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ON THE DIRECT SHOCK WAVE PROBLEM

Report No. II

"A Modified Theory (Final Report on Bodies with Spherical Nose)"

by

S. S. Shu

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Notations

$u = u(r, s)$ = the r-component of the velocity after the shock

$v = v(r, s)$ = the θ-component of the velocity after the shock

p = the pressure

ρ = the density

ψ = the stream function

(r, θ) = the spherical coordinates on the plane passing the axis of rotational symmetry

$s = \sin^2 \frac{\theta}{2}$

c = the maximum velocity of the flow

p_o and ρ_o are respectively the stagnation pressure and density

$$\bar{u} = u/c \,, \quad \bar{v} = v/c \, \frac{1}{\sqrt{s(1-s)}} \,, \quad p^* = \frac{p}{p_o}$$

$$\rho^* = \rho/\rho_o \quad \text{and} \quad L = \log p^*$$

The equation of Shock Wave:

$$r = g(s) = g_o + g_1 s + g_2 s^2 + \dots = \frac{g_o\left[1+(G_1-2\epsilon)s+(G_2-2\epsilon G_1)s^2+\dots\right]}{1-2\epsilon s}$$

where $\epsilon = \dfrac{1}{1+\sqrt{\dfrac{1-N\mu}{1-\mu}}} = \dfrac{1}{1+\sqrt{\dfrac{M^2-1}{M^2}}}$

$$g_o = 1 + \Delta$$

$$G_1 = \frac{g_1}{g_o} = 2+x, \quad G_2 = \frac{g_2}{g_o} = 4 + z$$

M = the free stream Mach number

γ = the ratio of specific heats

$$N = \frac{2}{\gamma - 1} \quad \frac{1}{M^2} + 1$$

$$\mu = \frac{\gamma - 1}{\gamma + 1}$$

$$N\mu = y$$

α, β, δ, η, and λ are parameters occurring in the coefficients of Taylor expansion of flow quantities in the neighborhood of the stagnation point.

The radius of the spherical body is normalized as unity.

I. Introduction

The present report is the continuation of Report I. The formulation of the problem has been given there (pp. 1-15). The method of attack (for the general body problem) will be modified as follows:

(1) To find exact relations between the parameters (in the sense that the results do not depend on the number of terms taken in the expansion). Using the same notations as in Report I, the following results are obtained:

$$\beta = \frac{1}{4}\alpha \tag{I}$$

$$\delta = \frac{-1}{32}\alpha - \frac{1}{2}\alpha^3 - \frac{1}{2}\alpha^5 \tag{II}$$

$$\alpha + \eta = \frac{(G_1-2)^2 (1-y)^2}{4N^{3/2}\mu g_o^2 (1-N\mu)^{\frac{1+\mu}{2\mu}}} \tag{III}$$

and a fourth relation involving λ, $\frac{g_1}{g_o}$, $\frac{g_2}{g_o}$ besides those which appear above. The results are obtained by utilizing the isentropic property on a stream line and the Bernoulli's equation involving the stream function ψ.

(2) The final solution of expressing all the parameters in terms of M and μ is obtained by the matching conditions of u and v from the body to the shock by the power series expansions in a transformed variable. The expansion has to converge fast enough in order to yield accurate results. Because of this consideration, we have modified our procedure in the following two ways:

(a) Apply the transformation

$$r^* = \log r$$

remove the possible singularity due to the coordinates at $r = o$ and reduce the body equation to $r^* = o$.

(b) For high free-stream Mach numbers, the convergence is seen to be quite good, while for low supersonic Mach numbers $\leqslant 2$, the convergence is worse (the series becomes oscillating; namely, the terms are alternatively positive and negative, while their magnitudes become the same order with the preceding term). Fortunately, in the present case, when free stream Mach number is down to 1.2, the detached shock distance is 0.8, the power series method still yields quite good results providing a transformation of independent variables is applied. In our calculation, we use

$$\xi = \frac{r^*}{N(y-\mu)r^* + \frac{y}{A}}$$

or $$r^* = \frac{\frac{y}{A}\,\xi}{1 - N(y-\mu)\,\xi}$$

where $$y = N\mu = \frac{\gamma - 1}{\gamma + 1} + \frac{2}{\gamma + 1}\,\frac{1}{M^2}$$

$$A = 2\alpha\sqrt{N} = 2\alpha\sqrt{1 + \frac{2}{\gamma - 1}\,\frac{1}{M^2}}$$

This transformation is motivated by the requirement that no change of variable is necessary at high Mach number M and a singularity inside the body near the boundary may occur at low supersonic Mach number M. This is seen from the following considerations: The fundamental equations are

$$\bar{u}\,\bar{u}_{r*} + \bar{v}\,\bar{v}_s\, s(1-s) - \bar{v}^2 s(1-s) = -\frac{\mu}{1+\mu}[1-\bar{u}^2-\bar{v}^2 s(1-s)]\,L_{r*} \tag{I}$$

$$\bar{u}\,\bar{v}_{r*} + \bar{v}\,\bar{v}_s\, s(1-s) + \bar{v}^2\left(\tfrac{1}{2}-s\right) + \bar{u}\,\bar{v} = -\frac{\mu}{1+\mu}[1-\bar{u}^2-\bar{v}^2 s(1-s)]\,L_s \tag{II}$$

$$2\bar{u} + \bar{v}(1-2s) + \bar{u}_{r*} + \bar{v}_s s(1-s) + \frac{1-\mu}{1+\mu}[\bar{u}L_{r*} + \bar{v}s(1-s)L_s] = o \tag{III}$$

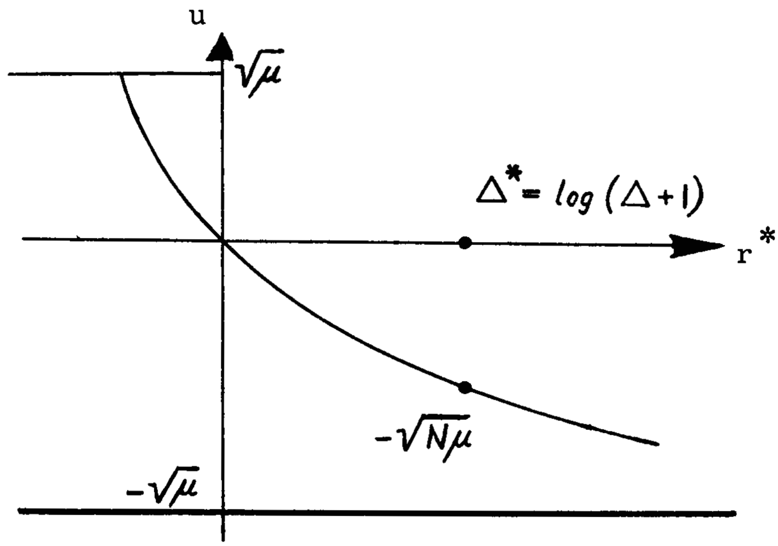
where $r* = \log r$, $\quad s = \sin^2\frac{\theta}{2}$

Note that the logarithmic pressure $L = \log\frac{p}{p_o}$ where p_o is the stagnation pressure and the velocity components $\bar{u} = u/c$ and $\bar{v} = \frac{v}{c}/s(1-s)$ are different from the notations used in Report I.

When $\underline{s = o}$, the equations (I) and (III) yield

$$(\bar{u}^2 - \mu)\frac{d\bar{u}}{dr*} = \mu\,(2\bar{u} + \bar{v})$$

The singularity will likely occur when $\bar{u} = \pm\sqrt{\mu}$



Since, for $M > 1$, $N\mu^2 = \mu\left[\frac{\gamma-1}{\gamma+1} + \frac{2}{\gamma+1}\ \frac{1}{M^2}\right] < \mu$

hence $\sqrt{N\mu} < \sqrt{\mu}$.

