tau\tau'})_0 \qquad (33)$$

As mentioned previously, in order to obtain definite results we arbitrarily assume that the microscales for the correlations are equal to λ. Thus, in Eqs. (26) let $\overline{u_3u_3'} = \overline{pu_3'} = \overline{\tau u_3'} = \overline{\tau\tau'} = \overline{\tau p'} = 0$ for $\xi^* = \xi/\lambda = 1$. (Note that the actual correlations are not zero at those points, but only the inscribed parabolas given by Eqs. (26).) Then $(\overline{u_3u_3'})_2$, $(\overline{\tau u_3'})_2$, $(\overline{\tau\tau'})_2$, $(\overline{pu_3'})_2$, and $(\overline{\tau p'})_2$ in Eqs. (29) to (33) are replaced respectively by $-(\overline{u_3u_3'})_0$, $-(\overline{\tau u_3'})_0$, $-(\overline{\tau\tau'})_0$, $-(\overline{pu_3'})_0$, and $-(\overline{\tau p'})_0$. (If the microscales are not taken as equal, $(\overline{u_3u_3'})_2$, for instance, would equal a negative constant times $(\overline{u_3u_3'})_0$.)

Next, let

$$\left.\begin{aligned}
(\overline{u_3u_3'})_0 &= (\overline{u_3u_3'})_{00} + (\overline{u_3u_3'})_{01}\, r_3^* + (\overline{u_3u_3'})_{02}\, r_3^{*2} + (\overline{u_3u_3'})_{03}\, r_3^{*3} \\
(\overline{pu_3'})_0 &= (\overline{pu_3'})_{00} + (\overline{pu_3'})_{01}\, r_3^* + (\overline{pu_3'})_{02}\, r_3^{*2} + (\overline{pu_3'})_{03}\, r_3^{*3} \\
(\overline{\tau u_3'})_0 &= (\overline{\tau u_3'})_{00} + (\overline{\tau u_3'})_{01}\, r_3^* + (\overline{\tau u_3'})_{02}\, r_3^{*2} + (\overline{\tau u_3'})_{03}\, r_3^{*3} \\
(\overline{\tau\tau'})_0 &= (\overline{\tau\tau'})_{00} + (\overline{\tau\tau'})_{01}\, r_3^* + (\overline{\tau\tau'})_{02}\, r_3^{*2} + (\overline{\tau\tau'})_{03}\, r_3^{*3} \\
(\overline{\tau p'})_0 &= (\overline{\tau p'})_{00} + (\overline{\tau p'})_{01}\, r_3^* + (\overline{\tau p'})_{02}\, r_3^{*2} + (\overline{\tau p'})_{03}\, r_3^{*3}
\end{aligned}\right\} \qquad (34)$$

Substituting Eqs. (34) into Eqs. (29) to (33) and setting the coefficient of r_3^{*0} and of r_3^{*1} in each equation equal to zero give (with $(\overline{u_3u_3'})_2 = -(\overline{u_3u_3'})_0$, etc.) a set of ordinary differential equations in $(x_3)_m^*$. In Eqs. (34) we let $(\overline{u_3u_3'})_0$, $(\overline{pu_3'})_0$, etc. equal 0 for $r_3^* = \pm 1$ to obtain $(\overline{u_3u_3'})_{02} = -(\overline{u_3u_3'})_{00}$, $(\overline{u_3u_3'})_{03} = -(\overline{u_3u_3'})_{01}$, $(\overline{pu_3'})_{02} = -(\overline{pu_3'})_{00}$, $(\overline{pu_3'})_{03} = -(\overline{pu_3'})_{01}$, etc.

The quantities $(\overline{u_3u_3'})_{00}$, etc. can be expanded as follows:

$$\left.\begin{aligned}
(\overline{u_3u_3'})_{00} &= (\overline{u_3u_3'})_{000} + (\overline{u_3u_3'})_{002}(x_3)_m^{*2} \\
(\overline{pu_3'})_{00} &= (\overline{pu_3'})_{000} + (\overline{pu_3'})_{001}(x_3)_m^{*} + (\overline{pu_3'})_{002}(x_3)_m^{*2} + (\overline{pu_3'})_{003}(x_3)_m^{*3} \\
(\overline{pu_3'})_{01} &= (\overline{pu_3'})_{010} + (\overline{pu_3'})_{011}(x_3)_m^{*} + (\overline{pu_3'})_{012}(x_3)_m^{*2} + (\overline{pu_3'})_{013}(x_3)_m^{*3} \\
(\overline{\tau p'})_{00} &= (\overline{\tau p'})_{000} + (\overline{\tau p'})_{001}(x_3)_m^{*} + (\overline{\tau p'})_{002}(x_3)_m^{*2} + (\overline{\tau p'})_{003}(x_3)_m^{*3} \\
(\overline{\tau p'})_{01} &= (\overline{\tau p'})_{010} + (\overline{\tau p'})_{011}(x_3)_m^{*} + (\overline{\tau p'})_{012}(x_3)_m^{*2} + (\overline{\tau p'})_{013}(x_3)_m^{*3} \\
(\overline{\tau u_3'})_{00} &= (\overline{\tau u_3'})_{000} + (\overline{\tau u_3'})_{002}(x_3)_m^{*2} \\
(\overline{\tau u_3'})_{01} &= (\overline{\tau u_3'})_{010} + (\overline{\tau u_3'})_{011}(x_3)_m^{*} + (\overline{\tau u_3'})_{012}(x_3)_m^{*2} + (\overline{\tau u_3'})_{013}(x_3)_m^{*3} \\
(\overline{\tau\tau'})_{00} &= (\overline{\tau\tau'})_{000} + (\overline{\tau\tau'})_{002}(x_3)_m^{*2}
\end{aligned}\right\} \quad (35)$$

The odd powers of $(x_3)_m^*$ are omitted in the expressions for $(\overline{u_3u_3'})_{00}$, $(\overline{\tau u_3'})_{00}$, and $(\overline{\tau\tau'})_{00}$ in order to make those one-point correlations symmetric about $(x_3)_m^* = 0$. Substituting Eqs. (35) into the ordinary differential equations in $(x_3)_m^*$ (not shown) and setting the coefficients of $(x_3)_m^{*0}$ equal to zero give (with $(\overline{u_3u_3'})_{02} = -(\overline{u_3u_3'})_{00}$, etc.).

$$0 = -(\overline{pu_3'})_{001} + 2(\overline{pu_3'})_{010} + (\overline{u_3u_3'})_{002} - 12(\overline{u_3u_3'})_{000} + 2(\overline{\tau u_3'})_{000} \tag{36}$$

$$0 = (\overline{u_3u_3'})_{000}\left[\frac{N_t}{Pr} - Pr(\overline{\tau u_3'})_{000}\right] - \frac{1}{2}(\overline{\tau p'})_{001} - (\overline{\tau p'})_{010} + \frac{1}{2}\left(1 + \frac{1}{Pr}\right)(\overline{\tau u_3'})_{002}$$

$$+ \left(1 - \frac{1}{Pr}\right)(\overline{\tau u_3'})_{011} - 6\left(\frac{1}{Pr} + 1\right)(\overline{\tau u_3'})_{000} + (\overline{\tau\tau'})_{000} \tag{37}$$

$$0 = 2(\overline{\tau u_3'})_{000}\left[\frac{N_t}{Pr} - Pr(\overline{\tau u_3'})_{000}\right] + \frac{1}{Pr}\left[(\overline{\tau\tau'})_{002} - 12(\overline{\tau\tau'})_{000}\right] \tag{38}$$

$$\frac{1}{2}(\overline{pu_3'})_{002} - (\overline{pu_3'})_{011} - 6(\overline{pu_3'})_{000} = -(\overline{\tau u_3'})_{010} \tag{39}$$

$$\frac{1}{2}(\overline{\tau p'})_{002} + (\overline{\tau p'})_{011} - 6(\overline{\tau p'})_{000} = 0 \tag{40}$$

$$0 = -\frac{1}{2}(\overline{\tau p'})_{011} + 2(\overline{\tau p'})_{000} + \frac{1}{2}\left(1 + \frac{1}{Pr}\right)(\overline{\tau u_3'})_{012} - 10\left(\frac{1}{Pr} + 1\right)(\overline{\tau u_3'})_{010} \tag{41}$$

