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TECHNICAL SUMMARY REPORT

for

STUDY TO INVESTIGATE THE STABILITY IN ORBIT DETERMINATION

(16 April 1964 - 15 April 1965)

Contract No. NAS 5-3811

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Goddard Space Flight Center
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ABSTRACT

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The stability of the Kalman filter as it is applied to orbit determination, and the dependence of its behavior on the transition matrix, the initial covariance matrix, the types of observations, and the covariance of the noise in the measurements is explored using a basic simple harmonic motion model with zero and negative damping.

Author

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SECTION I

INTRODUCTION

In a series of by now well-known papers [1-3], R. E. Kalman describes an optimal filter applicable to noisy, time-varying, linear systems. This filter, which is essentially a minimum variance linear estimator, is particularly suitable for those orbit-determination problems in which estimates of state variables, which are based on noisy measurements and random initial conditions, are desired as rapidly as possible.

The Kalman filter was first applied to an on-board space-navigation problem for a circumlunar mission by NASA at Ames [4,5], and shortly thereafter by MIT [6]. Republic [7-10], NASA at Goddard [11-13] and undoubtedly others have applied it to orbit-determination programs that utilize ground-based tracking data as well as on-board observational data.

In general, in any search for optimal filters and controls in orbit-determination and guidance problems, the question remains, even after optimality has been achieved, whether the resultant filter or control system is stable. An answer in the negative, of course, would destroy the usefulness of the optimal formulation because small disturbances would grow without bound in spite of the observations or control inputs. The question of finite time stability is also of significance here, particularly for orbit determination problems where a long time interval compared to the dynamics of the models may not be available for observation.

This question of the stability of the Kalman filter in its application to orbit-determination problems and the dependence of such stability on the transition matrix, the covariance matrix of initial conditions, the types of observations, and the covariance of the noise in the measurements does not appear to have been investigated, in spite of the fact that the convergence of any orbit-determination scheme is dependent upon it, and as far as the author is aware, no general proof of the stability or convergence for the filter in this application exists.

Kalman [3] has explored the question of stability for a linear system and has obtained sufficient conditions for asymptotic stability, requiring complete controllability and complete observability. In the basic orbit determination problem, however, complete controllability is usually missing and furthermore the dynamic systems to which the linear filters are being applied are basically nonlinear.

To gain some insight and experience with the operation of the Kalman filter as it is applied in the orbit determination problem, it was decided, before undertaking the investigation of the general problem, to look at some very basic simple models which might in some sense approximate a vehicle in orbit and yet be of sufficiently low order to be handled easily. We chose to look at a simple harmonic motion model in one dimension with both zero and negative damping and attempted to explore its behavior both for asymptotic stability and finite time stability as functions of various parameters such as the choice of observations, the time between observations, and the covariance matrix of initial conditions.

The investigation of this simple linear model, which was planned to be a preliminary exercise to the study of more general and nonlinear models was more interesting than anticipated. It is expected that the more general and nonlinear models will be explored in further studies. This report will be concerned with the results of the linear model investigation.

In general, for notation, we try to use lower case Latin letters for vectors, upper case Latin and Greek letters as matrices and lower case Greek, and Latin letters with subscripts, as scalars, except that t and k are always scalars denoting time. Exceptions are either obvious or noted in the text. The transpose of vectors and matrices is denoted by the vector or matrix primed.

SECTION II

FORMULATION OF THE PROBLEM

A. Solution of the Filtering Problem

The general filtering problem is essentially one of finding information (about a random process) that is based on observed values of another related random process. The most general solution to this problem is the conditional probability distributions of the unknown random process. The conditional mean is that nonlinear function of the observations that minimizes the mean-square error in the estimate.

For a Gaussian process, the conditional probability distribution is completely determined by its mean and covariance so that, in this case, it suffices to calculate these quantities for a complete solution. Furthermore, the conditional mean turns out to be a linear function of the observations whereas the conditional covariances are independent of the observations.

If, in addition, the process is assumed to be Markovian (for example, a linear dynamical system excited by Gaussian white noise), it suffices to know the means and the covariances at one instant of time. The conditional means are computed by putting the observed values through a linear filter whereas the conditional covariances that are necessary for computing the conditional means may be found independently.

It is the solution of the filtering and prediction problem of the linear Gauss-Markovian process that is presented by Kalman. The basic model of this process (or sequence in the case of discrete intervals of time) is given by

$$x(k+1) = \Phi(k+1,k)\, x(k) + \Gamma(k+1,k)\, u(k)\ ,$$
$$z(k) = M(k)\, x(k) + v(k), \qquad k = 0,1,2,\ldots\ , \tag{II-1}$$

for discrete time, where x(k) is an n-dimensional state vector; z(k) is a p-dimensional observation of the state corrupted by noise; Φ, the transition matrix, Γ, and M, the measurement or observation matrix are given matrix functions of time; and u(k) and v(k) are Gaussian white-noise sequences with $E[u(k)] = E[v(k)] = 0$, $E[u(k)\, u'(m)] = \delta_{km} U(k)$, $E[v(k)\, v'(m)] = \delta_{km} R(k)$, $E[u(k)\, v'(m)] = \delta_{km} C(k)$ for all k and m, where δ_{km} is the Kronecker delta, and for simplicity of notation, we have suppressed the t in the arguments of our functions and have written k for t_k. We assume also, in (II-1) that

$$\Phi(k,j)\,\Phi(j,m) = \Phi(k,m),\ \Phi(k,k) = I,$$

$$\Gamma(k,j)\,\Gamma(j,m) = \Gamma(k,m), \text{ and } \Gamma(k,k) = 0\ .$$

1. Prediction and Filtering

A solution to the prediction and filtering problem is the minimum variance estimate of x(k+1). This minimum variance estimate is the conditional mean of x(k+1), given the observations z(k), z(k-1), z(k-2),...z(0), which is denoted by $\hat{x}(k+1|k)$ and given by [1,3,14]

$$\hat{x}(k+1|k) = \Phi(k+1,k)\,\hat{x}(k|k-1) + K_1(k)\,[z(k) - M(k)\,\hat{x}(k|k-1)]\ , \qquad \text{(II-2)}$$

where the weighting function is defined as

$$K_1(k) = [\Phi(k+1,k)\,P(k|k-1)\,M'(k) + \Gamma(k+1,k)\,C(k)]\,[M(k)\,P(k|k-1)\,M'(k) + R(k)]^{-1}\ . \qquad \text{(II-3)}$$

The error in the estimate is defined as

$$\tilde{x}(k+1|k) = x(k+1) - \hat{x}(k+1|k)\ , \qquad \text{(II-4)}$$

and the covariance matrix of the error is

$$P(k+1|k) = E[\tilde{x}(k+1|k)\,\tilde{x}'(k+1|k)] = \Phi(k+1,k)\,P(k|k-1)\,\Phi'(k+1,k) - K_1(k)\,[M(k)\,P(k|k-1)\,\Phi'(k+1,k) + C'(k)\,\Gamma'(k+1,k)] + \Gamma(k+1,k)\,U(k)\,\Gamma'(k+1,k)\ . \qquad \text{(II-5)}$$

With the initial estimates of the state and the covariance matrices given by $\hat{x}(0|-1) = E[x(0)] = 0$ and $P(0|-1) = E[x(0)x'(0)]$, one may successively compute the estimate $\hat{x}(k+1|k)$ and the associated covariance matrix $P(k+1|k)$.

The noise input u becomes zero when there are no random perturbations associated with the dynamic equations. This implies that C = U = 0, and, in this case, Eqs. (II-2), (II-3), and (II-5) reduce to

$$\hat{x}(k+1|k) = \Phi(k+1,k)\, \hat{x}(k|k-1) + K_1(k)\, [z(k) - M(k)\, x(k|k-1)] , \tag{II-6}$$

$$K_1(k) = \Phi(k+1,k)\, P(k|k-1)\, M'(k)\, [M(k)\, P(k|k-1)\, M'(k) + R(k)]^{-1} , \tag{II-7}$$

$$P(k+1|k) = [\Phi(k+1,k) - K_1(k)\, M(k)]\, P(k|k-1)\, \Phi'(k+1,k) . \tag{II-8}$$

2. Filtering Only

For the filtering-only problem we obtain

$$\hat{x}(k|k) = \hat{x}(k|k-1) + K(k)\, [z(k) - M(k)\, \hat{x}(k|k-1)] , \tag{II-9}$$

$$K(k) = P(k|k-1)\, M'(k)\, [M(k)\, P(k|k-1)\, M'(k) + R(k)]^{-1} , \tag{II-10}$$

$$P(k|k) = [I - K(k)\, M(k)]\, P(k|k-1) , \tag{II-11}$$

which are useful only with the updating equations, valid for C = 0,

$$\hat{x}(k|k-1) = \Phi(k,k-1)\, \hat{x}(k-1|k-1) , \tag{II-12}$$

$$P(k|k-1) = \Phi(k,k-1)\, P(k-1|k-1)\, \Phi'(k,k-1) + \Gamma(k,k-1) U(k-1) \Gamma'(k,k-1) . \tag{II-13}$$

B. Application to Orbit Determination

In the usual orbit determination problem, we make discrete noisy measurements of variables related to the state of a vehicle whose motion is uniquely determined by its unknown initial state, and we ask, on the basis of the noisy measurements, for the "best" estimate of the state at any time.

For application to this orbit determination problem, Eqs. (II-9) to (II-13) are utilized in somewhat modified form, with $u = U = 0$. Instead of Eq. (II-12) in addition, the equations of motion are integrated from one observation time to another so that we obtain

$$\hat{x}(k|k-1) = f(k;\hat{x}(k-1|k-1),\ k-1)\ . \tag{II-14}$$

In addition, if we define $x^{(a)}(k)$ as the actual state, the measured observations $z^{(m)}(k)$ may be defined by

$$z^{(m)}(k) = g(x^{(a)}(k)) + v(k)\ , \tag{II-15}$$

where $v(k)$ is Gaussian white noise, and $E[v(k)\,v'(m)] = \delta_{km}\,R(k)$. Computed observations $z^{(c)}(k)$ are given by

$$z^{(c)}(k) = g\,(\hat{x}(k|k-1))\ . \tag{II-16}$$

The estimate of the state $\hat{x}(k|k)$, Eq. (II-9), is then taken to be

$$\hat{x}(k|k) = \hat{x}(k|k-1) + K(k)\,[z^{(m)}(k) - z^{(c)}(k)]\ . \tag{II-17}$$

Eqs. (II-13) for up-dating the covariance matrix becomes

$$P(k|k-1) = \Phi(k,k-1)\ P(k-1|k-1)\ \Phi'(k,k-1)\ , \tag{II-18}$$

while Eqs. (II-10) and (II-11) for the weighting function, and covariance matrix correction remain the same.

Since the equations of motion for the orbit determination problem are nonlinear, and the total observations (in which the state may enter nonlinearly) are used in Eq. (II-17), some explanation of the application of the linear filter theory to this problem is in order.

The justification lies in the assumption that deviations and the estimates of these deviations of the actual trajectory from an assumed known reference trajectory are small, so that in terms of the deviations, the estimates of the deviations, and the errors in the estimates, the equations are linear. The integrated equations of motion for both the actual and estimated trajectories may thus be considered equivalent to the integration of the known reference trajectory plus the unknown linear perturbations. Thus, if we define $x^{(r)}(k)$ to be the state of the reference trajectory, we may write

$$x^{(a)}(k) = x^{(r)}(k) + \delta x(k) ,$$

$$\hat{x}(k|k-1) = x^{(r)}(k) + \delta \hat{x}(k|k-1) , \qquad \text{(II-19)}$$

$$\hat{x}(k|k) = x^{(r)}(k) + \delta \hat{x}(k|k) . \qquad \text{(II-20)}$$

Errors in the estimates, also assumed small, are given by

$$\tilde{x}(k|k-1) = x^{(a)}(k) - \hat{x}(k|k-1) = \delta x(k) - \delta \hat{x}(k|k-1) ,$$

$$\tilde{x}(k|k) = x^{(a)}(k) - \hat{x}(k|k) = \delta x(k) - \delta \hat{x}(k|k) . \qquad \text{(II-21)}$$

If we expand the integrated equations of motion for both the actual and estimated trajectories about the reference trajectory, we obtain

$$x^{(a)}(k) = f(k;x^{(a)}(k-1), k-1) = x^{(r)}(k) + \delta x(k)$$

$$= f(k;x^{(r)}(k-1), k-1) + \Phi(k,k-1)\, \delta x(k-1) , \qquad \text{(II-22)}$$

and

$$\hat{x}(k|k-1) = f(k;\hat{x}(k-1|k-1), k-1) = x^{(r)}(k) + \delta \hat{x}(k|k-1)$$

$$= f(k;x^{(r)}(k-1), k-1) + \Phi(k,k-1)\, \delta \hat{x}(k-1|k-1) , \qquad \text{(II-23)}$$

where

$$\Phi(k,k-1) = \partial f(k;x^{(r)}(k-1),k-1)/\partial x^{(r)}$$

$$\approx \partial f(k;\hat{x}(k-1|k-1),\ k-1)/\partial \hat{x}\ . \qquad \text{(II-24)}$$

The deviations and the estimates of the deviations thus satisfy the linear equations

$$\delta(k) = \Phi(k,k-1)\ \delta x(k-1)\ , \qquad \text{(II-25)}$$

$$\delta \hat{x}(k|k-1) = \Phi(k,k-1)\ \delta \hat{x}(k-1|k-1)\ . \qquad \text{(II-26)}$$

If we similarly expand $z^{(m)}(k)$ and $z^{(c)}(k)$, we obtain

$$z^{(m)}(k) - z^{(c)}(k) = v(k) + M(k)\ \tilde{x}(k|k-1) \qquad \text{(II-27)}$$

where

$$M(k) \approx \partial g(\hat{x}(k|k-1))/\partial \hat{x}\ . \qquad \text{(II-28)}$$

From Eq. (II-21), this can also be written as

$$z^{(m)}(k) - z^{(c)}(k) = \delta z(k) - M(k)\ \delta \hat{x}(k|k-1) \qquad \text{(II-29)}$$

where

$$\delta z(k) = M(k)\ \delta x(k) + v(k)\ . \qquad \text{(II-30)}$$

Thus, using Eqs. (II-23) and (II-29), the estimate for the state, Eq. (II-17) may be written as

$$\hat{x}(k|k) = x^{(r)}(k) + \delta \hat{x}(k|k) = x^{(r)}(k) + \delta \hat{x}(k|k-1)$$

$$+ K(k)\ [\delta z(k) - M(k)\ \delta \hat{x}\ (k|k-1)] \qquad \text{(II-31)}$$

or

$$\delta \hat{x}(k|k) = \delta \hat{x}(k|k-1) + K(k)\ [\delta z(k) - M(k)\ \delta \hat{x}(k|k-1)]\ . \qquad \text{(II-32)}$$

From Eqs. (II-32), (II-30), and (II-23) leading to (II-26), we conclude then that although we are integrating nonlinear equations of motion Eq. (II-14) and dealing with total observations in which the state enters nonlinearly, (Eqs. (II-15) and (II-16), for small deviations and estimates of deviations from some reference trajectory, we are in fact applying the linear filter theory only to the deviations which satisfy linear equations.

We should note, however, that in this case, the transition matrix Φ, Eq. (II-24), and the observation matrix M, Eq. (II-28), are functions of the total estimated trajectory state. This is the fundamental difference between applying the filter to a linear system, and to the deviations of a nonlinear system.

C. Development of the Error Equations

In investigating the stability of the Kalman filter as applied to orbit determination, we explore the behavior of the error in the estimate (together with the covariance matrix of the error) rather than the estimate itself. If Eq. (II-23) is subtracted from Eq. (II-22), with the errors as defined in Eq. (II-21), we obtain a linear equation for the up-dating of the errors,

$$\tilde{x}(k|k-1) = \Phi(k,k-1)\,\tilde{x}(k-1|k-1)\ . \tag{II-33}$$

The linear equation for the corrections to the errors as a result of the observations, from Eqs. (II-21), (II-27), (II-29), and (II-32) becomes

$$\tilde{x}(k|k) = \tilde{x}(k|k-1) - K(k)\,[M(k)\,\tilde{x}(k|k-1) + v(k)]\ . \tag{II-34}$$

In summary, then, the equations for the errors and the covariance matrices of the errors to be investigated for stability may be written as

$$\tilde{x}(k|k) = \Psi(k,k-1)\,\tilde{x}(k-1|k-1) - K(k)\,v(k)\ , \tag{II-35}$$

$$\Psi(k,k-1) = [I - K(k)\,M(k)]\,\Phi(k,k-1)\ . \tag{II-36}$$

$$P(k|k) = E[\tilde{x}(k|k)\,\tilde{x}'(k|k)] = [I - K(k)\,M(k)]\,P(k|k-1)\ , \tag{II-37}$$

$$P(k|k-1) = E[\tilde{x}(k|k-1)\,\tilde{x}'(k|k-1)] = \Phi(k,k-1)\,P(k-1|k-1)\,\Phi'(k,k-1)\ , \tag{II-38}$$

$$K(k) = P(k|k-1)\,M'(k)\,[M(k)\,P(k|k-1)\,M'(k) + R]^{-1}\ . \tag{II-39}$$

Equation (II-35) is a linear nonautonomous vector difference equation so that its stability properties are independent of the input function v(k), and we may therefore study the stability of the free system given by

$$\tilde{x}(k|k) = \Psi(k,k-1)\,\tilde{x}(k-1|k-1)\ . \tag{II-40}$$

Equations (II-37) - (II-39) may be combined to give the nonlinear non-autonomous matrix difference equation

$$\begin{aligned} P(k|k) = \Phi(k,k-1)\,P(k-1|k-1)\,\Phi'(k,k-1)\,\{I \\ - M'(k)\,[M(k)\,\Phi(k,k-1)\,P(k-1|k-1)\,\Phi'(k,k-1)M'(k) \\ + R]^{-1}\,M(k)\,\Phi(k,k-1)\,P(k-1|k-1)\,\Phi'(k,k-1)\}\ . \end{aligned} \tag{II-41}$$

We note that the stability of the error (Eq. (II-40)) is determined by the behavior of the matrix Ψ which is a function of the covariance matrix but that the equation for the covariance matrix is independent of the error and so may be investigated first.

Some fundamental notions of stability and Liapunov's Method have been presented in [15] and [16]. We recall that given the free vector difference equation $x(t_{k+1}) = h(x(t_k), t_k)$, that a state $x^{(e)}$ is an equilibrium (or critical) state if $x^{(e)} = h(x^{(e)}, t_k)$ for all t_k. We observe from Eqs. (II-40) and (II-41) for the error and covariance matrix, that the origin is an equilibrium state for both the error, $\tilde{x}(k|k)$ and the covariance matrix $P(k|k)$. Additional equilibrium points, however, may arise under special conditions.

D. Positive-Definiteness of Covariance Matrix

Most of our investigations will be concerned with the behavior of the covariance matrix $P(k|k)$ for the particular models we have set up. In general, however, from the error equation

$$\tilde{x}(k|k) = \Psi(k|k-1)\,\tilde{x}(k-1|k-1) + K(k)\,v(k) \ ,$$

and the resulting covariance equation,

$$P(k|k) = E[\tilde{x}(k|k)\,\tilde{x}'(k|k)] \ ,$$

$$= \Psi\,P(k-1|k-1)\,\Psi' + KRK' \ ,$$

we note that if the noise covariance matrix R and the initial error covariance matrix $P(0|0)$ are symmetric and positive-definite, then the covariance matrix $P(k|k)$ is always symmetric and positive-definite, except, perhaps in the limit, at the equilibrium point at the origin where the norm of the matrix will be zero. (In general, for any positive-definite matrix A, the expression BAB' is positive-definite for all non-zero B, and sums of positive-definite matrices are positive-definite.)

SECTION III

SIMPLE HARMONIC MOTION MODEL

A. Dynamic Equations

The dynamic equations for simple harmonic motion are given by

$$\dot{x} = A\,x\ ,$$

where the two-component vector x and the two by two matrix A are defined by

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Here, the motion is periodic, in one dimension. With observations assumed every Δ units of time, the solution is given by

$$x(k+1) = \Phi(k+1,k)\,x(k)\ , \qquad \text{(III-1)}$$

$$\Phi(k+1,k) = \Phi(\Delta) = \begin{bmatrix} \cos\Delta & \sin\Delta \\ -\sin\Delta & \cos\Delta \end{bmatrix}, \qquad \text{(III-2a)}$$

$$\Delta = t_{k+1} - t_k\ ,$$

where $\Phi(\Delta)$ is the constant transition matrix. Observations,

$$z_i = M_i\,x + v_i\ ,\quad i = 1,2,\ldots\ ,$$

for such a system may take the forms listed:

range:	$M_1 = [1 \quad 0]$,
range-rate:	$M_2 = [0 \quad 1]$,
range and range-rate:	$M_3 = I$,
on-board observation of angle:	$M_4 = -[a \quad 0]$,
ground-based observation of angle:	$M_5 = +[a \quad 0]$,
	$a = b/(b^2 + x_1^2)$; $b > 0$.

The angle observation matrices arise from the following considerations. For the on-board observation, a landmark b units perpendicular to the line of motion is observed from the vehicle, and deviations of the angle $\theta = \arctan (b/x_1)$ (so that $\delta\theta = M_4\ \delta x$) are measured. Similarly, for the ground-based observation, a site again b units from the line of motion observes the vehicle, and measures deviations of the angle $\phi = \arctan (x_1/b)$ (so that $\delta\phi = M_5\ \delta x$).

The observation noise vector v_i is white and Gaussian with zero mean and components and variances defined as follows:

$$v_i = \epsilon_i \ , \qquad R = \sigma_i^2 \ , \qquad i = 1,2,4,5 \ .$$

$$v_3 = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} , \qquad R = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} .$$

B. Error and Variance Equations

The error equations for simple harmonic motion reduce to those of a two component state error vector and a three element two by two symmetric covariance matrix. The equations for the covariance matrix are independent of the error and so may be investigated first. If we define the covariance matrix at t_k due to measurements up to t_k as

$$P_i(k|k) = \begin{pmatrix} \xi_i(k) & \eta_i(k) \\ \eta_i(k) & \zeta_i(k) \end{pmatrix} , \qquad \text{(III-2b)}$$

and the covariance matrix at t_k due to measurements up to t_{k-1} as

$$P_i(k|k-1) = \begin{pmatrix} \xi_i^*(k-1) & \eta_i^*(k-1) \\ \eta_i^*(k-1) & \zeta_i^*(k-1) \end{pmatrix} \quad , \tag{III-2c}$$

and further define the vectors

$$p_i(k) = \begin{bmatrix} \xi_i(k) \\ \eta_i(k) \\ \zeta_i(k) \end{bmatrix} ; \quad p_i^*(k-1) = \begin{bmatrix} \xi_i^*(k-1) \\ \eta_i^*(k-1) \\ \zeta_i^*(k-1) \end{bmatrix}$$

$$i = 1, 2, \ldots \quad ,$$

we obtain the equations

$$p_i^*(k-1) = B\, p_i(k-1) \quad , \qquad i = 1, 2, \ldots \tag{III-3}$$

where

$$B = \begin{bmatrix} \cos^2 \Delta & \sin 2\Delta & \sin^2 \Delta \\ -\frac{1}{2} \sin 2\Delta & \cos 2\Delta & \frac{1}{2} \sin 2\Delta \\ \sin^2 \Delta & -\sin 2\Delta & \cos^2 \Delta \end{bmatrix} \tag{III-4}$$

and, for i = 1, and 2,

$$\begin{aligned} p_1(k) &= B\, p_1(k-1) - F(p_1^*(k-1)) \quad , \\ p_2(k) &= B\, p_2(k-1) - G(p_2^*(k-1)) \quad , \end{aligned} \tag{III-5}$$

where

$$F\,[p_1^*(k-1)] = [1/(\xi_1^*(k-1) + \sigma_1^2)] \begin{bmatrix} [\xi_1^*(k-1)]^2 \\ \xi_1^*(k-1)\, \eta_1^*(k-1) \\ [\eta_1^*(k-1)]^2 \end{bmatrix} , \tag{III-6}$$

and

$$G[p_2^*(k-1)] = [1/(\zeta_2^*(k-1) + \sigma_2^2)] \begin{bmatrix} [\eta_2^*(k-1)]^2 \\ \eta_2^*(k-1)\,\zeta_2^*(k-1) \\ [\zeta_2^*(k-1)]^2 \end{bmatrix} . \tag{III-7}$$

In terms of the components of $p_i(k)$ and $p_i^*(k-1)$, Eq. (III-5) becomes

$$\begin{aligned} \xi_1(k) &= \xi_1^*(k-1)\,[1 - \xi_1^*(k-1) / (\xi_1^*(k-1) + \sigma_1^2)] , \\ \eta_1(k) &= \eta_1^*(k-1)\,[1 - \xi_1^*(k-1) / (\xi_1^*(k-1) + \sigma_1^2)] , \\ \zeta_1(k) &= \zeta_1^*(k-1) - [\eta_1^*(k-1)]^2 / (\xi_1^*(k-1) + \sigma_1^2) , \end{aligned} \tag{III-8a}$$

$$\begin{aligned} \xi_2(k) &= \xi_2^*(k-1) - [\eta_2^*(k-1)]^2 / (\zeta_2^*(k-1) + \sigma_2^2) , \\ \eta_2(k) &= \eta_2^*(k-1)\,[1 - \zeta_2^*(k-1) / (\zeta_2^*(k-1) + \sigma_2^2)] , \\ \zeta_2(k) &= \zeta_2^*(k-1)\,[1 - \zeta_2^*(k-1) / (\zeta_2^*(k-1) + \sigma_2^2)] . \end{aligned} \tag{III-8b}$$

1. Linear Behavior of Variance Equation

In the neighborhood of the equilibrium at the origin, the terms F and G in Eq. (III-5) may be expanded in a power series in the variables ξ_i^*, η_i^* and ζ_i^* resulting in beginning terms of second order. In this case, the reduced equations (or equations of the first approximation)

$$p_i(k) = B\, p_i(k-1) , \qquad i = 1,2, \tag{III-9}$$

obtained from Eq. (III-5) by linearization are called equations with significant behavior of the equilibrium if the matrix B has either only eigen-values whose logarithms have negative real parts, or has at least one eigen-value whose logarithm has a positive real part. The equations have critical behavior of the equilibrium if none of the eigen-values has a logarithm with positive real parts; however, eigen-values whose logarithms have vanishing real parts do occur. (We recall that an eigen-value whose logarithm has negative real part is one whose absolute value is less than one.)

The importance of the linear approximation is contained in the following theorem due to Liapunov for differential equations and Perron for difference equations [17]:

Theorem: If the stability behavior of the difference (differential) equation of the first approximation is significant, then the equilibrium of the complete difference (differential) equation has the same stability behavior as the equilibrium of the reduced equation. In critical cases, the stability behavior is not determined only by the first order terms.

In view of the above theorem, it is thus of interest to investigate the linear equations (III-9) and find the eigen-values of the matrix B. By solving $\det (B - \lambda I) = 0$, we find the characteristic equation for the eigen-values to be

$$\lambda^3 - \lambda^2 (4 \cos^2 \Delta - 1) + \lambda (4 \cos^2 \Delta - 1) - 1 = 0 ,$$

which has the roots $\lambda_1 = 1$, $\lambda_2, \lambda_3 = \cos 2\Delta \pm i \sin 2\Delta$. The eigen-values of B are thus all of absolute value one which means the behavior of the linearized equation in this case is critical, rather than significant with the stability determined by the higher order terms.

We note that the matrix B is a function of the elements of the transition matrix alone. Thus, it is the dynamic system, and not the type of measurements nor their weighting that determines whether the linearized equations for the covariance terms have significant behavior at equilibrium, thereby determining the stability at the origin of the complete system. For dynamic systems which are themselves asymptotically stable, or unstable, one might then expect significant behavior; for systems which are weakly stable, critical behavior would be most likely. (In general, for nonlinear equations of motion, the dynamic system we would be concerned with would be the variational equations about some reference motion with the transition matrix as a fundamental matrix solution.)

2. Reducing to Canonical Form

By computing the eigen-vectors of B, and transforming B into its diagonal canonical form, we may show that in this case, the origin is the only equilibrium point [18].

The non-singular matrix of eigen-vectors turns out to be

$$T = \begin{bmatrix} 1 & -1 & -1 \\ 0 & -i & +i \\ 1 & 1 & 1 \end{bmatrix} ,$$

while its inverse may be shown to be

$$T^{-1} = \frac{1}{4} \begin{bmatrix} 2 & 0 & 2 \\ -1 & 2i & 1 \\ -1 & -2i & 1 \end{bmatrix} .$$

If we now define the vectors $r_j(k-1) = T^{-1} p_j(k-1)$, $j = 1,2,$ in terms of the components of r_j, we have

$$r_j = \begin{bmatrix} \psi_j \\ \rho_j \\ \bar{\rho}_j \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2\xi_j + 2\zeta_j \\ \zeta_j - \xi_j + 2i\eta_j \\ \zeta_j - \xi_j - 2i\eta_j \end{bmatrix} , \quad j = 1,2,$$

where $\bar{\rho}_j$ is the complex conjugate of ρ_j. Since $p_j(k-1) = T\,r_j(k-1)$, we obtain

$$p_j^*(k-1) = B\,T\,r_j(k-1) , \quad j = 1,2$$

$$r_1(k) = \Lambda\, r_1(k-1) - T^{-1} F(BT\, r_1(k-1)) ,$$

$$r_2(k) = \Lambda r_2(k-1) - T^{-1} G(BT\, r_2(k-1)) ,$$

where $\Lambda = (T^{-1} BT)$ is the diagonal matrix

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \bar{\lambda} \end{bmatrix} ,$$

$$BT = T\Lambda = \begin{bmatrix} 1 & -\lambda & -\bar{\lambda} \\ 0 & -i\lambda & +i\bar{\lambda} \\ 1 & \lambda & \bar{\lambda} \end{bmatrix} ,$$

and we have written λ for λ_2, and $\bar{\lambda}$ for λ_3. The expressions for $T^{-1}F$ and $T^{-1}G$ become

$$T^{-1}F(BTr_1(k-1)) = -\frac{1}{4[\psi_1(k-1) - 2R(\lambda\rho_1(k-1)) + \sigma_1^2]} \begin{bmatrix} 2\,|\psi_1(k-1) - 2\lambda\rho_1(k-1)|^2 \\ -[\psi_1(k-1) - 2\lambda\rho_1(k-1)]^2 \\ -[\psi_1(k-1) - 2\bar{\lambda}\bar{\rho}_1(k-1)]^2 \end{bmatrix}$$

and

$$T^{-1}G(BTr_2(k-1)) = \frac{1}{4\,[\psi_2(k-1) + 2R(\lambda\rho_2(k-1)) + \sigma_2^2]} \begin{bmatrix} 2\,|\psi_2(k-1) + 2\lambda\rho_2(k-1)|^2 \\ [\psi_2(k-1) + 2\lambda\rho_2(k-1)]^2 \\ [\psi_2(k-1) + 2\bar{\lambda}\bar{\rho}_2(k-1)]^2 \end{bmatrix}$$

where $2R(\lambda\rho_j(k-1)) = \lambda\rho_j(k-1) + \bar{\lambda}\bar{\rho}_j(k-1)$.

