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DYNAMIC STABILITY ANALYSIS OF A FREE-FREE CIRCULAR CYLINDRICAL SHELL SUBJECTED TO A GIMBALED, PERIODICALLY-VARYING END THRUST

Research & Analysis Section Tech. Memo. No.101

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GEORGE C. MARSHALL SPACE FLIGHT CENTER
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1.0 LIST OF SYMBOLS

Term	Definition
$a = \dfrac{8\pi r\ L\ T_o\ K}{I\Omega^2}$	Dimensionless parameter of the Matheau equation that governs stable solutions of $\psi(\tau)$.
a_x, a_θ, a_z	Axial, circumferential, and normal components of acceleration of the shell element.
$\left.\begin{matrix} A_i \\ B_i \\ C_i \\ D_i \end{matrix}\right\}$	Boundary value coefficients determined from initial conditions.
$C_k^{(s)}$	The kth element in the matrix $\left\{C_k\right\}^{(s)}$.
C_n	Constants determined from nth mode shape of a uniform beam at the center of mass.
C_{11}^{j0}, C_{12}^{j0}, C_{21}^{j0}, C_{22}^{j0}	Constants in the determinantal expressions of $\overline{U}_{j0}(\tau)$ and $\overline{V}_{j0}(\tau)$.
$\left.\begin{matrix} C_{11}^{jk}, C_{12}^{jk}, C_{13}^{jk} \\ C_{21}^{jk}, C_{22}^{jk}, C_{23}^{jk} \\ C_{31}^{jk}, C_{32}^{jk}, C_{33}^{jk} \end{matrix}\right\}$	Constants in the determinantal expressions of $\overline{U}_{jk}(\tau)$, $\overline{V}_{jk}(\tau)$ and $\overline{W}_{jk}(\tau)$.
$f^*(\xi, \tau)$	Displacement function (from consideration of the shell as a beam) in the u direction.

$f^*(-1,\tau)$	Displacement function at $\xi = -1$ [left end of beam (cylinder)].
$f^*(+1,\tau)$	Displacement function at $\xi = +1$ [right end of beam (cylinder)].
$f^*(\xi, 0)$	Displacement function at $\tau = 0$ (initial conditions).
$f_{mn}(\xi,\theta)$	Assumed displacement function for the shell displacement in the u-direction that satisfies the boundary conditions at $\xi = -1$ and $\xi = +1$.
$f_1(x, \theta, t)$, $f_2(x, \theta, t)$, $f_3(x, \theta, t)$	Intermediate forcing functions due to acceleration components a_x, a_θ, a_z, determined for a moving coordinate system.
$f_1(\xi, \theta, \tau)$, $f_2(\xi, \theta, \tau)$, $f_3(\xi, \theta,\tau)$	Dimensionless intermediate forcing functions.
$F_1(x, \theta, t)$, $F_2(x, \theta, t)$, $F_3(x, \theta, t)$	Intermediate forcing functions from consideration of gravitational component and acceleration components.
$F_1(\xi, \theta, \tau)$, $F_2(\xi, \theta, \tau)$, $F_3(\xi, \theta, \tau)$	Dimensionless intermediate forcing functions.
g	Acceleration of gravity.

$g^*(\xi, \tau)$	Displacement function (from consideration of the shell as a beam) in the v-direction.
$g^*(-1, \tau)$	Displacement function at $\xi = -1$.
$g^*(+1, \tau)$	Displacement function at $\xi = +1$.
$g^*(\xi, 0)$	Displacement function at $\tau = 0$ (initial conditions).
$g_{mn}(\xi, \theta)$	Assumed displacement function for the shell displacement in the v-direction that satisfies the boundary conditions at $\xi = -1$ and $\xi = +1$.
$G_1(x, \theta, t)$, $G_2(x, \theta, t)$, $G_3(x, \theta, t)$	Final forcing functions
$G_1(\xi, \theta, \tau)$, $G_2(\xi, \theta, \tau)$, $G_3(\xi, \theta, \tau)$	Dimensionless final forcing functions.
h	Thickness of cylindrical shell
$h^*(\xi, \tau) = g^*(\xi, \tau)$ $h^*(-1, \tau) = g^*(-1, \tau)$ $h^*(+1, \tau) = g^*(+1, \tau)$ $h^*(\xi, 0) = g^*(\xi, 0)$	Defined earlier
$h_{mn}(\xi, \theta)$	Assumed displacement function for the shell displacement in the w-direction that satisfies the boundary conditions at $\xi = -1$ and $\xi = +1$.

i, j, k, m, n, p, r, s	Summing indices used in the report.
I	Mass moment of inertia of the shell about a line through the center of mass perpendicular to elements of the cylinder.
K	Directional control factor determining the direction of thrust.
$\mathcal{L}[\quad]$	Laplace Transform notation.
L	Half length of cylinder.
$\bar{m}$	Total mass of an equivalent beam 2L in length.
$\bar{M}$	Total mass of the cylindrical shell 2L in length.
M_1	Limit on the summing indices j and m.
$M_x, M_\theta, M_{x\theta}, M_{\theta x}$	Moment resultants in the shell.
n	Index defining the particular beam mode.
N_1	Limit on summing indices k and n.
$N_x, N_\theta, N_{x\theta}, N_{\theta x}$	Stress resultants in the shell.
N_ξ	Dimensionless stress resultant in the ξ-direction.

$\bar{N}_\xi = \frac{N_\xi}{\rho h L^2 \omega_1^2}$ Dimensionless parameter.

P_x Distributed surface force in the axial direction.

P_θ Distributed surface force in the circumferential direction.

P_Z Distributed surface force in the radial direction.

$P_{00}(\tau)$ Expression for the forcing terms associated with time displacement $U_{00}(\tau)$.

$P_{0k}(\tau)$ Forcing expression in equation of motion for $U_{0k}(\tau)$.

$\left[P_{j0}(\tau)\right]_1$ Forcing expression in equations of motion for $U_{j0}(\tau)$ and $V_{j0}(\tau)$.

$\left[P_{j0}(\tau)\right]_2$ Forcing expression in equations of motion for $U_{j0}(\tau)$ and $V_{j0}(\tau)$.