$$\frac{1}{2}(\overline{pu_3'})_{012} + 2(\overline{pu_3'})_{001} - 10(\overline{pu_3'})_{010} = \frac{1}{2}(\overline{\tau u_3'})_{011} + 2(\overline{\tau u_3'})_{000} \tag{42}$$

$$\frac{1}{2}(\overline{\tau p'})_{012} - 2(\overline{\tau p'})_{001} - 10(\overline{\tau p'})_{010} = -2(\overline{\tau\tau'})_{000} \tag{43}$$

$$(\overline{u_3u_3'})_{010} = (\overline{\tau\tau'})_{010} = 0 \tag{44}$$

Setting the coefficients of $(x_3)_m^{*1}$ equal to zero gives

$$\frac{3}{2}(\overline{pu_3'})_{001} - 2(\overline{pu_3'})_{012} - 6(\overline{pu_3'})_{001} = (\overline{\tau u_3'})_{002} - (\overline{\tau u_3'})_{011} \tag{45}$$

$$\frac{3}{2}(\overline{\tau p'})_{003} + 2(\overline{\tau p'})_{012} - 6(\overline{\tau p'})_{001} = (\overline{\tau\tau'})_{002} \tag{46}$$

$$\frac{3}{2}(\overline{pu_3'})_{013} + 4(\overline{pu_3'})_{002} - 10(\overline{pu_3'})_{011} = (\overline{\tau u_3'})_{012} \tag{47}$$

$$\frac{3}{2}(\overline{\tau p'})_{013} - 4(\overline{\tau p'})_{002} - 10(\overline{\tau p'})_{011} = 0 \tag{48}$$

$$0 = \Pr(\overline{u_3u_3'})_{000}(\overline{\tau u_3'})_{002} - (\overline{\tau p'})_{012} + 2(\overline{\tau p'})_{001} + \frac{3}{2}\left(1 + \frac{1}{\Pr}\right)(\overline{\tau u_3'})_{013}$$

$$-4\left(1 - \frac{1}{\Pr}\right)(\overline{\tau u_3'})_{002} - 10\left(\frac{1}{\Pr} + 1\right)(\overline{\tau u_3'})_{011} \qquad (49)$$

$$(\overline{u_3u_3'})_{011} = (\overline{\tau\tau'})_{011} = 0 \qquad (50)$$

To set the microscales associated with $(x_3)_m$ equal to λ, let $(\overline{u_3u_3'})_{00}$, $(\overline{pu_3'})_{00}$, etc. in Eqs. (35) equal zero when $(x_3)_m^* = \pm 1/2$. (We use $(x_3)_m^* = \pm 1/2$ instead of ± 1 as in the other cases, because of the difference between the definition of $(x_3)_m$ and that of r_3 or ξ. See Fig. 1.) Then $(\overline{u_3u_3'})_{002} = -4(\overline{u_3u_3'})_{000}$, $(\overline{pu_3'})_{002} = -4(\overline{pu_3'})_{000}$, $(\overline{pu_3'})_{003} = -4(\overline{pu_3'})_{001}$, etc. With these relations, Eqs. (36) to (50) form a determinate set of algebraic equations that can be solved simultaneously. Equations (40), (48), and (41) show that

$$(\overline{\tau p'})_{000} = (\overline{\tau p'})_{011} = (\overline{\tau u_3'})_{010} = 0 \qquad (51)$$

From Eqs. (42), (45), and (36),

$$(\overline{u_3u_3'})_{000} = \frac{3}{32}(\overline{\tau u_3'})_{000} - \frac{1}{128}(\overline{\tau u_3'})_{011} \qquad (52)$$

Combining Eqs. (43), (46), and (49) gives

$$(\overline{\tau u_3'})_{011} = -\frac{1}{4}\frac{\Pr}{\frac{1}{\Pr} + 1}(\overline{u_3u_3'})_{000}(\overline{\tau u_3'})_{000}$$

$$+ \frac{1}{16}\frac{1}{\frac{1}{\Pr} + 1}(\overline{\tau\tau'})_{000} + \frac{\left(1 - \frac{1}{\Pr}\right)}{\frac{1}{\Pr} + 1}(\overline{\tau u_3'})_{000} \qquad (53)$$

From Eq. (38)

$$(\overline{\tau\tau'})_{000} = \frac{1}{8}\Pr(\overline{\tau u_3'})_{000}\left[\frac{N_t}{\Pr} - \Pr(\overline{\tau u_3'})_{000}\right] \qquad (54)$$

Finally, Eqs. (43), (46), (52) to (54) and (37) combine to give

$$0 = \left(\frac{3}{32} - \frac{1}{128}\,\frac{Pr - 1}{Pr + 1}\right)\omega - \frac{1}{16,384}\,\frac{Pr^2}{Pr + 1}\,\omega^2 + \frac{3}{32}\,Pr\omega\gamma$$

$$- 8\,\frac{Pr + 1}{Pr}\,\gamma - \frac{1}{4}\,\frac{Pr - 1}{Pr + 1}\,Pr\left(\frac{3}{32} - \frac{1}{128}\,\frac{Pr - 1}{Pr + 1}\right)(\overline{\tau u_3'})_{000}$$

$$+ \frac{1}{65,536}\,\frac{Pr^3(Pr - 1)}{(Pr + 1)^2}\,(\overline{\tau u_3'})_{000}\,\omega + \frac{Pr - 1}{Pr}\,\gamma + \frac{1}{128}\,\frac{(Pr - 1)}{(Pr + 1)}\,\omega\gamma \qquad (55)$$

where

$$\omega = \frac{N_t}{Pr} - Pr(\overline{\tau u_3'})_{000}$$

and

$$\gamma = 1 - \frac{1}{512}\,\frac{Pr^2}{(1 + Pr)}\,(\overline{\tau u_3'})_{000}$$

Note that the original $(\overline{\tau u_3'})_{000}$ in the linear terms of Eq. (37) (that equation having come from Eq. (10), the equation for $\overline{\tau u_3'}$) canceled out of Eq. (55). However, Eq. (55) still yields values for $(\overline{\tau u_3'})_{000}$, which is the value of $\overline{\tau u_3'}$ for $\xi = r_3 = (x_3)_m = 0$, because the original equations were nonlinear in $(\overline{\tau u_3'})_{000}$.

Positive values of N_t correspond to negative temperature gradients (heating from below), and turbulent solutions should exist for sufficiently high values of N_t. Equation (55) was solved for $Pr = 0.7$ (corresponding approximately to gases) and for several values of N_t. As N_t increases, $(\overline{\tau u_3'})$, as well as the other correlations, becomes positive at $N_t = 88.1$. Thus that value of N_t can be thought of as a critical N_t above which turbulence might occur. For $N_t = 100$, $(\overline{\tau u_3'})_{000} = 26.4$, and for $N_t = 200$, $(\overline{\tau u_3'})_{000} = 251$. From Eqs. (52) to (54), for $N_t = 100$, $(\overline{u_3 u_3'})_{000} = 2.49$ and $(\overline{\tau\tau'})_{000} = 287$, and for

$N_t = 200$, $(\overline{u_3 u_3'})_{000} = 27.3$ and $(\overline{\tau\tau'})_{000} = 2,414$. Thus the correlations have the correct signs and the correct trends as N_t increases above its critical value. For values of N_t below the critical value, including negative N_t (positive temperature gradient), the correlations become negative according to Eqs. (52) to (54), so that those equations do not yield possible solutions. In that case we should use the other solution of Eqs. (21) to (25), that is, the no-turbulence solution. Note that above the critical N_t both the turbulent and the nonturbulent solutions are possible according to the equations; consequently, the fluid is not necessarily turbulent for all values of N_t above the critical value considered here.

One other quantity that should be considered is the eddy diffusivity for heat transfer ϵ_h, which is defined as

$$\epsilon_h \equiv - \frac{\overline{\tau u_3}}{\partial T / \partial x_3} = \frac{\overline{\tau u_3}}{\dfrac{q_3}{k} - \dfrac{\overline{\tau u_3}}{\alpha}} \tag{56}$$

In dimensionless form, at $(x_3)_m^* = 0$,

$$\epsilon_h^* = \frac{(\overline{\tau u_3'})_{000}}{\dfrac{N_t}{Pr} - Pr(\overline{\tau u_3'})_{000}}$$

From the preceding computed results, ϵ_h is positive and increases as N_t increases above its critical value so that the values obtained for ϵ_h are also reasonable.