Now, $u = -\sqrt{N\mu}$ at $r^* = \log(1+\Delta)$.

Therefore the singularities may occur at a point inside the body as well as beyond the shock. In order to improve the convergence of any expansion at the body when $\theta = o$, this singularity should be transformed into a point far away from the body $r^* = o$.

(3) The equation of shock wave in the original series form will be only valid in the neighborhood of the nose. In order to calculate the flow field in the large, we write the equation of shock wave as

$$r = g(s) = \frac{1}{1+\sqrt{\dfrac{1-N\mu}{1-\mu}}\,s}\left[\,1+(G_1-2\epsilon)\,s+(G_2-2\epsilon G_1)\,s^2+\ldots\right]$$

The denominator is introduced to insure that the surface of the shock will tend to the Mach cone at infinity. As the numerical results show, this expression converges quickly and yields the location of sonic point on the shock quite accurately even if only g_o and G_1 are taken into consideration. This is not a surprise as such will be a conic section. *

* See also Ref. 1.

II. Presentation of Analytical Results

The Match conditions for the velocity components $\bar{u}$ and $\bar{v}$ are applied (as it will be discussed in detail in V.), we find the following results:

(1)

$$x = G_1 - 2 = \frac{(1.5-2.5\mu)\,y - 1.927 + 3.914\mu}{1 - 2\mu} \tag{1}$$

This formula compares with Van Dyke's numerical results in excellent agreement. From his definition of shock, we have the following relation

$$-x = 2\left[\frac{R_b + \Delta}{R_s}\right]$$

where R_b = the radius of curvature of the body,

R_s = the radius of curvature of the shock (at $s = o$)

Δ = the detached shock distance

(2)

The detached shock distance is given by the equation:

$$\Delta^* = \frac{y}{\phi - \frac{y}{2}} + \frac{\frac{Q}{2}\Delta^{*2}}{\phi - \frac{y}{2}} \tag{2}$$

if the higher order terms in Δ^{*3} are neglected, where

$$y = N\mu = \frac{\gamma - 1}{\gamma + 1} + \frac{2}{\gamma + 1} \quad \frac{1}{M^2}$$

$$\phi = -x + y\,(2 + x)$$

$$x = G_1 - 2 \qquad \text{which is given by (1)}$$

and

$$Q = \frac{x^2 (1-y)^2 e^{-2\Delta^*}}{2y[\,1 - y\mu\,]^{\frac{1+\mu}{2\mu}}} \tag{3}$$

The equation is solved by successive approximation. The result of shock distance (Fig. 1) is obtained from the first approximation

$$\Delta^* = \frac{y}{\phi - \frac{y}{2}} + \frac{\frac{Q}{2}\,\Delta_o^{*2}}{\phi - \frac{y}{2}} \tag{2a}$$

where $\Delta_o^* = \dfrac{y}{\phi - \frac{y}{2}}$ and the quantity Δ^* in Q is also substituted by Δ_o^*.

(3) The Velocity Gradient along the Body at the Stagnation.

The equation for the determination of $A = \sqrt{N}\,2\,\alpha$ is

$$\phi = A + \left(\frac{Q}{A} - 1\right) \frac{\Delta^*}{(N-1)\,\Delta^* + \frac{1}{A}} +$$

$$+ \left[\frac{1-\mu}{4y} A + \frac{Q+A}{2A^2} + N\left(1 - \frac{1}{N}\right)\left(\frac{Q-A}{A}\right)\right] \frac{\Delta^{*2}}{[\,(N-1)\Delta^* + \frac{1}{A}\,]^2} \tag{6}$$

where

$$Q = \frac{(1-y)^2 x^2}{2 g_o^2 y [1 - y\mu]^{\frac{1+\mu}{2\mu}}} = \frac{(1-y)^2 x^2 e^{-2\Delta*^2}}{2y[1 - y\mu]^{\frac{1+\mu}{2\mu}}}$$

$$x = G_1 - 2, \qquad \phi = -x + y(2+x)$$

The first approximation is given by neglecting the term $\Delta*^2$:

$$A = \phi + \frac{\Delta* A}{A(N-1)\Delta* + 1} - \frac{Q\Delta*}{1 + A(N-1)\Delta*}$$

or

$$A^2 (N-1)\Delta* + A[1 - (N-1)\Delta*\phi - \Delta^*] - (\phi - Q\Delta^*) = 0$$

For the determination of the sonic point and the velocity field in the large, (for the latter, Jacobi's expansion is used) see Section VI.

III. The Calculation of the Entropy as a Function of ψ Based on the Shock Shape.

From the equation of continuity, one may define the stream function as follows:

$$\psi\ (r', \theta') = 2\pi\,[\,\int_\Gamma \rho\, r v \,\mathrm{Sin}\,\theta\, dr - \int_\Gamma \rho\, r^2 u \,\mathrm{Sin}\,\theta\, d\theta\,] \tag{7}$$

Where Γ is a path in the fluid from $(1, o)$ to (r', θ'). The constant factor 2π is added so that the ψ has the interpretation of mass flow for the case of rotational symmetry.

On the shock

$$r = g_o + g_1 s + g_2 s^2 + \ . \ . \ .$$

$$\psi = \rho_1 U \pi\ (r\,\mathrm{Sin}\,\theta)^2 = 4\pi \rho_1 U g_o^{\,2} [\ s + (2G_1 - 1)\, s^2 \tag{8}$$

$$+ (2G_2 + G_1^{\,2} - 2G)\, s^3 + . \ . \ . \ .]$$

where ρ_1 = the free stream density
U = the free stream velocity
G_i = g_i / g_o , $i = 1, 2, \ldots .$

Thus, on the shock,

$$s\ = \overline{\psi} - (2G_1 - 1)\overline{\psi}^2 + \{2(2G_1 - 1)^2 - 2G_2 - G_1^{\,2} + 2G_1\}\,\overline{\psi}^3 \tag{9}$$

where

$$\overline{\psi} = \frac{1}{4\pi\rho_1 U g_o^{\,2}}\ \psi \tag{10}$$

Define the dimensionless entropy as

$$C(\psi) = \frac{p^*}{\rho^{*\gamma}} = \left(\frac{p}{p_o}\right) / \left(\frac{\rho}{\rho_o}\right)^\gamma .$$

On the shock wave, both p^* and ρ^* are known functions of s, if one assumes the shock shape $r = g(s)$ or

$$r = \log g(s) = h(s) = \Delta^* + h_1 s + h_2 s^2 + \ldots$$

where $\Delta^* = \log(1 + \Delta) = \log g_o$

$$h_1 = G_1$$

$$h_2 = G_2 - \frac{G_1^2}{2}$$

Consequently, on the shock, $\log C(\psi)$ can be computed as a power series* of s:

$$\begin{aligned}
\log C(\psi) &= \log \frac{p^*}{\rho^{*\gamma}} = \log\left(\frac{p^*}{\rho^*}\right)^\gamma + \log p^{*\,1-\gamma} \\
&= \log\left[1-\bar{u}^2 - \bar{v}^2 s(1-s)\right]^{\frac{1}{1-N\mu^2}} \\
&+ \log\left[1-2s+2h's(1-s)\right]^{\frac{-2\mu}{1-\mu}} \\
&+ \log\left[\frac{-\bar{u} + \bar{v}\, s(1-s)h'}{\sqrt{N\mu}}\right]^{\frac{2\mu}{1-\mu}}
\end{aligned} \tag{11}$$