$\left[\bar{P}_{j0}(\tau)\right]_1$ Laplace transform of $\left[P_{j0}(\tau)\right]_1$.

$\left[\bar{P}_{j0}(\tau)\right]_2$ Laplace transform of $\left[P_{j0}(\tau)\right]_2$.

$q = \frac{\gamma a}{2}$ Dimensionless parameter of the Matheau equation that governs stable solutions of $\psi(\tau)$.

$q_A(\tau)$	Translation term of the beam (considered zero since equations of motion are written for a moving coordinate system).
$q_B(\tau)$	Rotation term of the undeflected axis of the beam (cylinder).
$q_n(\tau)$	Generalized coordinate associated with $\phi_n(\xi)$ defined later.
Q_x, Q_θ	Shear resultants in the shell.
$\left[Q_{jk}(\tau)\right]_1$	Forcing expression associated with $U_{jk}(\tau)$, $V_{jk}(\tau)$, and $W_{jk}(\tau)$.
$\left[Q_{jk}(\tau)\right]_2$	Forcing expression associated with $U_{jk}(\tau)$, $V_{jk}(\tau)$, and $W_{jk}(\tau)$.
$\left[Q_{jk}(\tau)\right]_3$	Forcing expression associated with $U_{jk}(\tau)$, $V_{jk}(\tau)$, and $W_{jk}(\tau)$.
$\left[\overline{Q}_{jk}(\tau)\right]_1$	Laplace transform of $\left[Q_{jk}(\tau)\right]_1$.
$\left[\overline{Q}_{jk}(\tau)\right]_2$	Laplace transform of $\left[Q_{jk}(\tau)\right]_2$.
$\left[\overline{Q}_{jk}(\tau)\right]_3$	Laplace transform of $\left[Q_{jk}(\tau)\right]_3$.
r	Radius of the cylindrical shell, also used as an index in the section STABILITY ANALYSIS.
R	Maximum value of the index r.
s	Summing index used in the section STABILITY ANALYSIS.
S	Maximum value of the summing index s.

t	Real time variable.
$T(t)$	Thrust function in time t.
$T(\tau)$	Thrust function in dimensionless time τ.
T_o	Magnitude of the steady state thrust per unit length applied around the bottom of the shell.
$\overline{T}_o = \dfrac{T_o}{\rho h L^2 \omega_1^2}$	Dimensionless parameter.
u	Axial deflection component referenced to the moving reference frame.
$\overline{u} = \dfrac{u}{L}$	Dimensionless component.
$u(x, \theta, t)$	Axial deflection component written for real variables x, θ, and t.
$u(\xi, \theta, \tau)$	Dimensionless axial deflection component.
$\tilde{u}(\xi, \theta, \tau)$	Assumed dimensionless displacement that satisfies the boundary conditions.
$u^*(\xi, \theta, \tau)$	Beam action contribution to $\tilde{u}(\xi, \theta, \tau)$.
u_{0k}	Value of $U_{0k}(\tau)$ at $\tau = 0$.
u_{j0}	Value of $U_{j0}(\tau)$ at $\tau = 0$.
u_{jk}	Value of $U_{jk}(\tau)$ at $\tau = 0$.

$U_{0k}(\tau)$	Orthogonal function of the generalized coordinate $U_{0n}(\tau)$ used in Galerkin procedure.
$U_{j0}(\tau)$	Orthogonal function of the generalized coordinate $U_{m0}(\tau)$ used in Galerkin procedure.
$U_{jk}(\tau)$	Orthogonal function of the generalized coordinate $U_{mn}(\tau)$ used in Galerkin procedure.
$U_{mn}(\tau)$	Generalized coordinate in $\tilde{u}(\xi, \theta, \tau)$.
$\bar{U}_{j0}(\tau)$	Laplace transform of $U_{j0}(\tau)$.
$\bar{U}_{jk}(\tau)$	Laplace transform of $U_{jk}(\tau)$.
v	Circumferential deflection component referenced to the moving reference frame.
$v(x, \theta, t)$	Circumferential deflection component written for real variables x, θ, and t.
$v(\xi, \theta, \tau)$	Dimensionless circumferential deflection component.
$\tilde{v}(\xi, \theta, \tau)$	Assumed dimensionless displacement component that satisfies the boundary conditions.
$v^*(\xi, \theta, \tau)$	Beam action contribution to $\tilde{v}(\xi, \theta, \tau)$.
v_{0k}	Value of $V_{0k}(\tau)$ at $\tau = 0$.
v_{j0}	Value of $V_{j0}(\tau)$ at $\tau = 0$.

v_{jk}	Value of $V_{jk}(\tau)$ at $\tau = 0$.
$V_{0k}(\tau)$	Orthogonal function of the generalized coordinate $V_{0n}(\tau)$ used in Galerkin procedure.
$V_{j0}(\tau)$	Orthogonal function of the generalized coordinate $V_{m0}(\tau)$ used in Galerkin procedure.
$V_{jk}(\tau)$	Orthogonal function of the generalized coordinate $V_{mn}(\tau)$ used in the Galerkin procedure.
$V_{mn}(\tau)$	Generalized coordinate in $\tilde{v}(\xi, \theta, \tau)$.
$\bar{V}_{j0}(\tau)$	Laplace transform of $V_{j0}(\tau)$.
$\bar{V}_{jk}(\tau)$	Laplace transform of $V_{jk}(\tau)$.
w	Radial (positive inward) deflection component referenced to the moving reference frame.
$w(x, \theta, t)$	Radial deflection component written for real variables x, θ, and t.
$w(\xi, \theta, \tau)$	Dimensionless radial deflection component.
$\tilde{w}(\xi, \theta, \tau)$	Assumed dimensionless displacement component that satisfies the boundary conditions.
$w^*(\xi, \theta, \tau)$	Beam action contribution to $\tilde{w}(\xi, \theta, \tau)$.
w_{0k}	Value of $W_{0k}(\tau)$ at $\tau = 0$.
w_{j0}	Value of $W_{j0}(\tau)$ at $\tau = 0$.