Solving for the equilibrium points, we find, for $j = 1$,

$$\psi_1 = \psi_1 - \frac{|\psi_1 - 2\lambda\rho_1|^2}{2\,[\psi_1 - 2R(\lambda\rho_1) + \sigma_1^2]} ,$$

$$\rho_1 = \lambda\rho_1 + \frac{[\psi_1 - 2\lambda\rho_1]^2}{4\,[\psi_1 - 2R(\lambda\rho_1) + \sigma_1^2]} .$$

From the first equation, we require $\psi_1 - 2\lambda\rho_1 = 0$. But from the second equation, this implies $\rho_1 (1-\lambda) = 0$. Since for $0 < \Delta < 2\pi$, $\lambda \neq 1$, we must have $\rho_1 = 0$, and therefore ψ_1 (and of course $\bar{\rho}_1) = 0$. These same considerations apply to $j = 2$.

C. Contracting Mapping and Proof of Stability

The variance equations (III-3) and (III-5) may be thought of as a mapping from time $k-1 \equiv t_{k-1}$ to time $k \equiv t_k$, i.e.,

$$p_i(k) = H_i\,(p_i(k-1))\ , \quad i = 1,2,\ldots \quad .$$

The mapping is contracting if under a suitable norm (where norm $p_i(k) \equiv \| p_i(k) \|$), we have

$$\| p_i(k) \| = \| H_i\,(p_i(k-1)) \| < \| p_i(k-1) \| \quad .$$

In this case, the stability at the origin is asymptotic, for we may choose the norm itself as a positive-definite Liapunov function V, i.e.,

Liapunov function, $V\,(p_i(k)) = \| p_i(k) \|$, positive-definite,

in which case the change in the Liapunov function,

$$\Delta V = \| p_i(k) \| - \| p_i(k-1) \| \quad ,$$

is negative-definite. By one of Liapunov's theorems [15 - 17], under those conditions, the equilibrium point at the origin is asymptotically stable.

One way to find the conditions under which the mappings are contracting is to do it in two stages, i.e., first show that

$$\| p_i(k) \| < \| p_i{}^*(k-1) \| \quad , \tag{III-10}$$

and then that

$$\| p_i{}^*(k-1) \| \leq \| p_i(k-1) \| \quad , \tag{III-11}$$

so that by combining the two inequalities we obtain

$$\| p_i(k) \| < \| p_i(k-1) \| , \qquad i = 1,2,\ldots \quad . \tag{III-12}$$

For $i = 1,2,$ we can show, using the positive-definiteness of the co-variance matrices for non-zero elements that the components of $p_i(k)$ are less than the components of $p_i^*(k-1)$, i.e., that

$$[\xi_i(k)]^2 < [\xi_i^*(k-1)]^2 \quad ,$$
$$[\eta_i(k)]^2 < [\eta_i^*(k-1)]^2 \quad ,$$
$$[\zeta_i(k)]^2 < [\zeta_i^*(k-1)]^2 \quad .$$

If we use for a norm, then

$$\| p_i(k) \|^2 = p_i(k)' \, N \, p_i(k) \quad , \tag{III-13a}$$

$$\| p_i^*(k-1) \|^2 = p_i^*(k-1) \, N \, p_i^*(k-1) \quad , \tag{III-13b}$$

where N is a positive definite, diagonal matrix, with

$$N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & g \end{bmatrix} \qquad f > 0, \quad g > 0 \; , \tag{III-13c}$$

so that

$$\| p_i(k) \|^2 = [\xi_i(k)]^2 + f[\eta_i(k)]^2 + g[\zeta_i(k)]^2 \quad ,$$

and

$$\| p_i^*(k-1) \|^2 = [\xi_i^*(k-1)]^2 + f[\eta_i^*(k-1)]^2 + g[\zeta_i^*(k-1)]^2 \quad ,$$

we shall have

$$\| p_i(k) \| < \| p_i^*(k-1) \| \quad , \quad \text{for all } f,g > 0 \quad .$$

To find what values of f and g in the matrix N will cause

$$\| p_i{}^*(k-1) \| \leq \| p_i(k-1) \| \tag{III-11}$$

to hold requires somewhat more effort, for the identity will not. We prove (III-11) by computing the norm, Eq. (III-13) of both sides of

$$p_i{}^*(k-1) = B\, p_i(k-1)\ , \tag{III-3}$$

i.e.,

$$\| p_i{}^*(k-1) \| = \| B\, p_i(k-1) \| \quad .$$

For convenience, we define the following quantities:

$$\begin{aligned} c &= \cos \Delta \quad , & s &= \sin \Delta \quad , \\ c2 &= \cos 2\Delta \quad , & s2 &= \sin 2\Delta \quad , \\ c^2 &= \cos^2 \Delta \quad , & s^2 &= \sin^2 \Delta \quad . \end{aligned}$$

Eq. (III-3) in component form is thus (omitting the (k-1) arguments)

$$\begin{aligned} \xi_i{}^* &= c^2\, \xi_i + s2\, \eta_i + s^2\, \zeta_i \quad , \\ \eta_i{}^* &= -\,(s2/2)\, \xi_i + c2\, \eta_i + (s2/2)\, \zeta_i \quad , \\ \zeta_i{}^* &= s^2\, \xi_i - s2\, \eta_i + c^2\, \zeta_i \quad , \end{aligned}$$

so that

$$\begin{aligned} \| p_i{}^*(k-1) \|^2 &= [c^2 \xi_i + s2\eta_i + s^2\zeta_i]^2 + f[-(s2/2)\xi_i + c2\eta_i + (s2/2)\zeta_i]^2 \\ &\quad + g\, [s^2\xi_i - s2\eta_i + c^2\zeta_i]^2 \\ &= \xi_i{}^2\, [c^4 + fs^22/4 + gs^4] + \eta_i{}^2 [s^22 + fc^22 + gs^22] \\ &\quad + \zeta_i{}^2 [s^4 + fs^22/4 + gc^4] + \xi_i\eta_i [2c^2s2 - fs2c2 - g2s^2s2] \\ &\quad + \xi_i\zeta_i [2c^2s^2 - fs^22/2 + g2s^2c^2] + \eta_i\zeta_i [2s2s^2 + fc2s2 - g2s2c^2]. \end{aligned}$$

If we now let $g = 1$ and $f = 2$, we obtain

$$\|p_i^*(k-1)\|^2 = [\xi_i^2 + 2\eta_i^2 + \zeta_i^2] = \|p_i(k-1)\|^2 \quad , \qquad \text{(III-14)}$$

which is what we wanted to prove.

Thus, inequality (III-12) holds, the total mapping is contracting and the equilibrium at the origin of the variance vector, $p_i(k)$ is asymptotically stable.

We note that this norm, i.e.,

$$\|p_i(k-1)\|^2 = [\xi_i(k-1)]^2 + 2[\eta_i(k-1)]^2 + [\zeta_i(k-1)]^2 \quad , \qquad \text{(III-15a)}$$

$$\|p_i^*(k-1)\|^2 = [\xi_i^*(k-1)]^2 + 2[\eta_i^*(k-1)]^2 + [\zeta_i^*(k-1)]^2 \quad , \qquad \text{(III-15b)}$$

turns out to be one of the natural norms for a positive-definite matrix, i.e., the trace of the matrix $(P_i)^2$ which is equivalent to the sums of the squares of the eigen-values. Thus, Eq. (III-14) could also have been proven using Eq. (II-38) and (III-2), for

$$\text{trace}\ [P_i(k|k-1)]^2 = \text{trace}\left\{\Phi(\Delta)[P_i(k-1|k-1)]^2\,\Phi'(\Delta)\right\} \quad ,$$

since $\Phi'(\Delta) = \Phi^{-1}(\Delta)$. Upon expansion this results in

$$[\xi_i(k-1)]^2 + 2[\eta_i(k-1)]^2 + [\zeta_i(k-1)]^2 =$$

$$[\xi_i^*(k-1)]^2 + 2[\eta_i^*(k-1)]^2 + [\zeta_i(k-1)]^2 \quad .$$

D. Other Measurement Matrices

1. Identity Matrix

We have proven the asymptotic stability of the variance equation assuming simple harmonic motion, and two types of measurement matrices, $M_1 = [1 \quad 0]$ (implying range measurements) and $M_2 = [0 \quad 1]$ (implying range rate measurements). One would expect that the situation for M_3 equal to the two by two identity matrix, I, would be no worse. For $M_3 = I$, the equations affecting the covariance matrix are

$$K(k) = P(k|k-1) [P(k|k-1) + R]^{-1} ,$$

$$P(k|k) = [I - K(k)] P(k|k-1) ,$$

in addition to

$$P(k|k-1) = \Phi(k,k-1) P(k-1|k-1) \Phi'(k,k-1)$$

which is independent of M_3. With the noise covariance matrix defined by

$$R = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} ,$$

we obtain, for the elements of the covariance matrix, $P(k|k)$, the equations

$$\begin{aligned} \xi(k) &= (\sigma_1^2/\Delta) \left\{ \xi^*(k-1) [\zeta^*(k-1) + \sigma_2^2] - [\eta^*(k-1)]^2 \right\} , \\ \eta(k) &= \sigma_1^2 \sigma_2^2 \eta^*(k-1)/\Delta , \\ \zeta(k) &= (\sigma_2^2/\Delta) \left\{ \zeta^*(k-1) [\xi^*(k-1) + \sigma_1^2] - [\eta^*(k-1)]^2 \right\} , \end{aligned} \tag{III-15}$$

where

$$\Delta = [\xi^*(k-1) + \sigma_1^2] [\zeta^*(k-1) + \sigma_2^2] - [\eta^*(k-1)]^2 .$$

For these equations, we again have $\xi^*(k-1) > 0$, $\zeta^*(k-1) > 0$ and $\xi^*(k-1) \zeta^*(k-1) - [\eta^*(k-1)]^2 > 0$, and as a consequence $\xi(k) > 0, \zeta(k) > 0$, and $\xi(k) \zeta(k) - [\eta(k)]^2 > 0$. From Eq. (III-15), again it can be seen that

$$[\xi(k)]^2 < [\xi^*(k-1)]^2 ,$$

$$[\eta(k)]^2 < [\eta^*(k-1)]^2 ,$$

$$[\zeta(k)]^2 < [\zeta^*(k-1)]^2 ,$$

so that, using our norm (III-13), we again have

$$\|p(k)\|^2 < \|p^*(k-1)\|^2 .$$

Since Eq. (III-14) holds independently of the measurement matrix, we may again conclude that the variance equation for the identity measurement matrix is asymptotically stable for all initial conditions.

2. Angle Observations

The angle observation measurement matrices may be developed as follows. If b is a positive constant, and $x_1(t)$ is the position component of the state, then $\theta = \arctan(b/x_1(t))$ would represent an on-board observation of the point at b from the vehicle at $x_1(t)$, while $\phi = \arctan x_1(t)/b$ would be the equivalent ground-based observation of $x_1(t)$ from the point b. Differentiating both equations to obtain the angular deviations in terms of the deviations of the state, we find

$$\delta\theta = M_4\,\delta x,$$

$$\delta\phi = M_5\,\delta x,$$

where

$$M_4 = -[a \quad 0], \qquad M_5 = +[a \quad 0],$$

$$a = b/(b^2 + x_1^2), \text{ and } \quad \delta x' = [\delta x_1 \quad \delta x_2].$$

Substituting these measurement matrices into the equations for the covariance matrix we obtain

$$\xi_i(k) = \xi_i^*(k-1)\left[1 - \frac{\xi_i^*(k-1)}{\xi_i^*(k-1) + \sigma_i^2/a^2}\right]$$

$$\eta_i(k) = \eta_i^*(k-1)\left[1 - \frac{\xi_i^*(k-1)}{\xi_i^*(k-1) + \sigma_i^2/a^2}\right]$$

$$\zeta_i(k) = \zeta_i^*(k-1) - \frac{\left[\eta_i^*(k-1)\right]^2}{\xi_i^*(k-1) + \sigma_i^2/a^2} \quad ; \quad i = 4,5 .$$

Comparing these equations with those, for $p_i(k)$ [Eq. III-8a], for the range-only observation, we note that they are identical except for the noise covariance term which is now

$$\sigma_i^2/a^2 = (\sigma_i r)^2 \, (r/b)^2 \quad ,$$

with $r = (b^2 + x_1^2)^{\frac{1}{2}}$, the observation range. Since $r > b$, we see that using angle observation matrices is equivalent to using range measurements with correspondingly higher, and non-constant measurement errors. We note also that on-board and ground-based angular observations give identical results, and that both are thus also asymptotically stable for the simple harmonic model.

E. Behavior of Error Equation

Having determined the behavior of the variance equation for the measurement matrices discussed, one might ask then for the behavior of the linear error equation whose behavior is completely determined by the matrix

$$\Psi_i(k,k-1) = [I - K_i \, M_i \,] \, \Phi(k,k-1) \quad .$$

For $i = 1,2$, these are

$$\Psi_1(k,k-1) =$$

$$\frac{1}{[\xi_1{}^*(k-1) + \sigma_1^2]} \begin{bmatrix} \sigma_1^2 \cos \Delta & \sigma_1^2 \sin \Delta \\ & \\ -\eta_1{}^*(k-1) \cos \Delta & -\eta_1{}^*(k-1) \sin \Delta \\ -[\xi_1{}^*(k-1) + \sigma_1^2] \sin \Delta & +[\xi_1{}^*(k-1) + \sigma_1^2] \cos \Delta \end{bmatrix} ,$$

$$\Psi_2(k,k-1) =$$

$$\frac{1}{[\zeta_2{}^*(k-1) + \sigma_2^2]} \begin{bmatrix} \eta_2{}^*(k-1) \sin \Delta & -\eta_2{}^*(k-1) \cos \Delta \\ +[\zeta_2{}^*(k-1) + \sigma_2^2] \cos \Delta & +[\zeta_2{}^*(k-1) + \sigma_2^2] \sin \Delta \\ & \\ -\sigma_2^2 \sin \Delta & \sigma_2^2 \cos \Delta \end{bmatrix} .$$

We note that for these matrices, and for Ψ_i in general, that as $k \to \infty$, and $P_i \to 0$, we have (because $K_i \to 0$), $\Psi_i \to \Phi$, the transition matrix of the system, with, in the case of simple harmonic motion, eigen-values all of magnitude one. The stability of the error equations would then be that of the original dynamical system, and in our case, weakly stable. One would expect, however, that for every finite t, the eigen-values of Ψ would have magnitudes less than one so that one would again expect asymptotic stability. The analytic proof of these facts and the implications, considering that the matrix is time-varying have not yet been investigated, although computer simulations, which we shall discuss appear to demonstrate this.

F. Some Computer Studies

1. Parameter Variation

Equations for the orbit determination of the simple dynamical model without damping and with negative damping have been programmed in FORTRAN 4 for the IBM 7094 computer providing a relatively simple flexible tool for checking the analysis and analyzing those aspects of the filter behavior which become too complicated and tedious for analysis. In these runs, the effects of varying the time between observations, the covariance of the noise, the observation matrix, and the initial covariance matrix were observed. In all cases run without damping, stability at the origin was indicated with, however, the degree of stability very much a function of the parameter.

Table I lists the conditions for some representative runs made in which the parameters were varied as indicated. Phase plane plots of the resulting errors and diagonal elements of the covariance matrix are shown in Figures 1-16.

Comparing Figures (3-4) with (1-2) illustrates the degrading effect of increased noise covariance. Decreased initial covariance in position (Figures (5-6)) increases the initial excursions in the error but results in much smoother variance behavior. Increasing the initial covariance in velocity (Figures (7-8)) results in a much larger velocity error initially but again results in smooth variance decreases.

TABLE I

COMPUTER RUNS OF KALMAN FILTER
AND SIMPLE HARMONIC MOTION

$$x(0) = \begin{bmatrix} 100 \\ 0 \end{bmatrix}; \qquad \bar{x}(0|0) = \begin{bmatrix} 100 \\ -10 \end{bmatrix}$$

Figure No.	Run No.	M	P(0|0)	σ_i^2	$2n(t_k=k(\pi/n))$
1 - 2	16	$[1 \quad 0]$	diag $(10^4 \quad 10^2)$	2500	32
3 - 4	17			10^4	
5 - 6	20		$(10^2 \quad 10^2)$		
7 - 8	22		$(10^2 \quad 10^3)$		
9 - 10	24	$[0 \quad 1]$	$(10^4 \quad 10^2)$		
11 - 12	25		$(10^2 \quad 10^2)$		
13 - 14	28		$\begin{pmatrix} 10^4 & 10^2 \\ 10^2 & 10^2 \end{pmatrix}$		
15 - 16	29				8

Comparing Figure (9 - 10) with (1 - 2) illustrates the effect of measuring range-rate instead of range. For the conditions assumed, the solutions are not as stable. Figure (11 - 12) illustrates this effect even more dramatically, the errors being almost only weakly-stable. Finally Figure (13 - 14) and (15 - 16) illustrate the degrading effect of fewer observations per period.

2. Degree of Stability

To illustrate the degree of stability as a function of various parameters in a more quantitative way, a number of runs were made varying the number of observations per period and the initial covariance matrix with the noise covariance matrix normalized to one.

Recording the ratios of the magnitudes of final and initial errors, and final and initial covariance traces as functions of the parameters indicated in general that the larger the initial covariance matrices and the greater the number of observations per period, the greater the rate of decay of the errors and variances, i.e., the smaller was the ratio between the final and initial covariance matrix traces and error magnitudes at the end of one period. Figures [17 - 22] are plots of the ratio of the magnitudes of final and initial errors, and the ratio of final and initial covariance traces as functions of the logarithm of the initial velocity error variance with the position error variance as a parameter for 8, 16, and 48 observations per period. These runs are for an observation of range, i.e., $M_1 = [1 \quad 0]$. The runs for the velocity observation were identical.

SECTION IV

MOTION WITH NEGATIVE DAMPING

A. Dynamic Equations

As part of the objective of exploring the stability of the Kalman filter as a function of the transition matrix and the model used, simple harmonic motion with negative damping was investigated. The equations for this model are given by

$$\dot{x} = Dx \quad ,$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 0 & 1 \\ -1 & -2\alpha \end{bmatrix} ;$$

the solution to these equations is given by

$$x(k+1) = \Phi(k+1,k)\, x(k) \quad ,$$

where the transition matrix, $\Phi(k+1,k)$ is

$$\Phi(k+1,k) = \Phi(\Delta) = (e^{-\alpha\Delta}/\lambda) \begin{bmatrix} \lambda\cos\lambda\Delta + \alpha\sin\lambda\Delta & \sin\lambda\Delta \\ -\sin\lambda\Delta & \lambda\cos\lambda\Delta - \alpha\sin\lambda\Delta \end{bmatrix} ,$$

and

$$\Delta \equiv t_{k+1} - t_k \ ; \ \lambda = (1 - \alpha^2)^{\frac{1}{2}} \quad ; \quad -1 < \alpha < 0 \quad .$$

B. Variance Equations

We recall that the influence of the transition matrix is felt only in the so-called up-dating equations relating $p_i^*(k-1)$ to $p_i(k-1)$, i.e.,

$$p_i^*(k-1) = B\, p_i(k-1)$$

where now

$$B = \left(e^{-2\alpha\Delta}/\lambda^2 \right) \begin{bmatrix} \cos^2\theta + \alpha\lambda \sin 2\theta - \alpha^2 \cos 2\theta & \lambda \sin 2\theta + 2\alpha \sin^2\theta & \sin^2\theta \\ -\frac{\lambda}{2} \sin 2\theta - \alpha \sin^2\theta & \cos 2\theta - \alpha^2 & \frac{\lambda}{2} \sin 2\theta - \alpha \sin^2\theta \\ \sin^2\theta & -\lambda \sin 2\theta + 2\alpha \sin^2\theta & \cos^2\theta - \alpha\lambda \sin 2\theta - \alpha^2 \cos 2\theta \end{bmatrix}$$

with

$$\lambda = (1 - \alpha^2)^{\frac{1}{2}} \quad ; \quad -1 < \alpha < 0 \quad ; \quad \theta = \lambda \Delta \ .$$

The eigen-values of this matrix are given by

$$\mu_1 = e^{-2\alpha\Delta}$$

$$\mu_2, \mu_3 = e^{-2\alpha\Delta \pm i\, 2\theta} ,$$

whose absolute values are all greater than one. Thus, the behavior at the origin of the linearized portion of the total nonlinear equation for $p_i(k-1)$, Eq. (III-5), is significant with the stability determined by the eigen-values of the matrix B. In this case, we conclude that the origin is an unstable equilibrium point. However, in this case also, we might expect an equilibrium point for the covariance matrix other than the origin to exist.

C. Some Computer Results for Negative Damping

Figures 23-34 illustrate the unusual behavior one can expect with a model of this type. The conditions for these runs are given in Table II. In Figure 23, we observe what appears to be an unstable error phase plot over one period. (Time in this figure goes clockwise.) In Figure 24 appear increasing and apparently unstable diagonal terms of the covariance matrix as a function of time over one period. If we increase the initial covariance matrix by a factor of 10, we observe, in Figure 25, that the error curves are stable, while the variance curves of Figure 26 still are increasing and appear to be unstable. Finally, with a still larger initial covariance matrix, the error curves, in Figure 27, remain stable, while now in addition, in Figure 28, the covariances decrease as indicated.

TABLE II

MOTION WITH NEGATIVE DAMPING

$$x(0) = \begin{bmatrix} 100 \\ 0 \end{bmatrix}; \quad \tilde{x}(0|0) = \begin{bmatrix} 100 \\ -10 \end{bmatrix}, \quad R = \sigma_2^{\,2} = 1 \;.$$

$M = [0 \quad 1]$, $\alpha = -0.2$, $2n = 32$ (observations per period)

Figure No.	Run No.	P(0|0)
23 - 24	DHM 4	diag. (.01 .01)
25 - 26	DHM 5	(.1 .1)
27 - 28	DHM 6	(1. 1.)
29 - 30	DHM 4A	(.01 .01)
31 - 32	DHM 5A	(.1 .1)
33 - 34	DHM 6A	(1. 1.)

Longer time computer results over several periods show that the errors finally become stable and that the variances reach a steady state different from zero. Thus, in Figure 29, the error after increasing for two periods finally decreases and approaches the origin. At the same time, in Figure 30, the variances while still increasing initially, eventually approach a steady-state value of approximately 0.15. In Figure 31, for the higher initial covariance matrix, the errors curves remain stable while the covariances in Figure 32 again increase to a steady-state value. Finally in Figures 33 and 34, stability of both errors and variances are indicated with the variances in this case approaching the steady-state value of approximately 0.15 from above.

SECTION V

CONCLUSIONS AND RECOMMENDATIONS

As indicated in the Introduction, to gain some insight and experience with the operation of the Kalman filter as it is applied to the orbit problem, it was decided, before undertaking the investigation of the general and more complete problem, to look at some very basic simple models which might in some sense approximate a vehicle in orbit and yet be of sufficiently low order to be handled easily. We chose to look at a simple harmonic motion model in one dimension with both zero and negative damping and have explored its behavior both for asymptotic stability and finite time stability as functions of various parameters such as the choice of observations, the time between observations, and the covariance matrix of initial conditions.

Our results may be summarized as follows:

1. If the linear part of the variance equation has significant behavior, then the stability of the variance equation at the origin is completely determined by the transition matrix of the dynamic system, where the time between observations may, or may not enter. If the behavior is critical, then the observation matrix and noise covariance matrix play a role.

2. In the case of simple harmonic motion, we have proven asymptotic stability at the origin of the variance equation showing the mapping was contracting in two parts, the first affected solely by the transition matrix, and the second a function of the measurements and noise covariance.

3. Angle observations are equivalent to range observations with higher noise covariances.

4. The rate of decay of the error, which is of significance in orbit determination where not more than one period may be observed is very

much a function of the initial covariance matrix relative to the noise covariance, and the time between observations. In general, the larger the initial covariance matrix, and the greater the number of observations per period, the greater is the rate of decay of the errors and variances, i.e., the smaller is the ratio between final and initial quantities over one period.

5. For significant behavior of the variance equation at the origin which is unstable (e.g. our negatively damped model), one might expect other finite equilibrium points for the variance equation to exist and in fact, they are necessary for stability to be achieved.

As was also indicated in the Introduction, the investigation of the linear models studied here can only be considered the first part of the problem. The logical questions to be asked next are

a. Can anything be said about the stability of more general linear models applied to orbit determination, and

b. how does the filter behave when the dynamic system is nonlinear as it is in practical orbit determination problems.?

It is these questions which should be explored next.

SECTION VI

ACKNOWLEDGMENTS

The author would like to thank Mr. R. R. Ruggiero of the Applied Mathematics Section for his cooperation and very able assistance in the work on which this report is based. Mr. Ruggiero prepared and ran all of the computer simulation programs described in addition to assisting in some of the analysis.