It might again be mentioned that the preceding solutions are for the case where the microscales are all equal. Thus the results probably do not correspond numerically to those for particular boundary conditions.

They are given here to indicate that reasonable solutions for steady-state turbulence can be obtained from the correlation equations.

Equation (55) is quadratic in $(\overline{\tau u_3'})_{000}$ and even for unequal microscales the solution for $(\overline{\tau u_3'})_{000}$ is of the form

$$(\overline{\tau u_3'})_{000} = a + bN_t \pm \sqrt{cN_t^2 + eN_t + f} \tag{57}$$

where a, b, etc. are functions of Prandtl number and the microscale ratios. (The microscale ratios might in turn be functions of N_t, but those functions are probably slowly varying.) Setting $(\overline{\tau u_3'}) = 0$ in Eq. (57) gives two values for a critical N_t (for given values of a, b, etc.), but, of course, only one of those would be expected to be physically realizable.

SUSTAINED SHEAR-FLOW TURBULENCE

Correlation Equations

General two-point correlation equations for incompressible turbulent shear flow were obtained from the Navier-Stokes equations in reference 10 as

$$\frac{\partial}{\partial t}\overline{u_i u_j'} + \overline{u_k u_j'}\frac{\partial U_i}{\partial x_k} + \overline{u_i u_k'}\frac{\partial U_j'}{\partial x_k'} + (U_k' - U_k)\frac{\partial}{\partial r_k}\overline{u_i u_j'} + \frac{1}{2}(U_k + U_k')\frac{\partial}{\partial (x_k)_m}\overline{u_i u_j'}$$

$$+ \frac{1}{2}\frac{\partial}{\partial (x_k)_m}(\overline{u_i u_j' u_k'} + \overline{u_i u_k u_j'}) + \frac{\partial}{\partial r_k}(\overline{u_i u_j' u_k'} - \overline{u_i u_k u_j'})$$

$$= -\frac{1}{\rho}\left\{\frac{1}{2}\left[\frac{\partial}{\partial (x_i)_m}\overline{p u_j'} + \frac{\partial}{\partial (x_j)_m}\overline{u_i p'}\right] + \frac{\partial}{\partial r_j}\overline{u_i p'} - \frac{\partial}{\partial r_i}\overline{p u_j'}\right\}$$

$$+ \frac{1}{2}\nu\frac{\partial^2 \overline{u_i u_j'}}{\partial (x_k)_m \partial (x_k)_m} + 2\nu\frac{\partial^2 \overline{u_i u_j'}}{\partial r_k \partial r_k} \tag{58}$$

$$\frac{1}{\rho}\left[\frac{1}{4}\frac{\partial^2\overline{u_i p'}}{\partial(x_k)_m\partial(x_k)_m}+\frac{\partial^2\overline{u_i p'}}{\partial(x_k)_m\partial r_k}+\frac{\partial^2\overline{u_i p'}}{\partial r_k\partial r_k}\right] = -2\,\frac{\partial U_l'}{\partial x_k'}\left[\frac{1}{2}\frac{\partial\overline{u_i u_k'}}{\partial(x_l)_m}+\frac{\partial\overline{u_i u_k'}}{\partial r_l}\right]$$

$$-\frac{1}{4}\frac{\partial^2\overline{u_i u_l' u_k'}}{\partial(x_l)_m\partial(x_k)_m}-\frac{1}{2}\frac{\partial^2\overline{u_i u_l' u_k'}}{\partial(x_l)_m\partial r_k}-\frac{1}{2}\frac{\partial^2\overline{u_i u_l' u_k'}}{\partial(x_k)_m\partial r_l}-\frac{\partial^2\overline{u_i u_l' u_k'}}{\partial r_l\partial r_k} \qquad (59)$$

$$\frac{1}{\rho}\left[\frac{1}{4}\frac{\partial^2\overline{p u_j'}}{\partial(x_k)_m\partial(x_k)_m}-\frac{\partial^2\overline{p u_j'}}{\partial(x_k)_m\partial r_k}+\frac{\partial^2\overline{p u_j'}}{\partial r_k\partial r_k}\right] = -2\,\frac{\partial U_l}{\partial x_k}\left[\frac{1}{2}\frac{\partial\overline{u_k u_j'}}{\partial(x_l)_m}-\frac{\partial\overline{u_k u_j'}}{\partial r_l}\right]$$

$$-\frac{1}{4}\frac{\partial^2\overline{u_l u_k u_j'}}{\partial(x_l)_m\partial(x_k)_m}+\frac{1}{2}\frac{\partial^2\overline{u_l u_k u_j'}}{\partial(x_l)_m\partial r_k}+\frac{1}{2}\frac{\partial^2\overline{u_l u_k u_j'}}{\partial(x_k)_m\partial r_l}-\frac{\partial^2\overline{u_l u_k u_j'}}{\partial r_l\partial r_k} \qquad (60)$$

where u_i is the fluctuating part of an instantaneous velocity component, U_i is a mean velocity component, and the other quantities have the same meanings as in the equations in the preceding sections. As before, the unprimed quantities are measured at point P and the primed ones at P'. The vector configuration in Fig. 1 applies to the present equations. A one-point equation given in reference 10 is

$$\frac{\partial U_i}{\partial t} + U_k\frac{\partial U_i}{\partial x_k} = -\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x_i} + \frac{\partial}{\partial x_k}\left(\nu\,\frac{\partial U_i}{\partial x_k} - \overline{u_i u_k}\right) \qquad (61)$$

It is suggested in reference 10 that Eqs. (58) to (61), together with higher-order equations, should constitute a solution to the turbulent shear-flow problem. Further evidence that this is the case is given herein.

In this analysis it is assumed that the mean velocity is in the x_1 direction and that changes in the mean quantities can take place only in the x_2 direction; consequently, we have plane shear layers. If the turbulence is weak enough for triple correlations to be neglected and if we let $i = j = 2$ in Eq. (58), that equation becomes

$$\frac{\partial}{\partial t}\overline{u_2u_2'} = -(U_1' - U_1)\frac{\partial}{\partial r_1}\overline{u_2u_2'} - \frac{1}{\rho}\left\{\frac{1}{2}\left[\frac{\partial}{\partial(x_2)_m}\overline{pu_2'} + \frac{\partial}{\partial(x_2)_m}\overline{pu_2'}(-\vec{r})\right]\right.$$

$$\left. + \frac{\partial}{\partial r_2}(\overline{pu_2'})(-\vec{r}) - \frac{\partial}{\partial r_2}\overline{pu_2'}\right\} + \frac{1}{2}\nu\frac{\partial^2\overline{u_2u_2'}}{\partial(x_2)_m\partial(x_2)_m} + 2\nu\frac{\partial^2\overline{u_2u_2'}}{\partial r_k\partial r_k} \qquad (62)$$

Similarly, if $i = 1$ and $j = 2$ in Eqs. (58) to (60), we obtain

$$\frac{\partial}{\partial t}\overline{u_1u_2'} = -\overline{u_2u_2'}\frac{\partial U_1}{\partial x_2} - (U_1' - U_1)\frac{\partial}{\partial r_1}\overline{u_1u_2'} - \frac{1}{\rho}\left[\frac{1}{2}\frac{\partial}{\partial(x_2)_m}\overline{u_1p'}\right.$$

$$\left. + \frac{\partial}{\partial r_2}\overline{u_1p'} - \frac{\partial}{\partial r_1}\overline{pu_2'}\right] + \frac{1}{2}\nu\frac{\partial^2\overline{u_1u_2'}}{\partial(x_2)_m\partial(x_2)_m} + 2\nu\frac{\partial^2\overline{u_2u_2'}}{\partial r_k\partial r_k} \qquad (63)$$

$$\frac{1}{\rho}\left[\frac{1}{4}\frac{\partial^2\overline{pu_2'}}{\partial(x_2)_m\partial(x_2)_m} - \frac{\partial^2\overline{pu_2'}}{\partial(x_2)_m\partial r_2} + \frac{\partial^2\overline{pu_2'}}{\partial r_k\partial r_k}\right] = 2\frac{\partial U_1}{\partial x_2}\frac{\partial\overline{u_2u_2'}}{\partial r_1} \qquad (64)$$

and

$$\frac{1}{\rho}\left[\frac{1}{4}\frac{\partial^2\overline{u_1p'}}{\partial(x_2)_m\partial(x_2)_m} + \frac{\partial^2\overline{u_1p'}}{\partial(x_2)_m\partial r_2} + \frac{\partial^2\overline{u_1p'}}{\partial r_k\partial r_k}\right] = -2\frac{\partial U_1'}{\partial x_2'}\frac{\partial\overline{u_1u_2'}}{\partial r_1} \qquad (65)$$

