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Dipole Resonances in a Homogeneous Plasma in a Magnetic Field

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ABSTRACT

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Radar observations recently made from a satellite orbiting above the ionosphere provide evidence for resonances of a plasma in a magnetic field which may be excited and detected by a dipole. The plasma may be said to be resonant, for a particular mode and frequency, if the group velocity is zero. These resonances are studied theoretically on the assumption that the dipole is of infinitesimal extent and that the plasma is excited by a charge or current impulse. The former assumption restricts the validity of the results to the asymptotic response of the plasma. The latter assumption is not a restriction.

There are two electromagnetic resonances at the positive-frequency roots of $\omega^2 \pm \Omega\omega - \Pi^2 = 0$, where Ω is the electron gyro frequency and Π the electron plasma frequency, which occur at $k = 0$. There is also a resonance at $\omega = \Omega$, $k = \infty$, but this cannot be treated by the infinitesimal-dipole approximation and is deferred for separate study.

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"Electrostatic" resonances are treated in the quasi-electrostatic approximation. The resonant frequencies are Π, $(\Pi^2+\Omega^2)^{1/2}$, and $n\Omega$, $n = 2,3,\ldots$, and $k = 0$ at resonance. The infinitesimal-dipole model breaks down for the $n = 2,3,4$ cyclotron harmonics, but the infinitesimal line-dipole model and a line-charge model do not.

The analysis shows that the oscillations decay asymptotically as an inverse power of time. The analysis also indicates that the response would be significantly stronger than is observed if measurements were made with a stationary dipole, indicating that the observed duration of the resonances is to be ascribed to the finite velocity of the satellite with respect to the exospheric plasma.

1. INTRODUCTION

Experiments carried out with the Alouette Topside Sounder[1] have shown, in addition to the records which clearly deal with propagation and reflection phenomena, signals which have been termed "spikes" or "resonances." These signals are of such a character that the plasma may be said to "ring" when excited at one of a discrete set of frequencies. If a plasma is excited in a localized region, the power lost from that region depends upon the group velocity of the excited waves. Hence if, for a particular frequency, the group velocity is zero, the plasma is resonant in the sense that there is no energy lost by propagation. At the heights in the earth's atmosphere at which Alouette is operated, the collision frequency of electrons is sufficiently small that such resonances show up very clearly.

The sense in which a plasma may be said to be resonant at frequencies corresponding to modes of zero group velocity has been previously discussed,[2] and some of the frequencies (those corresponding to electrostatic modes in a cold plasma in a magnetic field) identified: $\omega = \Omega, \Pi, (\Omega^2+\Pi^2)^{1/2}$, where Ω is the electron gyro frequency and Π is the electron plasma frequency. If the same criterion is applied to electromagnetic modes of a cold plasma and to electrostatic modes of a hot plasma (taking into account the effect of electron temperature), one obtains the full set of resonances observed by Alouette, as we shall see in the course of this article.

One now faces the problem of calculating the magnitude and time variation of the response of the plasma when disturbed in such a way as to excite one of the resonances. This is the problem investigated in this report.

The technique used is based upon a "small-antenna approximation." We suppose the plasma to be excited by a dipole antenna, the dimensions of the dipole being small compared to the wavelengths of all waves which contribute to the observed signal. The conditions for the validity of this approximation will vary from case to case and are best investigated *a posteriori*. It is sufficient to consider the antenna to be excited by a δ-function impulse in time, since an arbitrary wave packet can be synthesized from such impulses. Following such an impulse, the electric field at the antenna has finite values at finite times after the impulse, and it is the magnitude and direction of this field which we calculate. From the results of such a calculation, it would be possible to predict the signal measured by an antenna of given dimensions, following the transmission of a pulse of given characteristics.

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2. BASIC EQUATIONS

The calculations to be presented are based upon the theory of waves in plasma, as presented by Stix.[3] The notation is identical to that of Stix, except that it is considered more convenient to measure current in emu rather than esu. The basic equation which we need is that which relates the electric field $\underset{\sim}{E}(x,t)$ set up in a plasma to the currents $\underset{\sim}{j}(x,t)$ introduced into the plasma by means external to the plasma, namely, the antenna. We introduce Fourier transforms according to the notation

$$\varphi(\underset{\sim}{x},t) = \iint d^3\underset{\sim}{k}d\omega e^{i(\underset{\sim}{k}.\underset{\sim}{x}-\omega t)}\varphi(\underset{\sim}{k},\omega) \tag{2.1}$$

$$\varphi(\underset{\sim}{k},\omega) = (2\pi)^{-4}\iint d^3\underset{\sim}{x}dte^{-i(\underset{\sim}{k}.\underset{\sim}{x}-\omega t)}\varphi(\underset{\sim}{x},t) \tag{2.2}$$

One may infer from the context whether a function or its Fourier transform is intended when the arguments are not given explicitly. By a slight extension of the calculation leading to Eq. (S.1.17) (Eq. (17), Chap. I of Stix[3]), one arrives at the following basic relation between $\underset{\sim}{E}(k,\omega)$ and $\underset{\sim}{j}(k,\omega)$:

$$\underset{\sim}{k}\wedge\underset{\sim}{k}\wedge\underset{\sim}{E} + \frac{\omega^2}{c^2}\underset{\sim}{K}.\underset{\sim}{E} = -i\frac{4\pi\omega}{c}\underset{\sim}{j} \tag{2.3}$$