SECTION VII

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SECTION VIII

GLOSSARY

A. Latin Symbols

A	constant matrix
b	positive constant
B	variance up-dating matrix
C	covariance matrix of noise u and v
c	cos Δ
E	expectation
F, G, f, g,	vector functions
f, g	positive, scalar elements of norm, N
h	vector function
H	nonlinear mapping
I	identity matrix
K, K_1	weighting function
k	integer representing time values ($\equiv t_k$)
m	integer representing time value ($\equiv t_m$)
M_i	measurement or observation matrix
N	positive definite diagonal norm matrix
P	covariance matrix of errors
p_i, p_i^*	variance vectors
R	covariance matrix of noise, v
r	observation range
r	transformed variance vector
s	sin Δ
t	time
T, T^{-1}	matrix of eigen-vectors and inverse

u, v	Gaussian white noise sequence vectors
U	covariance matrix of noise u
V	Liapunov function
x	n-dimensional state vector
$\hat{x}$	estimate of state vector; conditional mean
$\tilde{x}$	errors in estimate of state vectors

B. Greek Symbols

α	negative damping factor $(-1 < \alpha < 0)$
Γ	matrix function of time
δ_{km}	Kronecker delta
$\delta(\ )$	deviation
Δ	time between observations $(\equiv t_{k+1} - t_k)$
Δ	determinant appearing in identity measurement equations
ϵ_i	noise component of v
ζ, ζ^*	covariance matrix elements
η, η^*	covariance matrix elements
θ	angle of observation
θ	phase angle in motion with negative damping $(= \lambda \Delta)$
λ	$(1 - \alpha^2)^{\frac{1}{2}}$
λ, μ	eigen-values
Λ	diagonal matrix
ξ, ξ^*	covariance matrix elements
ρ	component of transformed vector r
$\bar{\rho}$	complex conjugate of ρ
σ_i^2	noise variance
Φ	transition matrix
ϕ	angle of observation
Ψ	matrix relating error states
ψ	components of vector r

C. Superscripts and Subscripts

a	actual
c	computed
e	equilibrium
i	pertaining to i^{th} measurement matrix
k	integer index of time
m	measured
r	reference

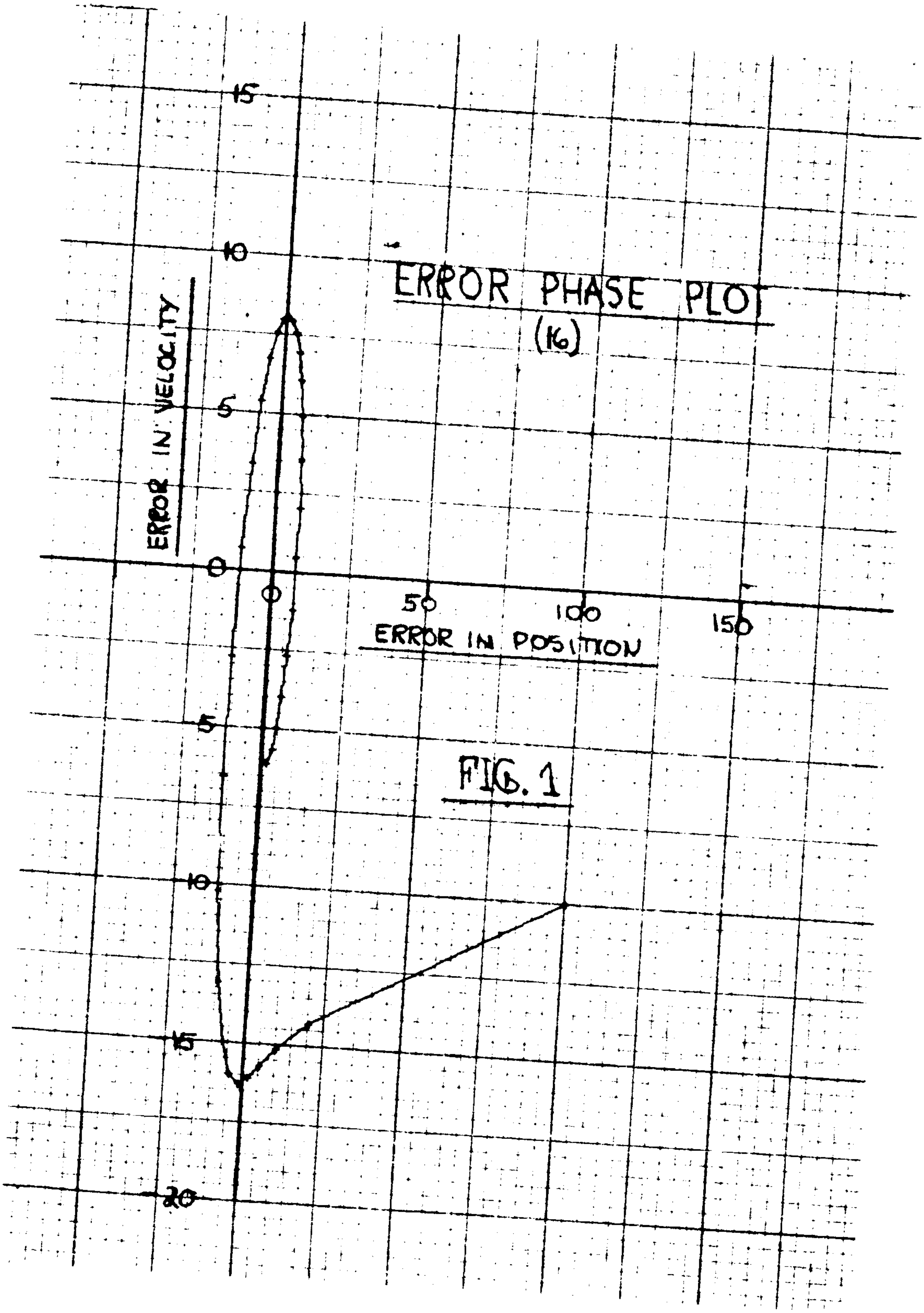
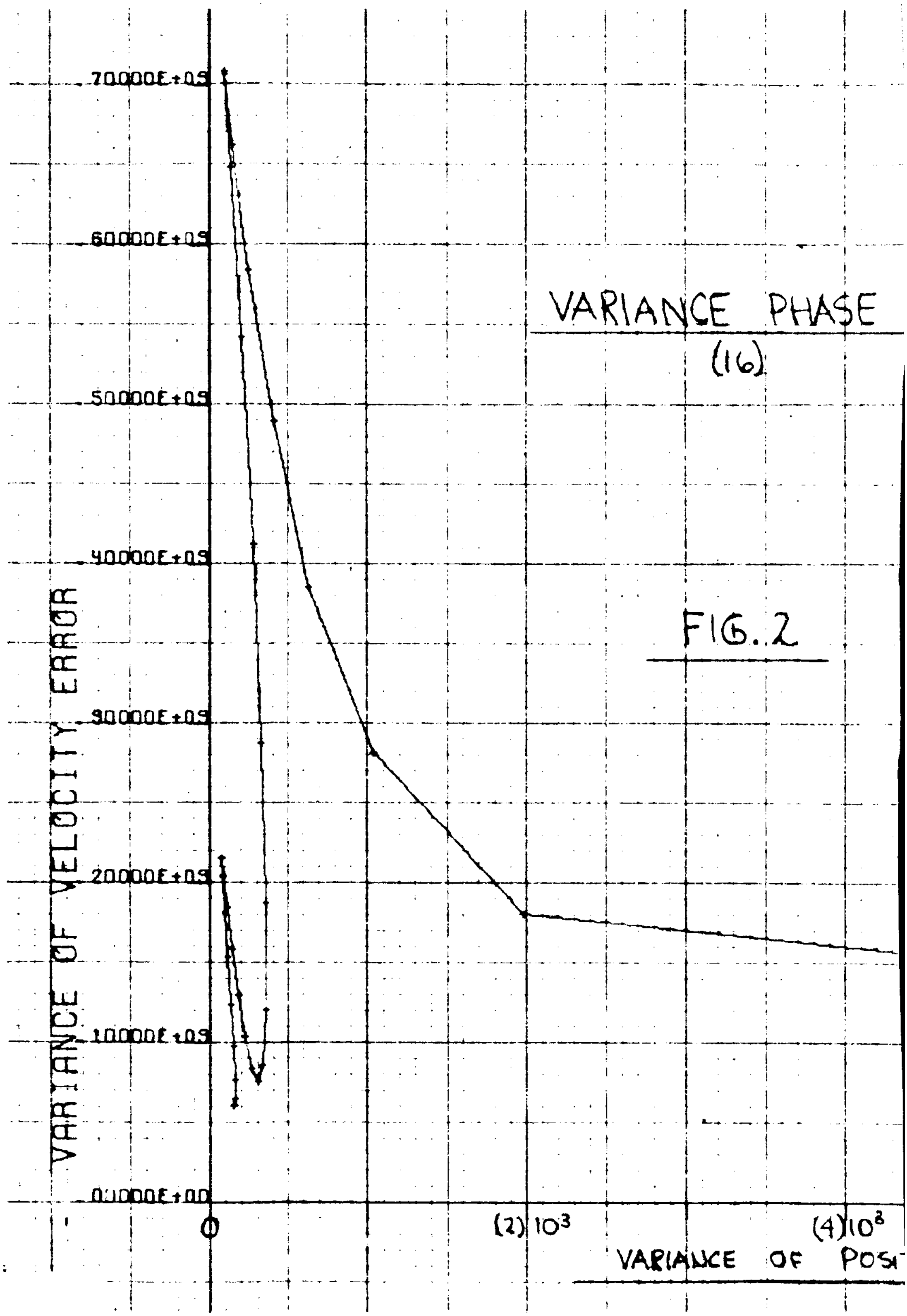
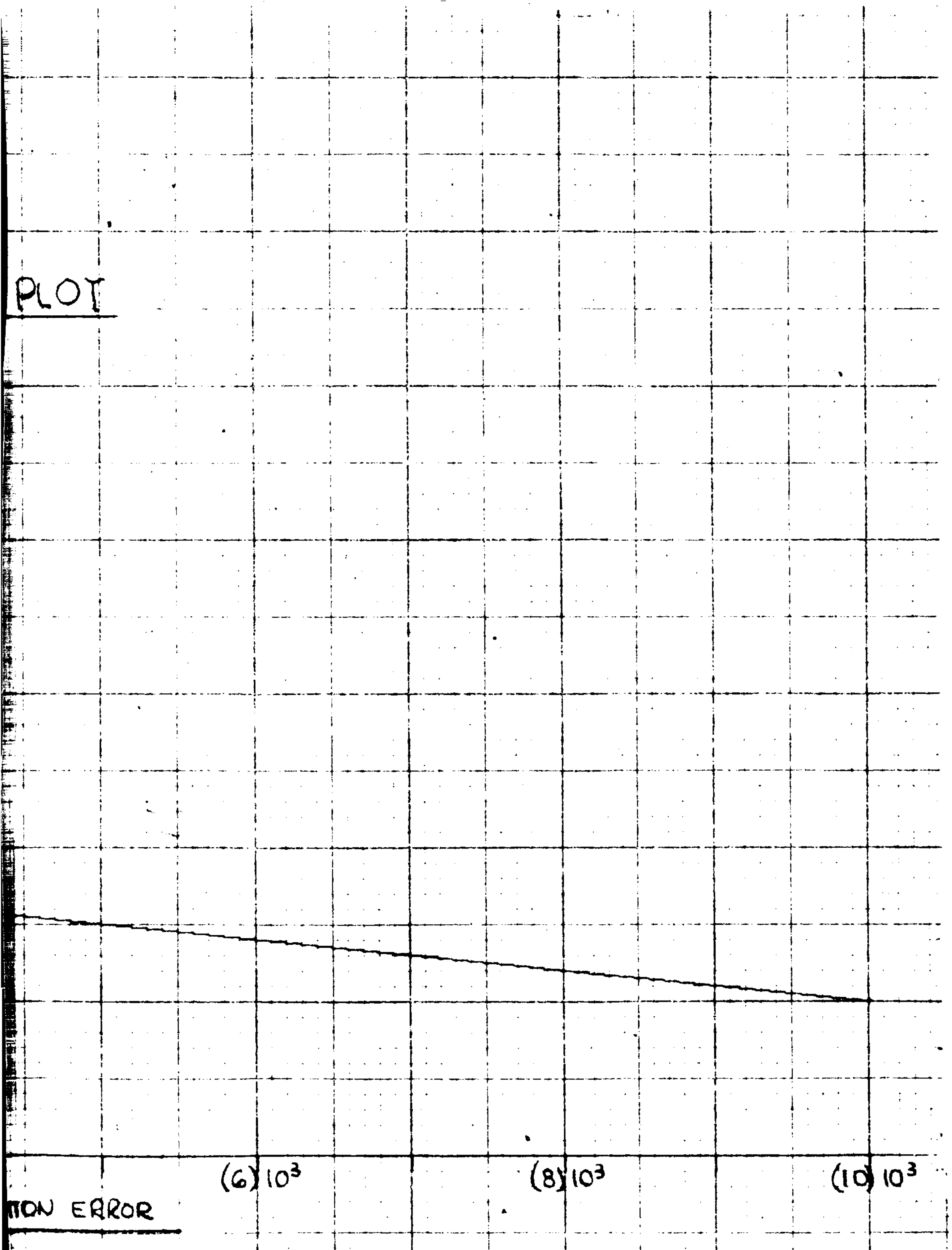

ERROR PHASE PLOT
(16)
ERROR IN VELOCITY
ERROR IN POSITION
15
10
5
0
5
10
15
20
0
50
100
150

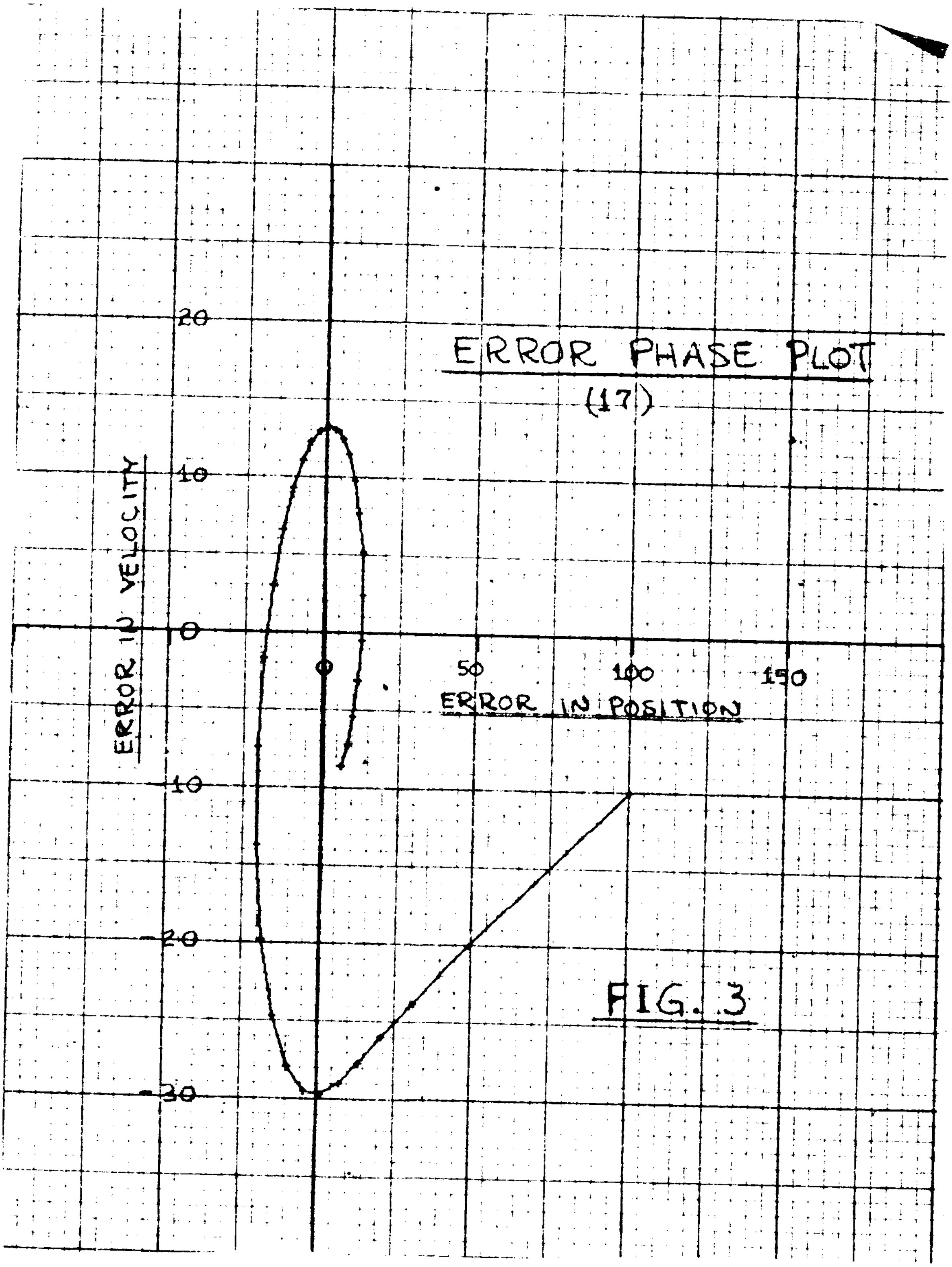
FIG. 1



VARIANCE PHASE
(16)
.70000E+05
.60000E+05
.50000E+05
.40000E+05
.30000E+05
.20000E+05
.10000E+05
0.0000E+00
VARIANCE OF VELOCITY ERROR
0
(2) 10³
(4)10³
VARIANCE OF POSI

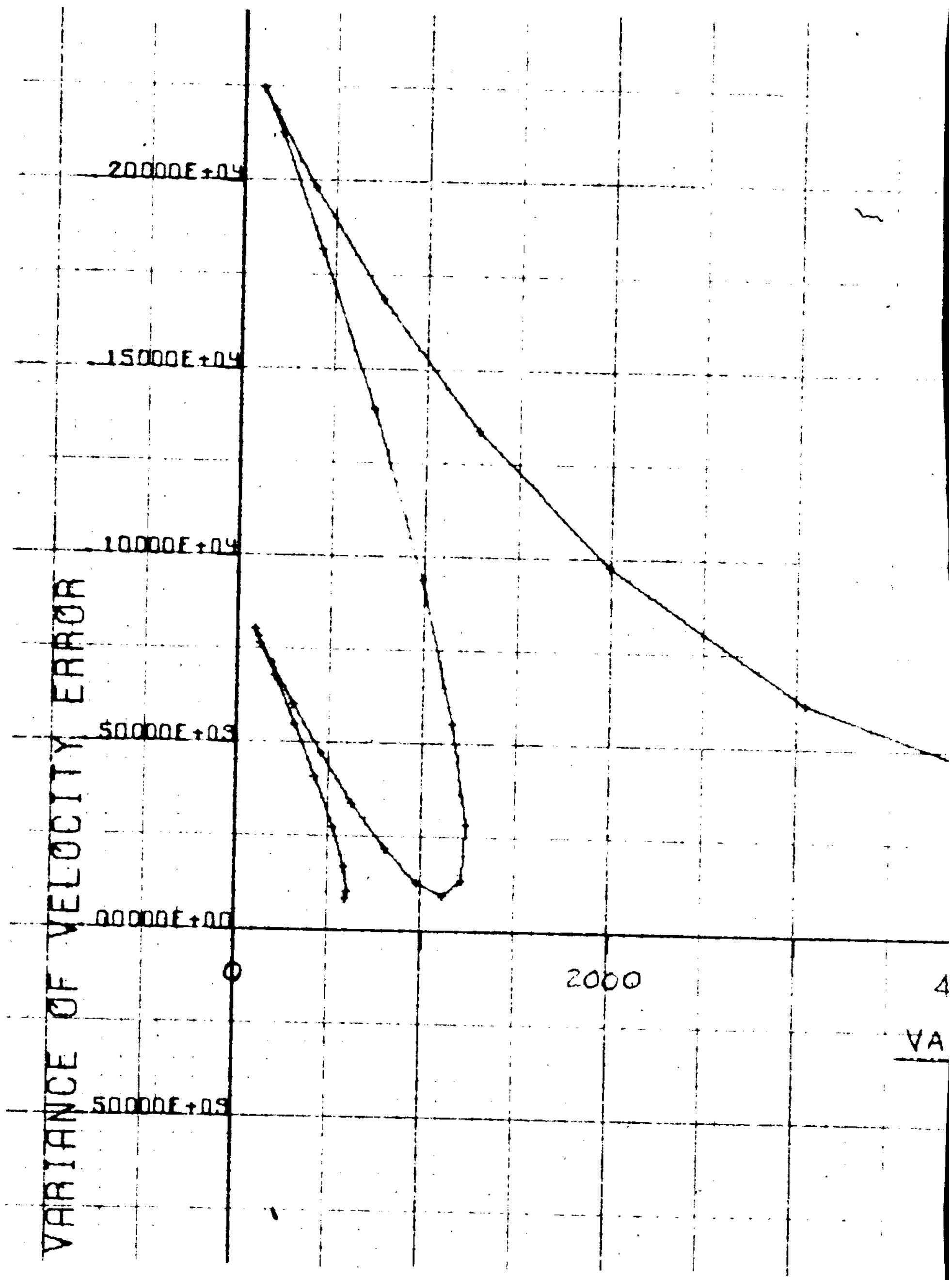
FIG. 2


PLOT
(6) 10^3
(8) 10^3
(10) 10^3
TION ERROR



ERROR PHASE PLOT
(17)
ERROR IN VELOCITY
ERROR IN POSITION
20
10
0
-10
-20
-30
50
100
150

FIG. 3


VARIANCE OF VELOCITY ERROR
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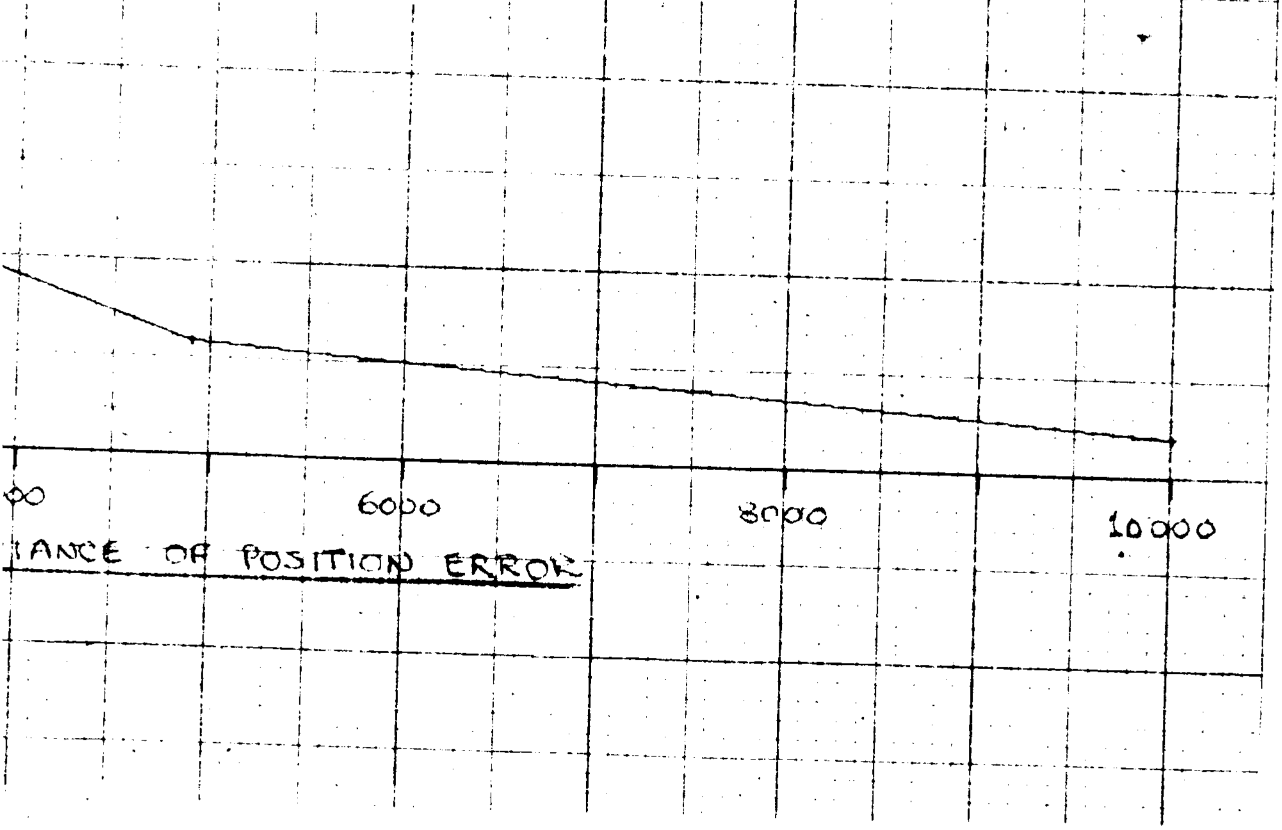
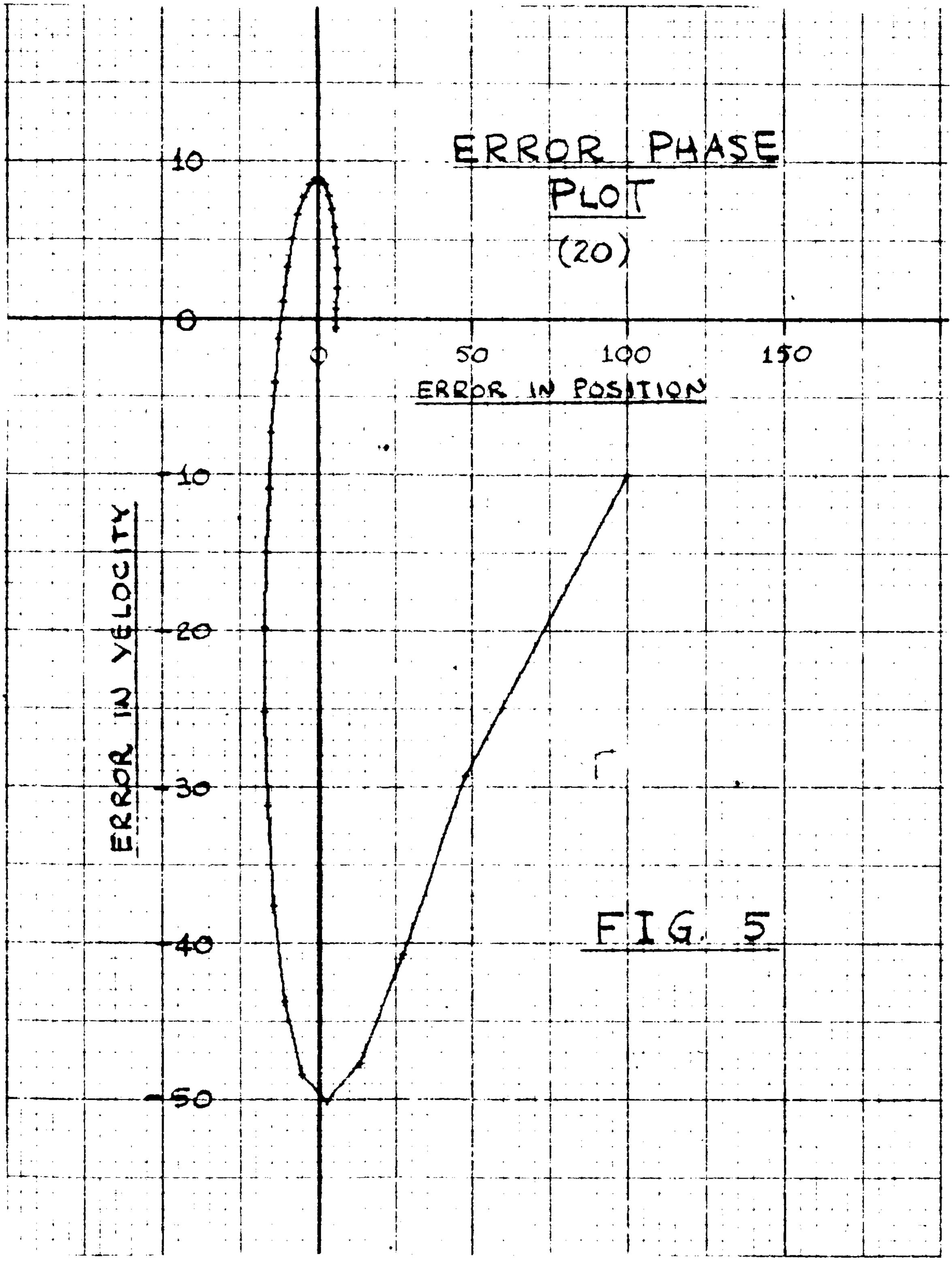
10000E+04
50000E+03
00000E+00
50000E+03
0
2000

VARIANCE PHASE PLOT
(17)

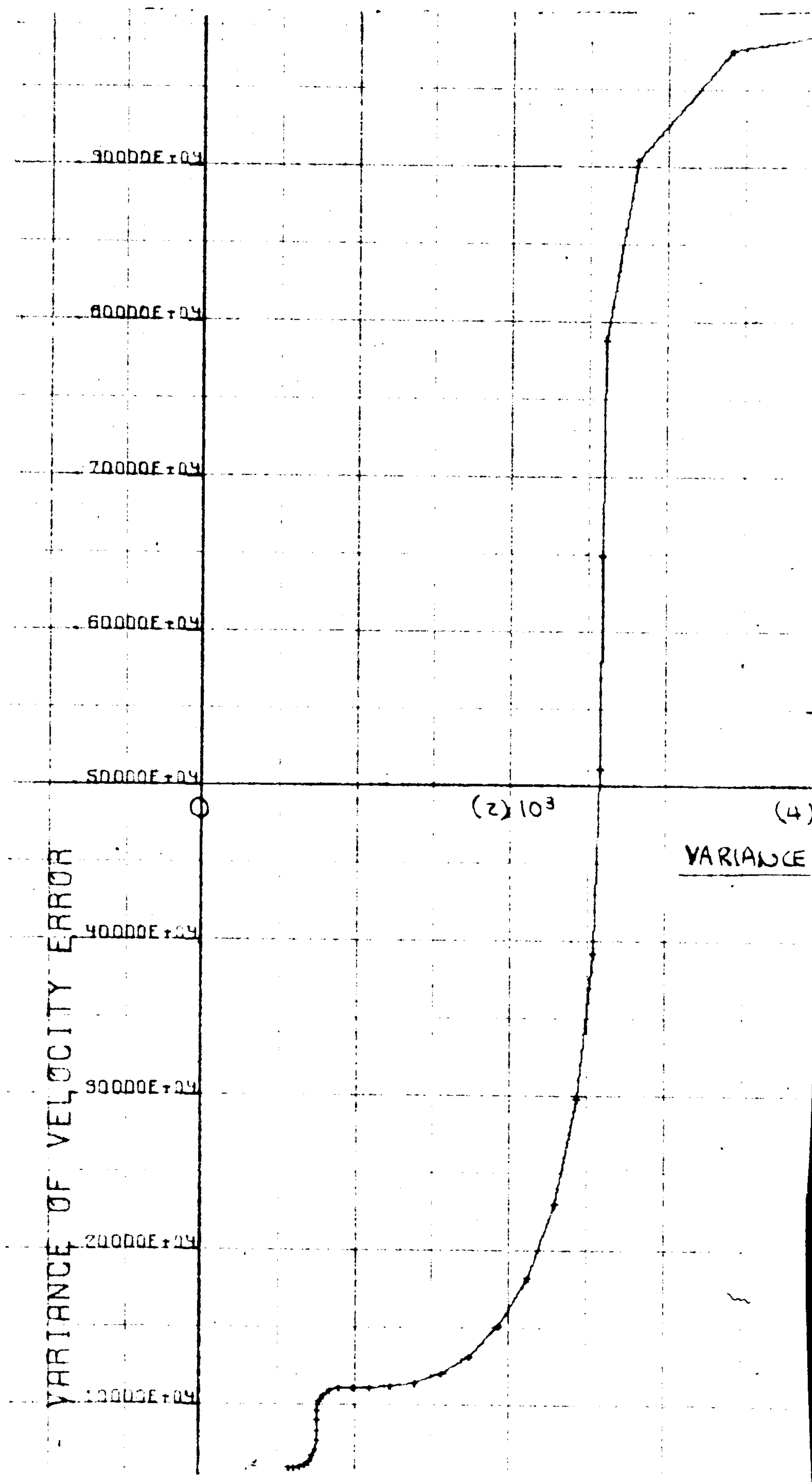
FIG. 4



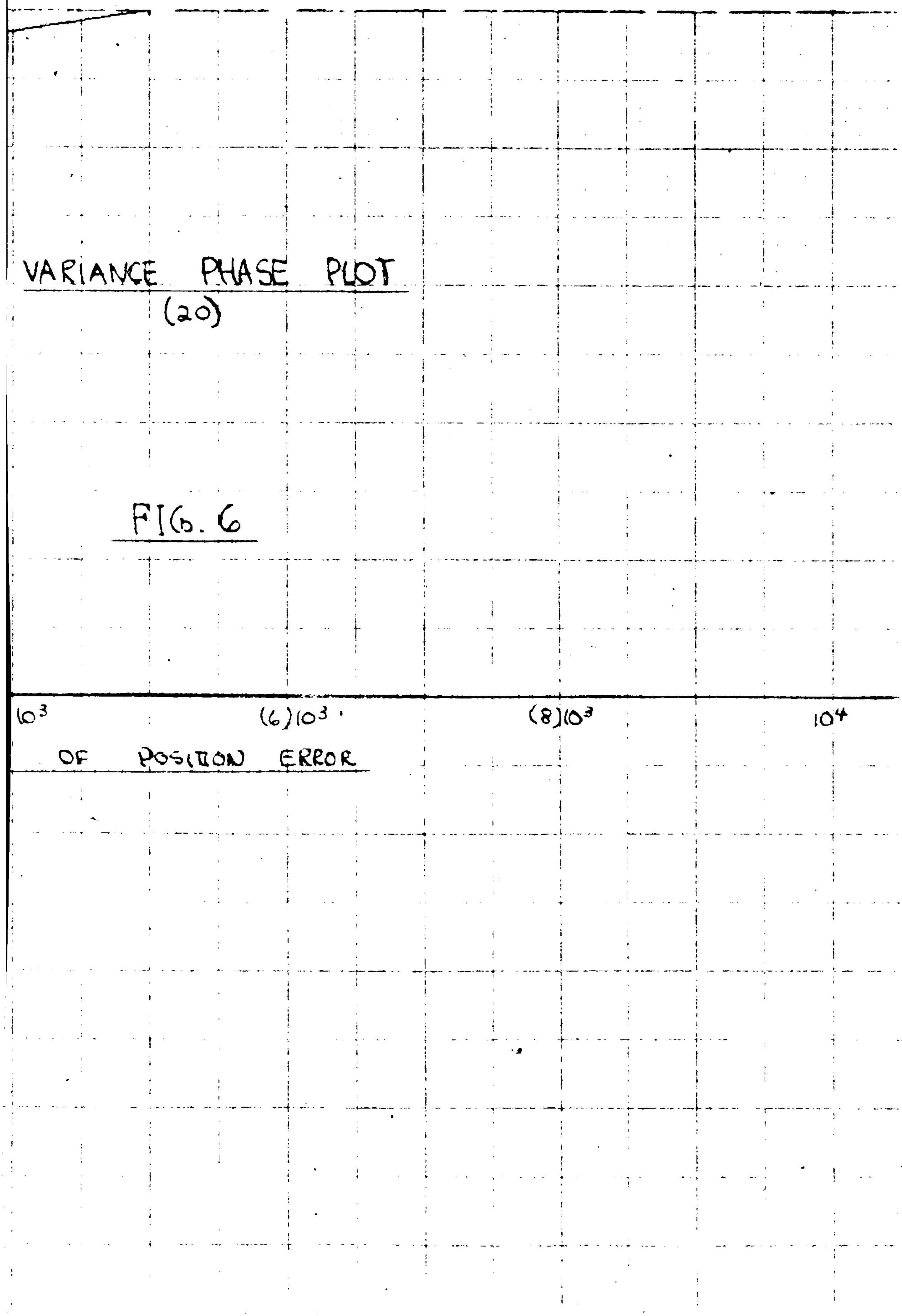


ERROR PHASE PLOT
(20)
10
0
-10
-20
-30
-40
-50
0
50
100
150
ERROR IN POSITION
ERROR IN VELOCITY

FIG. 5



.90000E+04
.80000E+04
.70000E+04
.60000E+04
.50000E+04
.40000E+04
.30000E+04
.20000E+04
.10000E+04
VARIANCE OF VELOCITY ERROR
0
(2) 10³
(4)
VARIANCE