w_{jk}	Value of $W_{jk}(\tau)$ at $\tau = 0$.
$W_{0k}(\tau)$	Orthogonal function of the generalized coordinate $W_{0n}(\tau)$ used in Galerkin procedure.
$W_{j0}(\tau)$	Orthogonal function of the generalized coordinate $W_{m0}(\tau)$ used in Galerkin procedure.
$W_{jk}(\tau)$	Orthogonal function of the generalized coordinate $W_{mn}(\tau)$ used in Galerkin procedure.
$W_{mn}(\tau)$	Generalized coordinate in $\bar{w}(\xi, \theta, \tau)$.
$\overline{W}_{j0}(\tau)$	Laplace transform of $W_{j0}(\tau)$.
$\overline{W}_{jk}(\tau)$	Laplace transform of $W_{jk}(\tau)$.
x	Coordinate of the shell in the axial direction.
X	Coordinate of the cylinder at the center of mass in the axial direction.
$X_0 = \frac{X}{L}$	Dimensionless coordinate of the cylinder at the center of mass in the axial direction.
X'	Displaced axis of the moving reference frame.
y	Coordinate of the shell in the circumferential direction.
Y	Coordinate of the cylinder at the center of mass.

$Y_0 = \frac{Y}{L}$	Dimensionless coordinate of the cylinder at the center of mass.
Y'	Displaced axis of the moving reference frame.
z	Coordinate of the shell in the radial direction.
Z'	Axis of the moving reference frame.

Greek Terms	Definition
β	Stability parameter of the solution for $\psi(\tau)$.
$\gamma = \frac{T_1}{T_o}$	Ratio of the magnitude of sinusoidal time varying thrust per unit length to T_o.
δ_{jk}	Kronecker delta.
$\Delta(\beta)$	Infinite determinant.
$\Delta(0)$	Value of infinite determinant $\Delta(\beta)$ for $\beta = 0$.
ε_1	Term used in writing the stability equations for the dynamic solution.
Σ	Summation symbol.
η	Dummy variable used in convolution integrals.
θ	Tangential shell coordinate.

$\lambda = \frac{L}{r}$	Dimensionless radius parameter.
$\mu = \frac{\rho L^2 \omega_1^2}{E}$	Dimensionless parameter.
ν	Poisson's ratio of shell material.
$\xi = \frac{x}{L}$	Dimensionless axial shell coordinate.
ξ_r	Elements of the infinite determinant $\Delta(\beta)$.
π	Constant of value 3.1416.
ρ	Mass per unit volume of the shell material.
$\sigma = \frac{h}{L}$	Dimensionless parameter.
$\tau = \omega_1 t$	Dimensionless time.
$\tau_1 = \frac{\Omega}{2} \frac{\tau}{\omega_1} = \frac{\Omega}{2} t$	Dimensionless time.
ϕ	Mode shape symbol.
ϕ_n	Mode shape of the nth vibration mode of the free-free beam.
$\chi(\beta)$	Function used in Matheau function consideration.
$\psi(\tau)$	Rigid body rotation of the cylinder about the center of mass.
ψ_o	Rotation of the cylinder at $\tau = 0$.
ω	Circular frequency symbol used in report.

$\omega_1, \omega_2, \omega_3, \ldots \omega_n$	Lateral bending frequencies of the free-free beam.
ω_{0k}	Natural frequencies of the cylinder involving only circumferential mode shapes.
ω_{j0}^{i}	Natural frequencies of the cylinder involving only longitudinal mode shapes.
ω_{jk}^{i}	Natural frequencies of the cylinder involving both longitudinal and circumferential mode shapes.
Ω	Circular frequency of the sinusoidal component of the applied frequency.
Ω_i	Solutions of the unstable values of Ω associated with the shell modes.

2.0 LIST OF FIGURES

3.0 LIST OF SPECIAL TERMS AND INTEGRALS

Certain terms and integrals appear repeatedly in the report so letters and numbers have been assigned to them. A listing of these terms and integrals appear below:

$$A_{jk} = \left\{ \left[C_{22}^{jk} - (\omega_{jk}^{m})^2\right] \left[C_{33}^{jk} - (\omega_{jk}^{m})^2\right] - (C_{23}^{jk})^2 \right\} \quad (3.0\text{-}1)$$

$$B_{jk} = \left\{ C_{12}^{jk} \left[C_{33}^{jk} - (\omega_{jk}^{m})^2\right] - C_{13}^{jk} C_{23}^{jk} \right\} \quad (3.0\text{-}2)$$

$$C_{jk} = \left\{ C_{13}^{jk} \left[C_{22}^{jk} - (\omega_{jk}^{m})^2\right] - C_{12}^{jk} C_{23}^{jk} \right\} \quad (3.0\text{-}3)$$

$$D_{jk} = \left\{ C_{12}^{jk} \left[C_{33}^{jk} - (\omega_{jk}^{m})^2\right] - C_{13}^{jk} C_{23}^{jk} \right\} \quad (3.0\text{-}4)$$

$$E_{jk} = \left\{ \left[C_{11}^{jk} - (\omega_{jk}^{m})^2\right] \left[C_{33}^{jk} - (\omega_{jk}^{m})^2\right] - (C_{13}^{jk})^2 \right\} \quad (3.0\text{-}5)$$

$$F_{jk} = \left\{ C_{23}^{jk} \left[C_{11}^{jk} - (\omega_{jk}^{m})^2\right] - C_{12}^{jk} C_{13}^{jk} \right\} \quad (3.0\text{-}6)$$

$$G_{jk} = \left\{ C_{13}^{jk} \left[C_{22}^{jk} - (\omega_{jk}^{m})^2\right] - C_{12}^{jk} C_{23}^{jk} \right\} \quad (3.0\text{-}7)$$

$$H_{jk} = \left\{ C_{23}^{jk} \left[C_{11}^{jk} - (\omega_{jk}^{m})^2\right] - C_{12}^{jk} C_{13}^{jk} \right\} \quad (3.0\text{-}8)$$