The one-point equation, Eq. (61), can be used to eliminate the mean velocities from these equations. If the turbulence is steady state, Eq. (61) becomes (for the plane shear layers considered here with no mean pressure gradient)

$$\frac{\partial}{\partial x_2}\left(\nu\frac{\partial U_1}{\partial x_2} - \overline{u_1u_2}\right) = 0$$

or

$$\nu \frac{\partial U_1}{\partial x_2} - \overline{u_1 u_2} = \frac{\tau}{\rho} \tag{66}$$

where τ is the total shear stress (made up of the laminar and the turbulent shear stress) and is independent of position. Then the velocity gradient at point P is

$$\frac{\partial U_1}{\partial x_2} = \frac{\tau}{\rho\nu} + \frac{1}{\nu}\overline{u_1 u_2} \tag{67}$$

and that at P' is

$$\frac{\partial U_1'}{\partial x_2'} = \frac{\tau}{\rho\nu} + \frac{1}{\nu}\overline{u_1' u_2'} \tag{68}$$

At a general point P'',

$$\frac{\partial U_1''}{\partial x_2''} = \frac{\tau}{\rho\nu} + \overline{u_1'' u_2''} \tag{69}$$

Integrating Eq. (69) between P and P', that is, from $(x_2)_m - r_2/2$ to $(x_2)_m + r_2/2$ (Fig. 1), gives

$$U_1' - U_1 = \frac{\tau}{\rho\nu} r_2 + \frac{1}{\nu}\int_{(x_2)_m - r_2/2}^{(x_2)_m + r_2/2} \overline{u_1'' u_2''}\, dx_2'' \tag{70}$$

The substitution of Eqs. (67), (68), and (70) into Eqs. (62) to (65) gives, for steady-state turbulence,

$$0 = -\left(\frac{\tau}{\rho\nu} r_2 + \frac{1}{\nu}\int_{(x_2)_m - r_2/2}^{(x_2)_m + r_2/2} \overline{u_1'' u_2''}\, dx_2''\right)\frac{\partial}{\partial r_1}\overline{u_2 u_2'}$$

$$- \frac{1}{\rho}\left\{\frac{1}{2}\left[\frac{\partial}{\partial (x_2)_m}\overline{p u_2'} + \frac{\partial}{\partial (x_2)_m}\overline{p u_2'(-\vec{r})}\right] + \frac{\partial}{\partial r_2}\overline{p u_2'(-\vec{r})} - \frac{\partial}{\partial r_2}\overline{p u_2'}\right\}$$

$$+ \frac{1}{2}\nu \frac{\partial^2 \overline{u_2 u_2'}}{\partial (x_2)_m \partial (x_2)_m} + 2\nu \frac{\partial^2 \overline{u_2 u_2'}}{\partial r_k \partial r_k} \tag{71}$$

$$0 = -\overline{u_2u_2'}\left(\frac{\tau}{\rho\nu} + \frac{1}{\nu}\overline{u_1u_2}\right) - \left(\frac{\tau}{\rho\nu}r_2 + \frac{1}{\nu}\int_{(x_2)_m - r_2/2}^{(x_2)_m + r_2/2} \overline{u_1''u_2''}\,dx_2''\right)\frac{\partial}{\partial r_1}\overline{u_1u_2'}$$

$$-\frac{1}{\rho}\left[\frac{1}{2}\frac{\partial}{\partial(x_2)_m}\overline{u_1p'} + \frac{\partial}{\partial r_2}\overline{u_1p'} - \frac{\partial}{\partial r_1}\overline{pu_2'}\right]$$

$$+\frac{1}{2}\nu\frac{\partial^2\overline{u_1u_2'}}{\partial(x_2)_m^2} + 2\nu\frac{\partial^2\overline{u_1u_2'}}{\partial r_k\partial r_k} \tag{72}$$

$$\frac{1}{\rho}\left[\frac{1}{4}\frac{\partial^2\overline{pu_2'}}{\partial(x_2)_m^2} - \frac{\partial^2\overline{pu_2'}}{\partial(x_2)_m\partial r_2} + \frac{\partial^2\overline{pu_2'}}{\partial r_k\partial r_k}\right] = 2\left(\frac{\tau}{\rho\nu} + \frac{1}{\nu}\overline{u_1u_2}\right)\frac{\partial}{\partial r_1}\overline{u_2u_2'} \tag{73}$$

and

$$\frac{1}{\rho}\left[\frac{1}{4}\frac{\partial^2\overline{u_1p'}}{\partial(x_2)_m^2} + \frac{\partial^2\overline{u_1p'}}{\partial(x_2)_m\partial r_2} + \frac{\partial^2\overline{u_1p'}}{\partial r_k\partial r_k}\right] = -2\left(\frac{\tau}{\rho\nu} + \frac{1}{\nu}\overline{u_1'u_2'}\right)\frac{\partial}{\partial r_1}\overline{u_1u_2'} \tag{74}$$

Equations (71) to (74) form a determinate set. A possible solution for these equations is that all the correlations are zero, as was the case for the equations for thermal turbulence. In that case no turbulence exists and the shear stress is produced by molecular action. We are here mainly interested in turbulent solutions that might be possible because of the nonlinear terms in the equations. Of these nonlinear terms the first term of Eq. (71) and the second term of Eq. (72) have been interpreted[10] as Fourier transforms of transfer terms that transfer energy between eddies of various sizes. The remainder of the nonlinear terms, that is, the first term on the right side of Eq. (72) and the last terms in Eqs. (73) and (74), might be interpreted as production terms. Some of the nonlinear terms here are more complicated than those for thermal turbulence, where they were all simple products of correlations. The equations can be written in dimensionless form by introducing the following dimensionless variables:

$$\overline{u_i u_j'}^{*} = \frac{\lambda^2 \overline{u_i u_j'}}{\nu^2}, \; N_s = \frac{\tau\lambda^2}{\rho\nu^2}, \; r_i^* = \frac{r_i}{\lambda}, \; (x_2)_m^* = \frac{(x_2)_m}{\lambda}, \; \overline{pu_j'}^{*} = \frac{\lambda^3 \overline{pu_j'}}{\rho\nu^3}$$

where the microscale λ is again used as a length scale because the solutions to be obtained are accurate only for small values of the space variables. The parameter N_s is a determining parameter for the shear-flow turbulence and has some similarity to the square of a Reynolds number. Equations (71) to (74) become, in dimensionless form,

$$0 = -\left(N_s r_2^* + \int_{(x_2)_m^* - r_2^*/2}^{(x_2)_m^* + r_2^*/2} \overline{u_1'' u_2''}\, dx_2''\right) \frac{\partial}{\partial r_1^*} \overline{u_2 u_2'}^{*} - \frac{1}{2} \frac{\partial}{\partial (x_2)_m^*} \overline{pu_2'}^{*}$$
$$- \frac{1}{2} \frac{\partial}{\partial (x_2)_m^*} \overline{pu'}^{*}(-\vec{r}^*) - \frac{\partial}{\partial r_2^*} \overline{pu_2'}^{*}(-\vec{r}^*)$$
$$+ \frac{\partial}{\partial r_2^*} \overline{pu_2'}^{*} + \frac{1}{2} \frac{\partial^2 \overline{u_2 u_2'}^{*}}{\partial (x_2)_m^{*2}} + 2 \frac{\partial^2 \overline{u_2 u_2'}^{*}}{\partial r_k^* \partial r_k^*} \tag{75}$$