VARIANCE PHASE PLOT
(20)

FIG. 6

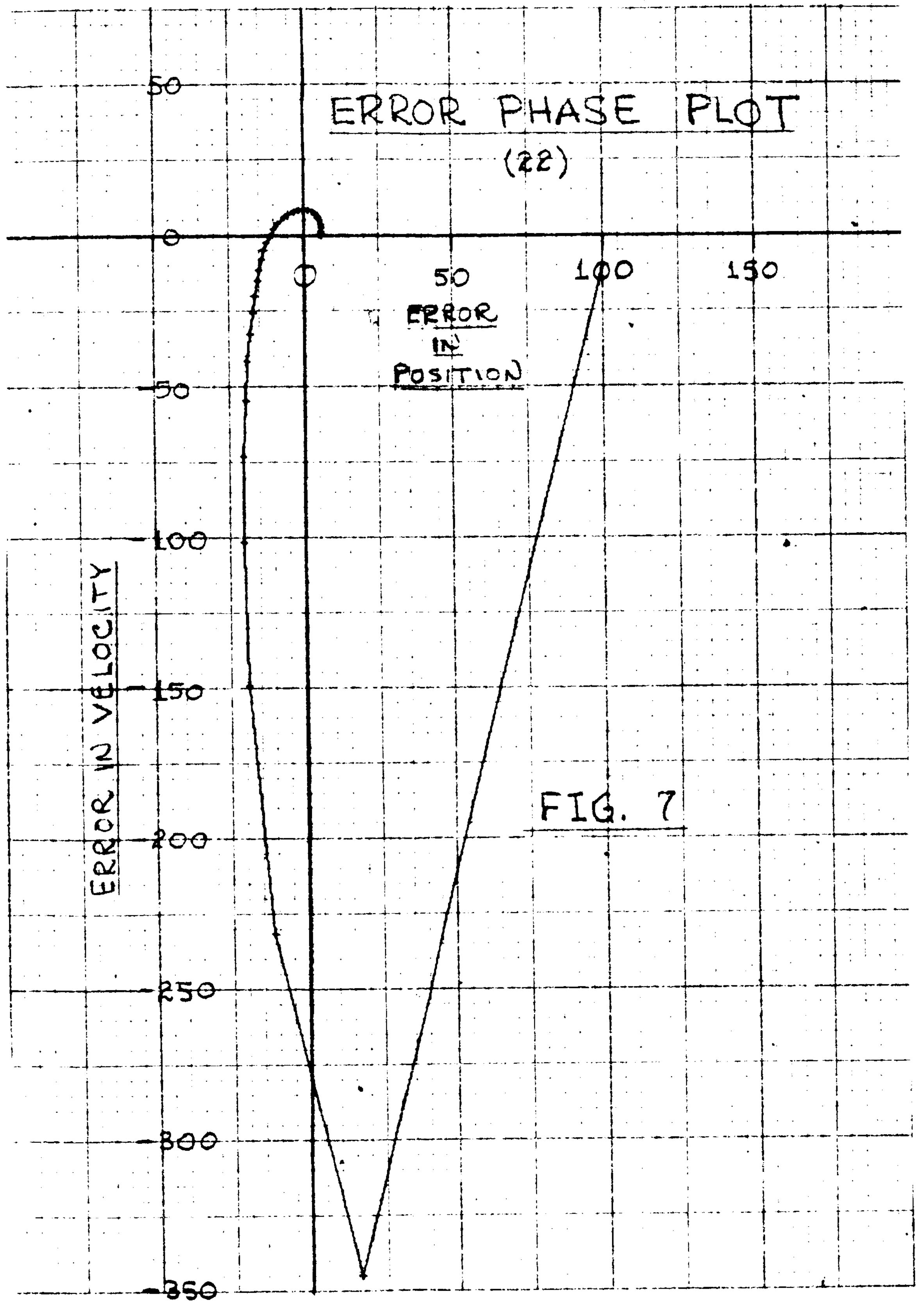

ERROR PHASE PLOT
(22)
50
0
-50
-100
-150
-200
-250
-300
-350
0
50
100
150
ERROR IN POSITION
ERROR IN VELOCITY

FIG. 7

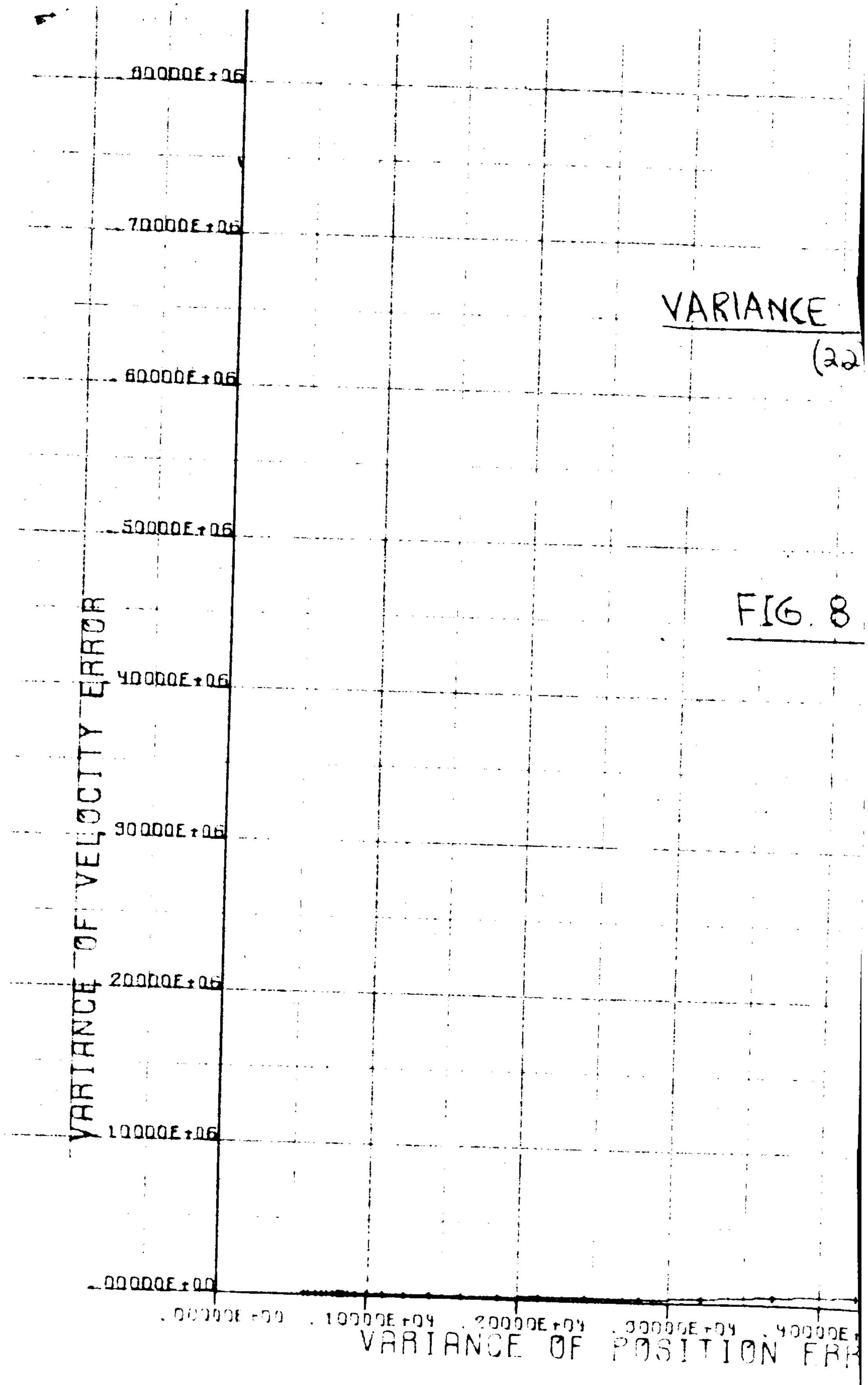
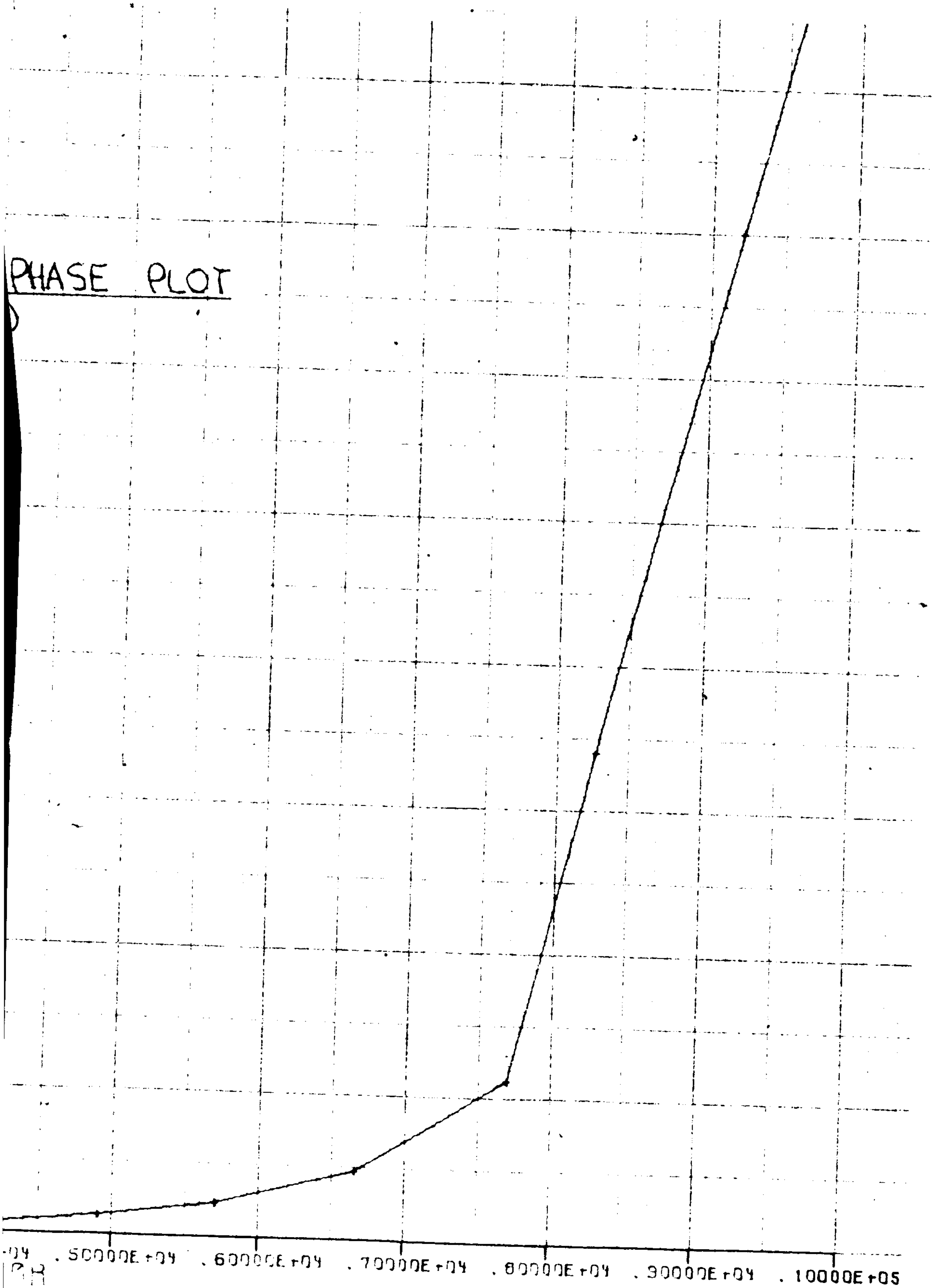

VARIANCE
(22
.80000E+06
.70000E+06
.60000E+06
.50000E+06
.40000E+06
.30000E+06
.20000E+06
.10000E+06
.00000E+00
VARIANCE OF VELOCITY ERROR
.00000E+00
.10000E+04
.20000E+04
.30000E+04
.40000E
VARIANCE OF POSITION ERR

FIG. 8



PHASE PLOT
.50000E+04
.60000E+04
.70000E+04
.80000E+04
.90000E+04
.10000E+05

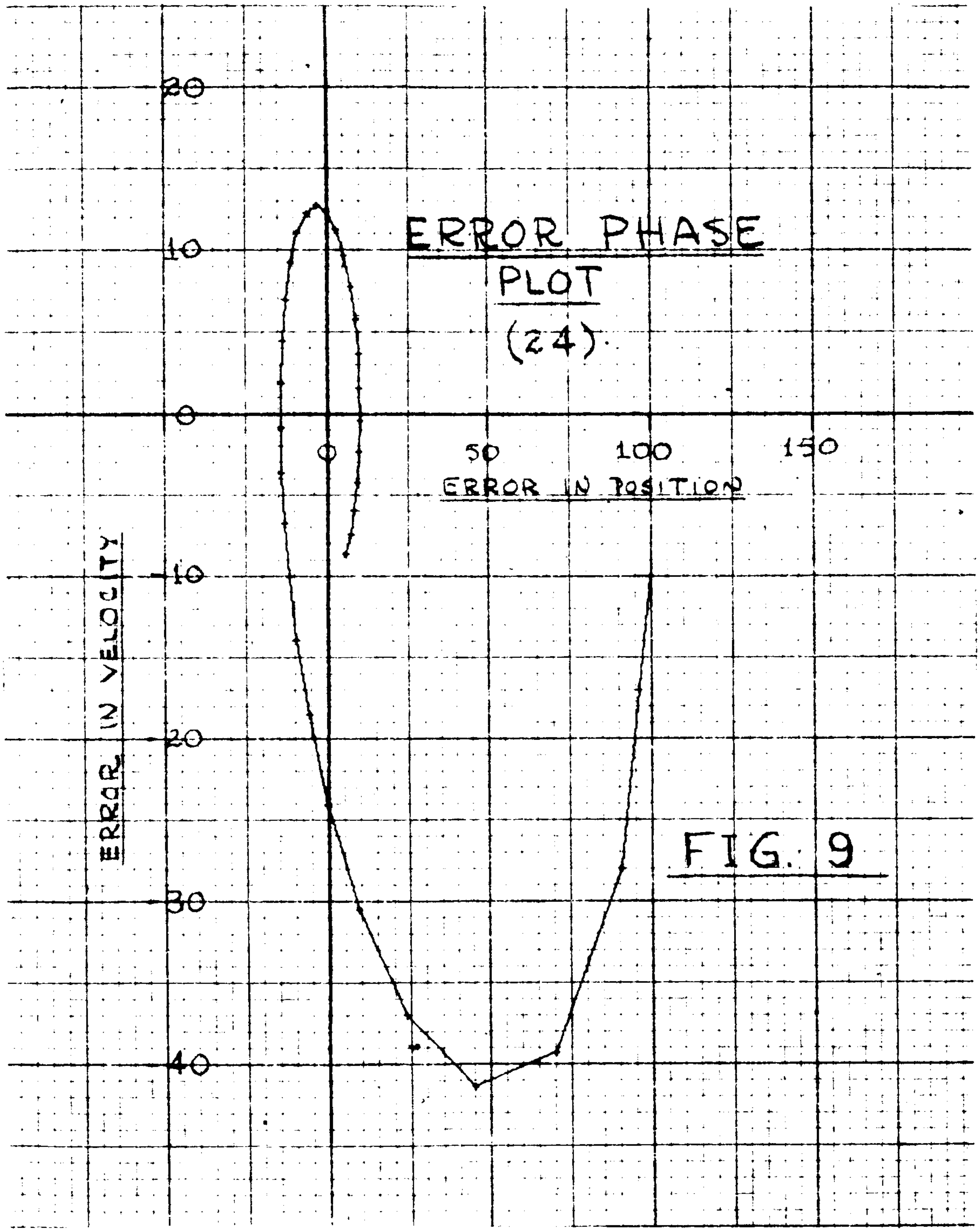
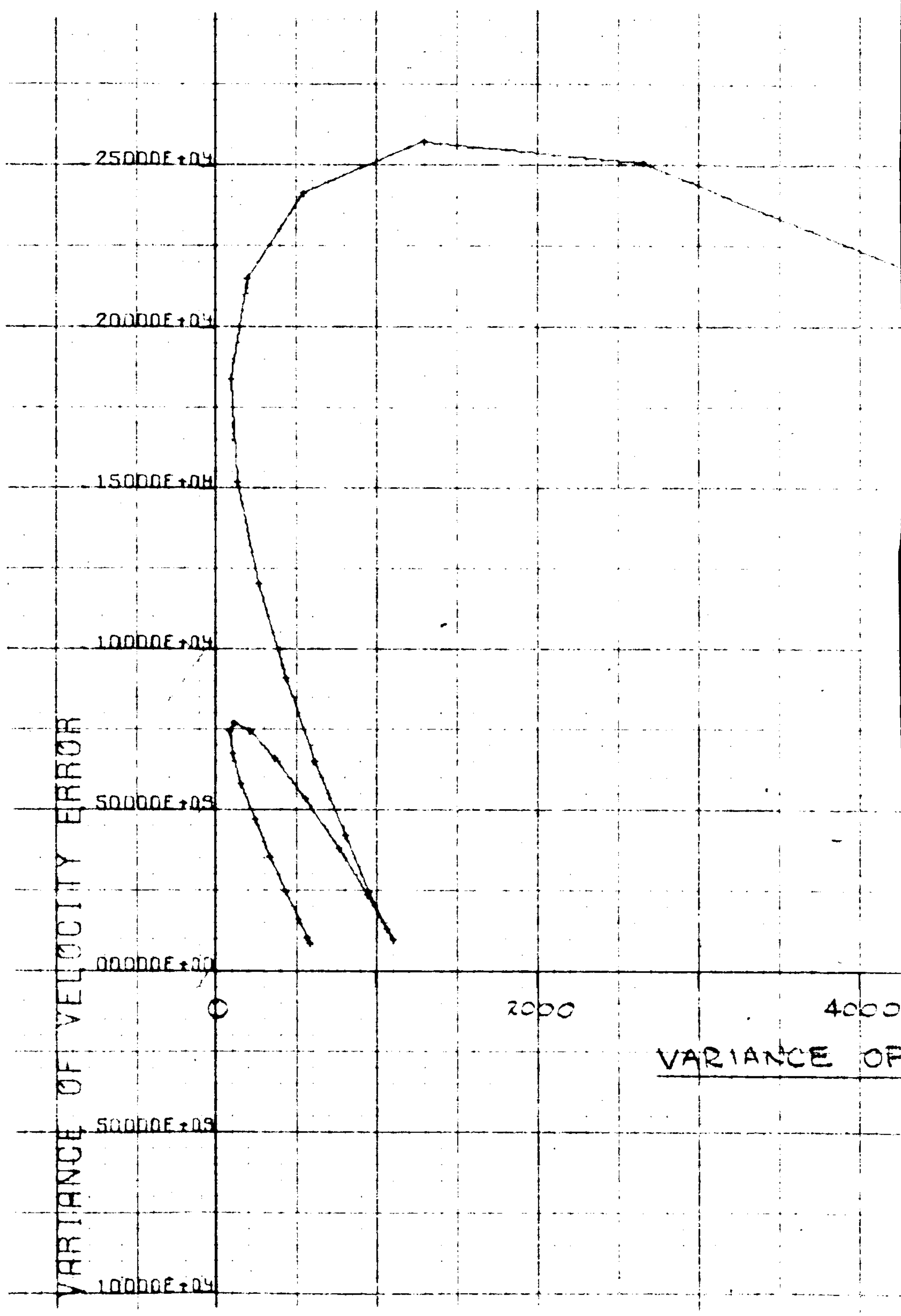

ERROR PHASE PLOT
(24)
20
10
0
-10
-20
-30
-40
0
50
100
150
ERROR IN POSITION
ERROR IN VELOCITY

FIG. 9



.25000E+04
.20000E+04
.15000E+04
.10000E+04
.50000E+03
.00000E+00
-.50000E+03
-.10000E+04
VARIANCE OF VELOCITY ERROR
0
2000
4000
VARIANCE OF

VARIANCE PHASE PLOT
(24)