$$I_{jk} = \left\{ \left[C_{11}^{jk} - (\omega_{jk}^{m})^2\right] \left[C_{22}^{jk} - (\omega_{jk}^{m})^2\right] - (C_{12}^{jk})^2 \right\} \quad (3.0\text{-}9)$$

$$P_{00}(\tau) = -\frac{1}{4\pi(1-\nu^2)} \left[\int_{-1}^{+1}\int_{0}^{2\pi} G_1(\xi,\ \theta,\ \tau)\,d\xi d\theta + \int_{-1}^{+1}\int_{0}^{2\pi} \frac{\partial^2[f^*(\xi,\ \tau)]}{\partial\xi^2}\,d\xi d\theta + \frac{(1+\nu)\lambda}{2} \int_{-1}^{+1}\int_{0}^{2\pi} \frac{[\partial^2\ g^*(\xi,\ \tau)\sin\theta]}{\partial\xi\partial\theta}\,d\xi d\theta - \nu\lambda \int_{-1}^{+1}\int_{0}^{2\pi} \frac{[\partial\ h^*(\xi,\ \tau)\ \cos\theta]}{\partial\xi}\,d\xi d\theta \right] \quad (3.0\text{-}10)$$

$$P_{0k}(\tau) = -\frac{1}{2\pi(1-\nu^2)\mu}\left[\int_{-1}^{+1}\int_{0}^{2\pi} G_1(\xi,\ \theta,\ \tau)\cos k\theta\ d\xi d\theta\right.$$

$$+ \int_{-1}^{+1}\int_{0}^{2\pi} \frac{\partial^2[f^*(\xi,\ \tau)]}{\partial\xi^2}\cos k\theta\ d\xi d\theta$$

$$- \frac{(1+\nu)\lambda}{2}\int_{-1}^{+1}\int_{0}^{2\pi} \frac{\partial^2[g^*(\xi,\ \tau)\sin\theta]}{\partial\xi\partial\theta}\cos k\theta\ d\xi d\theta$$

$$\left. + \nu\lambda\int_{-1}^{+1}\int_{0}^{2\pi} \frac{\partial[h^*(\xi,\ \tau)\cos\theta]}{\partial\xi}\cos k\theta\ d\xi d\theta\right] \qquad (3.0\text{-}11)$$

$$P_{j0}(\tau)_1 = -\frac{1}{2\pi(1-\nu^2)\mu}\left[\int_{-1}^{+1}\int_{0}^{2\pi} G_1(\xi,\ \theta,\ \tau)\cos\frac{j\pi}{2}(\xi+1)\ d\xi d\theta\right.$$

$$+ \int_{-1}^{+1}\int_{0}^{2\pi} \frac{\partial^2[f^*(\xi,\ \tau)]}{\partial\xi^2}\cos\frac{j\pi}{2}(\xi+1)\ d\xi d\theta$$

$$+ \frac{(1+\nu)\lambda}{2}\int_{-1}^{+1}\int_{0}^{2\pi} \frac{\partial^2[g^*(\xi,\ \tau)\sin\theta]}{\partial\xi\partial\theta}\cos\frac{j\pi}{2}(\xi+1)\ d\xi d\theta$$

$$- \nu\lambda\int_{-1}^{+1}\int_{0}^{2\pi} \frac{\partial[h^*(\xi,\ \tau)\cos\theta]}{\partial\xi}\cos\frac{j\pi}{2}(\xi+1)\ d\xi d\theta$$

$$(3.0\text{-}12)$$

$$P_{j0}(\tau)_2 = -\frac{1}{2\pi(1-\nu^2)\mu}\left[G_3(\xi,\ \theta,\ \tau)\sin\frac{j\pi}{2}(\xi+1)\ d\xi d\theta\right.$$

$$+ \nu\lambda\int_{-1}^{+1}\int_{0}^{2\pi} \frac{\partial[f^*(\xi,\ \tau)]}{\partial\xi}\sin\frac{j\pi}{2}(\xi+1)\ d\xi d\theta$$

$$+ \lambda^2\int_{-1}^{+1}\int_{0}^{2\pi} \frac{\partial[g^*(\xi,\ \tau)\sin\theta]}{\partial\theta}\sin\frac{j\pi}{2}(\xi+1)\ d\xi d\theta$$

$$- \lambda^2 \int_{-1}^{+1}\int_{0}^{2\pi} [h^*(\xi, \tau) \cos\theta] \sin \frac{j\pi}{2} (\xi+1) \, d\xi d\theta$$

$$- \frac{\sigma^2}{12} \int_{-1}^{+1}\int_{0}^{2\pi} \bar{\nabla}^4 [h^*(\xi, \tau) \cos\theta] \sin \frac{j\pi}{2} (\xi+1) \, d\xi d\theta \qquad (3.0\text{-}13)$$

$$Q_{jk}(\tau)_1 = \frac{\lambda \delta_{1k}}{2(1+\nu)\mu} \int_{-1}^{+1} \frac{\partial g^*(\xi, \tau)}{\partial \xi} \cos \frac{j\pi}{2} (\xi+1) \, d\xi$$

$$Q_{jk}(\tau)_2 = - \frac{1}{\pi(1-\nu^2)\mu} \int_{-1}^{+1}\int_{0}^{2\pi} G_2(\xi, \theta, \tau) \sin \frac{j\pi}{2} (\xi+1) \sin k\theta \, d\xi d\theta$$

$$- \frac{\lambda}{2(1+\nu)\mu} \int_{-1}^{+1}\int_{0}^{2\pi} \frac{\partial^2 [g^*(\xi, \tau) \sin \theta]}{\partial \xi^2} \sin \frac{j\pi}{2} (\xi+1) \sin k\theta \, d\xi d\theta$$

$$(3.0\text{-}14)$$

$$Q_{jk}(\tau)_3 = - \frac{1}{\pi(1-\nu^2)\mu} \int_{-1}^{+1}\int_{0}^{2\pi} G_3(\xi, \theta, \tau) \sin \frac{j\pi}{2}(\xi+1) \cos k\theta \, d\xi d\theta$$

$$- \frac{\nu\lambda}{\pi(1-\nu^2)\mu} \int_{-1}^{+1}\int_{0}^{2\pi} \frac{\partial f^*(\xi, \tau)}{\partial \xi} \sin \frac{j\pi}{2} (\xi+1) \cos k\theta \, d\xi d\theta$$