$$0 = -\overline{u_2 u_2'}^{*}(N_s + \overline{u_1 u_2'}^{*}) - \left(N_s r_2^* + \int_{(x_2)_m^* - r_2^*/2}^{(x_2)_m^* + r_2^*/2} \overline{u_1'' u_2''}\, dx_2''\right) \frac{\partial}{\partial r^*} \overline{u_1 u_2'}^{*}$$
$$- \frac{1}{2} \frac{\partial}{\partial (x_2)_m^*} \overline{u_1 p'}^{*} - \frac{\partial}{\partial r_2^*} \overline{u_1 p'}^{*}$$
$$+ \frac{\partial}{\partial r_1^*} \overline{pu'}^{*} + \frac{1}{2} \frac{\partial^2 \overline{u_1 u_2'}^{*}}{\partial (x_2)_m^{*2}} + 2 \frac{\partial^2 \overline{u_1 u_2'}^{*}}{\partial r_k^* \partial r_k^*} \tag{76}$$

$$\frac{1}{4}\frac{\partial^2\overline{pu_2'}^*}{\partial(x_2)_m^{*2}} - \frac{\partial^2\overline{pu_2'}^*}{\partial(x_2)_m^*\partial r_2^*} + \frac{\partial^2\overline{pu_2'}^*}{\partial r_k^*\partial r_k^*} = 2\left(N_s + \overline{u_1u_2}^*\right)\frac{\partial}{\partial r_1^*}\overline{u_1u_2'}^* \qquad (77)$$

$$\frac{1}{4}\frac{\partial^2\overline{u_1p'}^*}{\partial(x_2)_m^{*2}} + \frac{\partial^2\overline{u_1p'}^*}{\partial(x_2)^*\partial r_2^*} + \frac{\partial^2\overline{u_1p'}^*}{\partial r_k^*\partial r_k^*} = -2\left(N_s + \overline{u_1'u_2'}^*\right)\frac{\partial}{\partial r_1^*}\overline{u_1u_2'}^* \qquad (78)$$

Series Expansions

The expansion in power series of the correlation equations for shear flow is similar to that for thermal turbulence. Most of the discussion at the beginning of the section "Series Expansions" for sustained thermal turbulence therefore applies also to shear-flow turbulence. The main difference between the two cases is that we had the three independent variables ξ, r_3, and $(x_3)_m$ for thermal turbulence, whereas for shear flow we have the four variables r_1, r_2, r_3, and $(x_2)_m$. This is because there is no axial symmetry for shear-flow turbulence as there was for thermal turbulence. The final expanded correlations will then have four subscripts instead of three. As before, all of the microscales will at the beginning, for the sake of definiteness, be assumed equal to λ. In the present case, however, it will be found necessary to later modify that assumption to obtain reasonable results.

We will first obtain expressions for the nonlinear terms in Eqs. (75) to (78) in terms of the expanded correlations. The quantities $\overline{u_1u_2}^*$ and $\overline{u_1'u_2'}^*$ can be expanded in a Taylor series to give, for small r_2^*,

$$\overline{u_1u_2}^* = (\overline{u_1u_2'})_{000} - \frac{r_2^*}{2}\frac{\partial}{\partial(x_2)_m^*}(\overline{u_1u_2'})_{000} \qquad (79)$$

and

$$\overline{u_1'u_2'}^{*} = (\overline{u_1u_2'})_{000} + \frac{r_2^*}{2}\frac{\partial}{\partial(x_2)_m}(\overline{u_1u_2'})_{000} \tag{80}$$

where $(\overline{u_1u_2'})_{000}$ is the value of $\overline{u_1u_2'}$ at the point $(x_2)_m^*$ with $r_3^* = r_2^* = r_1^* = 0$. The quantity $(\overline{u_1u_2'})_{000}$, in turn, can be expanded as

$$(\overline{u_1u_2'})_{000} = (\overline{u_1u_2'})_{0000} + (\overline{u_1u_2'})_{0002}(x_2)_m^{*2} \tag{81}$$

where $(\overline{u_1u_2'})_{0000}$ is the value of $\overline{u_1u_2'}$ for $(x_2)^* = r_3^* = r_2^* = r_1^* = 0$. The odd powers of $(x_2)_m^*$ are omitted to make $(\overline{u_1u_2'})_{000}$ symmetric about $(x_2)_m^* = 0$. To relate $(\overline{u_1u_2'})_{0002}$ to $(\overline{u_1u_2'})_{0000}$, set the microscale for $(\overline{u_1u_2'})_{0000}$ and $(x_2)_m$ equal to λ by letting $(\overline{u_1u_2'})_{000} = 0$ for $(x_2)_m^* = 1/2$. This gives $(\overline{u_1u_2'})_{0002} = -4(\overline{u_1u_2'})_{0000}$. Then Eqs. (79) to (81) yield

$$\overline{u_1u_2}^{*} = (\overline{u_1u_2'})_{0000}\left[1 + 4(x_2)_m^* r_2^*\right] \tag{82}$$

$$\overline{u_1'u_2'}^{*} = (\overline{u_1u_2'})_{0000}\left[1 - 4(x_2)_m^* r_2^*\right] \tag{83}$$

For a general point x_2'',

$$\overline{u_1''u_2''} = (\overline{u_1u_2'})_{0000}\left[1 - 4x_2''r_2^*\right] \tag{84}$$

Then

$$\int_{(x_2)_m^*-r_2^*/2}^{(x_2)_m^*+r_2^*/2} \overline{u_1''u_2''}\,dx_2'' = (\overline{u_1u_2'})_{0000}r_2^* \tag{85}$$

Let

$$\left.\begin{aligned}\overline{u_2u_2'}^{*} &= (\overline{u_2u_2'})_0 + (\overline{u_2u_2'})_2\, r_3^{*2}\\ \overline{u_1u_2'}^{*} &= (\overline{u_1u_2'})_0 + (\overline{u_1u_2'})_2\, r_3^{*2}\\ \overline{pu_2'}^{*} &= (\overline{pu_2'})_0 + (\overline{pu_2'})_2\, r_3^{*2}\\ \overline{u_1p'}^{*} &= (\overline{u_1p'})_0 + (\overline{u_1p'})_2\, r_3^{*2}\end{aligned}\right\} \tag{86}$$

where the quantities in parenthesis are independent of r_3^*, and $(\overline{u_2u_2'})_0$, for instance, is the value of $\overline{u_2u_2'}$ for $r_3^* = 0$. The odd powers of r_3^* are omitted because of symmetry. Substituting Eqs. (82), (83), (85), and (86) into Eqs. (75) to (78) and equating to zero the coefficient of r_3^{*0} in each equation give

$$\begin{aligned}0 = &- \left[N_s + (\overline{u_1u_2'})_{0000}\right] r_2^* \frac{\partial}{\partial r_1^*} (\overline{u_2u_2'})_0 - \frac{1}{2} \frac{\partial}{\partial (x_2)_m^*} (\overline{pu_2'})_0\\ &- \frac{1}{2} \frac{\partial}{\partial (x_2)_m^*} (\overline{pu_2'})_0(-r_2^*, -r_1^*) - \frac{\partial}{\partial r_2^*} (\overline{pu_2'})_0(-r_2^*, -r_1^*)\\ &+ \frac{\partial}{\partial r_2^*} (\overline{pu_2'})_0 + \frac{1}{2} \frac{\partial^2}{\partial (x_2)_m^{*2}} (\overline{u_2u_2'})_0 + 2 \frac{\partial^2}{\partial r_1^{*2}} (\overline{u_2u_2'})_0\\ &+ 2 \frac{\partial^2}{\partial r_2^{*2}} (\overline{u_2u_2'})_0 + 4(\overline{u_2u_2'})_2\end{aligned} \tag{87}$$

$$\begin{aligned}0 = &-(\overline{u_2u_2'})_0\left[N_s + (1 + 4(x_2)_m^* r_2^*)(\overline{u_1u_2'})_{0000}\right] - \left[N_s + (\overline{u_1u_2'})_{0000}\right] r_2^* \frac{\partial}{\partial r_1^*} (\overline{u_1u_2'})_0\\ &- \frac{1}{2} \frac{\partial}{\partial (x_2)_m^*} (\overline{u_1p'})_0 - \frac{\partial}{\partial r_2^*} (\overline{u_1p'})_0\\ &+ \frac{\partial}{\partial r_1^*} (\overline{pu_2'})_0 + \frac{1}{2} \frac{\partial^2}{\partial (x_2)_m^{*2}} (\overline{u_1u_2'})_0\\ &+ 2 \frac{\partial^2}{\partial r_1^{*2}} (\overline{u_1u_2'})_0 + 2 \frac{\partial^2}{\partial r_2^{*2}} (\overline{u_1u_2'})_0 + 4(\overline{u_1u_2'})_2\end{aligned} \tag{88}$$