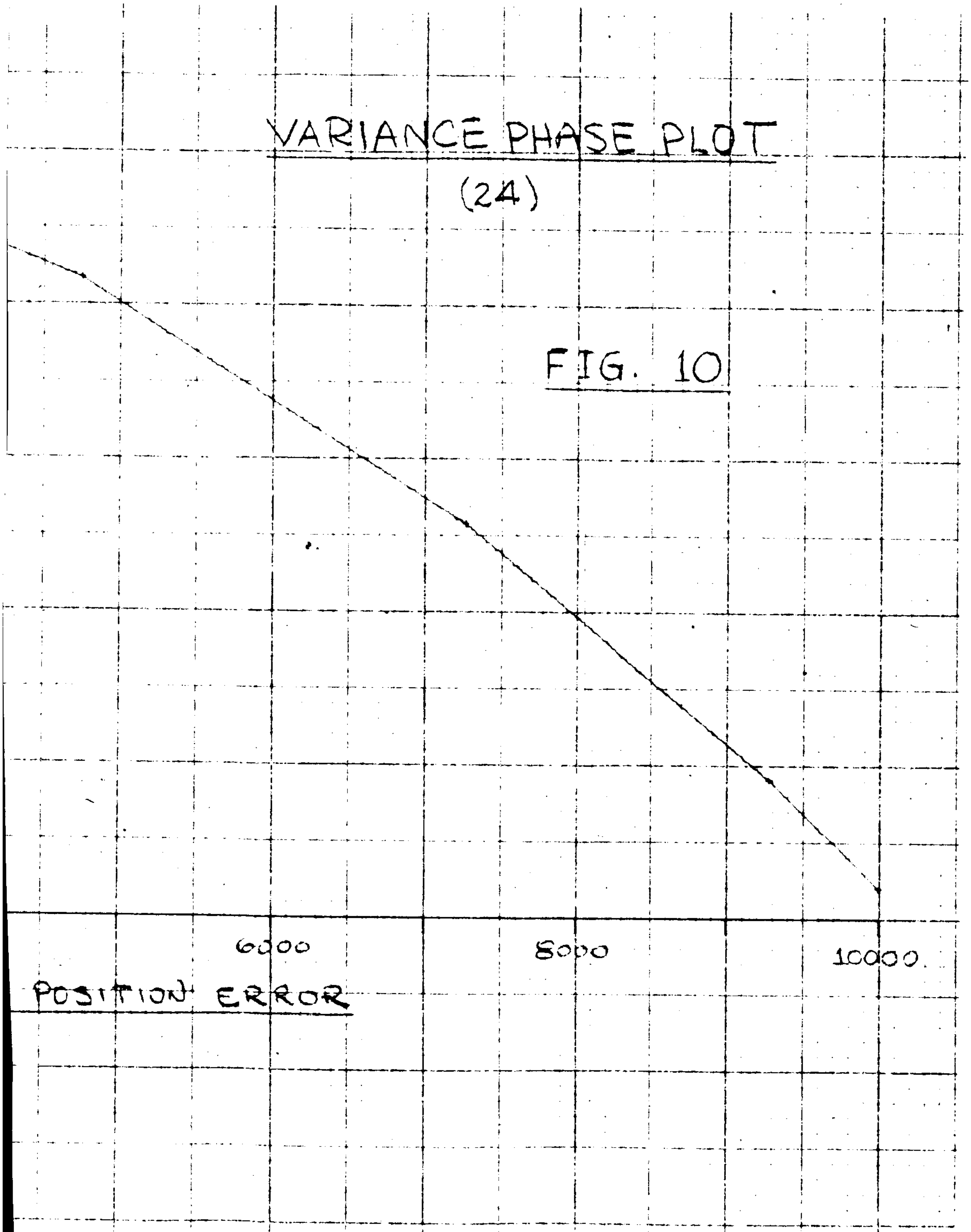


FIG. 10

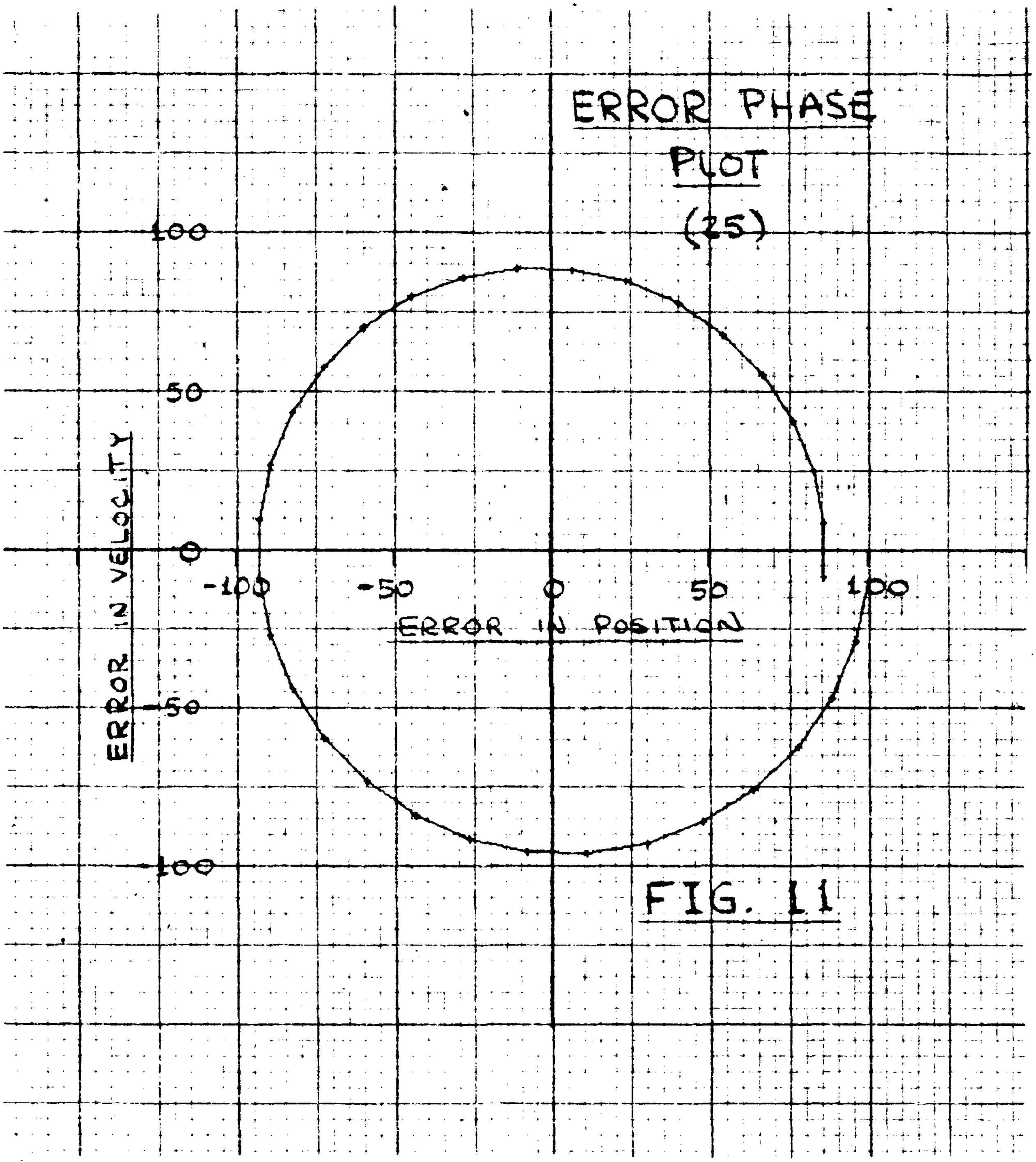

ERROR PHASE PLOT
(25)
100
50
0
-50
-100
-100
-50
0
50
100
ERROR IN POSITION
ERROR IN VELOCITY

FIG. 11

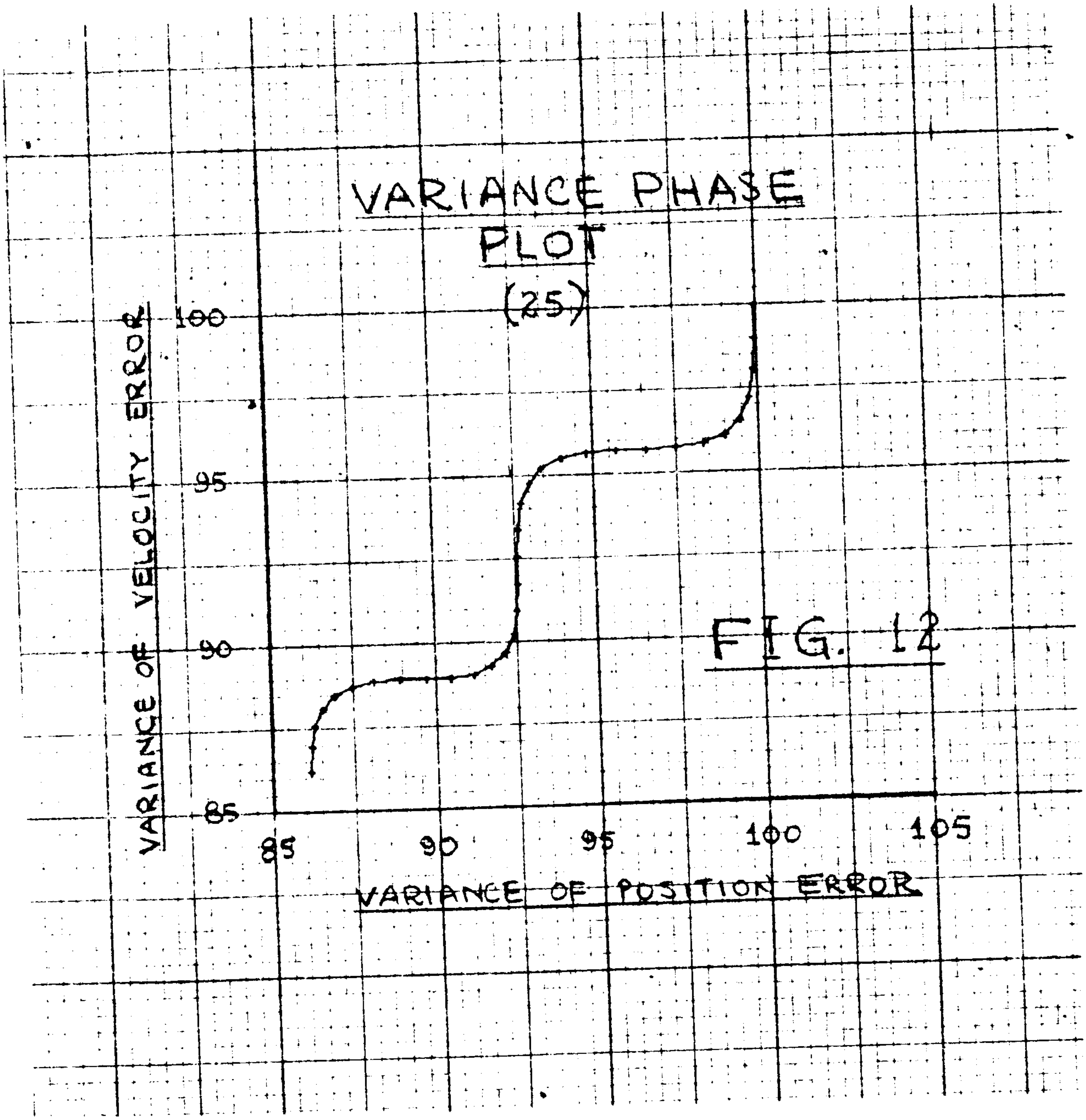

VARIANCE PHASE PLOT
(25)
VARIANCE OF VELOCITY ERROR
100
95
90
85
85
90
95
100
105
VARIANCE OF POSITION ERROR

FIG. 12

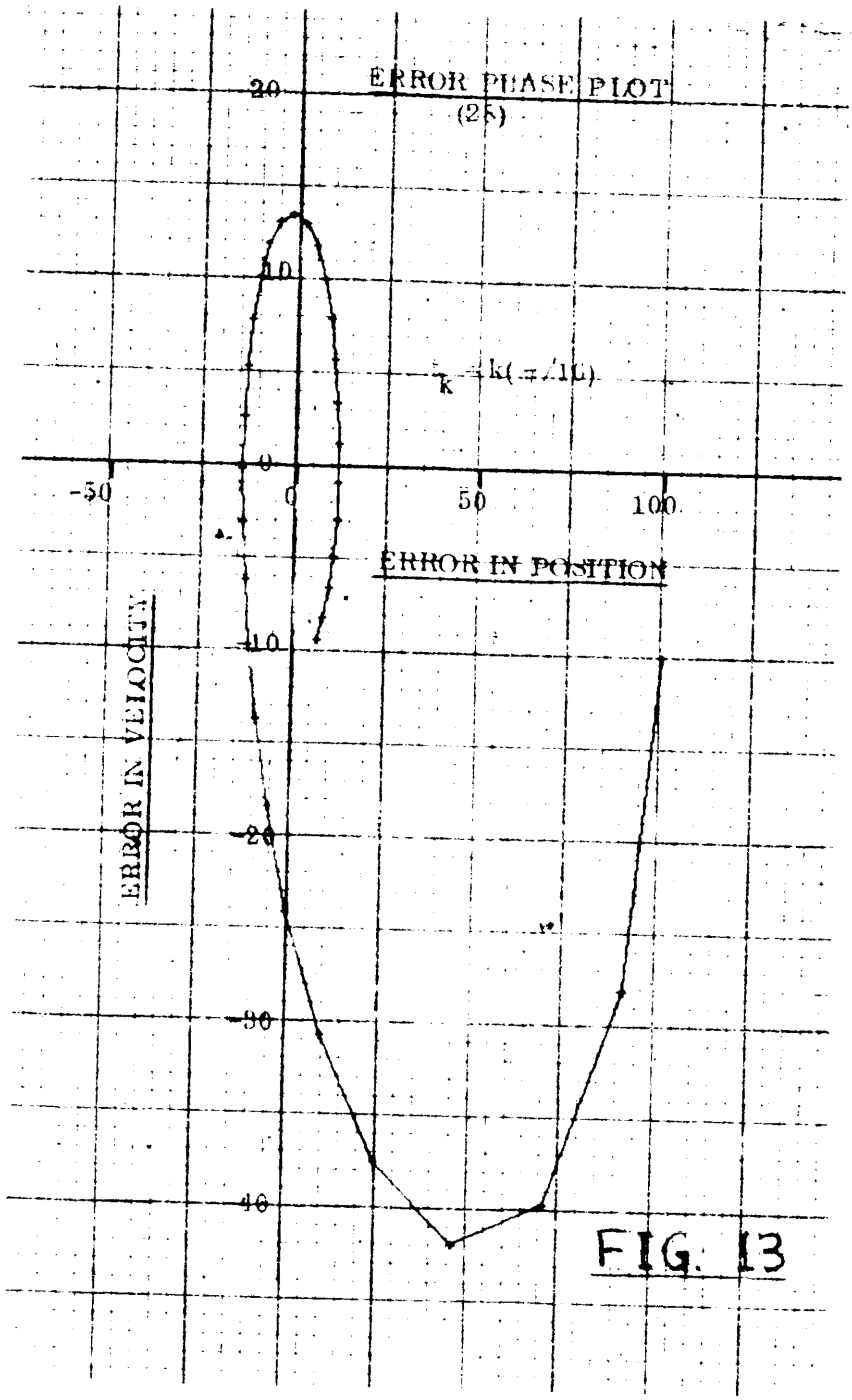
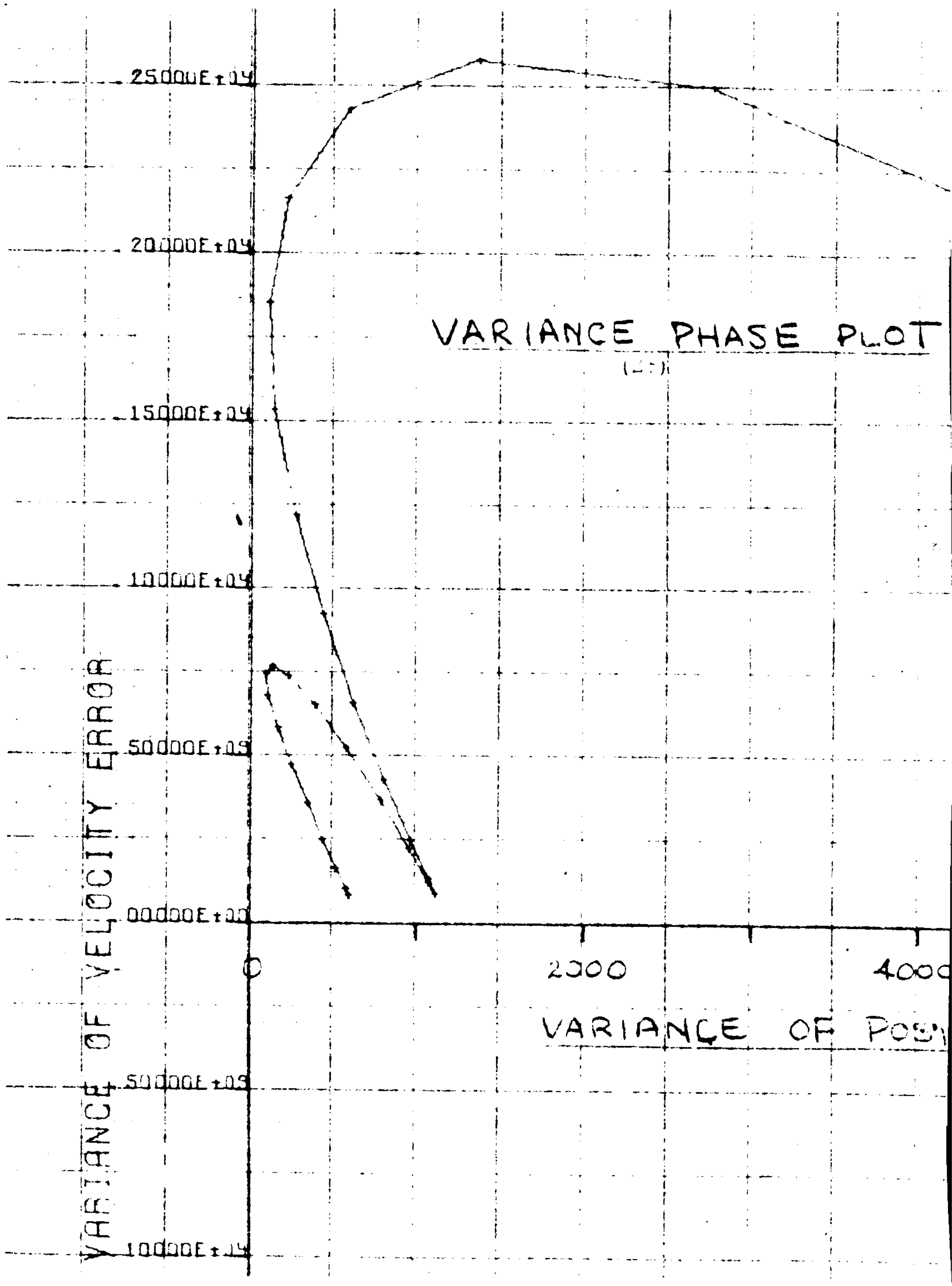

ERROR PHASE PLOT
(25)
ERROR IN VELOCITY
ERROR IN POSITION
20
10
0
-10
-20
-30
-40
-50
0
50
100

FIG. 13


VARIANCE PHASE PLOT
.25000E+04
.20000E+04
.15000E+04
.10000E+04
.50000E+03
.00000E+00
-.50000E+03
-.10000E+04
0
2000
4000
VARIANCE OF VELOCITY ERROR
VARIANCE OF POS

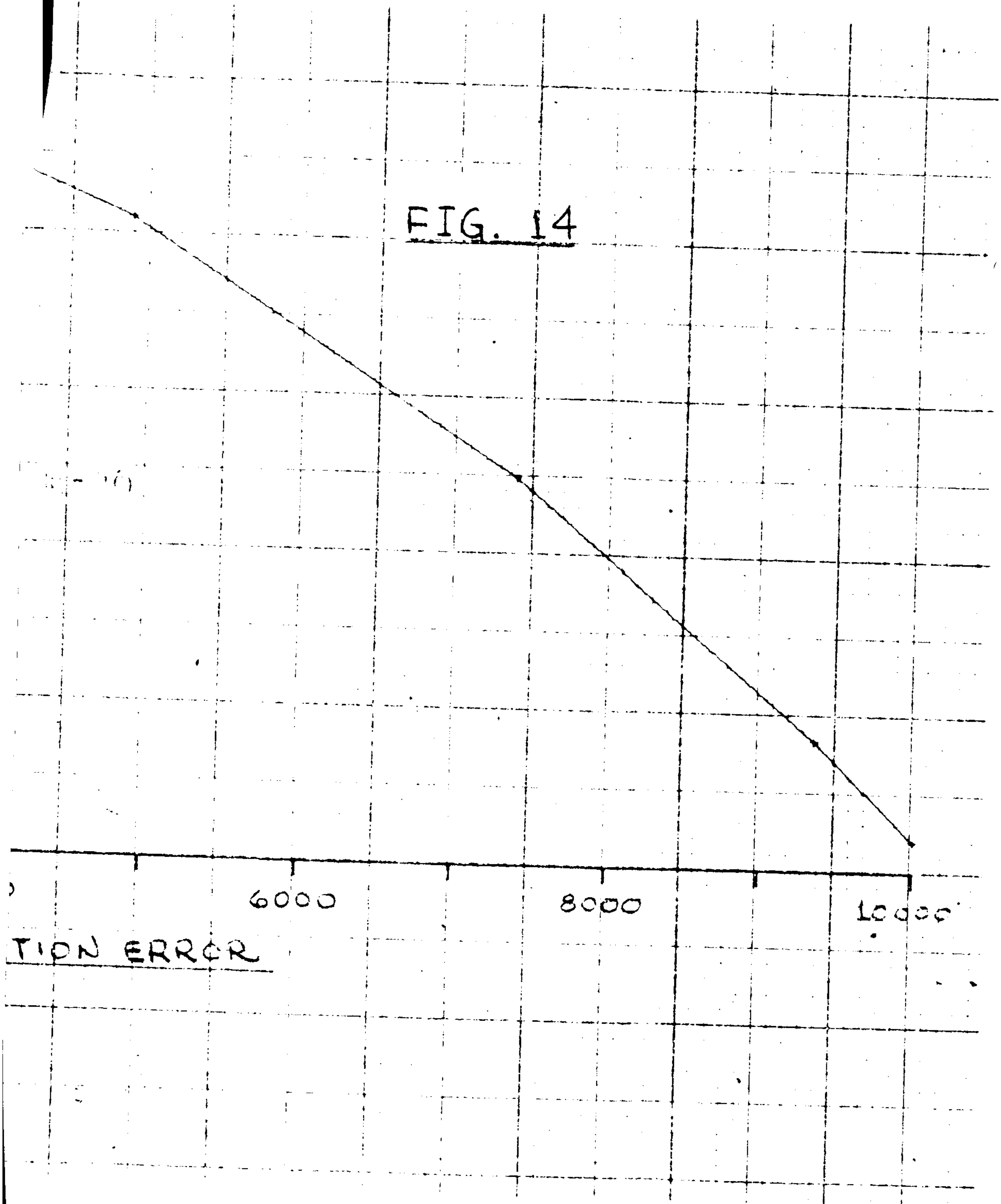


FIG. 14

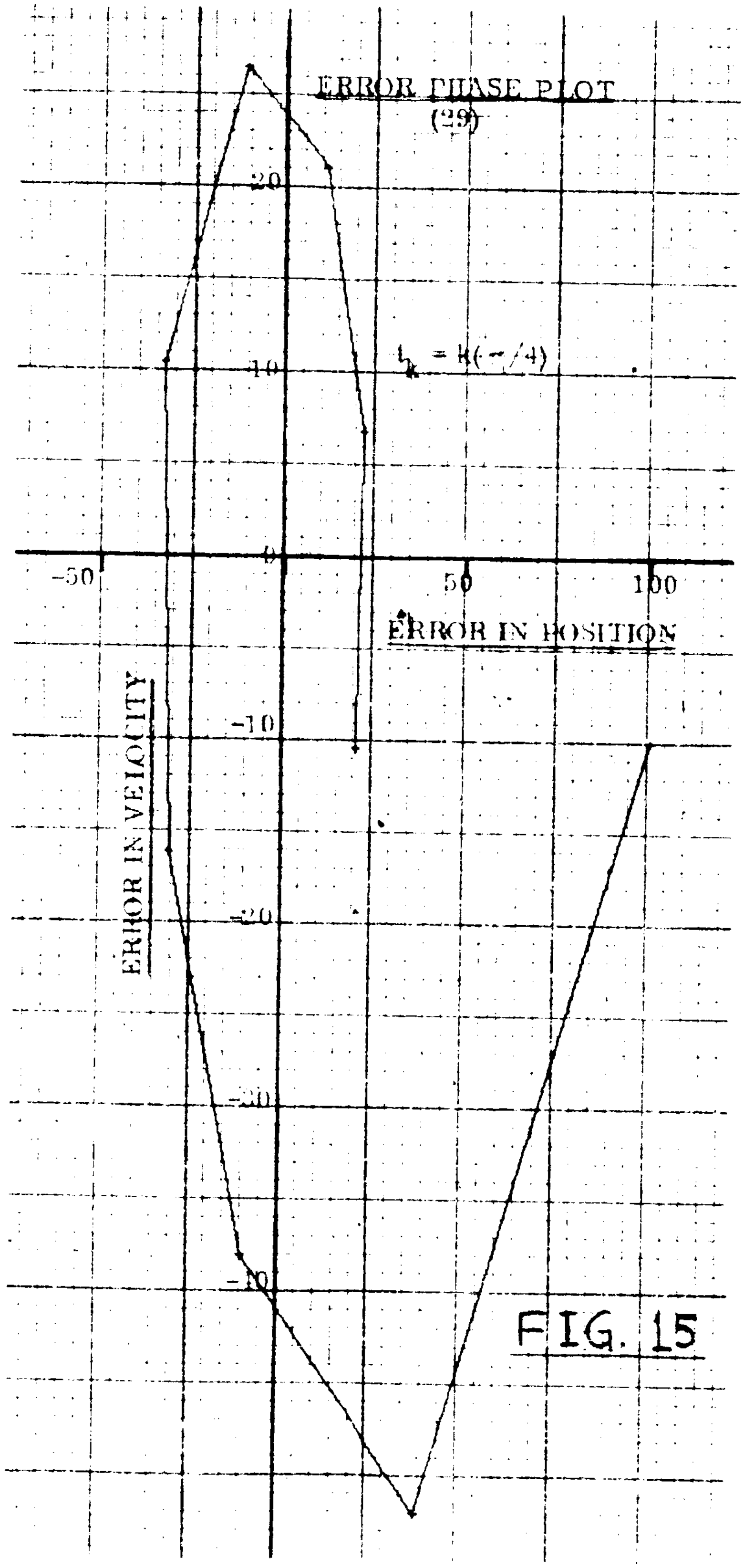
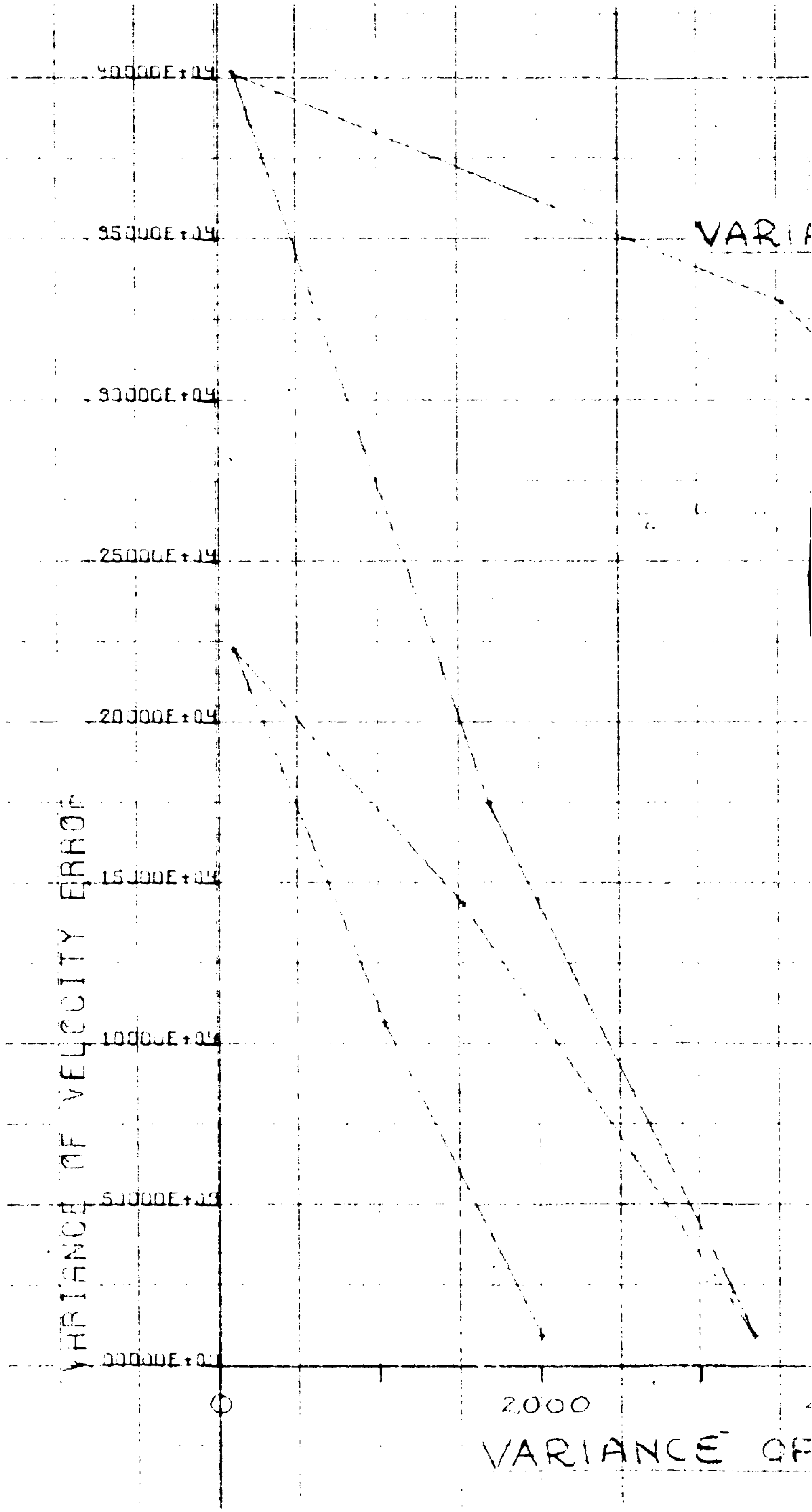

ERROR PHASE PLOT
(29)
20
10
0
-50
50
100
ERROR IN POSITION
ERROR IN VELOCITY
-10
-20

FIG. 15



.40000E+04
.35000E+04
.30000E+04
.25000E+04
.20000E+04
.15000E+04
.10000E+04
.50000E+03
.00000E+01
VARIANCE OF VELOCITY ERROR
0
2,000
VARIANCE OF
VARI