---

* The condition of L on the shock is given on p. 14 in Report I.

$$L = \log\left\{\frac{(\gamma-1)\rho_1 \bar{U}}{2\gamma p_o C}\left[1-\bar{u}^2-\bar{v}^2 s(1-s)\right]\frac{g\cos\theta + \frac{dg}{d\theta}\sin\theta}{\frac{dg}{d\theta}\bar{v}\sqrt{s(1-s)} - \bar{u}g}\right\}$$

By means of (9), it is expressed in terms of $\bar{\psi}$:

$$\log C(\psi) = A_1 \bar{\psi} \quad A_2 \bar{\psi}^2 + \dots \tag{12}$$

where

$$A_1 = -\frac{(1+\mu)\mu x^2 (1-y)^2}{y} \quad \frac{1}{(1-\mu)(1-y\mu)} \tag{13}$$

$$x = G_1 - 2, \qquad y = N\mu$$

From (11), and (9) and the power series expansion of $\bar{u}$ and $\bar{v}$ on the shock A_2 can be computed in terms of g_o, G_1, and G_2 as well as

$$\mu = \frac{\gamma - 1}{\gamma + 1} \quad \text{and} \quad N = 1 + \frac{2}{\gamma - 1} \; \frac{1}{M^2}$$

Finally, one may rewrite (7) as

$$\bar{\psi} = K_o \left[\int_\Gamma \rho^* \bar{u} \, (1+R)^2 \, ds - \int_\Gamma \rho^* \bar{v} \, s \, (1-s) \, (1+R) \, dR \right] \tag{14}$$

Where Γ is a curve connecting (o, o) to (R, s) in the Rs plane; $R = r-1$, $s = \sin^2 \frac{\theta}{2}$; $\bar{u} = \frac{u}{c}$ and $\bar{v} = \frac{v}{c} \Big/ \sqrt{s(1-s)}$ are functions of R and s;

$$K_o = \frac{1}{g_o^2} \; \frac{\rho_o C}{\rho_1 U} = \frac{1}{g_o^2} \; \frac{1}{\sqrt{N\mu} \, [\, 1 - N\mu^2 \,]^{\frac{1-\mu}{2\mu}}} \tag{15}$$

IV. Derivation of the Exact Relations

We shall now utilize the fact that the entropy is constant along the stream line. Thus we have

$$\log p^* - \gamma \log \rho^* = \log C(\psi) = \overline{A}_1 \overline{\Psi} + \overline{A}_2 \overline{\Psi}^2 + \ldots . \tag{16}$$

On the other hand, Bernoulli's equation in the dimensionless form along a stream line is

$$\frac{p^*}{\rho^*} = 1 - \overline{u}^2 - \overline{v}^2 \; s(1-s) \tag{17}$$

or

$$\log p^* - \log \rho^* = \log [\, 1 - \overline{u}^2 - \overline{v}^2 \; s(1-s)] \tag{18}$$

From (10) and (11), we have

$$L - \frac{1+\mu}{2\mu} \log [\, 1 - \overline{u}^2 - \overline{v}^2 \; s(1-s)]$$

$$= - \frac{1-\mu}{2\mu} [\, \overline{A}_1 \overline{\Psi} + \overline{A}_2 \overline{\Psi}^2 + \ldots .] \tag{19}$$

where $L = \log p^*$.

Now, consider (19) for small values of $R = r-1$ (namely, in the neighborhood of the body), the flow quantities L, $\overline{u}$, $\overline{v}$ can be expanded in power series of R:

$$L = L_o(s) + L_1(s) R + \frac{1}{2} L_2(s) R^2 + \ldots .$$

$$\overline{u} = u_1(s) R + \frac{1}{2} u_2(s) R^2 + \ldots$$

$$\overline{v} = v_o(s) + v_1(s) R + \frac{1}{2} v_2(s) R^2 + \ldots$$

By means of (17) and (9), the coefficients of $\frac{\partial \psi}{\partial R}$ can be explicitly expressed in terms of these coefficients. Differentiating (19) with respect to R and comparing the coefficient of R power, the first term yield the relation

$$L_1 + \frac{1+\mu}{2\mu} s(1-s)\, 2\, v_o v_1 \frac{1}{1-v_o^2 s(1-s)}$$
$$= -\frac{1-\mu}{2\mu} \overline{A}_1 v_o \frac{s(1-s)\left[1-s(1-s)v_o^2\right]^{\frac{1-\mu}{2\mu}}}{g_o^2 N^{1/2}\mu\left[1-N\mu^2\right]^{\frac{1-\mu}{2\mu}}} \tag{20}$$

From the fundamental equation (1) (Report I, p. 2)

$$L_1(s) = \frac{1+\mu}{\mu} \frac{s(1-s)\, v_o^2}{1-v_o^2 s(1-s)} \tag{21}$$

Thus, (20) and (21) give an exact relation

$$v_1(s) + v_o(s) = q\left[1-v_o^2 s\ (1-s)\right]^{\frac{1+\mu}{2\mu}} \tag{22}$$

which should be satisfied for all values of s, where

$$q = \frac{(1-y)^2 x^2}{2N^{3/2}\mu\, g_o^2\left[1-N\mu^2\right]^{\frac{1+\mu}{2\mu}}}$$

In particular, when $s = 0$, it yields

$$\alpha + \eta = \frac{q}{2}$$

The coefficient of s yields an identity, while that of s^2 gives:

$$-\delta = \frac{\alpha}{32} + \frac{1}{2}\alpha^3 + \frac{1}{2}\alpha^5 \tag{25}$$

Compare the coefficient of R^2 in (19), we have the relation:

$$\frac{1}{2} L_2 + \frac{1+\mu}{2\mu}\left[\frac{s(1-s)}{F}\left(\frac{u_1^2}{s(1-s)} + v_1^2 + v_o v_2\right) + \frac{s^2(1-s)^2}{F^2} 2 v_o^2 v_1^2\right]$$

$$= -\frac{1-\mu}{2\mu} A_1 K_o \frac{s(1-s)}{2F}\left(p_o^* v_1 + p_1^* v_o + p_o^* v_o + \frac{s(1-s)}{F} 2 v_o^2 v_1 p_1^*\right)$$

$$- A_2 K_o^2 p^{*2} v_o^2 s^2 (1-s)^2 \frac{1}{F^2} \tag{26}$$

where $F = 1 - v_o^2 s(1-s)$,

$$p^* = p_o^* + p_1^* R + \frac{1}{2} p_2^* R^2 + \dots$$

$$= e^L,$$

A_1 is given by (13), K_o by (15), and A_2 can be explicitly expressed in terms of g_o, G_1, G_2, N and μ.

Since (26) holds for all s (small enough), the coefficients yield successively, an identity,

$$\beta = \frac{1}{4}\alpha, \tag{27}$$

and a relation involving λ, n_2 and others already appeared. The last mentioned relation is not used in the present Report.

These exact relations (24), (25) and (26) are not found before.

V. Matching Conditions

The fundamental equations (1), (II), and (III) in the Introduction I, yield all the coefficients of $\bar{u}$, $\bar{v}$, and L as power series* of R in terms of $v_o(s)$ and its derivatives in s. The shock conditions yield all physical variables $\bar{u}$, $\bar{v}$ and L in terms of g(s) on the shock. In order to determine v_o and g, it is necessary to relate them by the matching conditions of $\bar{u}$ and $\bar{v}$ from some expansion (which is equivalent of the integration of the differential equation) from the body to the shock up to some powers of s besides those relations obtained from the constancy of entropy along a stream line.