$$\frac{1}{4}\frac{\partial^2}{\partial(x_2)_m^{*2}}(\overline{pu_2'})_0 - \frac{\partial^2}{\partial(x_2)_m^*\partial r_2^*}(\overline{pu_2'})_0 + \frac{\partial^2}{\partial r_1^{*2}}(\overline{pu_2'})_0 + \frac{\partial^2}{\partial r_2^{*2}}(\overline{pu_2'})_0 + 2(\overline{pu_2'})_2$$

$$= 2\left[N_s + (1 + 4(x_2)_m^* r_2^*)(\overline{u_1u_2'})_{0000}\right]\frac{\partial}{\partial r_1^*}(\overline{u_2u_2'})_0 \qquad (89)$$

$$\frac{1}{4}\frac{\partial^2}{\partial(x_2)_m^{*2}}(\overline{u_1p'})_0 + \frac{\partial^2}{\partial(x_2)_m^*\partial r_2^*}(\overline{u_1p'})_0 + \frac{\partial^2}{\partial r_1^{*2}}(\overline{u_1p'})_0 + \frac{\partial^2}{\partial r_2^{*2}}(\overline{u_1p'})_0$$

$$+ 2(\overline{u_1p'})_2 = -2\left[N_s + (1 - 4(x_2)_m^* r_2^*)(\overline{u_1u_2'})_{0000}\right]\frac{\partial}{\partial r_1^*}(\overline{u_1u_2'})_0 \qquad (90)$$

In order to eliminate r_2^* from the correlation equations, let

$$\left.\begin{aligned}
(\overline{u_2u_2'})_0 &= (\overline{u_2u_2'})_{00} + (\overline{u_2u_2'})_{01}\, r_2^* + (\overline{u_2u_2'})_{02}\, r_2^{*2} + (\overline{u_2u_2'})_{03}\, r_2^{*3} \\
(\overline{u_1u_2'})_0 &= (\overline{u_1u_2'})_{00} + (\overline{u_1u_2'})_{01}\, r_2^* + (\overline{u_1u_2'})_{02}\, r_2^{*2} + (\overline{u_1u_2'})_{03}\, r_2^{*3} \\
(\overline{pu_2'})_0 &= (\overline{pu_2'})_{00} + (\overline{pu_2'})_{01}\, r_2^* + (\overline{pu_2'})_{02}\, r_2^{*2} + (\overline{pu_2'})_{03}\, r_2^{*3} \\
(\overline{u_1p'})_0 &= (\overline{u_1p'})_{00} + (\overline{u_1p'})_{01}\, r_2^* + (\overline{u_1p'})_{02}\, r_2^{*2} + (\overline{u_1p'})_{03}\, r_2^{*3}
\end{aligned}\right\} \qquad (91)$$

where, for instance, $(\overline{u_2u_2'})_{00}$ is the value of $\overline{u_2u_2'}$ for $r_3^* = r_2^* = 0$. Substituting Eqs. (91) into (87) to (90) and equating to zero the coefficient of r_2^{*0} give, with $(\overline{u_2u_2'})_2 = -(\overline{u_2u_2'})_0$, etc., a set of partial differential equations with independent variables r_1^* and $(x_2)_m^*$. Setting the microscales for $(\overline{u_2u_2'})_0$, $(\overline{u_1u_2'})_0$, $(\overline{pu_2'})_0$, and $(\overline{u_1p'})_0$ in Eqs. (91) equal to λ by letting $(\overline{u_2u_2'})_0$, $(\overline{u_1u_2'})_0$, etc. $= 0$ for $r_2^* = \pm 1$, we find that $(\overline{u_2u_2'})_{02} = -(\overline{u_2u_2'})_{00}$, $(\overline{u_2u_2'})_{03} = -(\overline{u_2u_2'})_{01}$, $(\overline{u_1u_2'})_{02} = -(\overline{u_1u_2'})_{00}$, $(\overline{u_1u_2'})_{03} = -(\overline{u_1u_2'})_{01}$, etc.

To eliminate r_1^* from the correlation equations, let

$$
\begin{aligned}
(\overline{u_2u_2'})_{00} &= (\overline{u_2u_2'})_{000} + (\overline{u_2u_2'})_{001}\, r_1^* + (\overline{u_2u_2'})_{002}\, r_1^{*2} + (\overline{u_2u_2'})_{003}\, r_1^{*3} \\
(\overline{u_2u_2'})_{01} &= (\overline{u_2u_2'})_{010} + (\overline{u_2u_2'})_{011}\, r_1^* + (\overline{u_2u_2'})_{012}\, r_1^{*2} + (\overline{u_2u_2'})_{013}\, r_1^{*3} \\
(\overline{u_1u_2'})_{00} &= (\overline{u_1u_2'})_{000} + (\overline{u_1u_2'})_{001}\, r_1^* + (\overline{u_1u_2'})_{002}\, r_1^{*2} + (\overline{u_1u_2'})_{003}\, r_1^{*3} \\
\cdot \quad & \quad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \\
\cdot \quad & \quad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \\
\cdot \quad & \quad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \\
(\overline{u_1p'})_{01} &= (\overline{u_1p'})_{010} + (\overline{u_1p'})_{011}\, r_1^* + (\overline{u_1p'})_{012}\, r_1^{*2} + (\overline{u_1p'})_{013}\, r_1^{*3}
\end{aligned}
\tag{92}
$$

Substituting Eqs. (92) into the correlation equations with independent variables r_1^* and $(x_2)_m^*$ (not shown) and equating to zero the coefficients of r_1^{*0} and r_1^{*1} give (with $(\overline{u_2u_2'})_{02} = -(\overline{u_2u_2'})_{00}$, etc.) a set of ordinary differential equations in $(x_2)_m^*$.

Finally, to eliminate $(x_2)_m^*$, let

$$\left.\begin{aligned}
(\overline{u_2u_2'})_{000} &= (\overline{u_2u_2'})_{0000} + (\overline{u_2u_2'})_{0001}(x_2)^*_m + (\overline{u_2u_2'})_{0002}(x_2)^{*2}_m + (\overline{u_2u_2'})_{0003}(x_2)^{*3}_m \\
(\overline{u_2u_2'})_{001} &= (\overline{u_2u_2'})_{0010} + (\overline{u_2u_2'})_{0011}(x_2)^*_m + (\overline{u_2u_2'})_{0012}(x_2)^{*2}_m + (\overline{u_2u_2'})_{0013}(x_2)^{*3}_m \\
\cdot \quad & \quad \cdot \qquad\qquad \cdot \qquad\qquad \cdot \qquad\qquad \cdot \\
\cdot \quad & \quad \cdot \qquad\qquad \cdot \qquad\qquad \cdot \qquad\qquad \cdot \\
\cdot \quad & \quad \cdot \qquad\qquad \cdot \qquad\qquad \cdot \qquad\qquad \cdot \\
(\overline{u_1p'})_{011} &= (\overline{u_1p'})_{0110} + (\overline{u_1p'})_{0111}(x_2)^*_m + (\overline{u_1p'})_{0112}(x_2)^{*2}_m + (\overline{u_1p'})_{0113}(x_2)^{*3}_m
\end{aligned}\right\} \quad (93)$$

where, for instance, $(\overline{u_2u_2'})_{0000}$ is the value of $\overline{u_2u_2'}$ at $r_3^* = r_2^* = r_1^* = (x_2)^*_m = 0$. Substitution of Eqs. (93) into the ordinary differential equations in $(x_2)^*_m$ gives a set of 32 algebraic equations. Of these only the ones required for obtaining $(\overline{u_1u_2'})_{0000}$ and $(\overline{u_2u_2'})_{0000}$ will be given here. Setting the coefficients of $(x_2)^*_m$ equal to zero gives, with $(\overline{u_2u_2'})_{002} = -(\overline{u_2u_2'})_{000}$, $(\overline{u_2u_2'})_{003} = -(\overline{u_2u_2'})_{001}$, etc.,