$$+ \frac{\sigma^2}{12\pi(1-\nu^2)\mu} \int_{-1}^{+1}\int_{0}^{2\pi} \bar{\nabla}^4 [h^*(\xi, \tau) \cos\theta] \sin \frac{j\pi}{2} (\xi+1) \cos k\theta \, d\xi d\theta$$

$$(3.0\text{-}15)$$

$$e_{j0} = \int_{-1}^{+1} \cos \frac{j\pi}{2} (\xi+1) \, d\xi = 0 \quad \text{for all } j \qquad (3.0\text{-}16)$$

$$m_{j0} = \int_{-1}^{+1} \xi \sin \frac{j\pi}{2} (\xi+1) \, d\xi \qquad (3.0\text{-}17)$$

$$r_{jn} = \int_{-1}^{+1} \sin \frac{j\pi}{2} (\xi+1) \, d\xi \qquad (3.0\text{-}18)$$

$$s_{jn} = \int_{-1}^{+1} \phi_n(\xi) \sin \frac{j\pi}{2} (\xi+1) \, d\xi \qquad (3.0\text{-}19)$$

$$t_{jn} = \int_{-1}^{+1} \phi'_n(\xi) \sin \frac{j\pi}{2} (\xi+1)\, d\xi \qquad (3.0\text{-}20)$$

$$x_{jn} = \int_{-1}^{+1} \phi''_n(\xi) \sin \frac{j\pi}{2} (\xi+1)\, d\xi \qquad (3.0\text{-}21)$$

$$y_{jn} = \int_{-1}^{+1} \phi'''_n(\xi) \sin \frac{j\pi}{2} (\xi+1)\, d\xi \qquad (3.0\text{-}22)$$

$$z_{jn} = \int_{-1}^{+1} \phi'_n(\xi) \cos \frac{j\pi}{2} (\xi+1)\, d\xi \qquad (3.0\text{-}23)$$

$$I_1 = \int_0^\tau \left[P_{j0}(\eta)\right]_1 \sin \omega^1_{j0}(\tau-\eta)\, d\eta \qquad (3.0\text{-}24)$$

$$I_2 = \int_0^\tau \left[P_{j0}(\eta)\right]_1 \sin \omega^2_{j0}(\tau-\eta)\, d\eta \qquad (3.0\text{-}25)$$

$$I_3 = \int_0^\tau \left[P_{j0}(\eta)\right]_2 \sin \omega^1_{j0}(\tau-\eta)\, d\eta \qquad (3.0\text{-}26)$$

$$I_4 = \int_0^\tau \left[P_{j0}(\eta)\right]_2 \sin \omega^2_{j0}(\tau-\eta)\, d\eta \qquad (3.0\text{-}27)$$

$$J_1 = \int_0^\tau \left[Q_{jk}(\eta)\right]_1 \sin \omega^m_{jk}(\tau-\eta)\, d\eta \qquad (3.0\text{-}28)$$

$$J_2 = \int_0^\tau \left[Q_{jk}(\eta)\right]_2 \sin \omega^m_{jk}(\tau-\eta)\, d\eta \qquad (3.0\text{-}29)$$

$$J_3 = \int_0^\tau \left[Q_{jk}(\eta)\right]_3 \sin \omega^m_{jk}(\tau-\eta)\, d\eta \qquad (3.0\text{-}30)$$

4.0 INTRODUCTION

The dynamic response of a large rocket booster, idealized as a free-free, thin-walled cylinder shown in Figure (1) is investigated in this analysis. Powered flight simulation is produced by a gimbaled, time-varying end thrust controlled by sensors located at the center of mass as shown in Figures 2(a) and (b).

The analysis presented in this report represents a continuation of an earlier analysis by Hill (ref. 1), who in a response study investigated the simply-supported case. The additional instabilities reported in Section 10.0, STABILITY ANALYSIS, represent the coupling effect from consideration of beam action, not considered in the previously mentioned analysis.

5.0 EQUATIONS OF MOTION

5.1 Equations of Motion of the Shell Element

Using the sign convention of Figure (1), the equations of motion of the shell element as given by Langhaar (ref. 2) and as used by Hill (ref. 1) are as follows:

$$\frac{\partial N_x}{\partial x} + \frac{1}{r}\frac{N_{\theta x}}{\partial \theta} + P_x = \rho h\, a_x \qquad \text{(a)}$$

$$\frac{\partial N_{x\theta}}{\partial x} + \frac{1}{r}\frac{\partial N_\theta}{\partial \theta} - \frac{Q_\theta}{r} + P_\theta = \rho h\, a_\theta \qquad \text{(b)}$$

$$\frac{\partial Q_x}{\partial x} + \frac{1}{r}\frac{\partial Q_\theta}{\partial \theta} + \frac{1}{r} N_\theta + P_z = \rho h\, a_z \qquad \text{(c)}$$

$$\frac{\partial M_x}{\partial x} + \frac{1}{r}\frac{M_{\theta x}}{\partial \theta} - Q_x = 0 \qquad \text{(d)} \qquad (5.1\text{-}1)$$

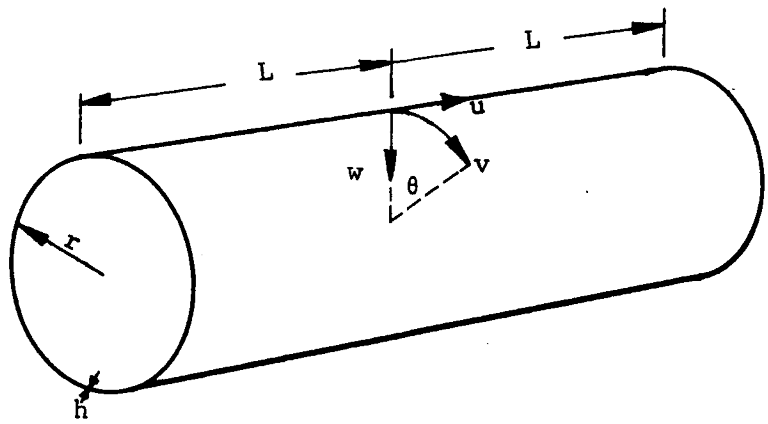
$$\frac{\partial M_{x\theta}}{\partial x} - \frac{1}{r}\frac{\partial M_\Theta}{\partial \theta} + Q_\theta = 0 \qquad \text{(e)}$$

$$N_{\theta x} - N_{x\theta} = \frac{1}{r} M_{\theta x} \qquad \text{(f)}$$

where the rotary inertia of the shell has been disregarded. All terms used in the above equations of motion are defined in the LIST OF SYMBOLS.