Following Stix, we assume that the magnetic field is oriented along the z-axis, and that the wave vector lies in the x-z plane, so that it may be written as

$$\underset{\sim}{k} = \frac{\omega}{c}n(\sin\theta,0,\cos\theta) \tag{2.4}$$

Then, using (S.1.20), we find that (2.3) may be expressed as

$$\begin{pmatrix} S-n^2\cos^2\theta & -iD & n^2\cos\theta\sin\theta \\ iD & S-n^2 & . \\ n^2\cos\theta\sin\theta & . & P-n^2\sin^2\theta \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = -i\frac{4\pi c}{\omega}\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} \tag{2.5}$$

In the cold-plasma approximation, which is appropriate for the discussion of electromagnetic resonances in the next section,

$$\left.\begin{aligned} &P = 1 - \frac{\Pi^2}{\omega^2} , \quad R = 1 - \frac{\Pi^2}{\omega(\omega-\Omega)} , \quad L = 1 - \frac{\Pi^2}{\omega(\omega+\Omega)} , \\ &S = \tfrac{1}{2}(R+L) , \quad D = \tfrac{1}{2}(R-L) \end{aligned}\right\} \qquad (2.6)$$

The determinant of the matrix occurring in (2.5) is expressible as

$$\mathcal{D} = An^4 - Bn^2 + C \qquad (2.7)$$

where

$$\left.\begin{aligned} A &= S\sin^2\theta + P\cos^2\theta \\ B &= RL\sin^2\theta + PS(1+\cos^2\theta) \\ C &= PRL \end{aligned}\right\} \qquad (2.8)$$

We are concerned with the solution of the dispersion relation

$$\mathcal{D}(\omega,n,\theta) = 0 \qquad (2.9)$$

for which the group velocity is zero. By referring to the formula for the components of group velocity in polar coordinates

$$\left.\begin{aligned} u_k &= \frac{-c(\partial\mathcal{D}/\partial n)}{\omega(\partial\mathcal{D}/\partial\omega) - n(\partial\mathcal{D}/\partial n)} \\ u_\theta &= \frac{-(c/n)(\partial\mathcal{D}/\partial\theta)}{\omega(\partial\mathcal{D}/\partial\omega) - n(\partial\mathcal{D}/\partial n)} \end{aligned}\right\} \qquad (2.10)$$

or by referring to the literature on waves in plasmas, one finds that these resonances fall into two classes, one with $n = 0$ and one with $n = \infty$. The first class is readily seen from (2.7) and (2.8) to correspond to values of ω for which

$$P = 0 , \quad R = 0 , \quad \text{or} \quad L = 0 \qquad (2.11)$$

that is (considering at this time only positive frequencies)

$$\omega = \Pi , \quad \omega = \omega_R , \quad \text{or} \quad \omega = \omega_L \qquad (2.12)$$

where ω_R and ω_L are the positive roots of $R = 0$ and $L = 0$, respectively. These are the resonances which we shall first investigate in Section 3.

The resonances $n = \infty$ will be discussed in Sections 4 to 7. The effects of thermal velocities are important to these modes (which are quasi-electrostatic), whereas thermal effects are unimportant for the resonances $n = 0$. One may see this by noting that the $n = 0$ modes have infinite phase velocities so that there will be no interaction between these modes and thermal motion of particles, whereas the $n = \infty$ modes have zero phase velocities, leading one to expect that the interaction of waves with particles will be significant in this case.

Although the dispersion relation (2.7) shows that there is an electromagnetic resonance at $\omega = \Pi$, we know there is also an electrostatic resonance at the same frequency, as may be seen by considering the limit $n \to \infty$ and the angle $\theta = 0$. When one comes to investigate these resonances one finds that they cannot be distinguished, since, if $\theta = 0$, $P = 0$ is the solution of the dispersion relation for all values of n. Hence it is essential to include the effect of temperature in the calculation of this resonance, and the principal contribution comes from high wave numbers, corresponding to electrostatic modes. It is therefore unnecessary to consider the "electromagnetic" resonance $P = 0$ in (2.11), so that the only electromagnetic resonances we need to consider are $R = 0$ and $L = 0$.

3. ELECTROMAGNETIC RESONANCES

One may verify from (2.5) that the modes $R = 0$ and $L = 0$ are purely transverse in the sense that $E_z = 0$. In consequence, the current which drives these resonances must have $j_z = 0$. We therefore consider the external current, which is to be localized in space and in time, to be of the form

$$\underset{\sim}{j}(\underset{\sim}{x},t) = \delta^3(\underset{\sim}{x})\delta(t)(\cos\varphi,-\sin\varphi,0) \tag{3.1}$$

Appropriate transformation of the x-y coordinates would result in a current vector parallel to the x-direction and a wave vector $\underset{\sim}{k}$ with arbitrary direction, with spherical polar angles k,θ,φ.

We find from (2.5) that the electric field produced by the current (3.1) may be written as

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = E_R e^{i\varphi}\begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \begin{pmatrix} E_x \\ E_y \end{pmatrix} = E_L e^{-i\varphi}\begin{pmatrix} 1 \\ -i \end{pmatrix} \tag{3.2}$$

for the two resonances $R = 0$, $L = 0$, respectively. The coefficients are given by

$$\left.\begin{aligned} E_R &= -i\,\frac{4\pi c}{\omega}\,(2\pi)^{-4}\,\frac{1}{2R-n^2(1+\cos^2\theta)} \\ E_L &= -i\,\frac{4\pi c}{\omega}\,(2\pi)^{-4}\,\frac{1}{2L-n^2(1+\cos^2\theta)} \end{aligned}\right\} \tag{3.3}$$

where we retain terms in R, L, and n to the lowest significant order only.

