PERTURBATIONS OF CIRCULAR SYNCHRONOUS SATELLITE ORBITS

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ABSTRACT

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Satellite drift maxima from a synchronous inclined circular orbit due to perturbations are investigated. A synchronous satellite, one which retraces its ground track each day, will tend to drift from the initially synchronous trajectory in time. Six, twelve, and twenty-four hour period orbits receive special attention.

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CONTENTS

I. INTRODUCTION

A constant ground track or synchronous satellite is one which ideally retraces its ground track each day. Assume that the satellite is initially over some fixed point A on the earth's surface. Assume also that this occurs at the ascending node. When the satellite reaches its node about one day later, Point A will again be there. As time goes on the satellite will tend to arrive early or late and thus miss passing over Point A. Assuming perfect guidance this miss is mainly caused by lunar and solar forces exerted on the orbit. The miss can be expressed by the angle ΔV between the radius vectors to Point A and to the satellite at the time Point A and the projection upon the earth of the orbits ascending node coincide. The purpose of this study is to obtain an upper bound for this miss angle for circular inclined orbits.

Besides the ΔV miss there is a deviation ΔL in maximum latitude. This is due to lunar and solar perturbations of the orbital inclination. An upper bound for this is obtained also. The upper bounds for ΔV and ΔL combine to give an upper bound for a satellite's deviation from a constant ground track trajectory.

II. THE NODAL PERIOD

A relation for the nodal period of a 24-hour satellite orbit with constant ground track is given by

$$\omega_e P_{Ne} = 2\pi + \Omega' P_{Ne} \tag{1}$$

where ω_e is angular rate of the earth's rotation, P_{Ne} is the nodal period, and Ω' is the drift rate of the satellite's equatorial node. The angles are measured in radians and time is measured in any convenient units.

The above equation states that while the satellite makes one revolution from node to node the earth rotates once relative to the node. This means that the satellite will be above the same point on the earth's surface at each nodal crossing.

Since the 12 hour and 6 hour constant ground track orbits are also of interest, it is desirable to generalize the above formula so that the satellite can make any number of revolutions per day. This can be done by replacing P_{Ne} in (1) by nP_{Ne}. Solving the resulting expression for P_{Ne} gives

$$P_{Ne} = \frac{2\pi}{n(\omega_e - \Omega')} \tag{2}$$

where n is the number of revolutions made by the satellite while the earth rotates once.

The term P_{Ne} can be interpreted either as the required nodal period for constant ground track or one nth the period required for the earth to rotate once relative to the ascending node.

To first order in J, neglecting higher order harmonics as well as solar lunar effects, the actual nodal period P_{Ns} of a circular orbit is given[1] by

$$P_{Ns} = P_k \left[1 - \frac{JR^2}{a^2} (3 - 2.5 \sin^2 i) \right] \tag{3}$$

where $P_k = 2\pi\sqrt{\frac{a^3}{GM}}$ is the Kepler period of the satellite, i is the orbital inclination to the equator, a is the semi-major axis of the orbit, R is the earth's equatorial radius, J = 0.001624 is the oblateness constant, and GM is the gravitational constant times the earth's mass.

A satellite is synchronous when $P_{Ns} = P_{Ne}$. The error or deviation ΔP_N in the nodal period is

$$\Delta P_N = P_{Ns} - P_{Ne} .$$

Assuming the satellite is exactly synchronous initially the error in nodal period at any time t is

$$\Delta P_{Nt} = dP_{Ns} - dP_{Ne} \tag{4}$$

where d denotes the differential or change in the quantity. These changes are small even over a period of several years.

The change in the nodal period dP_{Ns} of the satellite is obtained by differentiating (3). It is

$$dP_{Ns} = \frac{5P_k JR^2 \sin i \cos i \, d\, i}{a^2} \tag{5}$$

where a and hence P_k are assumed to be constant. This is a reasonable assumption since perturbations in a are mostly periodic except for low satellites where drag is important. In any event the error due to perturbations in a can be considered separately and added to the results which will be obtained from the above.