5.2 Newton's Laws of Plane Motion

Locating the moving coordinate system at the center of mass of the cylindrical shell shown in Figure (2) and using Newton's Laws of plane motion (note: the angle ψ is in the X-Y plane), the equations of motion are written



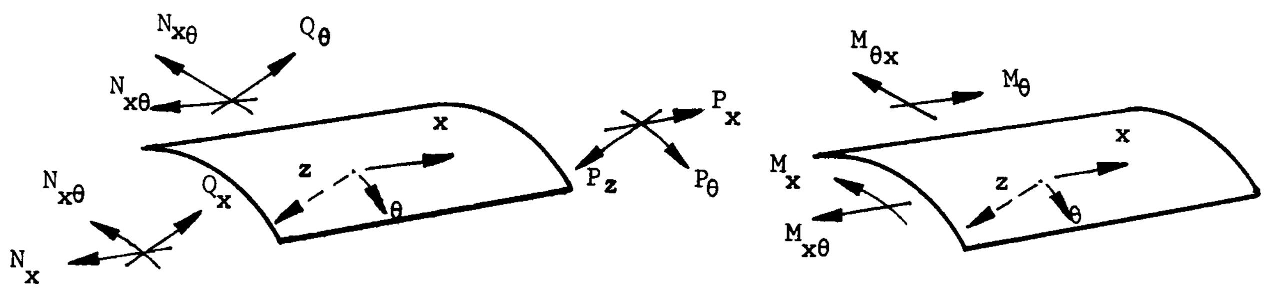


Figure 1 SIGN CONVENTION FOR COORDINATES, DISPLACEMENTS, STRESS RESULTANTS, AND MOMENT RESULTANTS IN THE THIN-WALLED CYLINDRICAL SHELL