If we write the solutions of $R = 0$ as $\omega = \omega_r$, and those of $L = 0$ as $\omega = \omega_l$, we see that

$$\omega_r = \omega_R\ ,\ -\omega_L\ , \qquad \omega_l = \omega_L\ ,\ -\omega_R \tag{3.4}$$

Since we shall be concerned with frequencies close to ω_r or ω_l, we write

$$\omega = \omega_r+\Delta\omega\ , \qquad \omega = \omega_l+\Delta\omega \tag{3.5}$$

for the two cases. We find from (2.6) that

$$\Delta R = \frac{(2\omega_r-\Omega)}{\Pi^2}\,\Delta\omega\ , \qquad \Delta L = \frac{(2\omega_l+\Omega)}{\Pi^2}\,\Delta\omega \tag{3.6}$$

Hence, formula (3.3) reduces to

$$\left.\begin{aligned} E_R &= -i(2\pi)^{-3}\,\frac{c\Pi^2}{\omega_r(2\omega_r-\Omega)}\,\frac{1}{\Delta\omega-\frac{1}{2}\,\dfrac{c^2\Pi^2k^2(1+\cos^2\theta)}{\omega_r^{\,2}(2\omega_r-\Omega)}} \\ E_L &= -i(2\pi)^{-3}\,\frac{c\Pi^2}{\omega_l(2\omega_l+\Omega)}\,\frac{1}{\Delta\omega-\frac{1}{2}\,\dfrac{c^2\Pi^2k^2(1+\cos^2\theta)}{\omega_l^{\,2}(2\omega_l+\Omega)}} \end{aligned}\right\} \tag{3.7}$$

It is now possible to evaluate the electric field at the antenna by means of the Fourier transformation (2.1). Since E_R, E_L, are independent of φ, this integration may be performed immediately, so that we obtain

$$E_R(0,t) = \iiint 2\pi k^2 \sin\theta dk d\theta d\omega e^{-i\omega t} E_R(k,\theta,\omega) \tag{3.8}$$

and a similar equation for E_L. The integration of ω may be performed immediately, the appropriate contour being determined by the requirement that $\underset{\sim}{E} = 0$ if $t < 0$. (See Eq. (A.1).) Hence,

$$E_R(0,t) = -\frac{1}{2\pi}\,\frac{c\Pi^2}{\omega_r(2\omega_r-\Omega)}\,e^{-i\omega_r t}\iint k^2\sin\theta dk d\theta \exp\left[-i\,\frac{1}{2}\,\frac{c^2\Pi^2k^2(1+\cos^2\theta)}{\omega_r^{\,2}(2\omega_r-\Omega)}\,t\right] \tag{3.9}$$

The integration over k may be carried out (Eq. (A.8)) and leads to

$$E_R(0,t) = 2^{-3/2}\pi^{-1/2}e^{i(\pi/4)}\,\frac{\left[\omega_r^{\,2}(2\omega_r-\Omega)\right]^{1/2}}{c^2\Pi t^{3/2}}\,e^{-i\omega_r t}\int_0^{\pi}\frac{d\theta\sin\theta}{(1+\cos^2\theta)^{3/2}} \tag{3.10}$$

The integration over θ may be carried out and the above equation then becomes

$$E_R(0,t) = \frac{1}{2}\,\pi^{-1/2}e^{i(\pi/4)}\,\frac{\left[\omega_r^{\,2}(2\omega_r-\Omega)\right]^{1/2}}{c^2\Pi}\,t^{-3/2}e^{-i\omega_r t}\ . \tag{3.11}$$

If, finally, we combine the four contributions corresponding to the four frequencies given by (3.4), and if we choose $\varphi = 0$ so that the dipole is parallel to the x-axis, we find that the components of electric field are given by

$$\left.\begin{aligned} E_x(0,t) &= \frac{\omega_R^{\,2}(2\omega_R-\Omega)^{1/2}}{\pi^{1/2}c^2\Pi t^{3/2}}\cos\left(\omega_R t-\frac{\pi}{4}\right)+\frac{\omega_L^{\,2}(2\omega_L+\Omega)^{1/2}}{\pi^{1/2}c^2\Pi t^{3/2}}\cos\left(\omega_L t-\frac{\pi}{4}\right) \\ E_y(0,t) &= \frac{\omega_R^{\,2}(2\omega_R-\Omega)^{1/2}}{\pi^{1/2}c^2\Pi t^{3/2}}\sin\left(\omega_R t-\frac{\pi}{4}\right)-\frac{\omega_L^{\,2}(2\omega_L+\Omega)^{1/2}}{\pi^{1/2}c^2\Pi t^{3/2}}\sin\left(\omega_L t-\frac{\pi}{4}\right) \end{aligned}\right\} \quad (3.12)$$

We see from the above formulas that the electric field is composed of two components, one rotating to the right with the frequency ω_R and the other rotating to the left with the frequency ω_L. It is curious, at first sight, that the magnitude of the electric field should vary <u>inversely</u> with Π As Π becomes smaller, there is less frequency spreading among neighboring wave numbers, and this effect proves to be more important than the term Π^2 in the numerator of Eqs. (3.7). We should note, moreover, that, for an antenna of finite dimensions, the formulas (3.12) are <u>asymptotic</u> and we may verify that the time at which these formulas become valid becomes longer as Π becomes smaller.