In the present report, if only α, β, δ, η, g_o (or Δ, or $\log(1+\Delta) = \Delta^*$) and G_1 (or x) are to be determined, three matching conditions are taken for both $\bar{u}(r^*, o)$ and $\bar{v}(r^*, o)$ and also for $\frac{\partial \bar{u}}{\partial s}$ at $r^* = \log(1+\Delta) = \Delta^*$, namely

$$\bar{u}(\Delta^*, o) = -\sqrt{N}\mu$$
$$= u_{o,o} + u_{1,o}\Delta^* + \frac{1}{2}(u_{2,o} + u_{1,o})\Delta^{*2}$$
$$+ \frac{1}{6}(u_{3,o} + 3u_{2,o} + u_{1,o})\Delta^{*3} + \ldots$$

Thus

$$-y = -A\Delta^* + \left(\frac{3}{2}A - \frac{Q}{2}\right)\Delta^{*2} + \left[-\frac{5}{12}(1-\mu)\frac{A^3}{y} - \frac{7}{6}A + \frac{1}{6}Q\right]\Delta^{*3} \quad (28a)$$

where $y = N\mu$

$A = \sqrt{N2}\,\alpha$

* cf. the Appendix

$$Q = \sqrt{N}q = \frac{(1-y)^2 x^2 e^{-2\Delta^*}}{2y[1-y\mu]^{\frac{1+\mu}{2\mu}}}$$

$$x = G_1 - 2$$

Similarly, from that for $\bar{v}$, we have

$$\phi \equiv -x+y(2+x) = A+(Q-A)\Delta^* + \left(\frac{1-\mu}{4y}A^3 + \frac{Q+A}{2}\right)\Delta^{*2} \quad (29)$$

The expression for $\frac{\partial \bar{u}}{\partial s}$ at $r^* = \Delta^*$ and $s = o$ is

$$u_s(\Delta^*, o) = u_1^o - (2+x)u_{r*}(\Delta^*, o)$$

since $$\frac{d\bar{u}^{(o)}}{ds} = \bar{u}_s + \bar{u}_{r*} h'(s)$$

where $$\left.\frac{d\bar{u}^{(o)}(s)}{ds}\right/_{s=o} = \bar{u}_1^o \text{ and } \bar{u}^{(o)}(s) = \bar{u}(h(s), s).$$

The subscript indicates the partial derivative.

Consequently, we have

$$-2x(2+x) + 2y(1+x) + \mu[2xy + x^2y + x^2]$$

$$= \frac{1}{2}\frac{A^3\Delta^*}{y}(1-\mu) + \left[Q + \frac{A^3}{y}\mu + \frac{1}{4}A^2\frac{3+\mu}{y}(Q-A)\right]\Delta^{*2} \quad (30)$$

The unknowns to be solved in (28a), (29) and (30) are $\Delta^* = \log(1+\Delta)$, $x = G_1 - 2$, and $A = \sqrt{N}2\alpha$.

From (28a) and (29), one derives

$$(\phi - \frac{y}{2}) - \frac{y}{\Delta^*} = \frac{Q}{2}\Delta^* + (-\frac{1-\mu}{6}\frac{A^3}{y} + \frac{1}{12}A + \frac{5}{12}Q)\Delta^{*2} \quad (31)$$

This expression is exact up to the order of Δ^{*2}. Hence, when the terms of Δ^{*3} are neglected, as compared to Δ^*, one obtains the following relation involving only Δ^* and **x**;

$$\Delta^* = \frac{y}{\phi - \frac{y}{2}} + \frac{Q}{2}\Delta^{*2}\frac{1}{\phi - \frac{y}{2}} \quad (32)$$

Since the numerical value of $\Delta^* = \log(1 + \Delta)$ turns out to be $\leqslant 0.6$ for $M \geqslant 1.2$, the equation (32) should be a good relation for a considerable range of values of M (provided that it is not too close to 1). With x to be determined later as functions of y and μ or M and γ, the detached shock wave distance Δ calculated from (32) is in excellent agreement with existing theoretical results [(1)] as well as experimental results [(11)] as indicated in the Figure 1.

The equation (29) yields A as a function of x, for which the approximation solution of x of (30) is obtained by solving x in terms of μ for various values of y or M ($y = \mu$ and $y = \frac{1}{2}$). This yields

$$x = \frac{(1.5 - 2.5\mu)y - 1.927 + 3.914\mu}{1 - 2\mu} \quad (33)$$

The comparison with Van Dyke's result gives quite good agreement for all $M \geqslant 1.2$. As expected, the result in A is also in good agreement (with a few per cents) except when $M < 1.8$, the discrepency becomes larger. This is not a surprise as all the expansions become alternative positive and negative while the Δ^* and the coefficients of its powers increase as M decreases (in the range $1.2 \leqslant M < 2$). This means the convergence of the power series becomes worse. The reason for this is due to the singularity of the solution, which occurs inside the body and gets closer to its boundary when M decreases. For this reason, as mentioned in the introduction, the transformation

$$\xi = \frac{r^*}{N(y-\mu)r^* + \frac{y}{A}}$$

is applied. The transformation is constructed based on the consideration that it reduces to identity transformation while $M = +\infty$ or $N = 1$, and it transforms a point close to the boundary inside the body into infinity for low supersonic M. The modified equation for A is given as in the introduction. While the modifications in x and Δ^* are small, we still use the expressions as obtained above.

The results thus obtained for A is used for

$$\alpha^* = \frac{A}{2\sqrt{N\mu}} = \frac{\alpha}{\sqrt{\mu}}$$

(the velocity gradient along the sphere at the stagnation) as a function of M. As shown in Figure 3, the agreement with the experimental values [11] is very good. The pressure distribution on the body is shown in Fig. 2 (for $\gamma = 1.4$).

VI. Discussion of the Results and Conclusions. (The Sonic Point and the Sonic Line)

As a further test for the present theory the sonic point and the Mach contour such as the sonic line are computed.

Since at the sonic point on the body $r^* = o$, the directionless velocity $v^* = \bar{v}\sqrt{s(1-s)}$ is a known value $\sqrt{\mu}$, thus one has the following relation for the determination of the sonic point or for the values of $\sigma = \sqrt{s(1-s)} = \frac{1}{2}\sin\theta$:

$$\frac{\sqrt{y}}{\sigma} = A + A\sigma^2 - A\left(\frac{1}{2} + \frac{2A^2}{N} + \frac{A^4}{2N^2}\right)\sigma^4 \tag{35}$$

when $y = N\mu$, $A = \sqrt{N}2\alpha$

or

$$\frac{\sqrt{y}}{A} = \sigma + \sigma^3 - \left(\frac{1}{2} + \frac{2A^2}{N} + \frac{A^4}{2N^2}\right)\sigma^5 \tag{35a}$$

For $M \geqslant 1.2$, $\frac{y}{A} < .5$. Thus the solution for σ in (35a) can be expressed as the power series of $\frac{\sqrt{y}}{A}$; i. e.,

$$\sigma = k - k^3 + 3k^4 + \left(\frac{-5}{2} + \frac{2A^2}{N} + \frac{A^4}{2N^2}\right)k^5,$$

where $k = \frac{\sqrt{y}}{A}$. The results thus obtained are plotted for various Mach numbers as in Figure 4, which is again in good agreement of the experiment.[11]