The change dP_{Ne} in the period P_{Ne} is obtained by differentiating (2). This gives

$$dP_{Ne} = \frac{2\pi \, d\Omega'}{n(\omega_e - \Omega')^2} \tag{6}$$

where dP_N and $d\Omega'$ denote small changes in the period and nodal drift rate respectively.

To first order in J, neglecting higher order harmonics and lunar-solar effects, Ω' is given[1] by

$$\Omega' = \frac{-2\pi JR^2 \cos i}{P_k a^2 (1 - e^2)^2} \quad \text{radians per unit time} \tag{7}$$

where e is the orbit's eccentricity.

For circular orbits, $e = 0$ is nearly constant. Then differentiating the above yields

$$d\Omega' = \frac{2\pi JR^2 \sin i \, d\, i}{P_k a^2} \tag{8}$$

where again a is assumed constant.

Combining (6) and (8), dP_{Ne} can be expressed as

$$dP_{Ne} = \frac{(2\pi)^2 JR^2 \sin i \, d\, i}{nP_k a^2 (\omega_e - \Omega')^2} \tag{9}$$

Substituting (5) and (9) in (4) yields

$$\Delta P_{Nt} = \left[5P_k \cos i - \frac{(2\pi)^2}{nP_k(\omega_e - \Omega')^2} \right] \frac{JR^2 \sin i \, d\, i}{a^2} \tag{10}$$

What is now needed is an estimate of how much the orbital inclination i changes with time.

III. ORBITAL INCLINATION

Since inclination is essentially unchanged by oblateness effects it is only necessary to consider lunar and solar forces in evaluating it. It is convenient to consider the lunar effects first.

Lunar perturbations on equatorial orbital parameters can best be determined by first considering perturbations on orbital parameters taken relative to the earth - moon plane. The lunar perturbation on the equatorial inclination i is then bounded[2] by

$$\left|di_1\right| \leq \frac{1}{\sin i}\left(\left|\Delta i_m\right| + \left|\sin I_m \sin i_m \sin \Omega_m \Delta \Omega_m\right|\right) \tag{11}$$

where $\left|\Delta i_m\right|$ and $\left|\Delta \Omega_m\right|$ indicate upper bounds for perturbations in inclination i_m and nodal position Ω_m respectively. The subscript m signifies that the orbital parameters are taken relative to the earth - moon plane. The parameter I_m is the inclination of the lunar orbit to the equator. The nodal position Ω_m is measured from the lunar equatorial node.

The upper bound given by (11) above is a simplified version of (36), Reference 2. This reference also provides a method for computing upper bounds for perturbations in i_m and Ω_m.

First order lunar perturbations in i_m and Ω_m can be expressed as the sum of a secular term and a periodic term. The periodic term contributes very little and can safely be omitted (see Reference 2).

The lunar perturbations in i_m and Ω_m per revolution of the satellite are

$$\Delta i_m = \frac{-15\pi a^3 H_m}{4\sqrt{1 - e^2}} \left(e^2 \sin 2\omega_m \sin 2i_m\right) \tag{12}$$

$$\Delta \Omega_m = \frac{-3\pi a^3 H_m \cos i_m}{\sqrt{1 - e^2}} \left(1 - e^2 + 5e^2 \sin^2 \omega_m\right) \tag{13}$$

where $H_m = \dfrac{M_m}{2a_m^3 M_e}$, M_m is the moon's mass, M_e is the earth's mass, a_m is the semi-major axis of the moon's orbit, and ω_m is the argument of perigee of the satellite's orbit.