We may see this by noting that the contribution to the integral (3.9) is modified by the antenna geometry for sufficiently large wave numbers. Hence, formulas (3.12) are valid only if the integral (3.9) is substantially unaffected when the integral is truncated at $k = k_A$, where k_A^{-1} is a characteristic dimension of the antenna. This will be the case only if $t \gg t_c$, where

$$\frac{c^2\Pi^2 k_A^{\,2}}{\omega_R^{\,2}(2\omega_R-\Omega)}\,t_c = 1 \quad\text{or}\quad \frac{c^2\Pi^2 k_A^{\,2}}{\omega_L^{\,2}(2\omega_L+\Omega)}\,t_c = 1 \qquad (3.13)$$

The values of the electric field which we obtain at $t = t_c$, if the formulas (3.12) are extrapolated to this time, are

$$\left.\begin{aligned} |E|_c &= \frac{c\Pi^2 k_A{}^3}{\pi^{1/2}\omega_R(2\omega_R - \Omega)} \\ |E|_c &= \frac{c\Pi^2 k_A{}^3}{\pi^{1/2}\omega_L(2\omega_L + \Omega)} \end{aligned}\right\} \qquad (3.14)$$

These formulas are, of course, not a reliable estimate of the magnitude of the field at this time; they are included merely as a verification that the response becomes small as the plasma density becomes small.

4. EQUATIONS FOR QUASI-ELECTROSTATIC RESONANCES IN A HOT PLASMA

The remaining resonances which we shall consider either do not occur in the theory of a cold electron plasma (harmonics of the cyclotron frequency) or cannot be evaluated on the basis of the cold-plasma model (plasma resonance and the hybrid resonance). For this reason, we must now set up equations for a plasma of nonzero temperature. Since the dependence of frequency on wave number will now be determined by the mean thermal speed rather than the speed of light, the significant range of phase velocities will be comparable in magnitude to the thermal speed. For this reason, we may adopt the quasi-electrostatic approximation which is equivalent to setting $c = \infty$.

In the quasi-electrostatic approximation, the rf magnetic field is of no significance and the rf electric field may be derived from a scalar electric potential φ. On noting that

$$\underset{\sim}{E} = -i\underset{\sim}{k}\varphi \qquad (4.1)$$

and

$$\underset{\sim}{k}\cdot\underset{\sim}{j} = \frac{\omega}{c}\,\rho \qquad (4.2)$$

we find that (2.3) leads to

$$\varphi = \frac{4\pi\rho}{F} \tag{4.3}$$

where

$$F = \underset{\sim}{k}.\underset{\sim}{K}.\underset{\sim}{k} \tag{4.4}$$

We find from Stix (S.9.103) that

$$F = \underset{\sim}{k}^2 + \frac{1}{2}\,\Pi^2 v^2 e^{-\lambda} \sum_{n=-\infty}^{\infty} I_n(\lambda) A_n \tag{4.5}$$

where v is defined by

$$\frac{1}{2}\, mv^2 = kT \tag{4.6}$$

$$\lambda = \frac{v^2({k_x}^2 + {k_y}^2)}{2\Omega^2} \tag{4.7}$$

$$A_n = 1 + i\,\frac{\omega}{vk_z}\,F_o(\alpha_n) \tag{4.8}$$

and

$$\alpha_n = \frac{\omega + n\Omega}{vk_z} \tag{4.9}$$

$I_n(\lambda)$ is a Bessel function in the notation of Watson[4] and $F_o(\alpha_n)$ is the function defined by (S.8.34):

$$F_o = \pi^{1/2}\,\frac{k_z}{|k_z|}\,\exp(-{\alpha_n}^2) + 2iS(\alpha_n) \tag{4.10}$$

where

$$S(z) = e^{-z^2}\int_o^z e^{t^2}\,dt \tag{4.11}$$

On expanding the function F so that for each resonance the thermal speed v appears only to lowest significant order, one arrives at

$$\begin{aligned} F = {} & k^2 - \frac{\Pi^2}{\omega^2}\,k^2\cos^2\theta - \frac{\Pi^2}{\omega^2-\Omega^2}\,k^2\sin^2\theta - \frac{3}{2}\,\frac{\Pi^2}{\omega^4}\,v^2k^4\cos^4\theta + \frac{1}{2}\,\frac{\Pi^2}{\Omega^2\omega^2}\,v^2k^4\cos^2\theta\sin^2\theta \\ & + \frac{1}{2}\,\frac{\Pi^2 v^2 k^4 \sin^4\theta}{\Omega^2(\omega^2-\Omega^2)} - \frac{1}{2}\,\frac{\Pi^2\omega^2(\omega^2+3\Omega^2)}{\Omega^2(\omega^2-\Omega^2)^3}\,v^2k^4\cos^2\theta\sin^2\theta \\ & - 4\Pi^2 \sum_{n=2}^{\infty} \frac{1}{2^{2n}n!}\,\frac{n^2\Omega^2}{\omega^2-n^2\Omega^2}\,\frac{v^{2n-2}k^{2n}\sin^{2n}\theta}{\Omega^{2n}} \end{aligned} \tag{4.12}$$

where we have now adopted polar coordinates for the wave vector $\underset{\sim}{k}$.