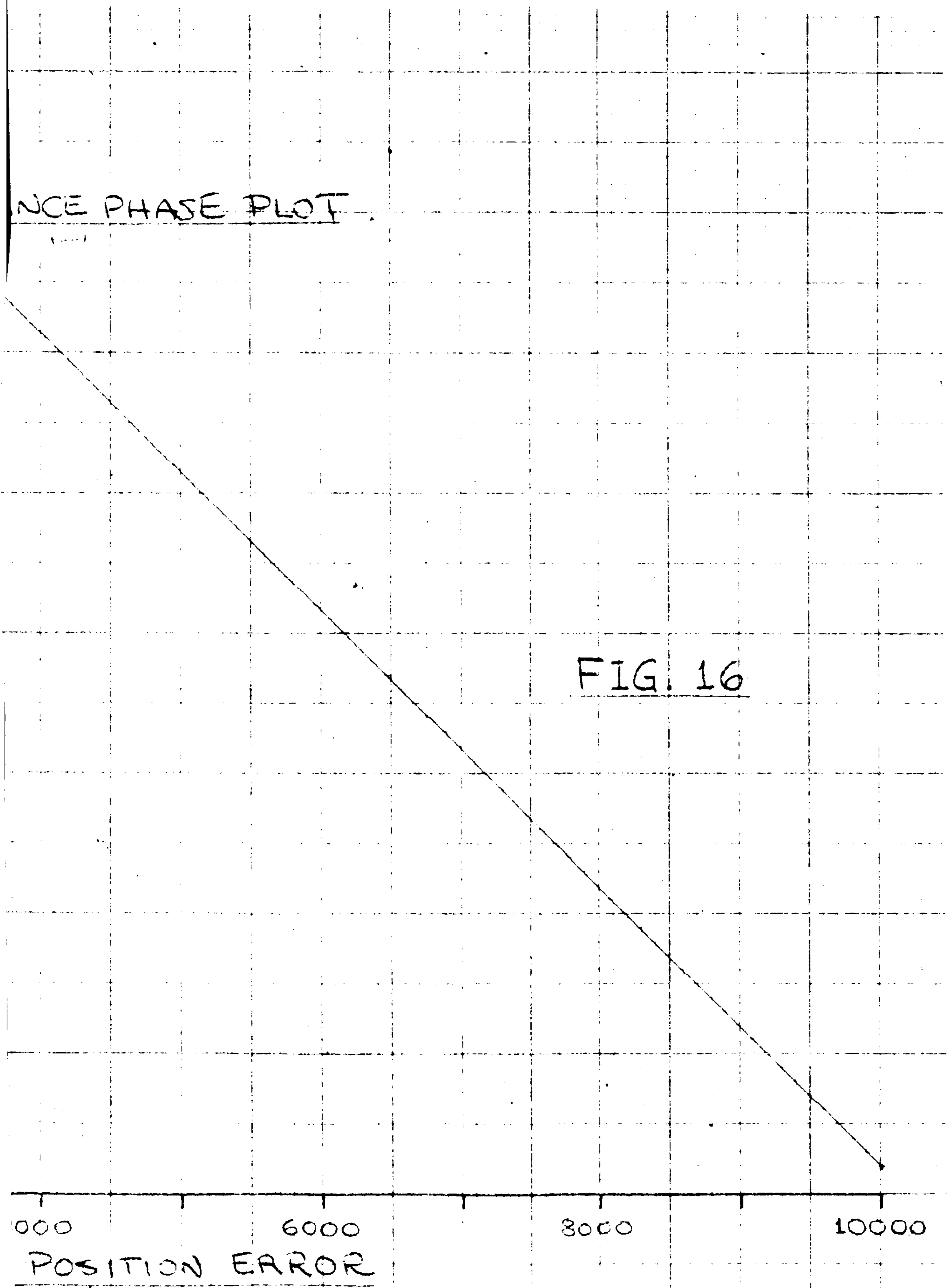
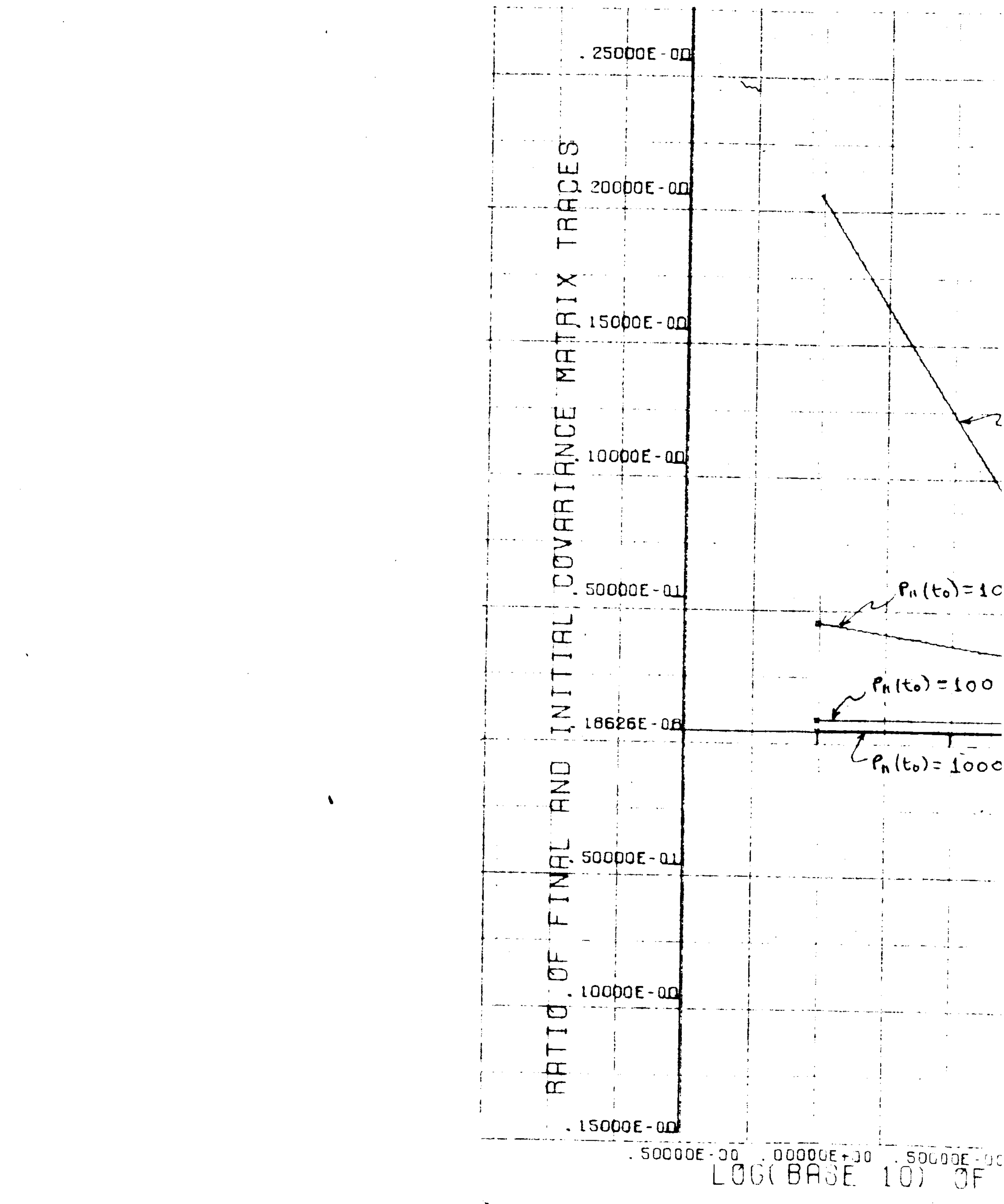
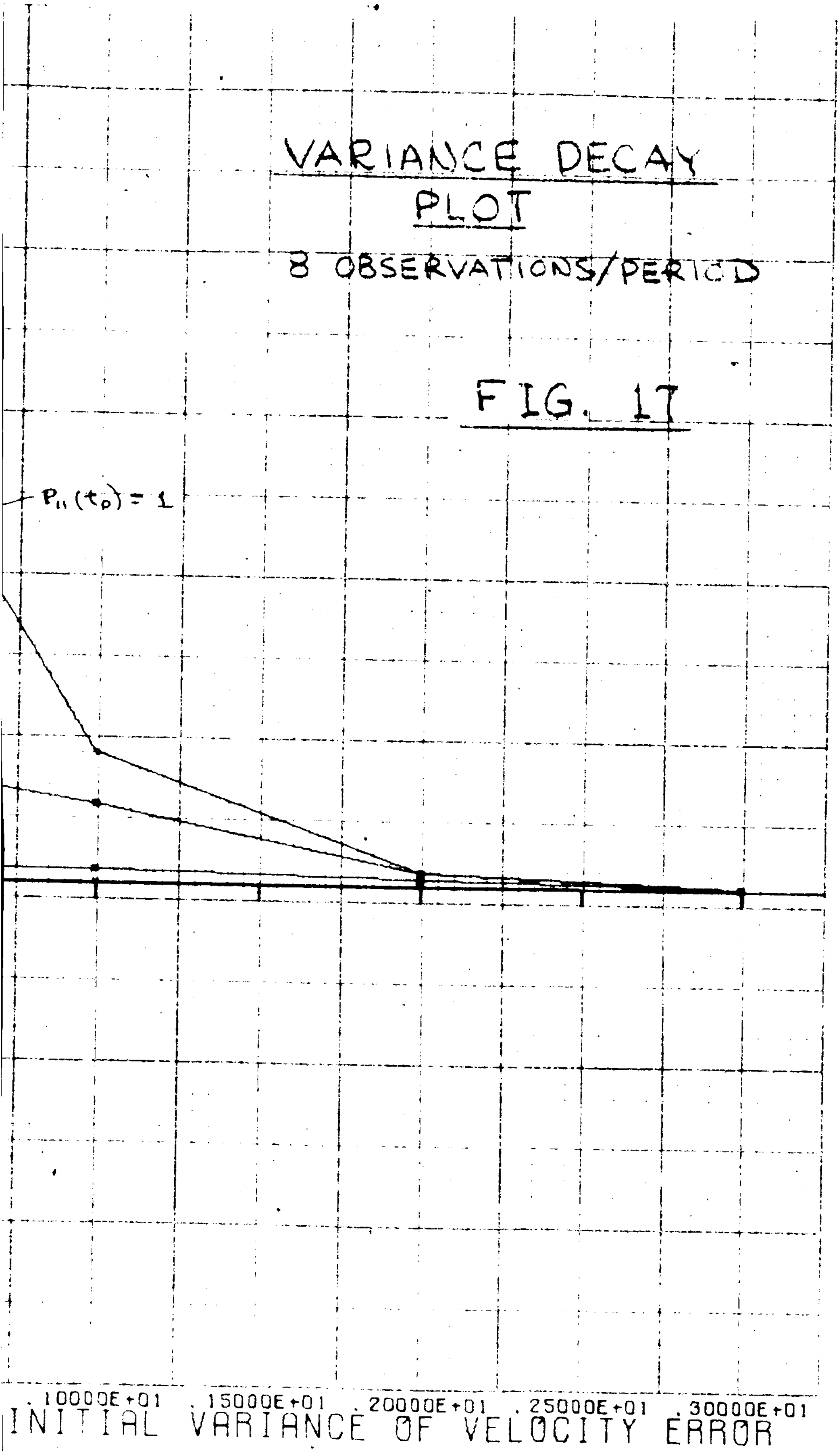


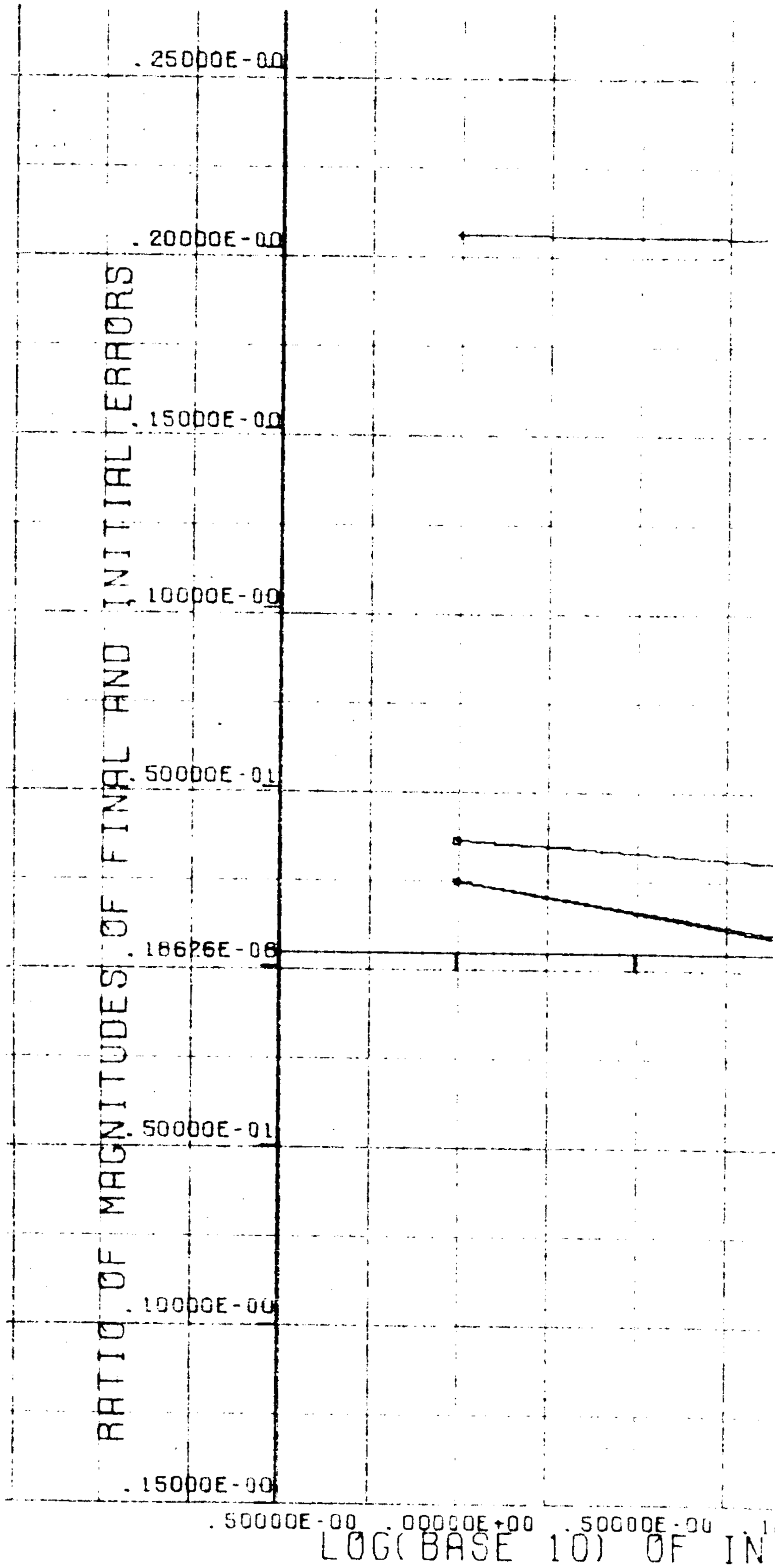
FIG. 16

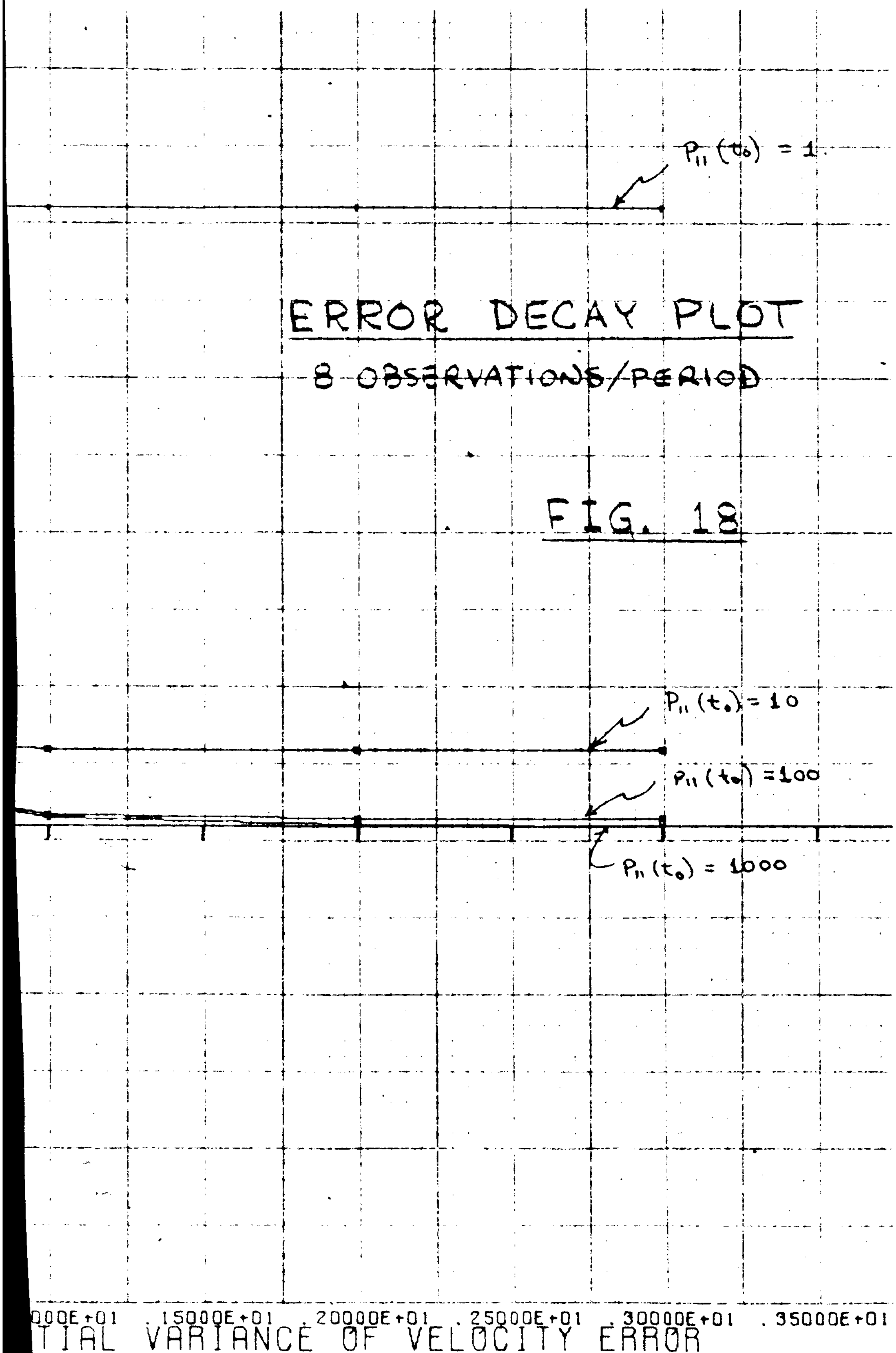

RATIO OF FINAL AND INITIAL COVARIANCE MATRIX TRACES
.25000E-00
.20000E-00
.15000E-00
.10000E-00
.50000E-01
.18626E-08
.50000E-01
.10000E-00
.15000E-00
$P_n(t_0) = 100$
.50000E-00
.00000E+00
LOG(BASE 10) OF



VARIANCE DECAY PLOT
8 OBSERVATIONS/PERIOD
$P_{11}(t_0) = 1$
.10000E+01
.15000E+01
.20000E+01
.25000E+01
.30000E+01
INITIAL VARIANCE OF VELOCITY ERROR

FIG. 17

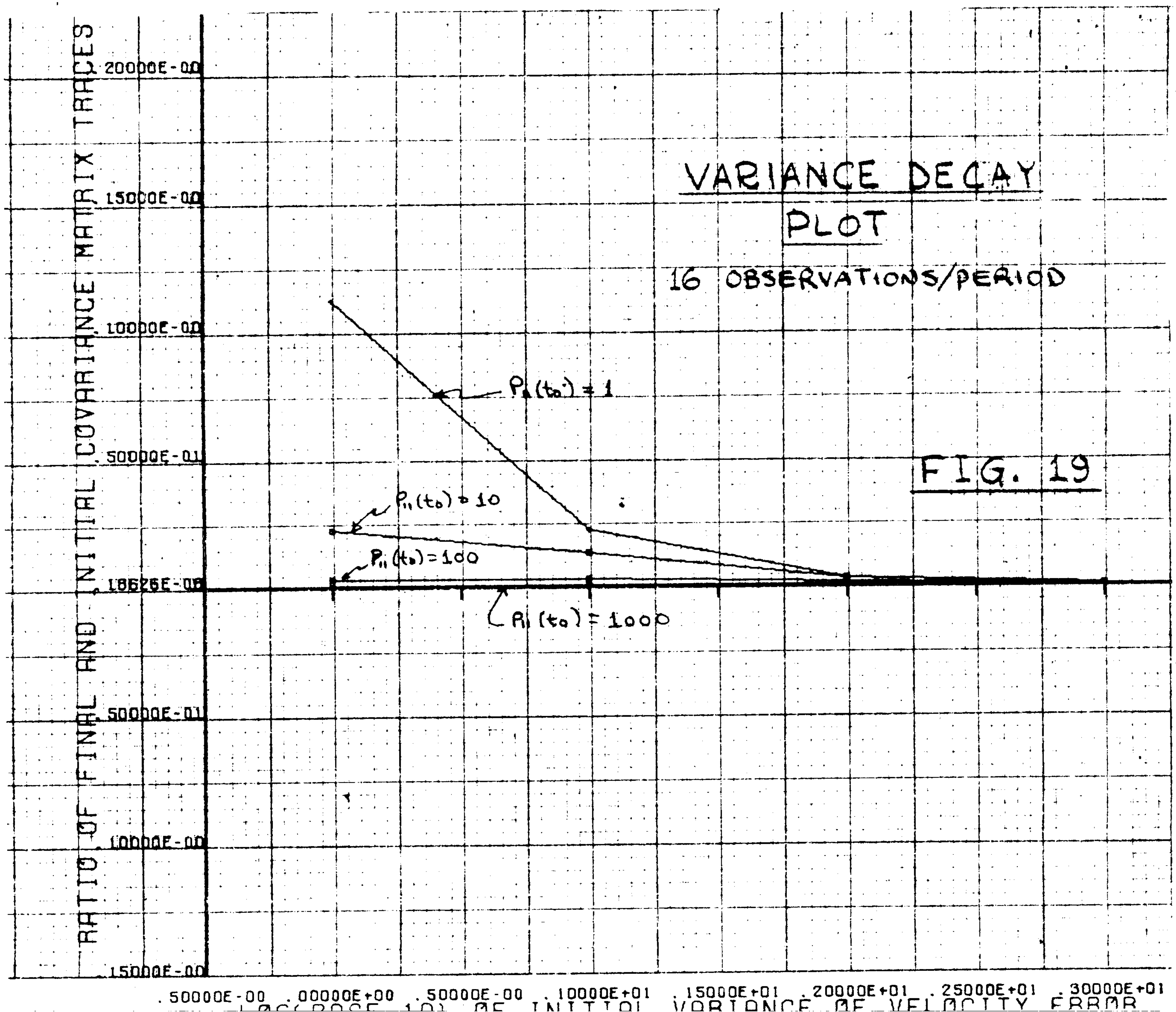

RATIO OF MAGNITUDES OF FINAL AND INITIAL ERRORS
.25000E-00
.20000E-00
.15000E-00
.10000E-00
.50000E-01
.18626E-08
.50000E-01
.10000E-00
.15000E-00
.50000E-00
.00000E+00
.50000E-00
LOG(BASE 10) OF IN



ERROR DECAY PLOT

8 OBSERVATIONS/PERIOD

FIG. 18



VARIANCE DECAY
PLOT
16 OBSERVATIONS/PERIOD
$P_{11}(t_0) = 1$
$P_{11}(t_0) = 10$
$P_{11}(t_0) = 100$
$P_{11}(t_0) = 1000$
RATIO OF FINAL AND INITIAL COVARIANCE MATRIX TRACES
.20000E-00
.15000E-00
.10000E-00
.50000E-01
.18626E-08
.50000E-01
.10000E-00
.15000E-00
.50000E-00
.00000E+00
.50000E-00
.10000E+01
.15000E+01
.20000E+01
.25000E+01
.30000E+01

FIG. 19

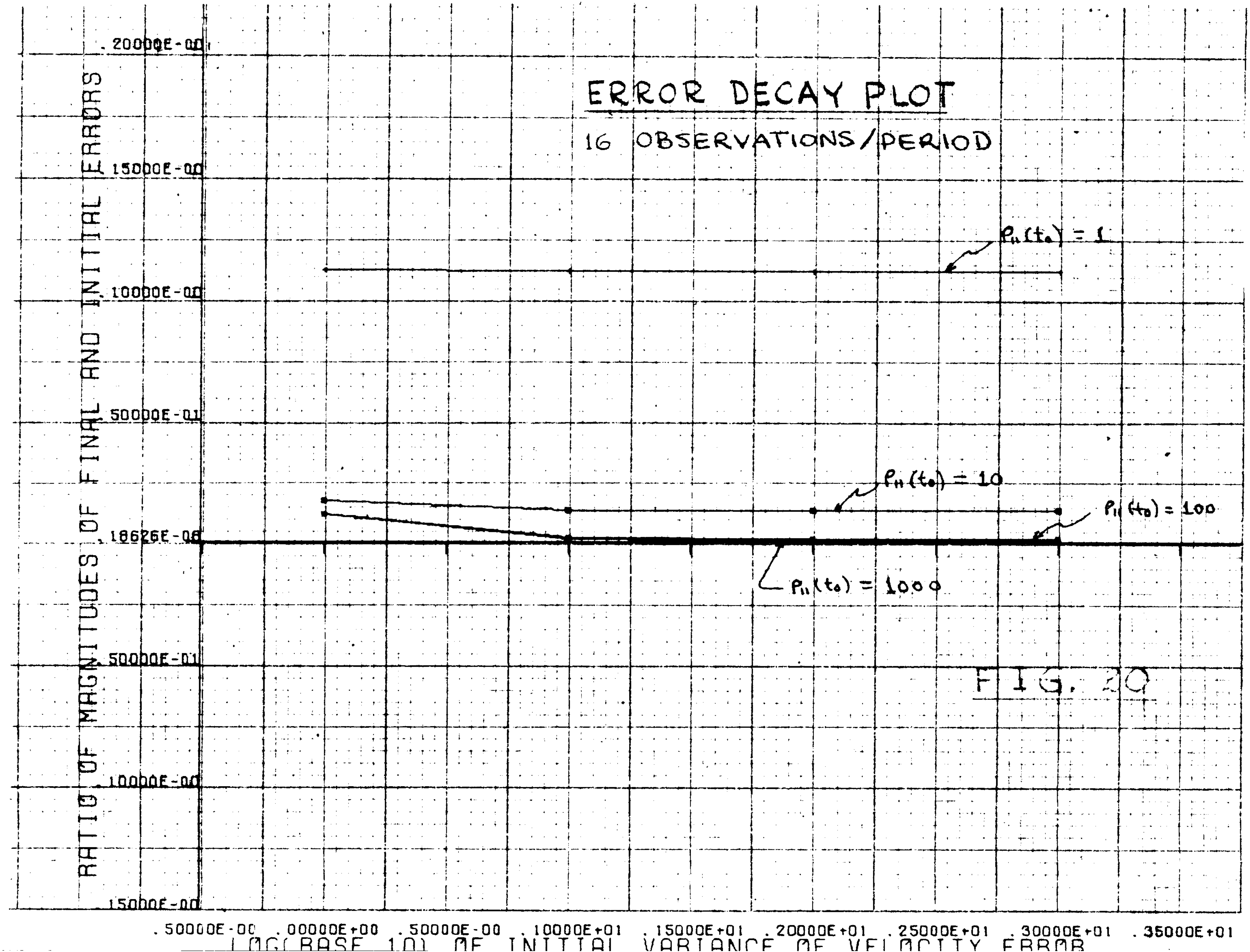
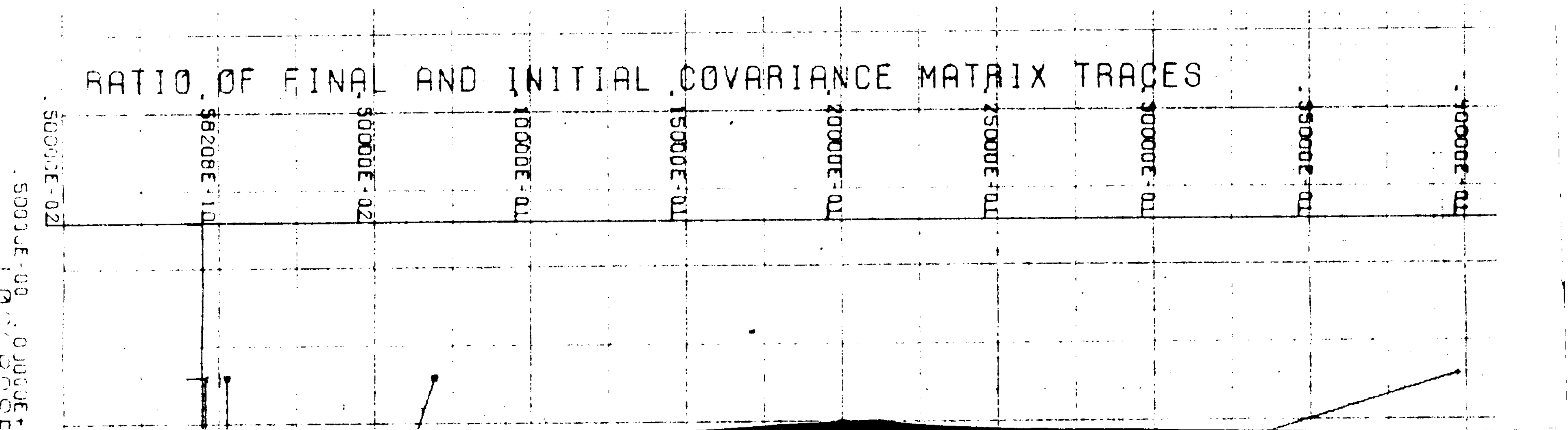

ERROR DECAY PLOT
16 OBSERVATIONS/PERIOD
$P_{11}(t_0) = 1$
$P_{11}(t_0) = 10$
$P_{11}(t_0) = 100$
$P_{11}(t_0) = 1000$
RATIO OF MAGNITUDES OF FINAL AND INITIAL ERRORS
.20000E-00
.15000E-00
.10000E-00
.50000E-01
.18626E-08
.50000E-01
.10000E-00
.15000E-00
.50000E-00
.00000E+00
.50000E-00
.10000E+01
.15000E+01
.20000E+01
.25000E+01
.30000E+01
.35000E+01
LOG(BASE 10) OF INITIAL VARIANCE OF VELOCITY ERROR