Formulas (12) and (13) correspond to (26) and (27) in Reference 2. For nearly circular orbits these reduce to

$$\Delta i_m = 0$$

$$\Delta\Omega_m = -3\pi a^3 H_m \cos i_m \text{ radians per revolution.} \tag{14}$$

Substituting these in (11) gives an upper bound to the lunar secular perturbation in i per revolution of the satellite. That is

$$\left| di_l \right| \leq \frac{3\pi a^3 H_m \cos i_m}{\sin i} \left| \sin I_m \sin i_m \sin \Omega_m \right|$$

Since $\left| \sin i_m \cos i_m \right| = \left| 0.5 \sin 2i_m \right| \leq 1/2$ the above simplifies to

$$\left| di_l \right| \leq \frac{3\pi a^3 H_m \sin I_m \left| \sin \Omega_m \right|}{2P_k \sin i} \text{ radians per unit time} \tag{15}$$

where division by the Kepler period P_k has converted the perturbation to radians per unit time.

Similarly it can be shown that an upper bound for the solar secular perturbation in i is

$$\left| di_s \right| \leq \frac{3\pi a^3 H_c \sin I_c \left| \sin \Omega_c \right|}{2P_k \sin i} \text{ radians per unit time.} \tag{16}$$

where $H_c = \dfrac{M_c}{2a_c^3 M_e}$, M_c is the sun's mass, M_e is the earth's mass, a_c is the semi-major axis of the earth's orbit, I_c is the inclination of the ecliptic to the equator, and Ω_c is the longitude of the orbit's ecliptic node.

The above can be combined to give an upper bound for the perturbation in i per unit time. That is

$$\left| di \right| \leq \frac{3\pi a^3}{2P_k \sin i} \left(H_m \sin I_m \left| \sin \Omega_m \right| + H_c \sin I_c \left| \sin \Omega_c \right| \right) \tag{17}$$

Since I_m is at most 28.5 degrees and I_c = 23.44 degrees, the above can be replaced by a simpler and more conservative upper bound which is

$$\left| di \right| \leq \frac{3\pi a^3 (H_m + H_c) \sin 28.5}{2P_k \sin i} \quad \text{radians per unit time.} \tag{18}$$

IV. TIME DRIFT

Substituting the above in (10) and multiplying by time T gives an upper bound to the error in period ΔP_{NT} at time T. It is

$$\left| \Delta P_{NT} \right| \leq \left| 5P_k \cos i - \frac{(2\pi)^2}{nP_k(\omega_e - \Omega')^2} \right| \frac{3\pi JR^2 a (H_m + H_c) T \sin 28.5}{2P_k} \tag{19}$$

where time and distance may be expressed in any convenient units.

If time is expressed in days and distance in nautical miles the following values for the constants hold:

$J = 0.001624$

$R = 3444$ n mi

$H_m = 0.687 \times 10^{-18} (\text{n mi})^{-3}$

$H_c = 0.316 \times 10^{-18} (\text{n mi})^{-3}$

The following approximations are also of use.

$$P_k = 1/n \text{ days}$$

$$\omega_e - \Omega' \doteq 2\pi \text{ radians per day}$$

Using these values (19) reduces to

$$\left| \Delta P_{NT} \right| \leq 13.8 \times 10^{-15} \pi \; a \; n \; T \left| \frac{5 \cos i}{n} - 1 \right| \text{days.} \tag{20}$$

Before proceeding further it is interesting to note that the upper bound is zero when

$$\cos i = \frac{n}{5} \; .$$

For 6 hour, 12 hour, and 24 hour orbits this zero occurs when the inclination i is about 37 degrees, 66 degrees, and 78 degrees respectively.

Returning to the general case the average error in nodal period during time T will be less than one-half the upper bound expressed by (20). Since the satellite makes about T/P_k revolutions the time miss Δt at time T will be bounded by

$$\left| \Delta t \right| \leq \frac{T \, \Delta P_{N\,max}}{2P_k} \tag{21}$$

where $\Delta P_{N\,max}$ denotes the upper bound expressed by (19) or (20).