5. RESONANCE AT THE PLASMA FREQUENCY

It is well known that the plasma has a resonance at $\omega = \Pi$, $\theta = 0$, $k = 0$. Hence, to investigate this resonance, we derive the following approximation for F, good for small values of $\omega^2-\Pi^2$, θ and k:

$$F = \frac{k^2}{\Pi^2}\left(\omega^2-\Pi^2+\frac{\Pi^2\Omega^2}{\Omega^2-\Pi^2}\,\theta^2-\frac{3}{2}\,v^2k^2\right) \tag{5.1}$$

This resonance is excited by a dipole in the same direction as the magnetic field, so we write

$$\rho(\underset{\sim}{x},t) = \delta(x)\delta(y)\,\frac{\partial\delta(z)}{\partial z}\,\delta(t) \tag{5.2}$$

which has the Fourier transform

$$\rho(\underset{\sim}{k},\omega) = (2\pi)^{-4}ik\cos\theta \tag{5.3}$$

The electric field components E_x and E_y vanish at the origin. We find, from (2.1), (4.3), and (5.3), that

$$E_z(0,t) = 4\pi(2\pi)^{-4}\iiint d\omega 2\pi k^2\sin\theta dk d\theta\,\frac{k^2\cos^2\theta}{F}\,e^{-i\omega t} \tag{5.4}$$

The integration over frequency may be carried out by using (A.2), and we obtain

$$E_z(0,t) = \Re\,\frac{i}{\pi}\,\Pi\iint dk k^2 d\theta\sin\theta\cos^2\theta\exp\left[i\left(\Pi-\frac{1}{2}\,\frac{\Pi\Omega^2}{\Omega^2-\Pi^2}\,\theta^2+\frac{3}{4}\,\frac{v^2k^2}{\Pi}\right)t\right] \tag{5.5}$$

where $\Re$ indicates the real part of a complex expression.

We may carry out the integrations over θ and k by using Eqs. (A.7) and (A.9). On noting that the θ integration contains two equal contributions, one for $\theta \approx \pi$ the other for $\theta \approx 0$, we arrive at

$$E_z(0,t) = \frac{4}{3^{3/2}\pi^{1/2}}\,\frac{\Pi^{3/2}(\Omega^2-\Pi^2)}{\Omega^2v^3}\,\frac{1}{t^{5/2}}\cos\left(\Pi t+3\frac{\pi}{4}\right) \tag{5.6}$$

6. THE HYBRID RESONANCE

We next investigate the hybrid resonance which occurs at the frequency $(\Pi^2+\Omega^2)^{1/2}$ for wave vectors with orientation $\theta \approx \pi/2$. For this reason we introduce the symbol

$$\psi = \frac{\pi}{2} - \theta \tag{6.1}$$

and select from (4.12) the relevant terms of lowest order in $\omega^2-\Pi^2-\Omega^2$, k, and ψ:

$$F = \frac{k^2}{\Pi^2}\left(\omega^2-\Pi^2-\Omega^2+\frac{\Omega^2\Pi^2}{\Pi^2+\Omega^2}\psi^2+\frac{1}{2}v^2k^2\right) \tag{6.2}$$

This resonance will be excited by a dipole normal to the magnetic field, so we adopt

$$\rho(\underset{\sim}{x},t) = \frac{\partial\delta(x)}{\partial x}\delta(y)\delta(z)\delta(t) \tag{6.3}$$

which has the Fourier transform

$$\rho(\underset{\sim}{k},\omega) = i(2\pi)^{-4}k\cos\psi\cos\varphi \tag{6.4}$$

when expressed in polar coordinates. Similarly, the x-component of the electric field will be given by

$$E_x(\underset{\sim}{k},\omega) = -i\varphi(\underset{\sim}{k},\omega)k\cos\psi\cos\varphi \tag{6.5}$$

Hence, we find that the nonzero component of electric field at the origin is given by

$$E_x(0,t) = (2\pi)^{-2}\iiint d\omega dk k^4 d\psi \cos^3\psi \frac{e^{-i\omega t}}{F} \tag{6.6}$$

when one carries out the integration over φ.

Integration over frequency is seen from (A.2) to give

$$E_x(0,t) = \Re\frac{i}{2\pi}\frac{\Pi^2}{(\Pi+\Omega^2)^{1/2}}\iint dk k^2 d\psi \exp\left\{i\left[(\Pi^2+\Omega^2)^{1/2} - \frac{1}{2}\frac{\Pi^2\Omega^2\psi^2}{(\Pi^2+\Omega^2)^{3/2}} - \frac{1}{4}\frac{v^2k^2}{(\Pi^2+\Omega^2)^{1/2}}\right]t\right\} \tag{6.7}$$

where we have made the approximation $\cos\psi \approx 1$. We may perform the integrations over k and ψ by using Eqs. (A.8) and (A.6) and so obtain

$$E_x(0,t) = 2^{1/2} \frac{\Pi(\Pi^2+\Omega^2)}{\Omega} \frac{1}{v^3} \frac{1}{t^2} \cos\left[(\Pi^2+\Omega^2)^{1/2} t - \frac{\pi}{2}\right] \tag{6.8}$$

7. HARMONICS OF THE CYCLOTRON FREQUENCY

The function F given by (4.12) has zeros near the harmonics of the cyclotron frequency $\omega = n\Omega$, $n \geq 2$, and we shall proceed to evaluate the asymptotic form of the excitation associated with these resonances. It is curious that the function does not give a resonance at $\omega = \Omega$. The reason for this may be traced to the fact that, for ω close to Ω, the dominant terms on the right-hand side of (4.12) are the third and the seventh, and that these have the same sign.