Finally, for the velocity field as a whole, we shall use the data thus obtained in conjunction of the use of Jacobi's expansion. For instance, as it is checked above that the velocity component $\bar{v}\sqrt{s(1-s)}$ along the body is given, at least, accurately up to the sonic point as a power series of s with the parameter M and μ, we may utilize the fact that $\bar{v}$ is also known at the sonic point $s = s_o$ to construct the Jacobi's expansion of $\bar{v}$. This partial sum of Jacobi's expansion is expected to give good approximation for s beyond the sonic point s_o. Thus, at $r^* = o$,

$$\sqrt{N}\, v_o(s) = \left(1 - \frac{s^3}{s_o^3}\right) A + \frac{s}{s_o}\left(1 - \frac{s^2}{s_o^2}\right) A - \frac{s^2}{s_o^2}\,\frac{1}{2}\left(3 + \frac{4A^2}{N} + \frac{A^4}{N^2}\right)$$

$$+ \left(\frac{s}{s_o}\right)^3 \sqrt{\frac{y}{s_o(1-s_o)}} \tag{37}$$

Since the exact relation

$$v_o + v_1 = q\, F^{\frac{1+\mu}{2\mu}} = q\left[1 - v_o^2 s(1-s)\right]^{\frac{1+\mu}{2\mu}} \tag{38}$$

holds, this gives v_1 immediately on any point on the body. Now, for a fixed value of s, utilizing the data of $\bar{v}$ and $\bar{v}_{r^*}$ at $r^* = 1$, and $\bar{v}$ on the shock as given by:*

$$\sqrt{N}\,\bar{v} = (1+\mu) - (1-\mu)\,\frac{h'(s)\,(1-2s)}{1+h'^2 s(1-s)} + \frac{1-\mu}{M^2}\,\frac{h'}{1-2s+2h's(1-s)}$$

$$+ (1-\mu)\,\frac{1-h'^2 s(1-s)}{1+h'^2 s(1-s)}$$

One can form the Jacobi expansion for $\bar{v}$ in r^*.

Similarly, with the data of $\bar{u}$, (i. e., u_1 and u_2) on the body and that of $\bar{u}$ on the shock given by

* See s'_2 or (12) on Report I.

$$\sqrt{N}\,\bar{u} = -\frac{1+\mu}{2}(1-2s) + \frac{1-\mu}{2}(1-2s)\,\frac{1-h'^2 s(1-s)}{1+h'^2 s(1-s)}$$

$$-(1-\mu)\frac{1}{M^2}\,\frac{1}{1-2s+2h's(1-s)}$$

$$+\ (1+\mu)\,\frac{2h's(1-s)}{1+h'^2 s(1-s)} \tag{40}$$

where
$$h' = \frac{\frac{dg}{ds}}{g}$$

$$g = \frac{1}{1+\sqrt{1-\frac{1}{M^2}}\,s}\left[1+\left(G_1-\sqrt{1-\frac{1}{M^2}}\right)s + \left(G_2-\sqrt{1-\frac{1}{M^2}}\right)s^2\right] + \ldots \tag{41}$$

The Jacobi's expansion for $\bar{u}$ in r^* can be formed. This will give the velocity field which is accurate enough in a region beyond the subsonic region.

As for the sonic line, its equation is given by

$$u^{*2} + v^{*2} \equiv \bar{u}^2 + \bar{v}^2\ s(1-s) = \mu. \tag{42}$$

Now, if one expresses s as function of r^* for (42):

$$s = s(r^*)$$

the data* of $s(o)$ and $s'(o)$, $s''(o)$ are computed from (42) so is s at $r^* = h(s_o)$ from the shock conditions and (42). The sonic line can then be plotted by Jacobi's expansions of s in r^*. The result is indicated in Figure 5.

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* The slope is very accurate provided s_o is because of the exact relation.

In conclusion, the present theory has established exact relations between the derivatives of flow velocity components and the shock shape parameters. Additional matching conditions on velocity components yield all parameters explicitly in terms of

$N = 1 + \frac{2}{\gamma - 1} \frac{1}{M^2}$ and $\mu = \frac{\gamma - 1}{\gamma + 1}$. The matching conditions yield

excellent results for all $M \geqslant 1.2$ if the power series of flow velocity components are expressed in an appropriately transformed variable ξ instead of r^*, which is motivated by the singularity inside the body (by analytic extension of the solution). Then the velocity field is constructed by the use of Jacobi's expansion, which is accurate enough for a domain covering the whole sub-sonic region as a sub-domain. A better result can immediately be obtained if the exact relation involving λ and G_2, and a matching condition on $v_s = \frac{\partial v}{\partial s}$ are taken into consideration. This consideration has not carried out in the present report.

VII. Appendix

(A) Derivatives on the Body

For the sake of self-containing the derivatives (at the body) of the flow velocity components and the logarithmic pressure $L = \log \frac{p}{p_o}$ up to the second powers are listed below*:

$$\bar{u} = \frac{u}{c} = u_o(s) + u_1(s)\,(r-1) + \frac{1}{2}\,u_2(s)\,(r-1)^2 + \ldots$$

$$v = \frac{v}{c}\;\frac{1}{\sqrt{s(1-s)}} = v_o(s) + v_1(s)\,(r-1) + \frac{1}{2}\,v_2(s)\,(r-1)^2 + \ldots$$

$$L = \log \frac{p}{p_o} = L_o(s) + L_1(s)\,(r-1) + \frac{1}{2}\,L_2(s)\,(r-1)^2 + \ldots$$

where $u_o(s) \equiv o$

$$u_1(s) = -2\alpha + (4\alpha - 16\beta \quad \frac{4\alpha^3(1-\mu)}{\mu})\,s^2$$

$$+ \frac{4(1-\mu)}{\mu}\,(20\,\alpha^3 - 3\alpha^3 + 6\beta\,\frac{2\,\mu}{1-\mu} + 4\alpha^5 - \frac{24\,\mu\,\delta}{1-\mu})\,s^2 + \ldots$$

$$u_2(s) = 2\,(3\,\alpha-\eta) + [\,64\,\beta - 4\,(3\,\alpha-\eta) + \frac{4\alpha^3(5\mu-1)}{\mu} + \frac{4\alpha^2\eta(3+\mu)}{\mu}]\,s$$

$$+ [\,96\delta - \frac{1+\mu}{\mu}\quad 192\,\alpha^2\beta - \frac{12\alpha^5(3+4\mu+5\mu^2)}{\mu^2}$$

$$-144\beta^2 + \frac{32(5+\mu)}{\mu}\,\alpha\beta\eta + \frac{(5+\mu)\,(1-\mu)}{\mu^2}\,4\,\alpha^4\eta$$

$$-192\,\beta - \frac{5\,\mu-1}{\mu}\,12\alpha^3 - \frac{\mu+3}{\mu}\,12\,\alpha^2\eta\,]\,s^2 + \ldots$$

$$v_o(s) = 2\,\alpha + 8\,\beta\,s + (32\,\delta - 8\beta)\,s^2 + \ldots$$

$$v_1(s) = 2\,\eta + (-\frac{1+\mu}{\mu}\,4\,\alpha^2\,\eta - \frac{1+\mu}{\mu}\,4\,\alpha^3 - 8\,\beta)\,s$$

* Several misprints in Report I are corrected.