FIG. 20



RATIO OF FINAL AND INITIAL COVARIANCE MATRIX TRACES
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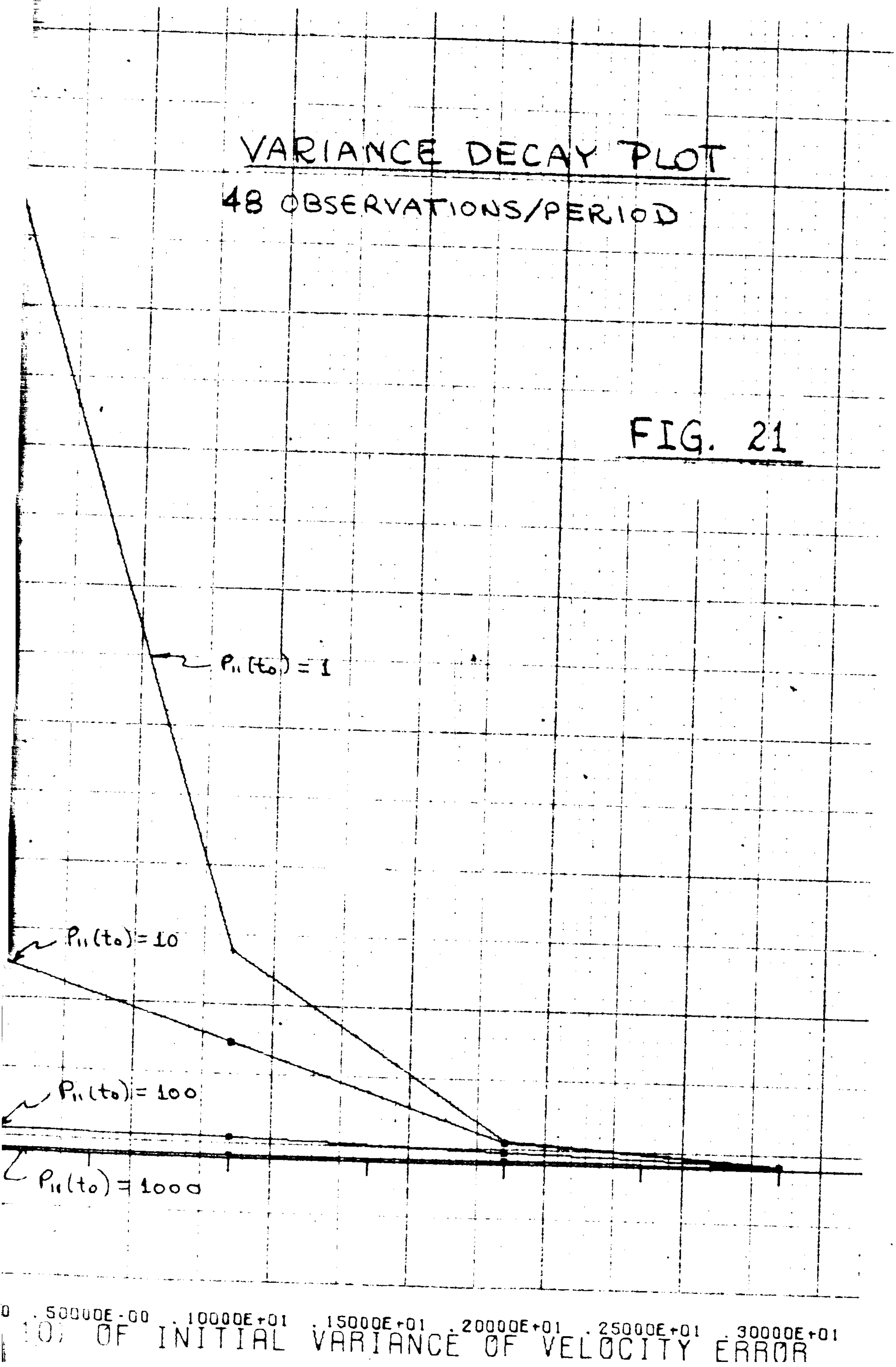
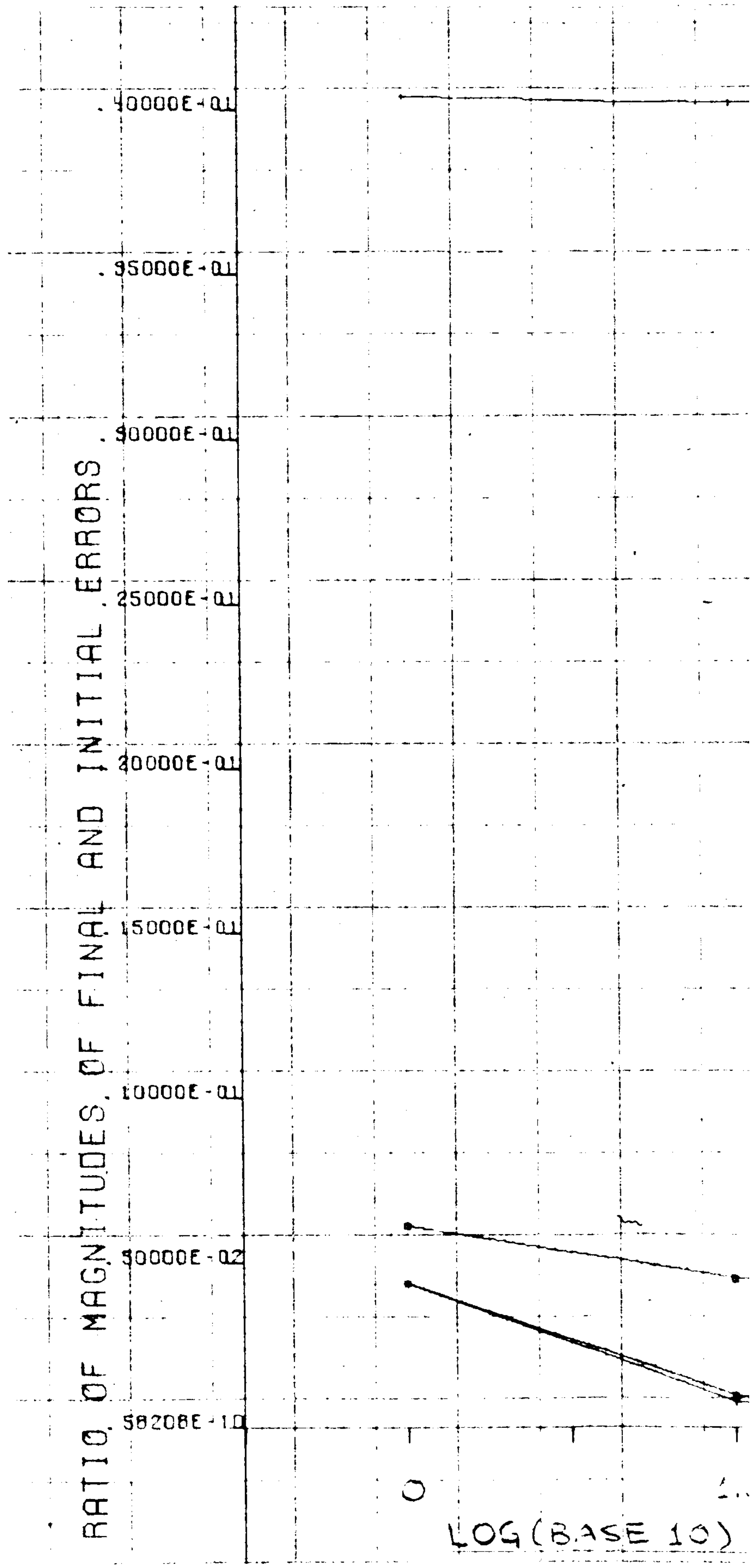
.50000E-02
.10000E-01
.15000E-01
.20000E-01
.25000E-01
.30000E-01
.35000E-01
.40000E-01
.50000E-00
LOG BASE

VARIANCE DECAY PLOT

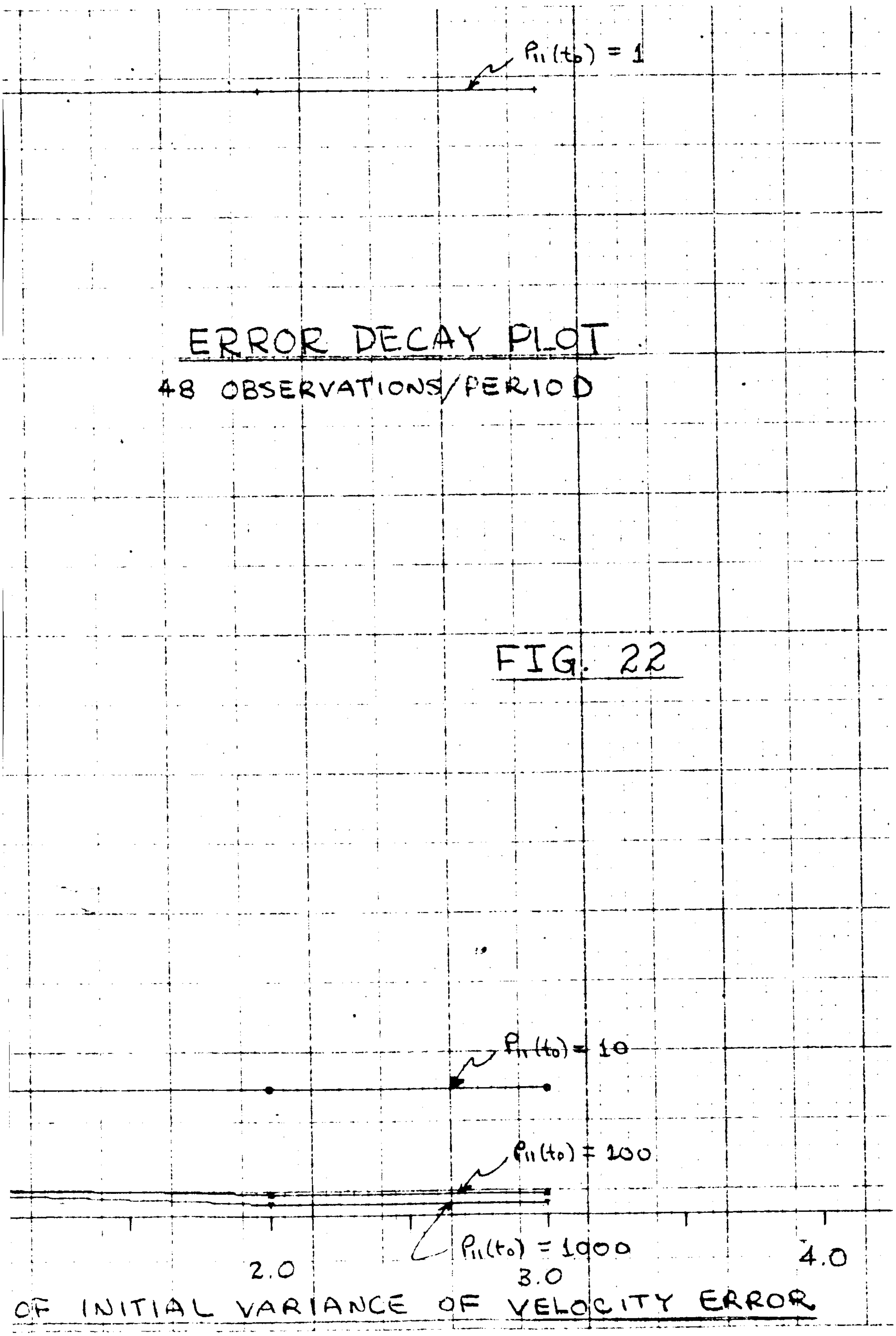
48 OBSERVATIONS/PERIOD

FIG. 21





.40000E-01
.35000E-01
.30000E-01
.25000E-01
.20000E-01
.15000E-01
.10000E-01
.50000E-02
.58208E-10
RATIO OF MAGNITUDES OF FINAL AND INITIAL ERRORS
0
LOG(BASE 10)



ERROR DECAY PLOT
48 OBSERVATIONS/PERIOD
$P_{11}(t_0) = 1$
$P_{11}(t_0) = 10$
$P_{11}(t_0) = 100$
$P_{11}(t_0) = 1000$
2.0
3.0
4.0
OF INITIAL VARIANCE OF VELOCITY ERROR

FIG. 22

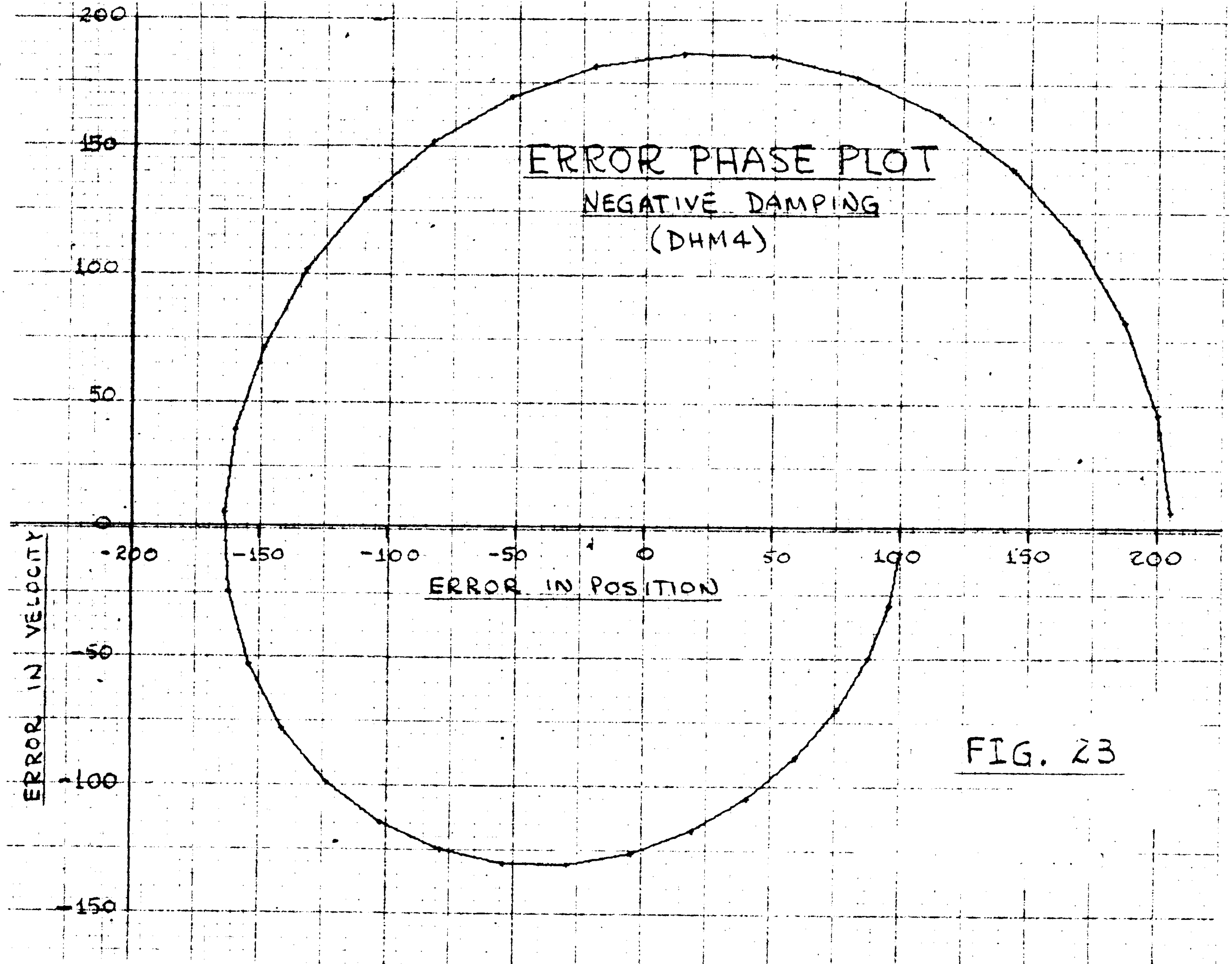

ERROR PHASE PLOT
NEGATIVE DAMPING
(DHM4)
ERROR IN POSITION
ERROR IN VELOCITY
-200
-150
-100
-50
0
50
100
150
200

FIG. 23

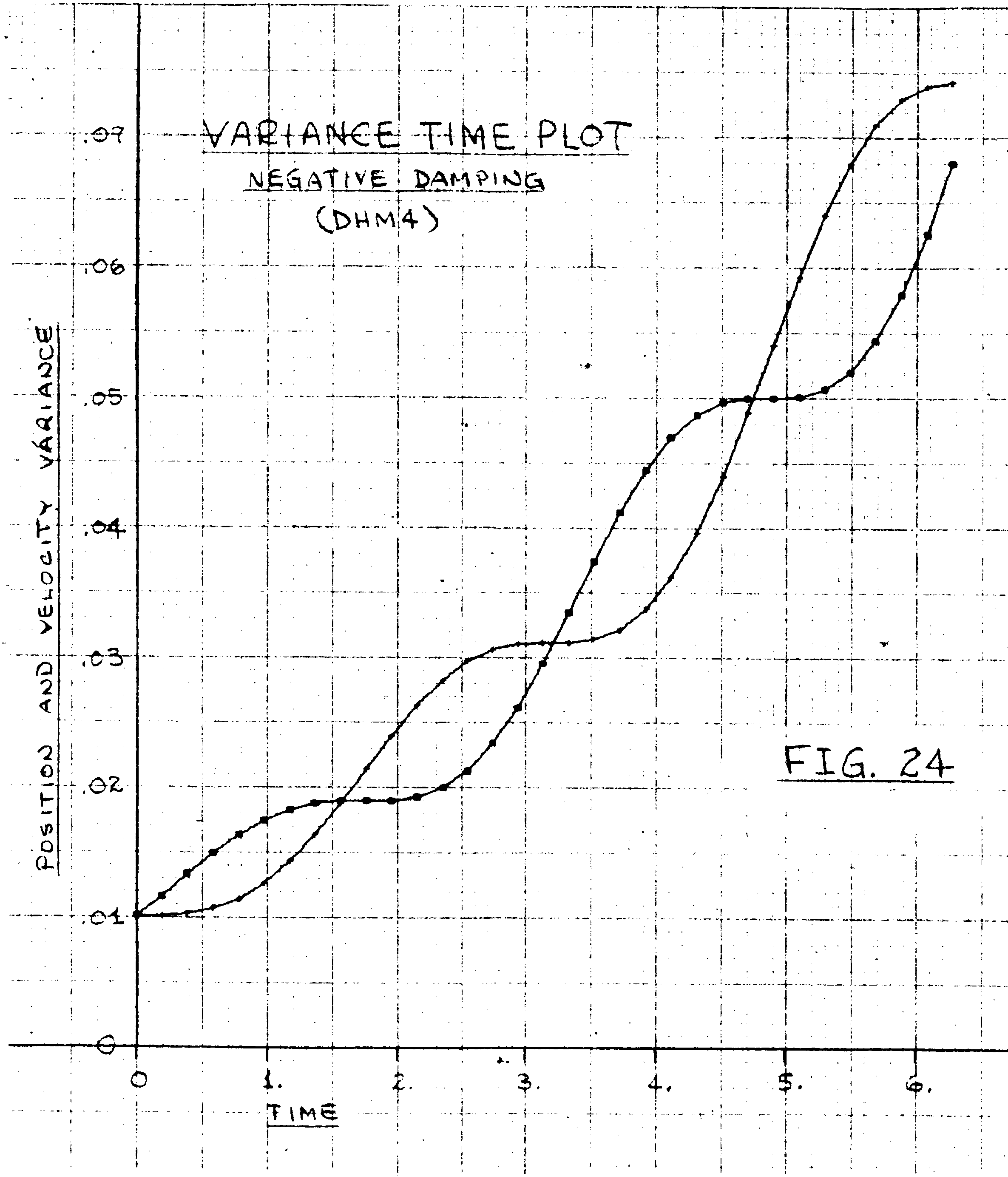
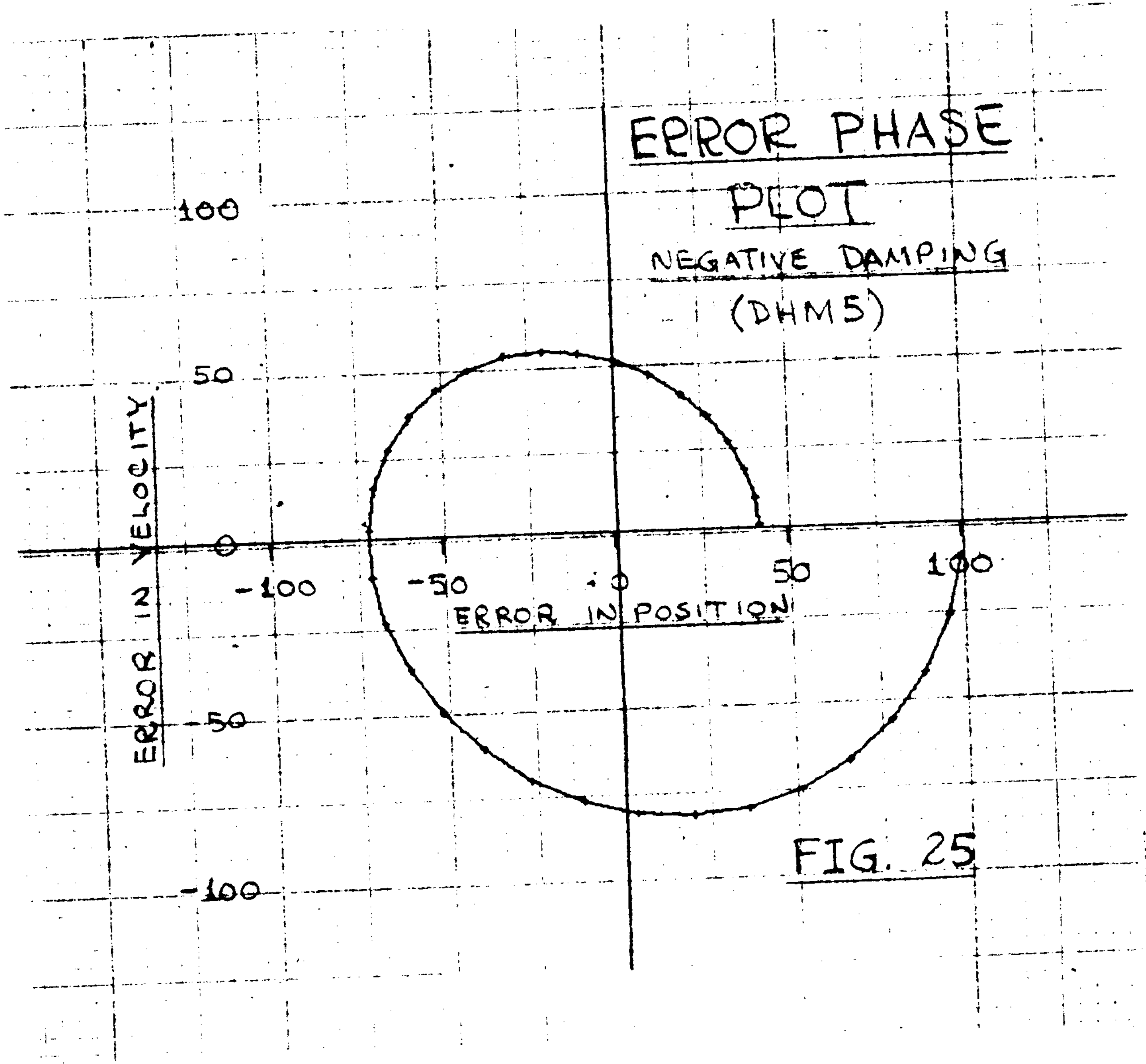

VARIANCE TIME PLOT
NEGATIVE DAMPING
(DHM4)
POSITION AND VELOCITY VARIANCE
.07
.06
.05
.04
.03
.02
.01
0
0
1.
2.
3.
4.
5.
6.
TIME

FIG. 24



ERROR PHASE
PLOT
NEGATIVE DAMPING
(DHM5)
100
50
0
-50
-100
-100
-50
0
50
100
ERROR IN VELOCITY
ERROR IN POSITION

FIG. 25

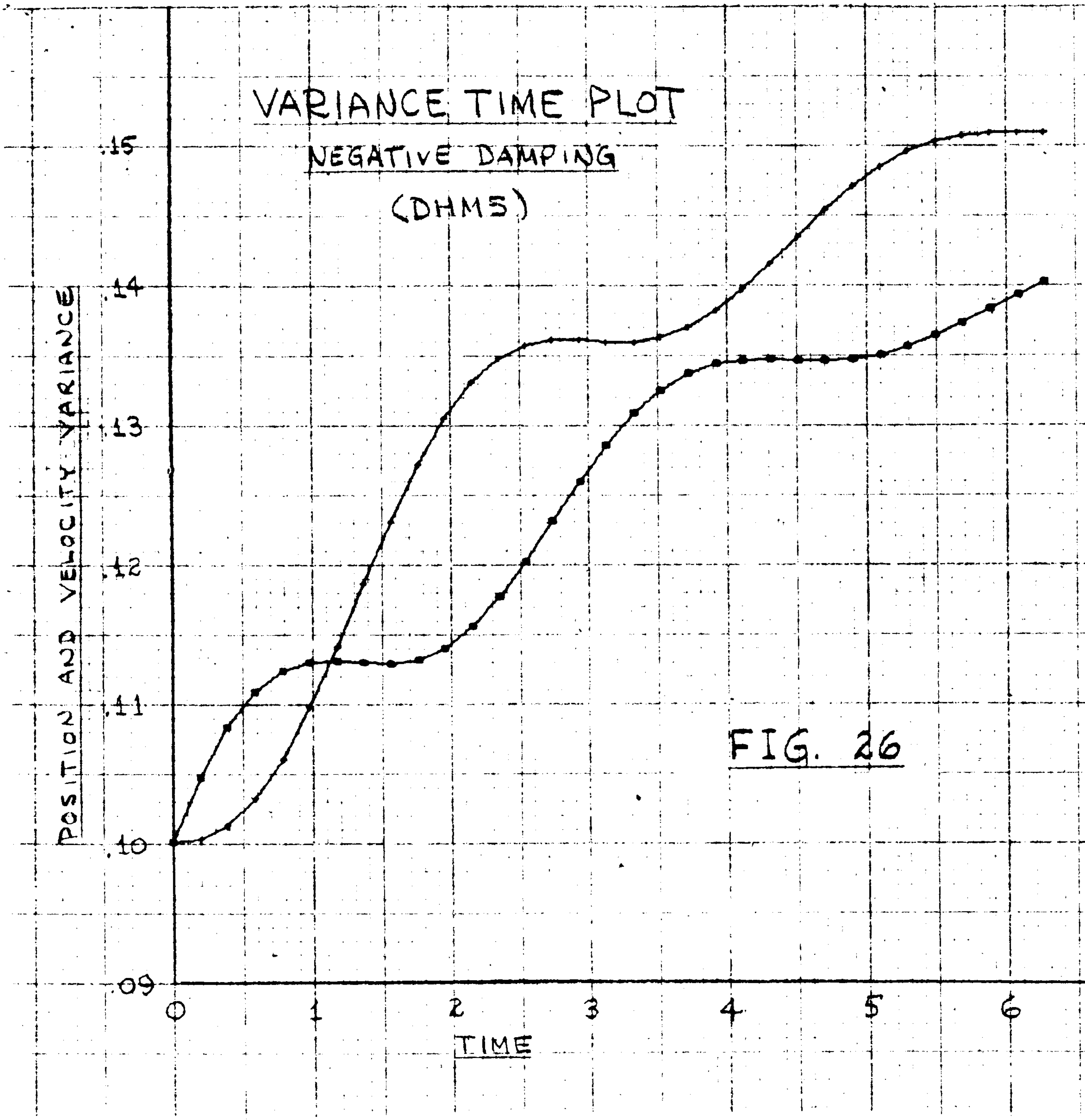
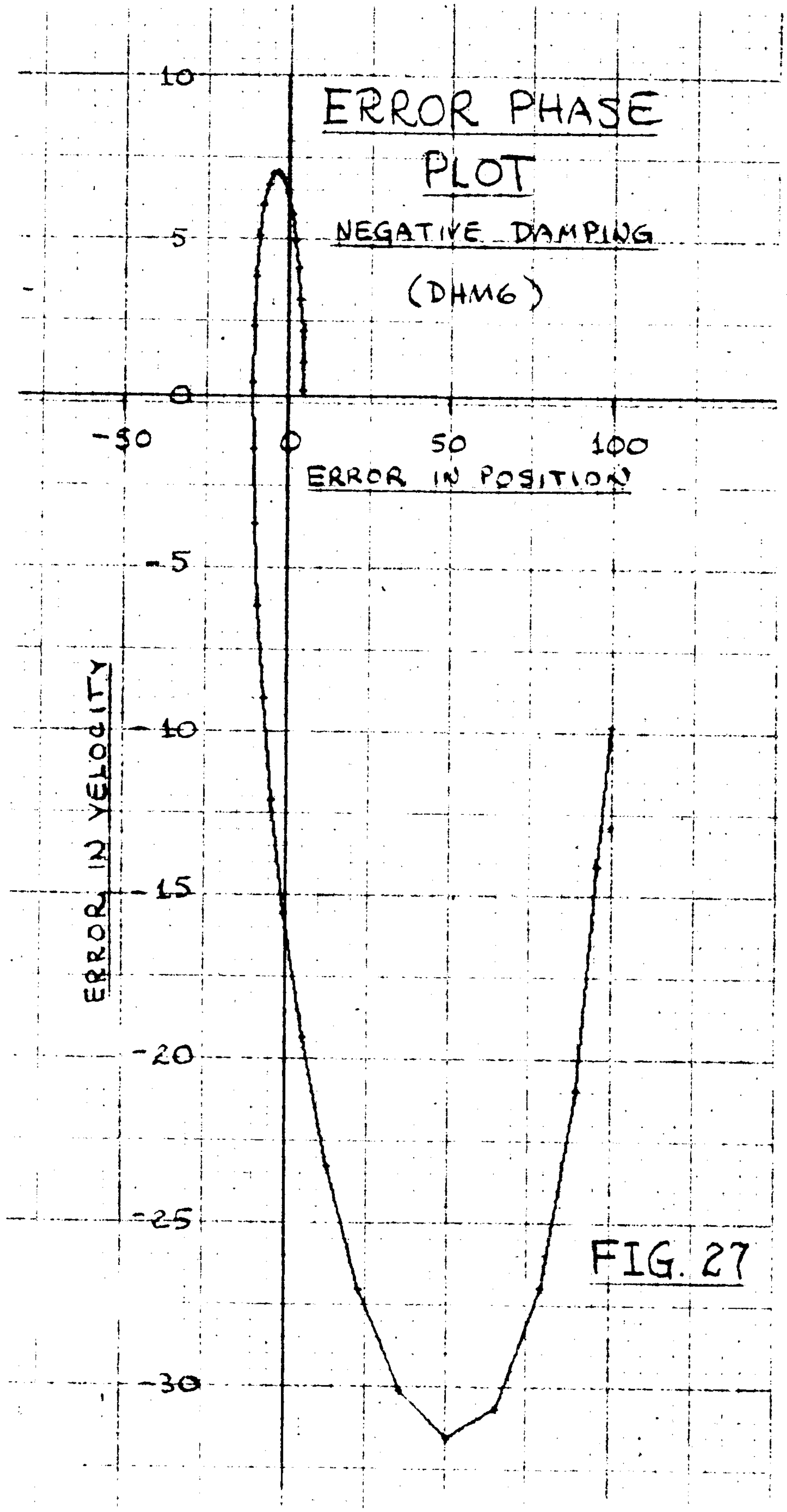
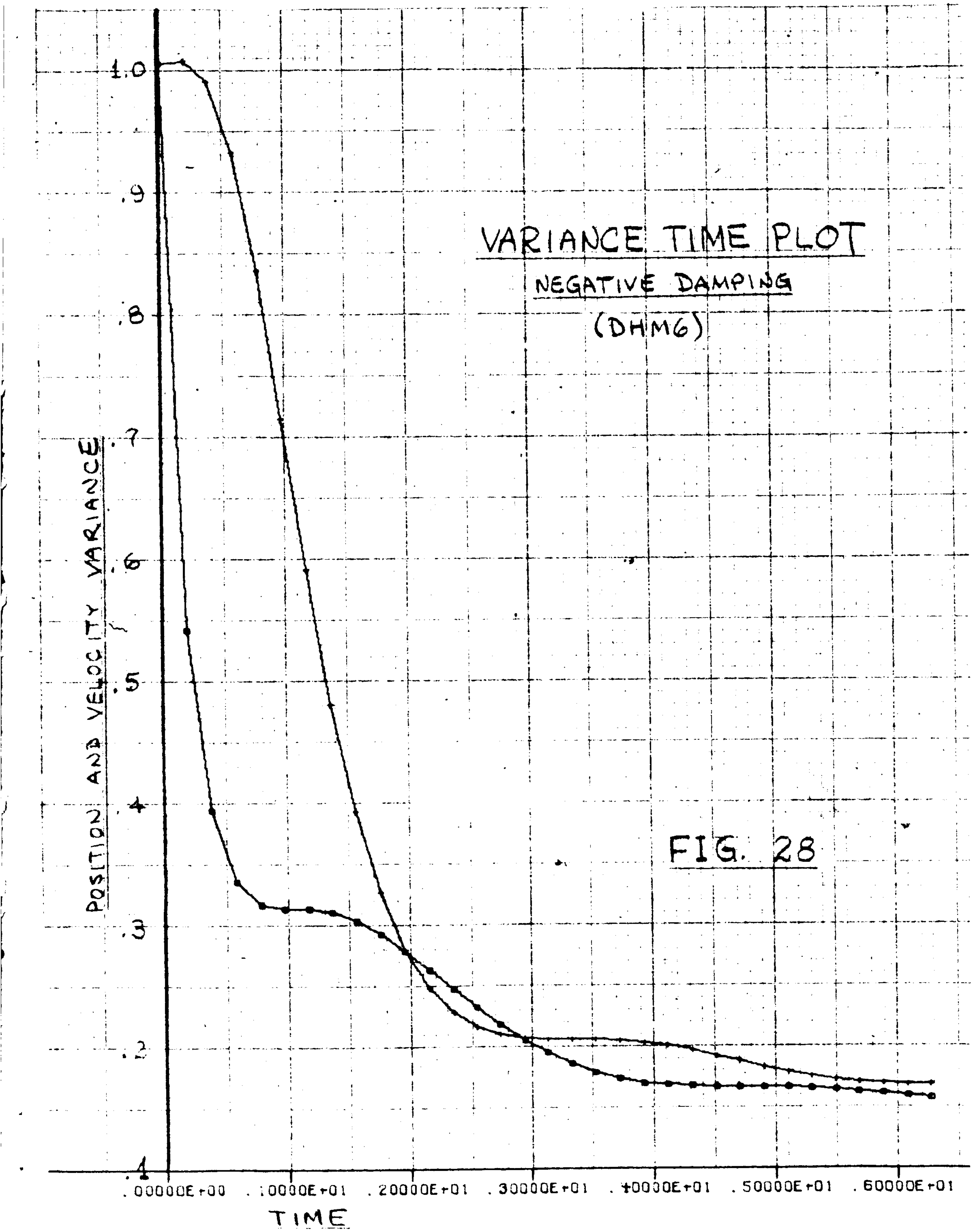