If we select from (4.12) the terms which are dominant for the nth harmonic, we arrive at

$$F = A(\theta)k^2 - \frac{Bk^{2n}\sin^{2n}\theta}{\omega^2-n^2\Omega^2} \tag{7.1}$$

where

$$A(\theta) = 1 - \frac{\Pi^2}{\Omega^2} \frac{n^2-\cos^2\theta}{n^2(n^2-1)} \tag{7.2}$$

and

$$B = \frac{n}{2^{2n-2}(n-1)!} \frac{\Pi^2 v^{2n-2}}{\Omega^{2n-2}} \tag{7.3}$$

If $3\Omega^2 > \Pi^2$, $A(\theta)$ has no zero; we shall assume that the parameters are restricted in this way.

If we again consider the dipole to be oriented as in (6.3), we obtain the following expression for the electric field at the origin:

$$E_x(0,t) = (2\pi)^{-2} \iiint d\omega dk k^4 d\theta \sin^3\theta \frac{e^{-i\omega t}}{F} \tag{7.4}$$

On integrating over frequency this expression becomes

$$E_x(0,t) = \Re \frac{i}{2\pi} \frac{B}{n\Omega} \iint dk d\theta \frac{k^{2n}\sin^{2n+3}\theta}{A^2} \exp\left[i\left(n\Omega + \frac{1}{2}\frac{B}{A}\frac{k^{2n-2}}{n\Omega}\sin^{2n}\theta\right)t\right] \tag{7.5}$$

We may perform the integration over k by using (A.11) and so obtain

$$E_x(0,t) = \Re \frac{i}{2\pi} \frac{B}{n\Omega} I_n e^{i\kappa_n} e^{in\Omega t} \int d\theta \frac{\sin^{2n+3}\theta}{A^2} \left(\frac{2An\Omega}{B\sin^{2n}\theta\, t}\right)^{\frac{2n+1}{2n-2}} \tag{7.6}$$

which becomes

$$E_x(0,t) = -\frac{1}{\pi}\left(\frac{2n\Omega}{B}\right)^{\frac{3}{2(n-1)}} I_n \int \frac{d\theta}{A^{\frac{2n-5}{2n-2}}\sin^{\frac{3}{n-1}}\theta} \frac{1}{t^{\frac{2n+1}{2n-2}}} \sin(n\Omega t + \kappa_n) \tag{7.7}$$

We see from the integral over θ that this integral will diverge if $n = 2,3,4$. Hence, for these harmonics, the simple dipole model for the antenna is inappropriate. However, the integral is convergent for $n \geq 5$. For these values of n, the formula (7.7) may be rewritten as

$$E_x(0,t) = -\frac{8}{\pi} I_n \left(\frac{2(n-1)!\Omega}{\Pi^2}\right)^{\frac{3}{2n-2}} \frac{\Omega^3}{V^3} \int_0^\pi \frac{d\theta}{A^{\frac{2n-5}{2n-2}}\sin^{\frac{3}{n-1}}\theta} \frac{1}{t^{\frac{2n+1}{2n-2}}} \sin(n\Omega t + \kappa_n) \tag{7.8}$$

by using (7.3).

A simple model for an antenna which yields a finite result for all harmonics of the cyclotron frequency is that of a line dipole, which may be regarded as a limiting case of an antenna formed by two long parallel wires. Accordingly, we consider the case that the plasma is excited by a charge distribution given by

$$\rho(\underset{\sim}{x},t) = \frac{\partial\delta(x)}{\partial x}\delta(y)\delta(t) \tag{7.9}$$

This assumes that the antenna is parallel to the magnetic field. The Fourier transform of this charge distribution is

$$\rho(\underset{\sim}{k},\omega) = (2\pi)^{-3} i k_x \delta(k_z) \tag{7.10}$$

On using (4.3), we obtain the following expression for the electric field at the origin:

$$E_x(0,t) = 4\pi(2\pi)^{-3} \iint d^3\underset{\sim}{k}\, d\omega \, \frac{k_x{}^2 \delta(k_z) e^{-i\omega t}}{F} \tag{7.11}$$

On using (7.1), introducing cylindrical polar coordinates $k,\ \psi$ for the $k_x k_y$ plane, and integrating over the angle ψ, this becomes

$$E_x(0,t) = \frac{1}{2\pi} \iint d\omega dk \, \frac{k(\omega^2 - n^2\Omega^2) e^{-i\omega t}}{A_o[\omega^2 - n^2\Omega^2 - (B/A_o)k^{2n-2}]} \tag{7.12}$$

where $A_o = A(0)$ so that

$$A_o = 1 - \frac{\Pi^2}{n^2\Omega^2} \tag{7.13}$$

We may integrate over frequency with the help of (A.2) and (A.4) to obtain

$$E_x(0,t) = \Re \, \frac{i}{n\Omega} \frac{B}{A_o{}^2} \int dk k^{2n-1} \exp\left[i\left(n\Omega + \frac{B}{2n\Omega A_o} k^{2n-2}\right)t\right] \tag{7.14}$$