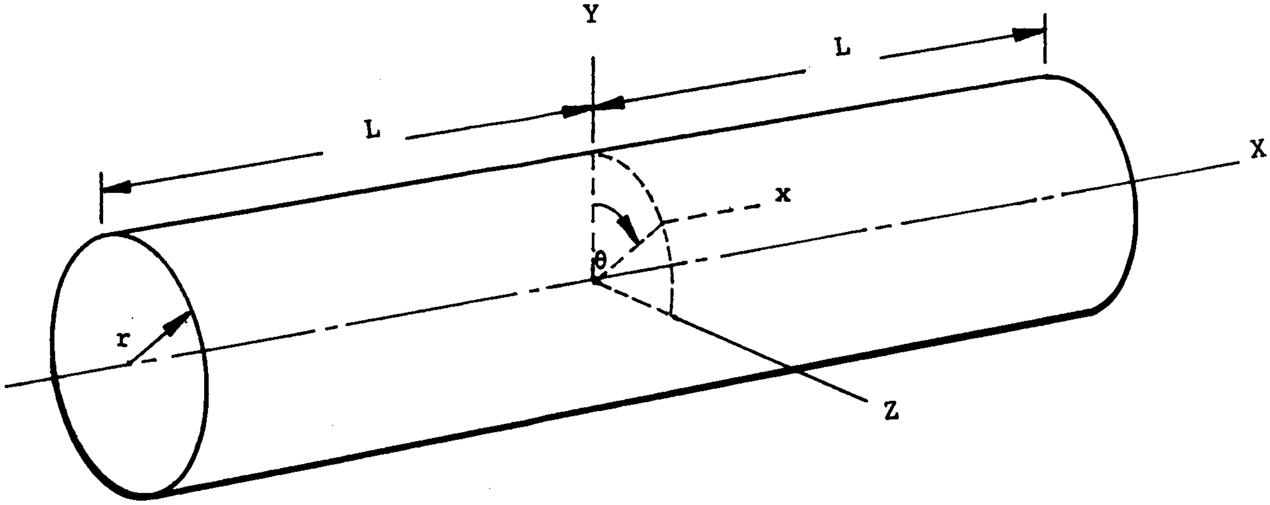


Figure 2(a) UNDEFLECTED CYLINDER IN THE MOVING (X, Y, Z) COORDINATE SYSTEM

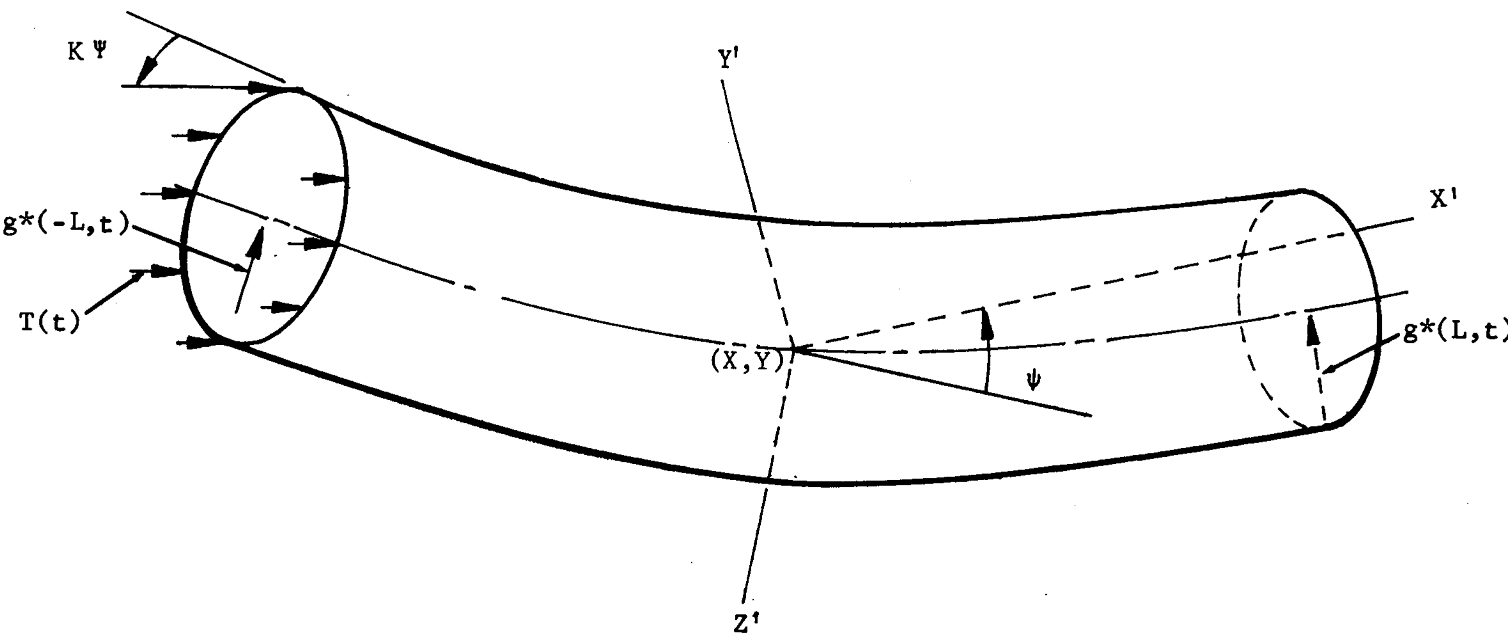


Figure 2(b) DEFLECTED CYLINDER IN THE MOVING (X', Y', Z') COORDINATE SYSTEM WITH ORIGIN ALWAYS AT THE CENTER OF MASS. (NOTE THAT THE X' Y' COORDINATES CONTINUE TO BE IN THE XY PLANE)

$$M\ddot{X} = 2\pi r \quad T(t) \cos (K + 1)\psi - Mg \qquad \text{(a)}$$

$$M\ddot{Y} = 2\pi r \quad T(t) \sin (K + 1)\psi \qquad \text{(b)} \quad (5.2\text{-}1)$$

$$I\ddot{\psi} = -2\pi r \; L\; T(t) \sin K\psi \qquad \text{(c)}$$

where the assumption is made that the deformation does not change the half length nor the mass moment of inertia of the cylindrical shell.

5.3 Expressions for the Axial, Circumferential, and Normal Components of Acceleration

Kinematics of a moving point will yield expressions for a_x, a_θ, and a_z in terms of u, v, and w, the shell displacements, and X, Y, and ψ, the coordinates of the reference frame, as follows:

$$a_x = \frac{\partial^2 u}{\partial t^2} + \ddot{X}\,\text{cox}\,\psi + \ddot{Y}\sin\psi - r\cos\theta\ddot{\psi} - \ddot{X}\dot{\psi}^2$$

$$\underline{+ 2\dot{\psi}\left(\frac{\partial v}{\partial t}\sin\theta + \frac{\partial w}{\partial t}\cos\theta\right)} \qquad \text{(a)}$$

$$a_\theta = \frac{\partial^2 v}{\partial t^2} + (r\cos\theta\dot{\psi}^2 - X\ddot{\psi} - \ddot{Y}\cos\psi + \ddot{X}\sin\psi)\sin\theta \qquad (5.3\text{-}1)$$

$$\underline{-2\dot{\psi}\frac{\partial u}{\partial t}\sin\theta} \qquad \text{(b)}$$

$$a_z = \frac{\partial^2 w}{\partial t^2} + (r\cos\theta\dot{\psi}^2 - X\ddot{\psi} - \ddot{Y}\cos\psi + \ddot{X}\sin\psi)\cos\theta$$

$$\underline{-2\dot{\psi}\frac{\partial u}{\partial t}\cos\theta} \qquad \text{(c)}$$

The underlined terms are Coriolis terms and will be neglected in the subsequent analysis.

Substituting the expressions for a_x, a_θ, and a_z into equations (5.1-1), eliminating the shearing forces Q_x and Q_θ in equations (5.1-1)(d) and (e) and adding the acceleration of gravity terms, renders the three following equations:

$$\frac{\partial N_x}{\partial x} + \frac{1}{r}\frac{\partial N_{\theta x}}{\partial \theta} + P_x = \rho h \frac{\partial^2 u}{\partial t^2} + f_1\ (x, \theta, t) + \rho h g \cos\psi \quad (a)$$

$$\frac{1}{r}\frac{\partial N_\theta}{\partial \theta} + \frac{\partial N_{x\theta}}{\partial x} + \frac{1}{r}\frac{\partial M_{x\theta}}{\partial x} - \frac{1}{r^2}\frac{\partial M_\theta}{\partial \theta} + P_\theta = \rho h \frac{\partial^2 v}{\partial t^2} + f_2\ (x, \theta, t)$$

$$+ \rho h g \sin\psi \sin\theta \quad (b) \qquad (5.3\text{-}2)$$

$$\frac{1}{r} N_\theta + \frac{1}{r}\frac{\partial^2 M_{\theta x}}{\partial x \partial \theta} + \frac{\partial^2 M_x}{\partial x^2} - \frac{1}{r}\frac{\partial^2 M_{x\theta}}{\partial x \partial \theta} + \frac{1}{r}\frac{\partial^2 M_\theta}{\partial \theta^2} + P_Z$$

$$= \rho h \frac{\partial^2 w}{\partial t^2} + f_3\ (x, \theta, t)$$

$$+ \rho h g \sin\psi \cos\theta \quad (c)$$

where

$$f_1(x, \theta, t) = \rho h(\ddot{X} \cos\psi + \ddot{Y} \sin\psi - r \cos\theta\ddot{\psi} - x\dot{\psi}^2) \quad (a)$$

$$f_2(x, \theta, t) = \rho h(\ddot{X} \sin\psi - \ddot{Y} \cos\psi + r \cos\theta\dot{\psi}^2 - x\ddot{\psi}) \sin\theta \quad (b) \qquad (5.3\text{-}3)$$

$$f_3(x, \theta, t) = \rho h(\ddot{X} \sin\psi - \ddot{Y} \cos\psi + r \cos\theta\dot{\psi}^2 - x\ddot{\psi}) \cos\theta \quad (c)$$