$$+ [- \frac{1+\mu}{\mu} 32\,\alpha\beta\eta + \frac{5\mu-1}{\mu} 32\,\alpha^2\beta + \frac{1-\mu^2}{\mu^2} 4\,\alpha^4\,\eta$$

$$+ \; 8\,\beta + 48\frac{\beta^2}{\alpha} + 64\delta + \frac{1+11\mu^2}{\mu^2} 4\,\alpha^5 + \frac{1+\mu}{\mu} 4\,\alpha^2\eta \,]\, s^2 + \ldots$$

$$v_2(s) = 8\alpha - 16\beta + \frac{(-5\mu+1)\,4\alpha^3}{\mu} + 64\,\alpha^2\beta + 8\lambda\, s + \ldots$$

$$L_o = -\frac{1+\mu}{\mu} 2\,\alpha^2 s + \frac{1+\mu}{\mu} (2\,\alpha^2 - 16\,\alpha\beta - 4\,\alpha^4)\, s^2 + \ldots$$

$$L_1 = \frac{1+\mu}{\mu} 4\,\alpha^2 s + 4\,\frac{1+\mu}{\mu} (8\,\beta\alpha - \alpha^2 + 4\,\alpha^4)\, s^2 + \ldots$$

$$L_2 = -\frac{4\,\alpha^2(1+\mu)}{\mu} + \frac{1+\mu}{\mu} (4\,\alpha^2 - 32\,\alpha\beta + 8\,\alpha\eta + \frac{8\alpha^4(1-3\mu)}{\mu})\, s + \ldots$$

From the fundamental equations, we have also the following relations:

$$L_o(s) = \frac{1+\mu}{2\mu} \log\,(1 - v_o^2 s(1-s))$$

$$L_1(s) = \frac{1+\mu}{\mu} \frac{v_o^2 s(1-s)}{1 - v_o^2 s(1-s)}$$

$$L_2(s) = -\frac{1+\mu}{\mu[\,1 - v_o^2 s(1-s)]} [\, u_1^2 + u_1'\, v_o s(1-s) + v_o^2 s(1-s)$$

$$- 2\,v_o v_1 s(1-s)\,] + 2\,v_1 v_o^3 s^2 (1-s)^2 \,\frac{1+\mu}{\mu}\, \frac{1}{1 - v_o^2 s(1-s)}$$

where $u_1' = \frac{d}{ds} u_1$

$$u_1 = -[\,v_o s(1-s)]' + \frac{1-\mu}{2\mu}\, v_o s(1-s)\, \frac{[\,v_o^2 s(1-s)\,]'}{1 - v_o^2\, s(1-s)} .$$

(B) Shock Conditions

Let the shock shape be given as:

$$r = g(s) = g_o \left[1 + G_1 s + G_2 s^2 + \ldots . \right]$$

$$= g_o \left[1 + (2+x)s + (4+z)s^2 + \ldots . \right]$$

On the shock,

$$\sqrt{N}\,\bar{u} = -y + (2y - 2x + 2yx - x^2 + x^2\mu)\,s$$

$$+ \left[-2(y-\mu)(-2z + 9x + 3x\mu) \right.$$

$$- (1-\mu)(4z - 18x + 4xz - 25x^2$$

$$\left. - 8x^3 - x^4) \right] s^2 + \ldots .$$

$$\sqrt{N}\,\bar{v} = -x + y(2 + x)$$

$$+ \left[(y-u)(2z - 10x - 3x^2) + (1-\mu)(-2z + 10x + 5x^2 + x^3) \right] s + \ldots$$

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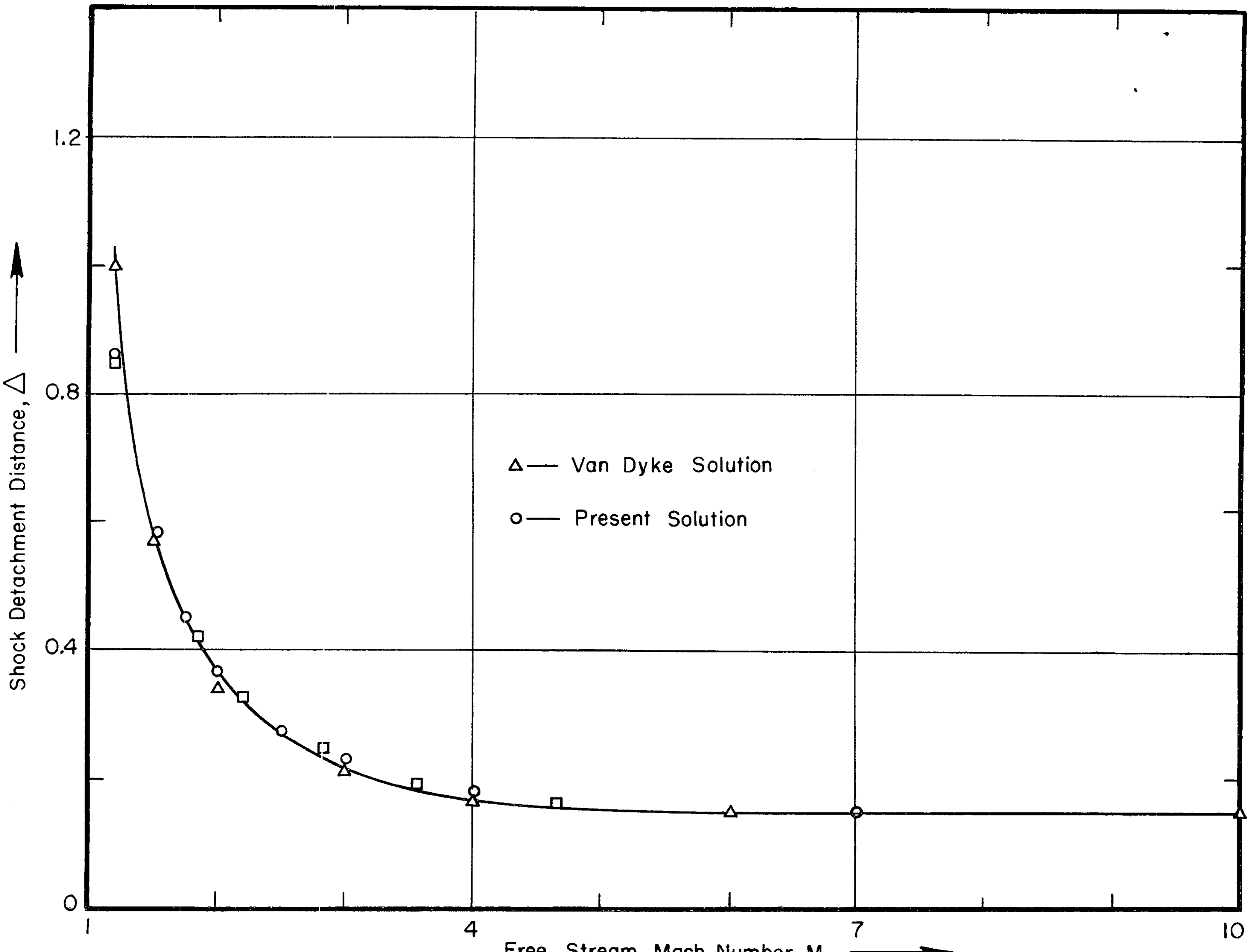