We may use the notation of (A.12) to evaluate (7.14) and obtain

$$E_x(0,t) = -8\left[\frac{2(n-1)!\Omega}{\Pi^2}\right]^{\frac{1}{n-1}} J_n \frac{\Omega^2}{v^2} \frac{1}{A_o^{\frac{n-2}{n-1}}} \frac{1}{t^{\frac{n}{n-1}}} \sin(n\Omega t + \lambda_n) \tag{7.15}$$

Finally, we shall investigate another model for the excitation of cyclotron harmonic resonances which is closer to the Alouette experiment than the previous two models. Since the dimensions of the wave packets are probably small compared with the length of the large antenna, we may approximate any short length of the antenna by a line charge. Hence, we consider the excitation

$$\rho(\underset{\sim}{x},t) = \delta(x)\delta(y)\delta(t) \tag{7.16}$$

with Fourier transform

$$\rho(\underset{\sim}{k},\omega) = (2\pi)^{-3}\delta(k_z) \tag{7.17}$$

We allow for the fact that the line charge may not be parallel to the magnetic field by now writing

$$\underset{\sim}{B} = (B\sin\Theta,\ 0,\ B\cos\Theta) \tag{7.18}$$

so that Θ is the angle between the line charge and the magnetic field. Equations (2.1) and (4.3) show that the electric potential at the line charge is given by

$$\varphi(0,0,z,t) = \iint d^3\underset{\sim}{k}\,d\omega\,4\pi(2\pi)^{-3}\delta(k_z)\,\frac{e^{i(k_z z-\omega t)}}{F} \tag{7.19}$$

confirming that the potential is independent of z.

It is convenient to introduce the polar coordinates

$$k_x = k\cos\psi\ , \qquad k_y = k\sin\psi \tag{7.20}$$

so that θ, the angle between the wave vector and the magnetic field, is given by

$$\cos\theta = \sin\Theta\cos\psi \tag{7.21}$$

Integrations over ω and k proceed without difficulty with the help of Eqs. (A.2) and (A.7), leading to

$$\varphi(0,t) = -\frac{1}{\pi}\,\frac{1}{n-1}\,\frac{\cos n\Omega t}{t}\int_0^{2\pi}\frac{d\psi}{1-\frac{\Pi^2}{\Omega^2}\,\frac{n^2-\sin^2\Theta\cos^2\psi}{n^2(n^2-1)}} \tag{7.22}$$

Evaluation of this integral is straightforward, and finally leads to

$$\varphi(0,t) = -\frac{2}{n-1}\,A^{-1/2}\left(\frac{\pi}{2}\right)A^{-1/2}\left(\frac{\pi}{2}-\Theta\right)\frac{\cos n\Omega t}{t} \tag{7.23}$$

8. DISCUSSION

The resonances listed and discussed in this article represent those observed on the Alouette satellite experiment, except for the observed resonance at the electron cyclotron frequency, which has not shown up in our calculations. This indicates that the fundamental cyclotron resonance is different in nature from the harmonic cyclotron resonances, an inference in accord with the recent experiments of Crawford, Kino, and Weiss[5] on cyclotron resonances in a mercury-vapor discharge. It is possible that this resonance is to be associated with the right-hand polarized electromagnetic wave which has the dispersion relation $n^2 = R$, that is,

$$c^2k^2 = \omega^2 - \frac{\Pi^2\omega}{\omega-\Omega} \qquad (8.1)$$

for $\theta = 0$, since the group velocity vanishes for $k = \infty$, $\omega = \Omega$. Since this resonance occurs for high wave numbers, it would be inappropriate to use the infinitesimal-dipole approximation, and it would be necessary to consider a model of finite dimensions. Another possibility will be mentioned later.

Data concerning the Alouette satellite are given by Thomas and Sader.[6] For present initial estimates, we consider the particular values $\Omega/2\pi = 10^6\ \mathrm{sec}^{-1}$, $\Pi/2\pi = 2.10^6\ \mathrm{sec}^{-1}$. For frequencies in this range (all but the higher cyclotron harmonics), the longer antenna, of 50 m, is used. The antenna impedance is given as 400 ohms and the transmitter power as 100 watts, so that the rms current is 0.05 emu. Since the pulse length is 10^{-4} sec, the equivalent strength of the current dipole, as used in Section 3, is 2.5×10^{-2} emu. At 1 Mc frequency, the rms charge flowing in the antenna is 250 esu, leading to an equivalent electric dipole, as used in

Sections 5 to 7, of strength 125 esu. These estimates are appropriate if the transmitter is at precisely the same frequency as the resonance, but should be reduced if the frequencies are not coincident, according to the spectrum of the transmitter pulse. The receiver sensitivity is quoted as 20 db above kTb. If we take $T = 10^3$ deg and the bandwidth b as 20 kc, the sensitivity is 3×10^{-14} watt or 3×10^{-6} volt, corresponding to an electric field strength of 2×10^{-12} esu.

The $R = 0$ resonance occurs at about 1.5 Mc for the figures quoted. On using formulas (3.12), and noting that these are to be multiplied by the current dipole moment, we find that if the transmitter and resonance frequencies are matched precisely, $E \sim 3\times10^{-12}t^{-3/2}$, so that the pulse should be detectable for 300 msec. This could conceivably be reduced to about 15 msec if the resonance is excited by a sideband of the transmitter pulse. However, a similar calculation for the plasma resonance based on Eq. (5.6) and using the above estimate for the electrostatic dipole moment and $v = 1.6\times10^6$ cm sec^{-1} leads to the conclusion that the resonance should have been observable for 16 sec, very much longer than that observed (about 5 msec).