$$0 = -(\overline{pu_2'})_{0001} + 2(\overline{pu_2'})_{0100} + (\overline{u_2u_2'})_{0002} - 12(\overline{u_2u_2'})_{0000} \quad (94)$$

$$0 = -(\overline{u_2u_2'})_{0000}\left[N_s + (\overline{u_1u_2'})_{0000}\right] - \frac{1}{2}(\overline{u_1p'})_{0001} - (\overline{u_1p'})_{0100} + (\overline{pu_2'})_{0010} + (\overline{u_1u_2'})_{0002} - 12(\overline{u_1u_2'})_{0000} \quad (95)$$

$$\frac{1}{2}(\overline{pu_2'})_{0102} + 2(\overline{pu_2'})_{0001} - 10(\overline{pu_2'})_{0100} = 2\left[N_s + (\overline{u_1u_2'})_{0000}\right](\overline{u_2u_2'})_{0110} \quad (96)$$

$$\frac{1}{2}(\overline{u_1p'})_{0102} - 2(\overline{u_1p'})_{0001} - 10(\overline{u_1p'})_{0100}$$

$$= -2\left[N_s + (\overline{u_1u_2'})_{0000}\right](\overline{u_1u_2'})_{0110} \qquad (97)$$

$$\frac{1}{2}(\overline{pu_2'})_{0012} - (\overline{pu_2'})_{0111} - 10(\overline{pu_2'})_{0010}$$

$$= -4\left[N_s + (\overline{u_1u_2'})_{0000}\right](\overline{u_2u_2'})_{0000} \qquad (98)$$

$$\frac{1}{2}(\overline{u_1p'})_{0012} + (\overline{u_1p'})_{0111} - 10(\overline{u_1p'})_{0010}$$

$$= 4\left[N_s + (\overline{u_1u_2'})_{0000}\right](\overline{u_1u_2'})_{0000} \qquad (99)$$

$$0 = 2\left[N_s + (\overline{u_1u_2'})_{0000}\right](\overline{u_2u_2'})_{0000} - (\overline{pu_2'})_{0111} - 4(\overline{pu_2'})_{0010}$$

$$+ (\overline{u_2u_2'})_{0112} - 28(\overline{u_2u_2'})_{0110} \qquad (100)$$

$$0 = -(\overline{u_2u_2'})_{0110}\left[N_s + (\overline{u_1u_2'})_{0000}\right] + 2\left[N_s + (\overline{u_1u_2'})_{0000}\right](\overline{u_1u_2'})_{0000} - \frac{1}{2}(\overline{u_1p'})_{0111}$$

$$+ 2(\overline{u_1p'})_{0010} - 2(\overline{pu_2'})_{0100} + (\overline{u_1u_2'})_{0112} - 28(\overline{u_1u_2'})_{0110} \qquad (101)$$

Equating the coefficients of $(x_2)_m^{*1}$ to zero gives

$$\frac{3}{2}(\overline{pu_2'})_{0003} - 2(\overline{pu_2'})_{0102} - 6(\overline{pu_2'})_{0001}$$

$$= 2\left[N_s + (\overline{u_1u_2'})_{0000}\right](\overline{u_2u_2'})_{0011} \qquad (102)$$

$$\frac{3}{2}(\overline{u_1p'})_{0003} + 2(\overline{u_1p'})_{0102} - 6(\overline{u_1p'})_{0001}$$

$$= -2\left[N_s + (\overline{u_1u_2'})_{0000}\right](\overline{u_1u_2'})_{0011} \qquad (103)$$

$$0 = -\left[N_s + (\overline{u_1u_2'})_{0000}\right](\overline{u_2u_2'})_{0011} + 3(\overline{u_2u_2'})_{0103} - 20(\overline{u_2u_2'})_{0101} \qquad (104)$$

$$0 = -(\overline{u_2u_2'})_{0101}\left[N_s + (\overline{u_1u_2'})_{0000}\right] - 4(\overline{u_2u_2'})_{0000}(\overline{u_1u_2'})_{0000}$$

$$- \left[N_s + (\overline{u_1u_2'})_{0000}\right](\overline{u_1u_2'})_{0011} - (\overline{u_1p'})_{0102} + 2(\overline{u_1p'})_{0001}$$

$$+ (\overline{pu_2'})_{0111} + 3(\overline{u_1u_2'})_{0103} - 20(\overline{u_1u_2'})_{0101} \tag{105}$$

$$0 = 3(\overline{u_2u_2'})_{0013} - 20(\overline{u_2u_2'})_{0011} \tag{106}$$

$$0 = -(\overline{u_2u_2'})_{0011}\left[N_s + (\overline{u_1u_2'})_{0000}\right] - (\overline{u_1p'})_{0012} - (\overline{u_1p'})_{0111} - 2(\overline{pu_2'})_{0001}$$

$$+ 3(\overline{u_1u_2'})_{0013} - 20(\overline{u_1u_2'})_{0011} \tag{107}$$

$$\frac{3}{2}(\overline{pu_2'})_{0113} + 4(\overline{pu_2'})_{0012} - 14(\overline{pu_2'})_{0111} = -4\left[N_s + (\overline{u_1u_2'})_{0000}\right](\overline{u_2u_2'})_{0101}$$

$$- 16(\overline{u_1u_2'})_{0000}(\overline{u_2u_2'})_{0000} \tag{108}$$

$$\frac{3}{2}(\overline{u_1p'})_{0113} - 4(\overline{u_1p'})_{0012} - 14(\overline{u_1p'})_{0111}$$

$$= 4\left[N_s + (\overline{u_1u_2'})_{0000}\right](\overline{u_1u_2'})_{0101} - 16(\overline{u_1u_2'})^2_{0000} \tag{109}$$

Equations (106) and (104) give

$$(\overline{u_2u_2'})_{0011} = (\overline{u_2u_2'})_{0101} = 0 \tag{110}$$

since $(\overline{u_2u_2'})_{0013}$ and $(\overline{u_2u_2'})_{0103}$ are proportional respectively to $(\overline{u_2u_2'})_{0011}$ and $(\overline{u_2u_2'})_{0101}$.

Equation (94) emphasizes the importance of the pressure-velocity correlations for shear-flow turbulence. In the absence of those correlations $(\overline{u_2u_2'})_{0000}$ would be zero (since $(\overline{u_2u_2'})_{0002}$ is proportional to $-(\overline{u_2u_2'})_{0000}$) and there would be no turbulence. This is not surprising, since the turbulent energy is fed into the turbulent field through the $\overline{u_1u_1'}$ component and is distributed between the various components of the energy by the pressure-velocity correlation terms.

As in the case of thermal turbulence, it is the presence of the non-linear or quadratic terms in some of Eqs. (94) to (109) that makes possible a nonzero solution of those equations. In the absence of those terms the equations would be linear and homogeneous and, in general, would have only a no-turbulence solution.

To relate $(\overline{u_2u_2'})_{0002}$ to $(\overline{u_2u_2'})_{0000}$, etc. let the microscales associated with $(x_2)_m^*$ in Eqs. (93) be equal to λ by letting, for instance, $(\overline{u_2u_2'})_{000} = 0$ for $(x_2)_m^* = \pm 1/2$. This gives $(\overline{u_2u_2'})_{0002} = -4(\overline{u_2u_2'})_{0000}$, $(\overline{u_2u_2'})_{0003} = -4(\overline{u_2u_2'})_{0001}$, $(\overline{u_2u_2'})_{0012} = -4(\overline{u_2u_2'})_{0010}$, etc. Using these relations we get, when substituting Eqs. (96), (98), (100), (102), (108), and (110) into (94),

$$-7168(\overline{u_2u_2'})_{0000} = (\overline{u_2u_2'})_{0000}\left[N_s + (\overline{u_1u_2'})_{0000}\right]\left[3N_s + (\overline{u_1u_2'})_{0000}\right] \qquad (111)$$