VARIANCE TIME PLOT
NEGATIVE DAMPING
(DHM5)
POSITION AND VELOCITY VARIANCE
.15
.14
.13
.12
.11
.10
.09
0
1
2
3
4
5
6
TIME

FIG. 26

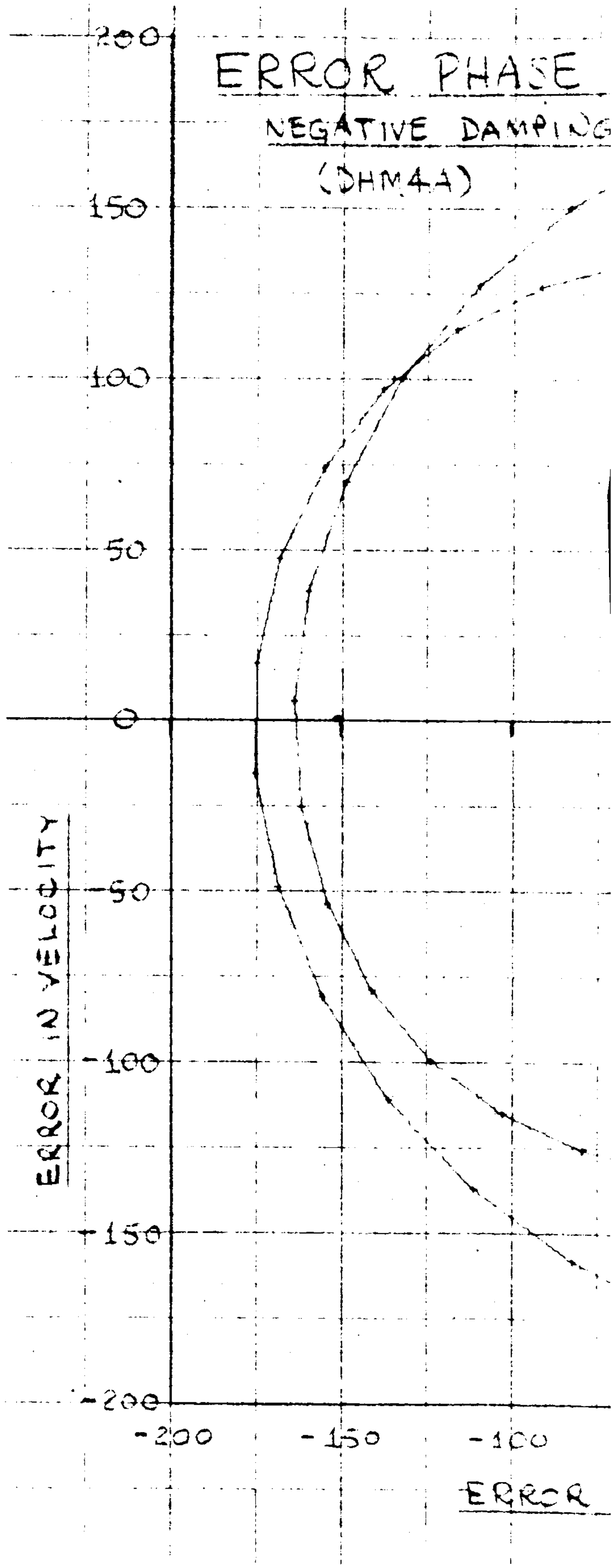


ERROR PHASE PLOT
NEGATIVE DAMPING
(DHM6)
10
5
0
-5
-10
-15
-20
-25
-30
-50
0
50
100
ERROR IN POSITION
ERROR IN VELOCITY

FIG. 27



VARIANCE TIME PLOT
NEGATIVE DAMPING
(DHM6)
FIG. 28
POSITION AND VELOCITY VARIANCE
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.4
.00000E+00
.10000E+01
.20000E+01
.30000E+01
.40000E+01
.50000E+01
.60000E+01
TIME



ERROR PHASE
NEGATIVE DAMPING
(DHM4A)
200
150
100
50
0
-50
-100
-150
-200
ERROR IN VELOCITY
-200
-150
-100
ERROR

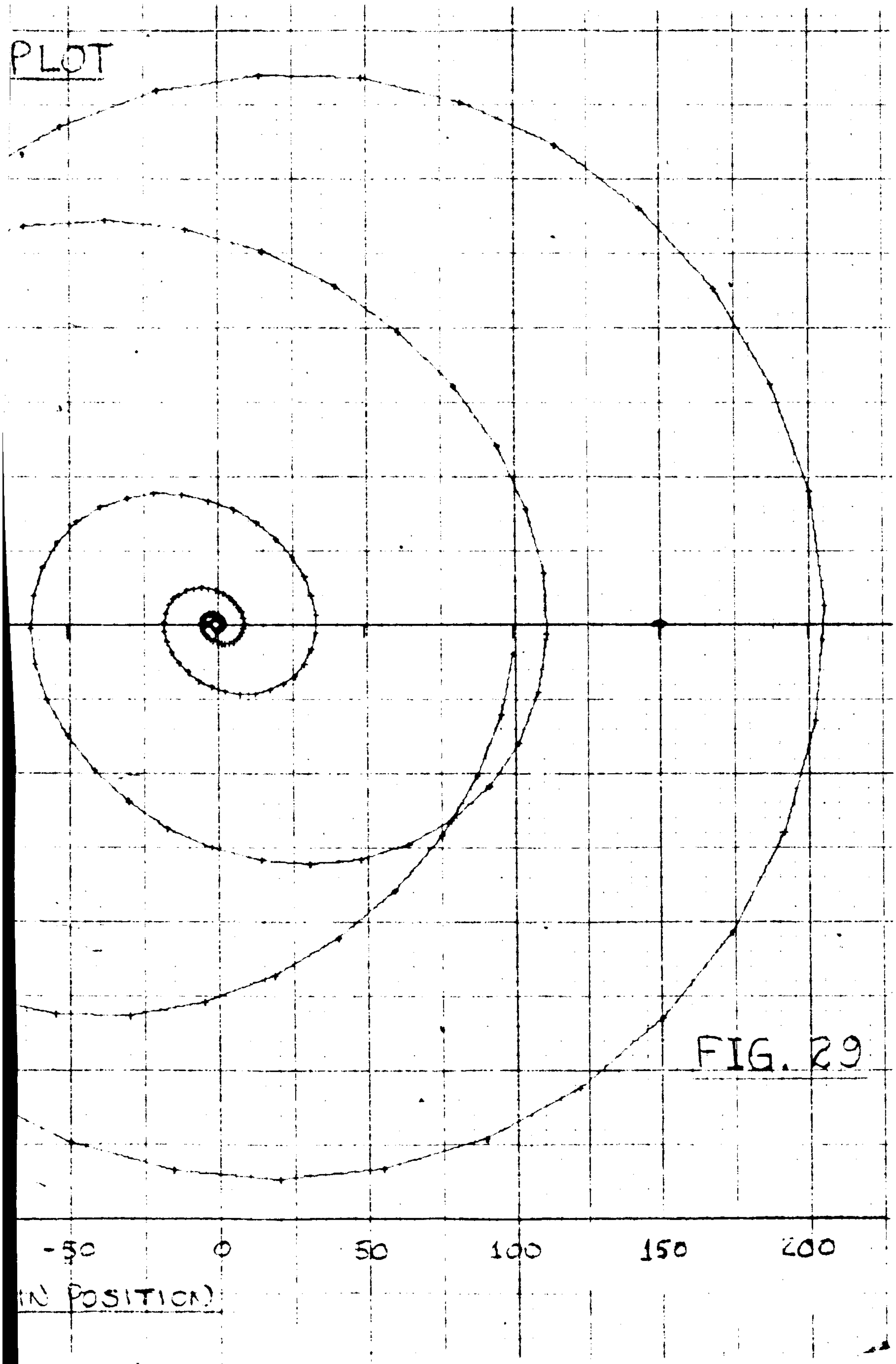


FIG. 29

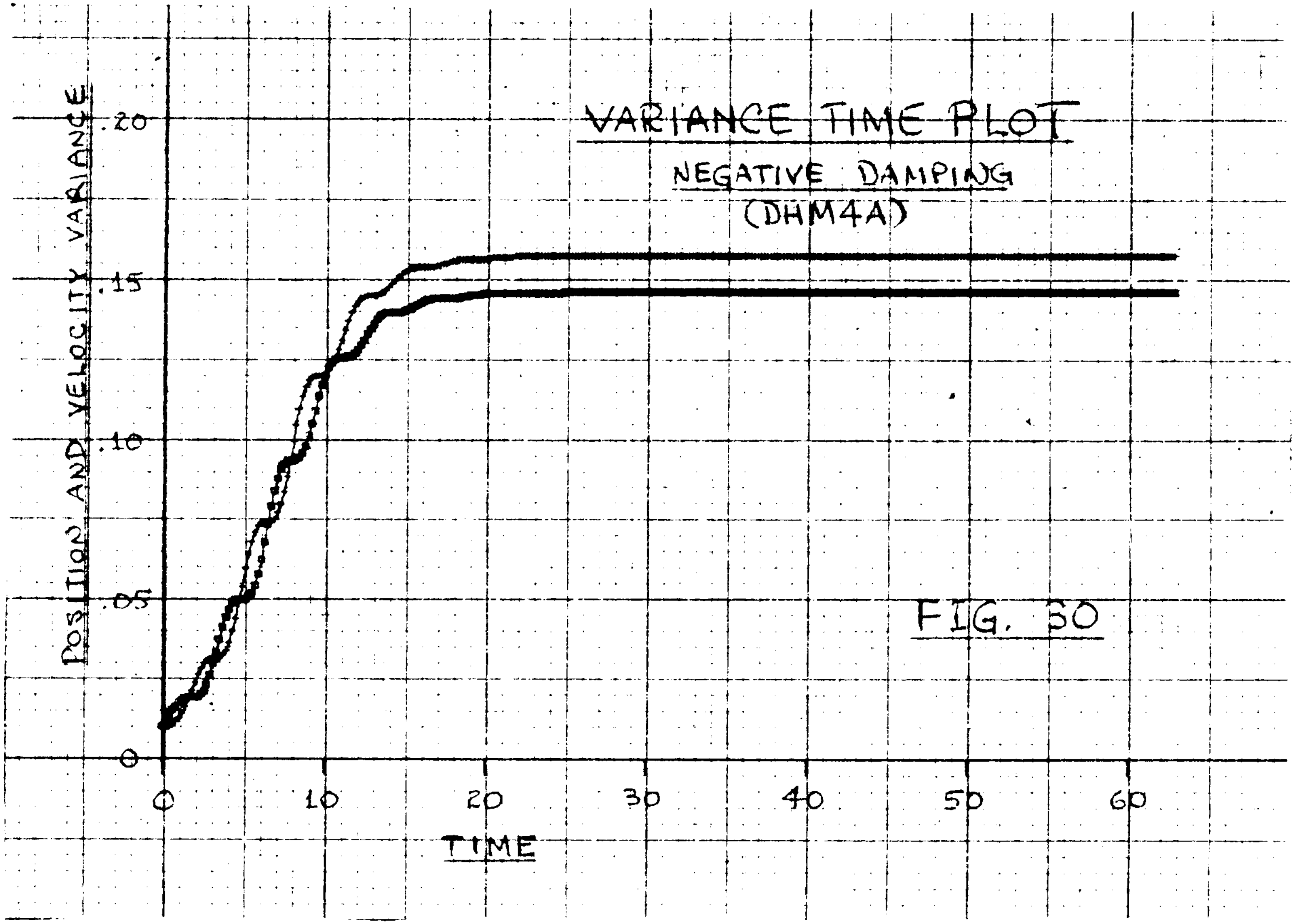

VARIANCE TIME PLOT
NEGATIVE DAMPING
(DHM4A)
POSITION AND VELOCITY VARIANCE
0
.05
.10
.15
.20
0
10
20
30
40
50
60
TIME

FIG. 30

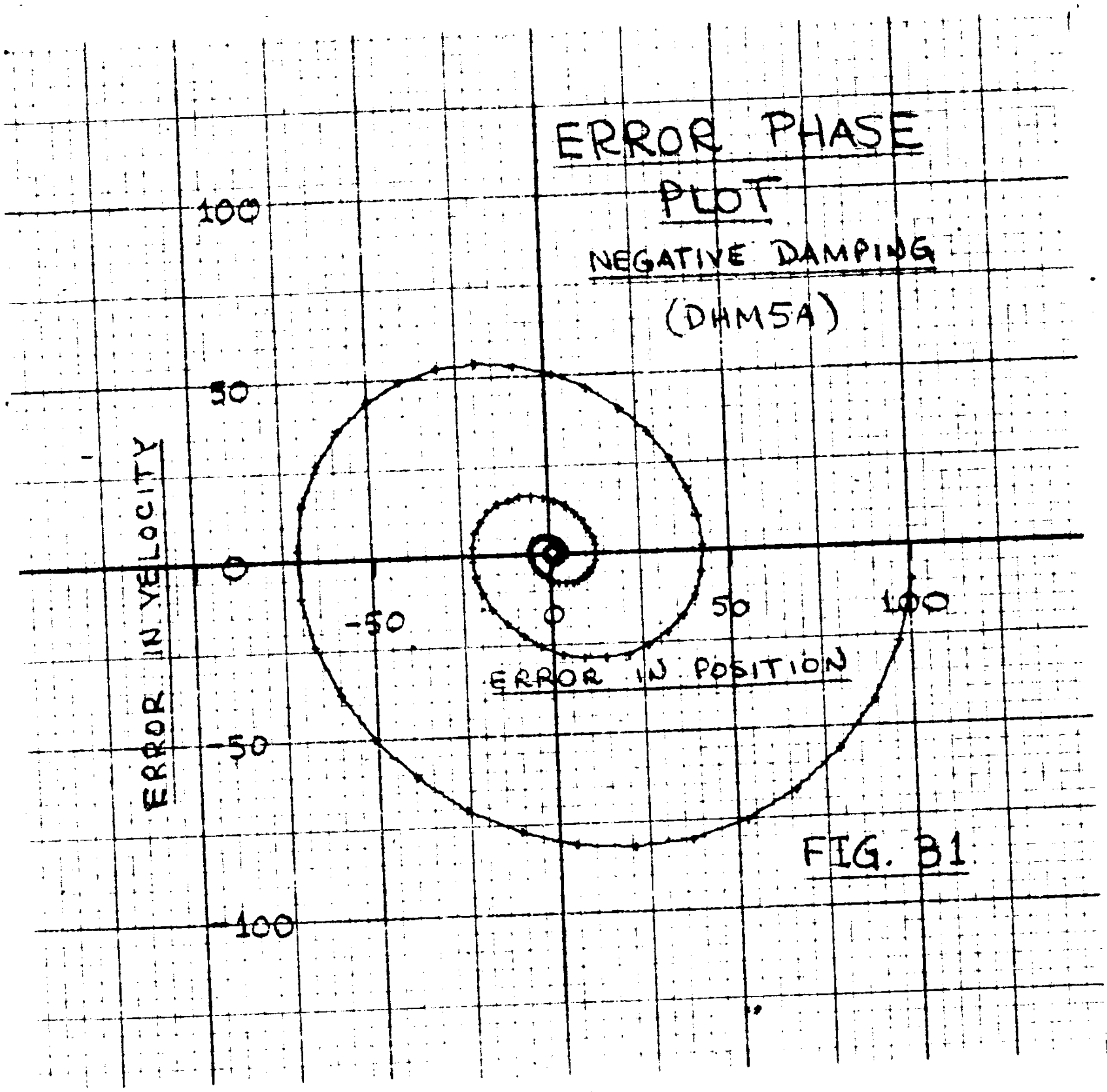
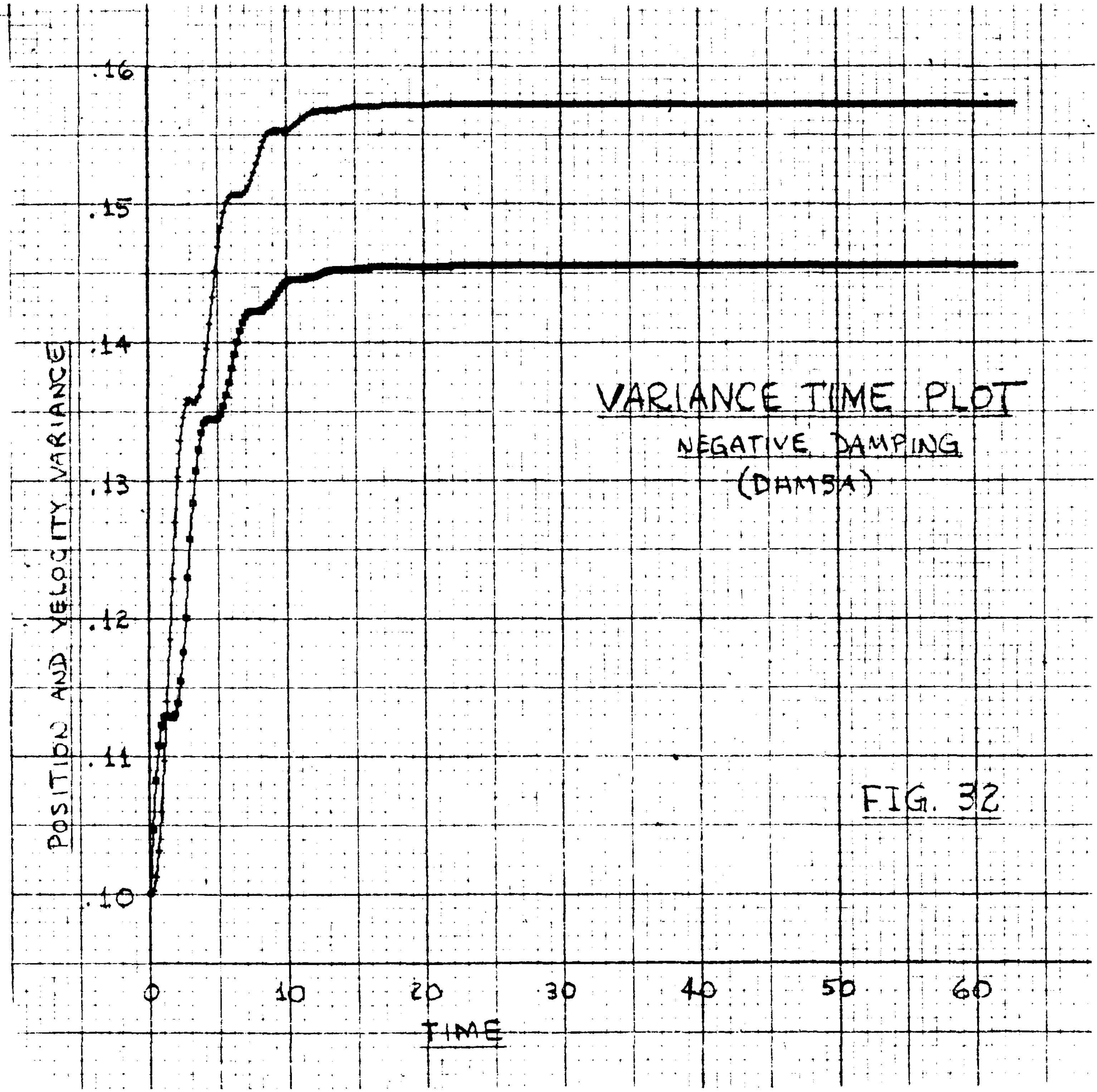
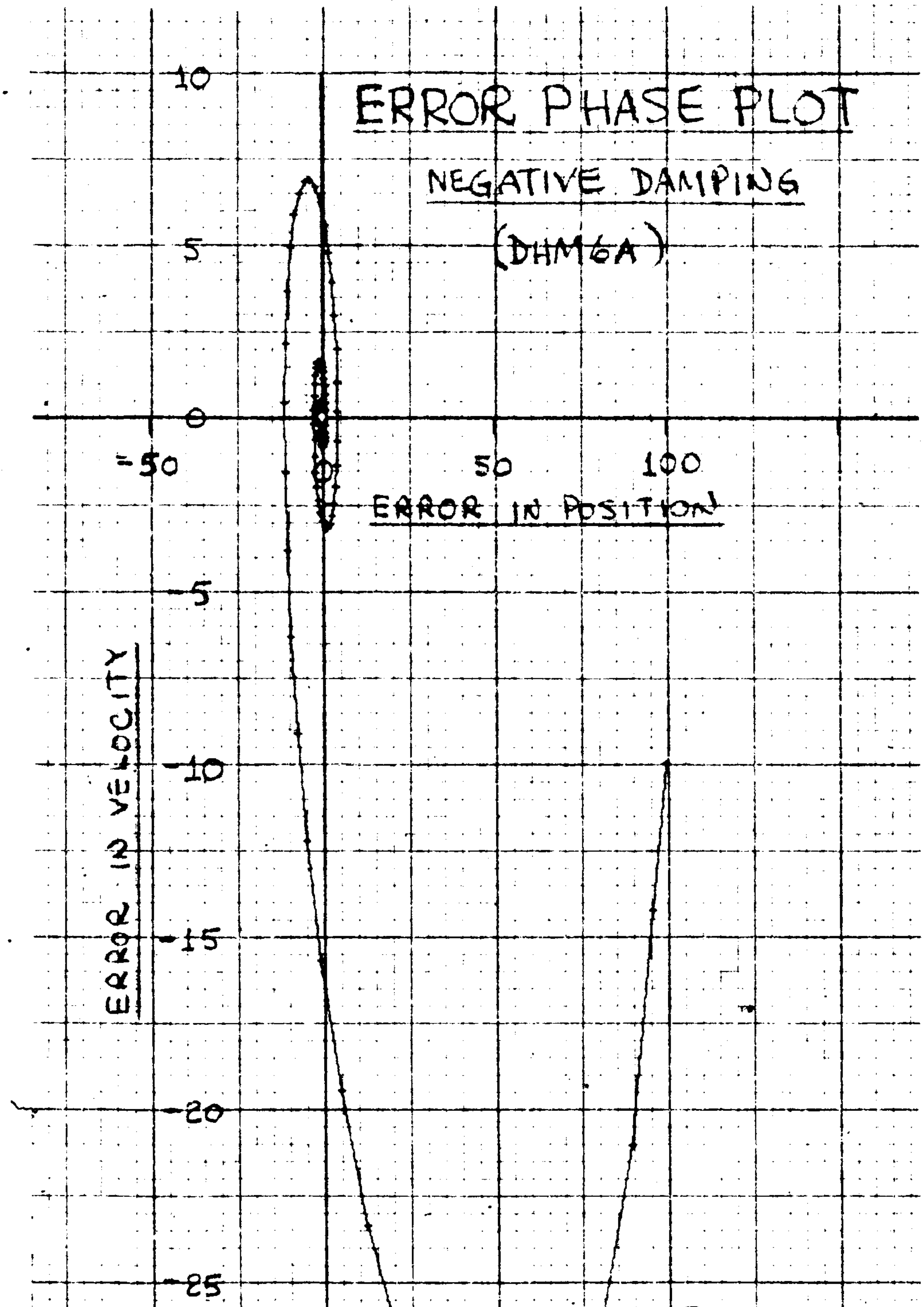

ERROR PHASE PLOT
NEGATIVE DAMPING
(DHM5A)
100
50
0
-50
-100
-50
0
50
100
ERROR IN VELOCITY
ERROR IN POSITION

FIG. B1



.16
.15
.14
.13
.12
.11
.10
POSITION AND VELOCITY VARIANCE
0
10
20
30
40
50
60
TIME
VARIANCE TIME PLOT
NEGATIVE DAMPING
(DAMSA)

FIG. 32



ERROR PHASE PLOT
NEGATIVE DAMPING
(DHM6A)
10
5
0
-5
-10
-15
-20
-25
-30
-50
50
100
ERROR IN POSITION
ERROR IN VELOCITY

FIG. 33

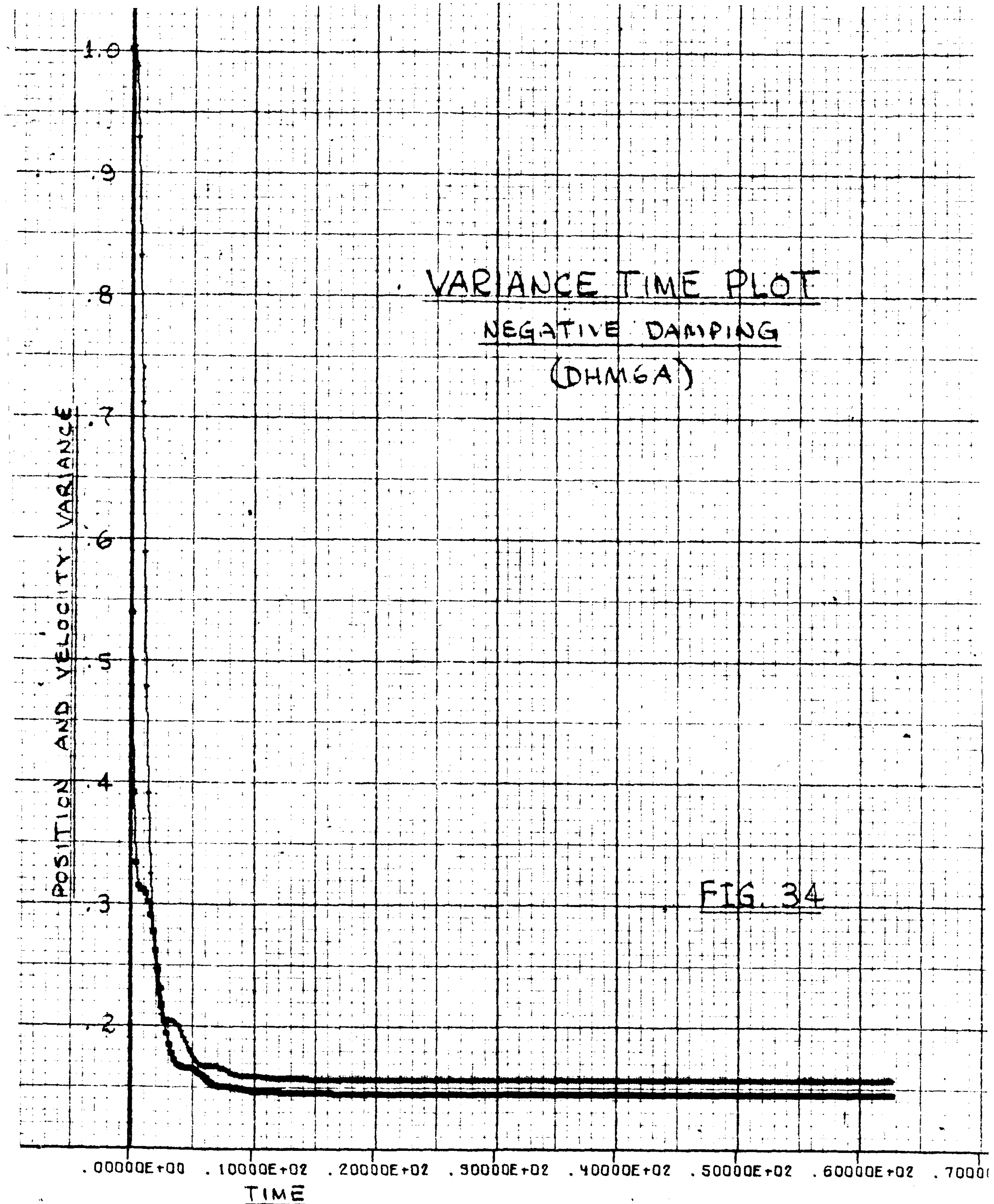

VARIANCE TIME PLOT
NEGATIVE DAMPING
(DHMGA)
POSITION AND VELOCITY VARIANCE
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.00000E+00
.10000E+02
.20000E+02
.30000E+02
.40000E+02
.50000E+02
.60000E+02
.7000
TIME

FIG. 34