V. MISS IN TRUE ANOMALY

A time miss Δt will cause a miss ΔV in true anomaly where ΔV can be expressed by

$$\Delta V = \frac{2\pi \, \Delta t}{P_k}$$

since the angular rate of the satellite is about $2\pi/P_k$ radians per unit time.

Then referring to (21) an upper bound for the miss ΔV at time T is given by

$$\left| \Delta V \right| \leq \frac{\pi T \ \Delta P_{N\,max}}{P_k^2} \quad \text{radians.}$$

Substituting the upper bound given in (20) for $\Delta P_{N\,max}$ an upper bound for the miss ΔV is given by

$$\left| \Delta V \right| \leq 13.8 \times 10^{-15} \ \pi^2 \ a n^3 \ T^2 \left| \frac{5 \cos i}{n} - 1 \right| \ \text{radians} \qquad (22)$$

where P_k is replaced by $1/n$. Also a and T must be in nautical miles and days respectively.

The above expression can be further simplified by expressing the semi-major axis a in terms of the number of revolutions per day n. If a is in nautical miles then[2]

$$P_k = 1/n = 2.9030 \times 10^{-7} \ a\sqrt{a} \ \text{days}$$

which is obtained from the standard formula for the Kepler period (see (3)).

Solving this for a in terms of n gives

$$a = 22{,}809 \ n^{-2/3} \ \text{n mi}$$

Substituting this into (22), converting radians to degrees, and combining all constants gives

$$\left| \Delta V \right| \leq 0.0178 \times 10^{-5} \ (n\,T)^2 \ \sqrt[3]{n} \ \left| \frac{5 \cos i}{n} - 1 \right| \ \text{deg} \qquad (23)$$

or

$$\left|\Delta V\right| \leq 0.0238\ (n\ Y)^2\ \sqrt[3]{n}\ \left|\frac{5 \cos i}{n} - 1\right|\ \text{deg} \tag{24}$$

where T and Y are the time of flight in days and years respectively.

Recall the assumption that the initial nodal period of the satellite is such that the trajectory is synchronous for the first day. For example, assume that a 12 hour synchronous satellite is injected into a circular orbit. Also assume that this satellite is exactly above Point A on the earth's surface the first and second time that A crosses the satellite's ascending node. Then each time A arrives at the ascending node the satellite will be directly overhead except for a small error ΔV.

The angle ΔV lies in the satellite orbital plane with its vertex at the earth's center. One side goes through Point A which is on the earth's surface and just crossing the node. The other side goes through the satellite. Assume further that the orbit is inclined at 30 degrees to the equator. Then during the first three years of the satellite's lifetime, the angle ΔV will be less than 1.3 degrees. The upper bound 1.3 is obtained by substituting

$n = 2$, $\sqrt[3]{n} = 1.26$, $Y = 3$, and $i = 30^{\circ}$ into (24).

Returning to the general case, assume that the initial nodal period of the satellite is biased in such a way that the satellite passes directly above Point A at the beginning and end of the time interval T. Then the maximum miss angle is bounded by one-fourth the amount given by (23) or (24). This is because the maximum possible error in period as expressed by (20) is reduced by a factor of one-half and the time during which ΔV is increasing or decreasing is also reduced by a factor of one-half.

Hence the upper bound for the miss ΔV of the 12-hour satellite discussed above would be 0.32 degree. This is for $i = 30$ degrees and $Y = 3$ years.

In addition to the miss ΔV there is a miss due to the change in maximum latitude which in turn is due to a change in the inclination i.

VI. DRIFT IN LATITUDE

In addition to the angular drift ΔV, constant ground track satellites are subject to an error in maximum latitude due to perturbations in the orbital inclination i. An upper bound for the perturbation in i per unit time is given by (18). This leads to an upper bound for the change ΔL in maximum latitude L during time T. This is

$$\left|\Delta L\right| \leq \frac{3\pi a^3 (H_m + H_c) (\sin 28.5)\, T}{2P_k \sin i} \text{ radians} \tag{25}$$

where the time after injection T and period P_k must be in the same units. See (8) and (11) for the definitions of H_m and H_c which must be expressed in units consistent with those used for the semi-major axis a.