Note that $(\overline{u_2u_2'})_{0000}$ cancels out of this equation leaving a quadratic equation in $(\overline{u_1u_2'})_{0000}$. Thus we get the somewhat unexpected result that Eq. (94), which was originally obtained from Eq. (62) (the equation for $\overline{u_2u_2'}$), gives, when combined with the other equations, a solution for $(\overline{u_1u_2'})_{0000}$ rather than for $(\overline{u_2u_2'})_{0000}$. Similarly, Eq. (95) will give a solution for $(\overline{u_2u_2'})_{0000}$ rather than for $(\overline{u_1u_2'})_{0000}$. Solution of Eq. (111) gives

$$(\overline{u_1u_2'})_{0000} = -2N_s \pm \sqrt{N_s^2 - 7168} \qquad (112)$$

This equation can give a negative value of $\overline{u_1u_2'}$ for positive N_s and a positive $\overline{u_1u_2'}$ for negative N_s, as it should. However, the critical N_s (the value of N_s for $(\overline{u_1u_2'})_{0000} = 0$) is imaginary. Moreover, if we calculate the eddy diffusivity for momentum transfer from

$$\epsilon = - \frac{\overline{u_1 u_2}}{\partial U_1/\partial x_2} = - \frac{\overline{u_1 u_2}}{\frac{\tau}{\rho\nu} + \overline{u_1 u_2}}$$

which in dimensionless form at $(x_2)^*_m$ becomes

$$\epsilon^* = - \frac{(\overline{u_1 u_2'})_{0000}}{N_s + (\overline{u_1 u_2'})_{0000}} \tag{113}$$

we find that Eq. (112) will not give a positive solution for ϵ as required. This only indicates that all the microscales cannot be equal as assumed. If, for instance, we take the microscale for $(\overline{pu_2'})_{010}$ in Eqs. (93) as $\lambda/2$ instead of λ, we get $(\overline{pu_2'})_{0102} = -16(\overline{pu_2'})_{0100}$ and $(\overline{pu_2'})_{0103} = -16(\overline{pu_2'})_{0101}$. If the other microscales are taken equal to λ,

$$(\overline{u_1 u_2'})_{0000} = -2N_s \pm \sqrt{N_s^2 + 17{,}024} \tag{114}$$

where the positive sign is taken for positive N_s and the negative one for negative N_s. Equation (114) gives a critical value of N_s (at which $(\overline{u_1 u_2'}) = 0$) of ±75.3. Also, substitution of Eq. (114) into (113) gives a positive eddy diffusivity, which has the correct trends with increasing N_s. Physically realizable results for $(\overline{u_1 u_2'})_{0000}$ and ϵ were also obtained by letting the microscales for $(\overline{pu_2'})_0$, $(\overline{pu_2'})_{00}$, and $(\overline{pu_2'})_{000}$ in Eqs. (91), (92), and (93) be equal to 4λ, the other microscales being set equal to λ as before.

To obtain $(\overline{u_2 u_2'})_{0000}$ as a function of N_s, we solve Eqs. (103) to (110) and Eq. (114) simultaneously. The values for microscales used for obtaining Eq. (114) were used here. For $(\overline{u_2 u_2})_{0000} = 0$, $N_s = \pm 75.3$, as was obtained for $(\overline{u_1 u_2'})_{0000} = 0$ in Eq. (114). For $N_s = \pm 100$, $(\overline{u_2 u_2'})_{0000} = 93$, and for $N_s = \pm 150$, $(\overline{u_2 u_2'})_{0000} = 197$. Thus $(\overline{u_2 u_2'})_{0000}$, as well as $(\overline{u_1 u_2'})_{0000}$ and ϵ, has the correct trends and signs. Note that possible solutions are obtained for either positive or negative N_s, whereas for the case of thermal turbulence solutions were obtained only for positive N_t. This is to be expected, since the turbulence should not be affected by the sign of the shear stress. For thermal turbulence, on the other hand, the turbulence should exist only for heat transfer in the positive direction (negative temperature gradients).

As in the case of thermal turbulence, the results obtained in the preceding paragraphs probably do not correspond numerically to those for particular boundary conditions because the microscale ratios would be different. They are given to show that reasonable results can be obtained from the correlation equations. In the general case, Eq. (111) can be written as

$$A = \left[N_s + (\overline{u_1 u_2'})_{0000}\right]\left[aN_s + (\overline{u_1 u_2'})_{0000}\right] \tag{115}$$

where A and a are functions of the microscale ratios (which might in turn be weak functions of N_s). Solving Eq. (115) for $(\overline{u_1 u_2'})_{0000}$ we get

$$(\overline{u_1 u_2'})_{0000} = -\frac{(1 + a)}{2} N_s \pm \frac{(1 - a)}{2} \sqrt{N_s^2 + \frac{4A}{(1 - a)^2}} \tag{116}$$

For $(\overline{u_1 u_2'})_{0000} = 0$, $N_s = \pm\sqrt{A/a}$. Thus A and a should both be either positive or negative, at least in the vicinity of $(\overline{u_1 u_2'})_{0000} = 0$. Comparison of Eq. (116) to (57) shows that the expressions for both shear-flow and thermal turbulence are solutions of quadratic equations, although the forms differ somewhat.

CONCLUDING REMARKS

By expanding the two-point nonlinear correlation equations in power series, reasonable solutions were obtained for both thermal and shear-flow turbulence. By reasonable it is meant that the correlations and eddy diffusivities had the correct signs and trends. Moreover, critical values of the determining parameters, below which unphysical turbulent solutions occurred, were obtained. Below the critical values the no-turbulence solution of the correlation equations was therefore appropriate. Above the critical values the equations showed that the fluid could be either turbulent or nonturbulent. For shear-flow turbulence the same solutions are obtained for either positive or negative shear stress, whereas for thermal turbulence it was necessary for the heat transfer to be positive (negative temperature gradient). These results are, of course, to be expected if the equations yield reasonable solutions. The results obtained would not be expected to correspond numerically to those for particular boundary conditions inasmuch as the microscale ratios corresponding to those conditions were not determined. To determine those it would be necessary to carry higher order terms in the expansions and apply the particular boundary conditions of interest. The presence of pressure-velocity correlations in the equations was found to be indispensible if steady-state shear-flow turbulence is to exist. The pressure-velocity correlations were of less importance for thermal turbulence.

FOOTNOTES

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12. The action in sustained turbulence is somewhat similar to that of a clock, a violin bow, or an electronic oscillator in that in each of these a steady flow of energy is converted into oscillating energy by a nonlinear mechanism. Turbulence differs from the others, of course, in that its motion is random and has an infinite number of degrees of freedom.

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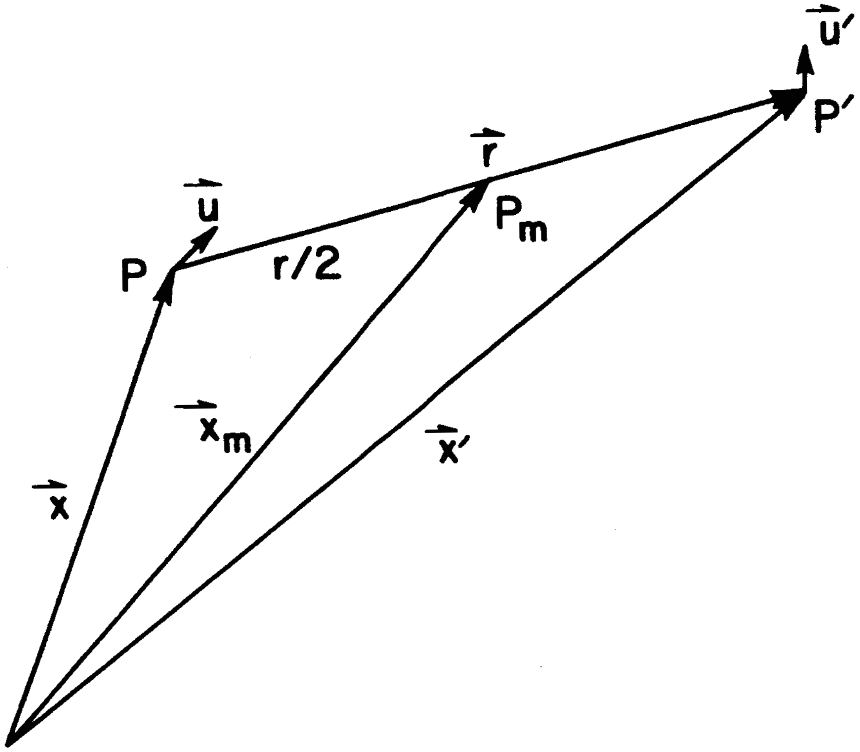

Figure 1. - Vector configuration for two-point correlation equations.