FIG. 1 Shock Detachment Distance vs Free Stream Mach Number, $\gamma = 1.4$

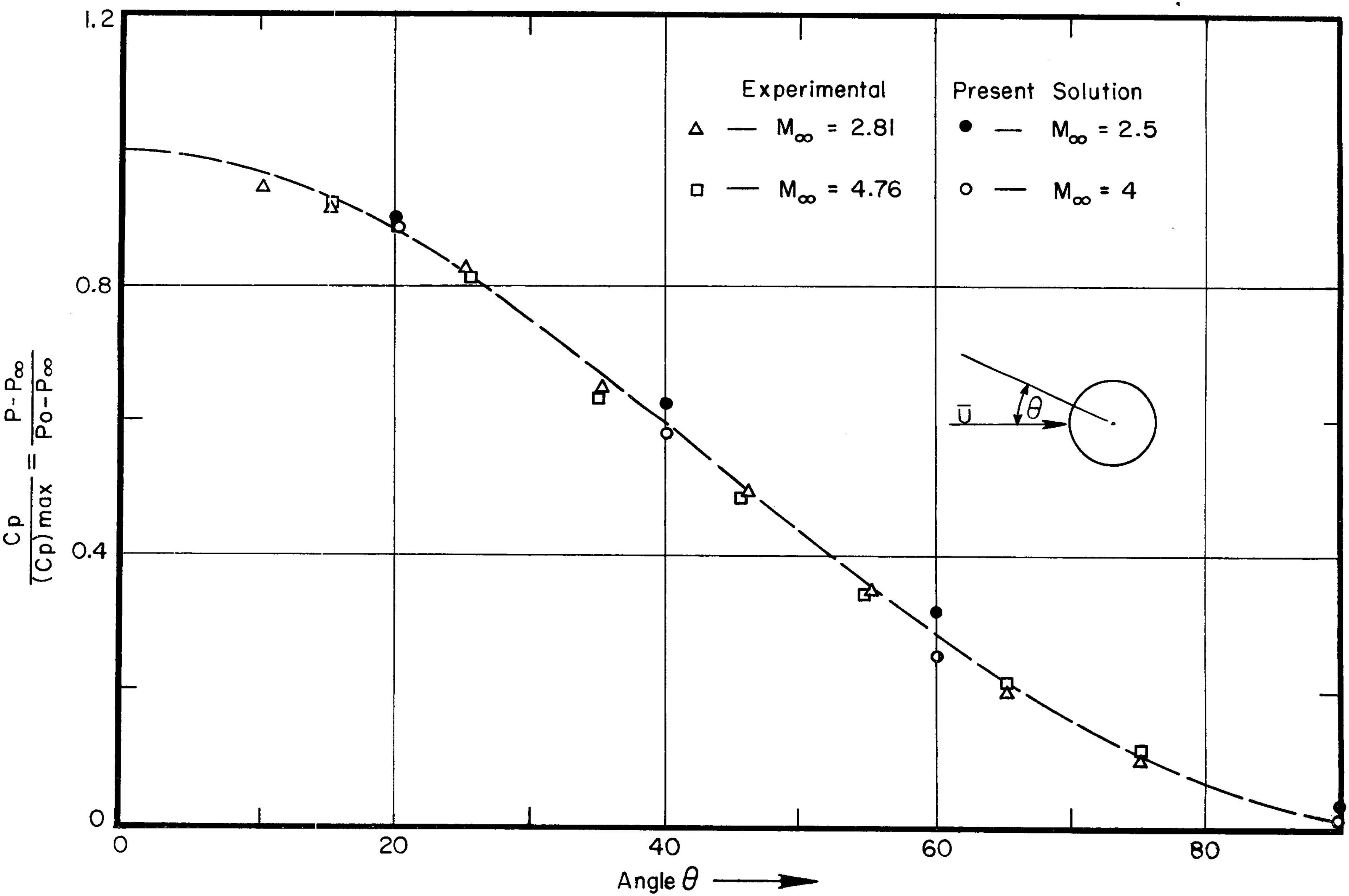

FIG. 2 Pressure Distribution on a Sphere $\gamma = 1.4$

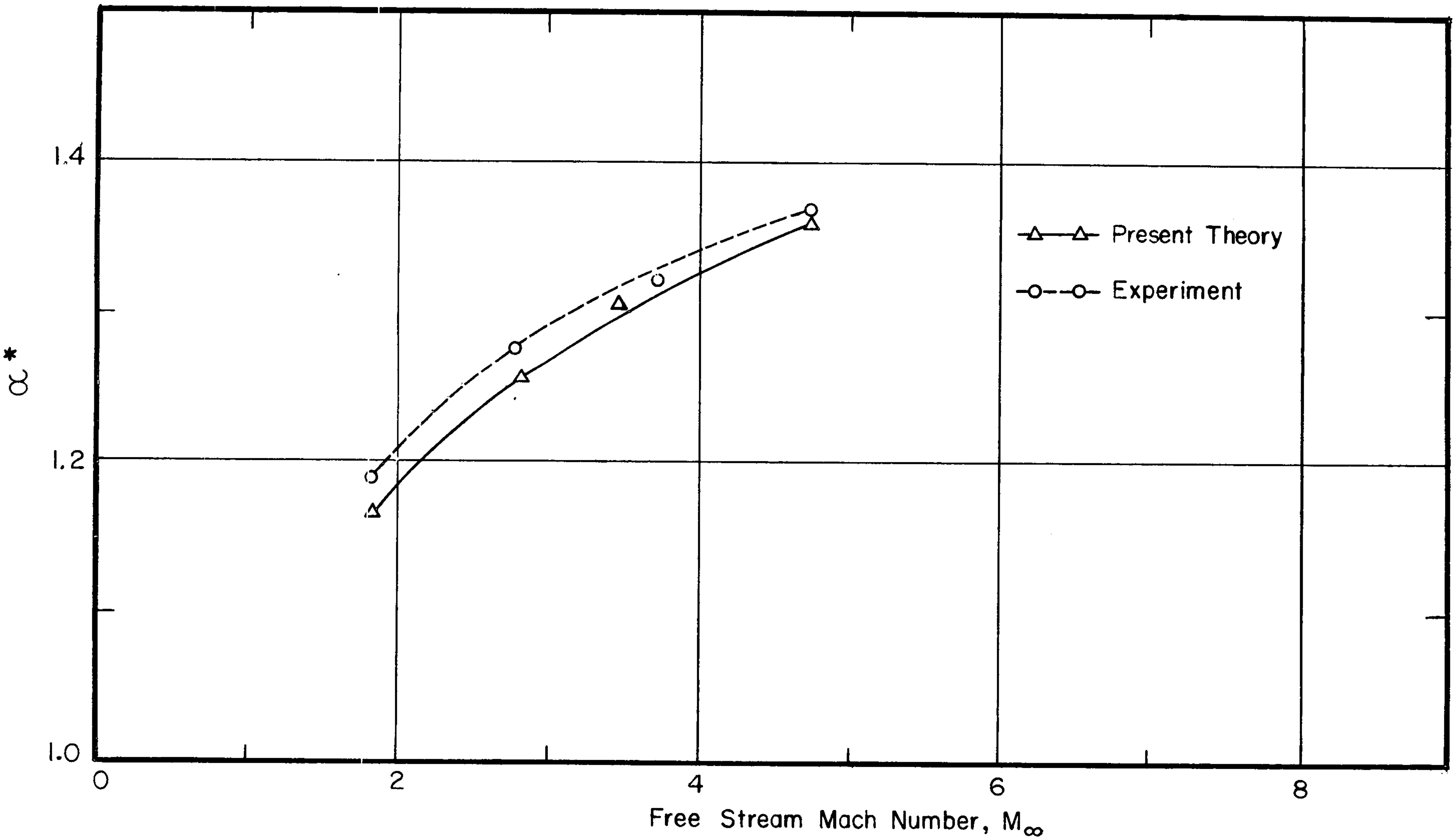