Converting the right side of (25) to degrees and then combining the various constants the upper bound for the change in maximum latitude becomes

$$\left|\Delta L\right| \leq \frac{0.129 \times 10^{-15} a^3 T}{P_k \sin i} \text{ deg.} \tag{26}$$

where the semi-major axis a must be in nautical miles. Also the time after injection T and the period P must be in the same units.

For the twelve hour orbit previously discussed, $a = 14,342$ nautical miles, $T = 1096$ days, and $P = 0.5$ days. Assuming that the equatorial inclination i is 30 degrees, the upper bound for the deviation in latitude is 1.67 degrees.

If a and P_k are expressed in terms of n, and time T is expressed in years, then (26) becomes

$$\left|\Delta L\right| \leq \frac{0.56\, Y}{n \sin i} \text{ deg.} \tag{27}$$

VII. SUMMARY

The principal result of this memo is the determination of an upper bound for the deviation ΔV in true anomaly V of a constant ground track satellite with a nearly circular orbit. As a byproduct, an upper bound for the deviation ΔL in maximum latitude L of the satellite is also obtained.

These results are expressed by (24) and (27) which are

$$\left|\Delta V\right| \leq 0.0238\,(n\,Y)^2\,\sqrt[3]{n}\,\left|\frac{5\cos i}{n} - 1\right|\,\text{deg} \qquad (24)$$

$$\left|\Delta L\right| \leq \frac{0.56\,Y}{n\sin i}\,\text{deg} \qquad (27)$$

where n is the number of revolutions the satellite makes per day, Y is the time after injection in years, a is the semi-major axis in nautical miles, i is the equatorial inclination of the orbit, and T is the time after injection in any unit consistent with the units expressing the period P of the orbit.

The upper bound for ΔV assumes that the initial nodal period is such that the satellite is exactly on target initially. This upper bound is also subject to the assumption that the semi-major axis a is constant except for small periodic fluctuations.

If the initial nodal period is such that the trajectory is exactly on target at first and the last nodal crossing of a given period of Y years, then the upper bound for the error ΔV in the true anomaly is

$$\left|\Delta V\right| \leq 0.006\,(n\,Y)^2\,\sqrt[3]{n}\,\left|\frac{5\cos i}{n} - 1\right|\,\text{deg.} \qquad (28)$$

The error in maximum latitude can be cut in half by biasing the initial inclination, then

$$\left| \Delta L \right| \leq \frac{0.28\ Y}{n \sin i} \quad \text{deg.} \tag{29}$$

All of the above upper bounds in this memo are subject to the following restrictions:

1. $e \leq 0.05$
2. $\sin i \geq 0.05$
3. $n \leq 6$
4. $Y \geq 2$ years

These restrictions are approximate only. This study was made with a near circular, 12 hour satellite orbit with $i = 30$ degrees in mind. Therefore no investigation was made as to exactly what the restrictions should be.

The third restriction is necessary because drag was assumed to be nearly zero. This would not be true for a low orbit. The fourth restriction is due to the fact that neglected periodic terms are important for shorter time intervals.

Circular orbits of 6 hour, 12 hour, and 24 hour periods are of special interest. If $i = 30$ degrees, $Y = 3$ years, and the initial inclination and nodal period are not biased, the upper bounds for the 6, 12, and 24 hour orbits are

$$\left| \Delta V \right| \leq 0.45,\ 1.3,\ \text{and } 0.71 \text{ deg;}$$

$$\left| \Delta L \right| \leq 0.84,\ 1.7,\ \text{and } 3.4 \text{ deg;}$$

and

$$\left| \Delta L + \Delta V \right| \leq 1.0,\ 2.3,\ \text{and } 3.6 \text{ deg}$$

respectively.