The above comparison strongly suggests that the pulse lengths of the resonances are not determined by the decay process treated in this article, so that one must consider other processes for decay. One is the effect of collisions between electrons and other particles but the mean electron collision frequency is very slow (less than 1 sec^{-1}) at the heights at which the satellite operates, so that one cannot explain the rapid decay in this way. Another obvious source of damping is the motion of the satellite, since the receiver is rapidly moving away from the region excited by the transmitter. Hence, one can explain the characteristic decay time of about 5 msec by assuming that the wave packet is of dimensions of about 50 m, since the satellite speed is about 10^6 cm sec^{-1}. Since this distance is comparable with the antenna dimensions, it seems very likely that the rapid decay of resonances is, in fact, due to the speed of the satellite. However, in order to make a more precise check of this suggestion, it would be necessary to extend the theory given in this article to evaluate the electric field strength as a function of both time and position, following excitation of the plasma by a dipole. It would, of course, be desirable to include also the finite geometry of the antenna.

We see from (5.5) (and similar equations for other cases) that the infinitesimal dipole approximation is valid only for times greater than some critical time t_c, since the main contribution to the integral should come from a range of wave numbers smaller than the wave number k_A characteristic of the antenna. From (5.5), these quantities are related approximately by

$$\frac{v^2 k_A^2}{\Pi} t_c = 1 \tag{8.2}$$

Since $k_A \approx 10^{-3}$ cm^{-1}, this leads to $t_c \approx 50$ msec. This demonstrates clearly the necessity for considering the finite antenna geometry in future calculations.

The calculations of Section 7 show clearly that harmonics of the cyclotron frequency should be excited, and that the response will decrease only slowly with harmonic number. The line-charge model, which is most closely related to the Alouette experiments, leads to estimates of the observable lifetime of the resonances of order 10^3 sec. Once again we are led to ascribe the observed duration of the pulse (~1 msec) to the finite velocity of the satellite. A pulse duration of 1 msec indicates a packet size of 10 m, which is about 20 r_G, where r_G is the electron gyro radius. This is a reasonable estimate for the packet dimensions, suggesting that it may be possible to estimate more accurately the observable decay of the cyclotron harmonics without considering the finite antenna geometry.

It is now worth noting that, if the calculations are extended in such a way as to allow for a finite streaming velocity of the plasma with respect to the antenna, the degeneracy which gave a finite value of F at the cyclotron frequency (see formula (4.12)) will be removed. We should then find a resonance at a frequency close to the electron gyro frequency.

Although these calculations were prompted by the Alouette experiment, it appears that they will be more relevant to experimental studies of similar resonances in laboratory plasmas, for which the effect of the plasma streaming velocity might be negligible. Such experiments have been made by Crawford, Kino, and Weiss.[5]

The theory of Section 7 indicates clearly that electrostatic modes at harmonics of the electron cyclotron frequency can be strongly excited by current sources in a plasma. It seems very likely that this fact underlies the high noise emission observed at these frequencies in laboratory plasmas.[7]

APPENDIX

The following integrals are used in the text:

$$\int \frac{d\omega e^{-i\omega t}}{\omega-\omega_0} = -2\pi i e^{-i\omega_0 t} \tag{A.1}$$

$$\int \frac{d\omega e^{-i\omega t}}{\omega^2-\omega_0^{\,2}} = -\frac{2\pi \sin\omega_0 t}{\omega_0} \tag{A.2}$$

$$\int \frac{d\omega\omega e^{-i\omega t}}{\omega^2-\omega_0^{\,2}} = -2\pi i \cos\omega_0 t \tag{A.3}$$

$$\int \frac{d\omega\omega^2 e^{-i\omega t}}{\omega^2-\omega_0^{\,2}} = -2\pi\omega_0 \sin\omega_0 t \tag{A.4}$$

The appropriate contour of integration, in the above integrals, is that which runs above the singularities, since the boundary conditions on the problem require that the integral has value zero if $t < 0$.

$$\int_0^\infty dx e^{ix^2} = \frac{1}{2}\pi^{1/2} e^{i\pi/4} \tag{A.5}$$

$$\int_{-\infty}^\infty dx e^{ix^2} = \pi^{1/2} e^{i\pi/4} \tag{A.6}$$

$$\int_0^\infty dx x e^{ix^2} = \frac{1}{2} i \tag{A.7}$$

$$\int_0^\infty dx x^2 e^{ix^2} = -\frac{1}{4}\pi^{1/2} e^{-i\pi/4} \tag{A.8}$$

$$\int dx x^2 e^{ix^2} = -\frac{1}{2}\pi^{1/2} e^{-i\pi/4} \tag{A.9}$$

$$\int_0^\infty dx x^3 e^{ix^2} = -\frac{1}{2} \tag{A.10}$$

The following notation is used:

$$\int_0^\infty dx x^{2n} e^{ix^{2n-2}} = I_n e^{i\kappa_n} \tag{A.11}$$

$$\int_0^\infty dx x^{2n-1} e^{ix^{2n-2}} = J_n e^{i\lambda_n} \tag{A.12}$$

The asymptotic values are

$$I_n,\ J_n \to \frac{1}{2n}\,, \qquad \kappa_n,\ \lambda_n \to \frac{\pi}{2} \quad \text{as} \quad n \to \infty \tag{A.13}$$

FOOTNOTES

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