FIG. 3 Stagnation – Point Velocity Gradient ; α^* vs Mach Number

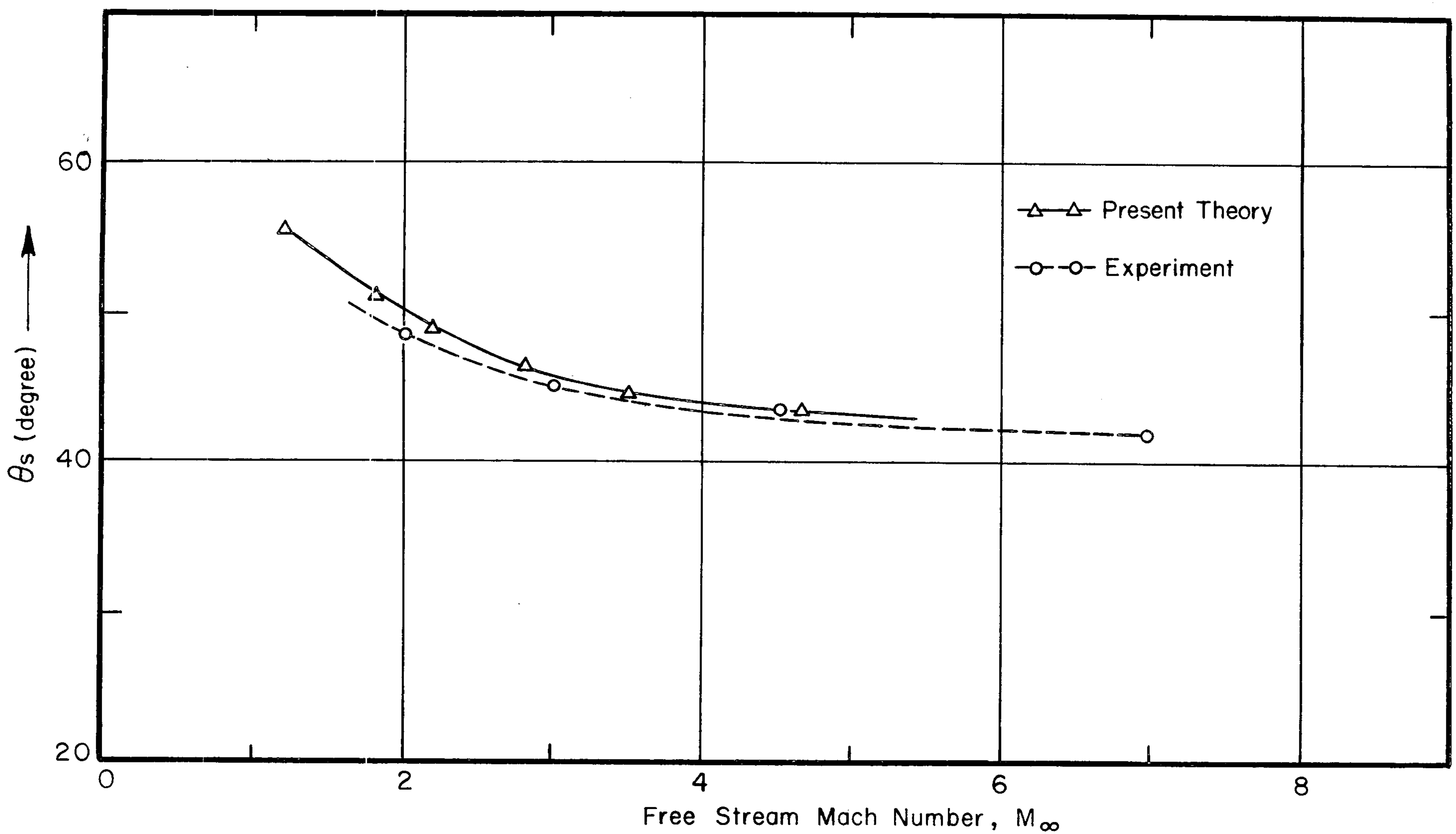

FIG. 4 The Angular Location of Sonic Point on The Body ($\gamma = 1.4$)

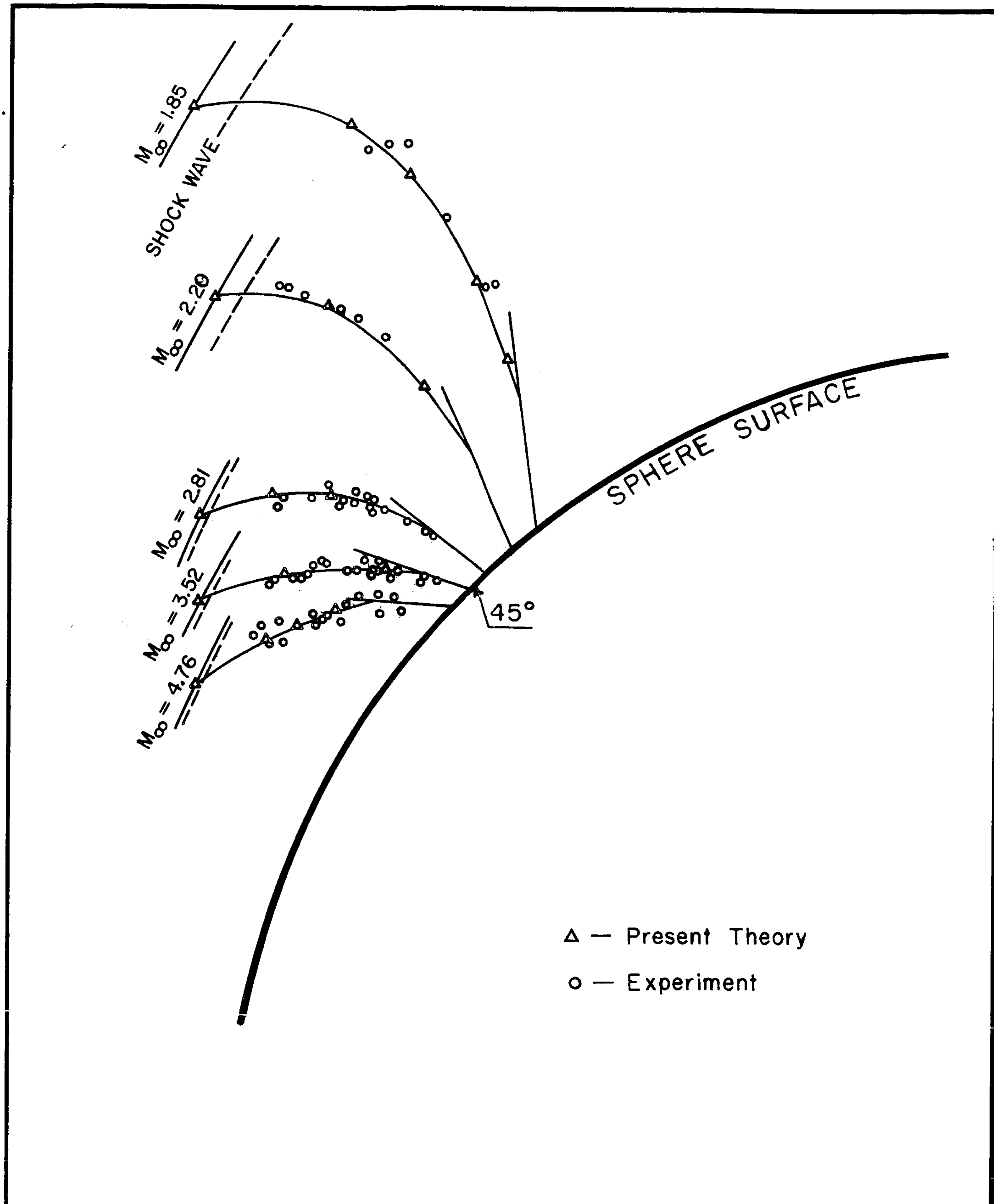

FIG. 5 Sonic Line on a Sphere for Several Mach Numbers.