For the 12-hour orbit the above says that the satellite is never more than 2.3 degrees out of position during the first three years. Note that the two errors act at right angles when the latitude error is maximum. Hence 2.3 degrees, not 3.0 degrees, is the upper bound for the combined error.

If the initial nodal periods and inclinations of the orbits are biased as discussed in connection with (28) and (29) above then the upper bounds for the 6, 12, and 24 hour circular, 30 degree orbits are

$$\left| \Delta V \right| \leq 0.11,\ 0.31,\ \text{and } 0.18 \text{ deg;}$$

$$\left| \Delta L \right| \leq 0.42,\ 0.85,\ \text{and } 1.8 \text{ deg;}$$

and

$$\left| \Delta L + \Delta V \right| \leq 0.45,\ 0.9,\ \text{and } 1.9 \text{ deg}$$

respectively.

The upper bounds for changes in inclination were derived under the assumption that the satellite's node was initially at the most unfavorable position. Since no allowance was made for nodal drift the upper bounds for changes in i are conservative especially over long periods of time. This in turn makes the upper bounds for perturbations in true anomaly and latitude conservative.

Actually lunar and solar perturbations in inclination are periodic functions of Ω_m and Ω_c respectively. These parameters are closely related to the satellite's equatorial node Ω. This parameter has a period of 3.4, 15, and 87 years for the 6 hour, 12 hour, and 24 hour orbits respectively. This is obtained from (7) with i taken equal to 30 degrees.

LIST OF SYMBOLS

a	=	semi-major axis of the satellite's orbit
a_m	=	semi-major axis of the moon's orbit about the earth
a_c	=	semi-major axis of the earth's orbit about the sun
d	=	differential of a parameter
e	=	eccentricity of the satellite's orbit
i, i_m	=	inclination of the satellite's orbit to the equatorial or lunar plane respectively
$\left\|\Delta i_m\right\|$	=	an upper bound to perturbations in i_m due to lunar forces
$d\,i_l$	=	change in i due to lunar forces
$d\,i_s$	=	change in i due to solar forces
I_m	=	inclination of the lunar orbit to the equatorial plane
I_c	=	inclination of the earth's orbit to the equatorial plane
H_m, H_c	=	perturbation constants: see (13), (16) and (19)
J	=	oblateness constant ($J = 0.001624$)
M_m, M_c, M_e	=	mass of the moon, sun, and earth respectively
n	=	satellite revolutions per day
P_k	=	Kepler period of the satellite
P_{Ne}	=	one nth of the time required for the earth to rotate once relative to the satellite's node; see (2)
P_{Ns}	=	nodal period of the satellite
R	=	earth's equatorial radius
V	=	satellite's angular position measured from the ascending equatorial node
ΔV	=	error or change in V

Y	=	time in years
Δ	=	change or perturbation in a parameter
ω_e	=	earth's angular rate of rotation relative to inertial space
Ω	=	longitude of the satellite's equatorial node
Ω'	=	drift rate of the satellite's equatorial node
Ω_m	=	angle from the lunar equatorial node to the satellite ascending lunar node
Ω_c	=	longitude of the satellite ascending ecliptic node
$\Delta\Omega_m$	=	perturbation in Ω_m due to lunar forces
$\Delta\Omega_c$	=	perturbation in Ω_c due to solar forces

FOOTNOTES

1. L. Blitzer, "On the Motion of a Satellite in the Gravitational Field of the Oblate Earth," GM-TM-0165-00279, Space Technology Laboratories, Inc., 5 September 1958.

2. R. L. Dunn, "Some Perturbations of a Class of Critically Inclined Orbits," 9861.3-105, Space Technology Laboratories, Inc., 14 November 1961.