5.4 Approximate Relations Between Stress and Moment Resultants and Displacements

The following approximate relations between the stress resultants and the displacements will be used to write equations (5.3-2) in terms of the displacements u, v, and w.

$$N_x = \frac{Eh}{1-V^2}\left[\frac{\partial u}{\partial x} + \frac{v}{r}\left(\frac{\partial v}{\partial \theta} - w\right)\right] \quad (a)$$

$$N_\theta = \frac{Eh}{1-V^2}\left[\frac{1}{r}\left(\frac{\partial v}{\partial \theta} - w\right) + \nu \frac{\partial u}{\partial x}\right] \qquad (b)$$

$$N_{x\theta} = N_{\theta x} = Gh\left[\frac{1}{r} \quad \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x}\right] \qquad (c)$$

$$M_x = -D\left[\frac{\partial^2 w}{\partial x^2} + \frac{\nu}{r^2} \quad \frac{\partial^2 w}{\partial \theta^2}\right] \qquad (d) \quad (5.4\text{-}1)$$

$$M_\theta = -D\left[\frac{1}{r^2} \quad \frac{\partial^2 w}{\partial \theta^2} + \nu \frac{\partial^2 w}{\partial x^2}\right] \qquad (e)$$

$$M_{x\theta} = -M_{\theta x} = D \frac{1-\nu}{r}\left(\frac{\partial^2 w}{\partial x \partial \theta}\right) \qquad (f)$$

Substituting equations (5.4-1) into equations (5.3-2) renders the following:

$$\frac{\partial^2 u}{\partial x^2} + \frac{(1-\nu)}{2r^2} \quad \frac{\partial^2 u}{\partial \theta^2} + \frac{1+\nu}{2r} \quad \frac{\partial^2 v}{\partial x \partial \theta} - \frac{\nu}{r} \quad \frac{\partial w}{\partial x} = \frac{(1-\nu^2)}{Eh}\left[\rho h \frac{\partial^2 u}{\partial t^2} + f_1(x, \theta, t) - P_1\right] \qquad (a)$$

$$\frac{(1+\nu)}{2r} \quad \frac{\partial^2 u}{\partial x \partial \theta} + \frac{(1-\nu)}{2} \quad \frac{\partial^2 v}{\partial x^2} + \frac{1}{r^2} \quad \frac{\partial^2 v}{\partial \theta^2} - \frac{1}{r^2} \frac{\partial w}{\partial \theta}$$
$$+ \frac{h^2}{12r^2}\left(\frac{\partial^3 w}{\partial x^2 \partial \theta} + \frac{\partial^3 w}{r^2 \partial \theta^3}\right) + \frac{h^2}{12r^2}\left[(1-\nu) \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{r^2 \partial \theta^2}\right] \qquad (b) \quad (5.4\text{-}2)$$
$$= \frac{(1-\nu^2)}{Eh}\left[\rho h \frac{\partial^2 v}{\partial t^2} + f_2(x, \theta, t) - P_2\right]$$

$$\frac{\nu}{r}\frac{\partial u}{\partial x}+\frac{1}{r^2}\frac{\partial v}{\partial \theta}-\frac{w}{r^2}-\frac{h^2}{12r}\left(r\frac{\partial^4 w}{\partial x^4}\ \frac{2}{r}\frac{\partial^4 w}{\partial x^2\partial\theta^2}+\frac{\partial^4 w}{r^3\partial\theta^4}\right)$$

(c)

$$-\frac{h^2}{12r}\left(\frac{2-\nu}{r}\ \frac{\partial^3 v}{\partial x^2\partial\theta}+\frac{\partial^3 v}{r^3\partial\theta^3}\right)=\frac{(1-\nu^2)}{E}\left[\rho h\frac{\partial^2 w}{\partial t^2}+f_3(x,\theta,t)-P_3\right]$$

5.5 Simplification of the Equations of Motion In Terms of the Shell Displacements

As advanced by Donnell (ref. 3), and as recently considered by Yu (ref. 4) the components u and v are considered to be of the order of magnitude $w\sqrt{h/r}$ then the last two terms on the left side of equation (5.4-2b) and the last term on the left side of equation (5.4-2c) are of a higher order than all of the others and may be neglected. The equations are therefore reduced to the following simpler form:

$$\frac{\partial^2 u}{\partial x^2}+\frac{(1-\nu)}{2r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{(1+\nu)}{2r}\frac{\partial^2 v}{\partial x\partial\theta}-\frac{\nu}{r}\frac{\partial w}{\partial x}=\frac{(1-\nu^2)}{Eh}\left[\rho h\frac{\partial^2 u}{\partial t^2}+f_1(x,\theta,t)-P_x\right]\qquad\text{(a)}$$

$$\frac{(1+\nu)}{2r}\frac{\partial^2 u}{\partial x\partial\theta}+\frac{(1-\nu)}{2}\frac{\partial^2 v}{\partial x^2}+\frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2}-\frac{1}{r^2}\frac{\partial w}{\partial\theta}=\frac{(1-\nu^2)}{Eh}\left[\rho h\frac{\partial^2 v}{\partial t^2}+f_2(x,\theta,t)-P_\theta\right]\qquad\text{(b)}$$

(5.5-1)

$$\frac{\nu}{r}\frac{\partial u}{\partial x}+\frac{1}{r^2}\frac{\partial v}{\partial\theta}-\frac{w}{r^2}-\frac{h^2}{12}\left[\frac{\partial^4 w}{\partial x^4}+\frac{2}{r^2}\frac{\partial^4 w}{\partial x^2\partial\theta^2}+\frac{1}{r^4}\frac{\partial^4 w}{\partial\theta^4}\right]=\frac{(1-\nu^2)}{Eh}\left[\rho h\frac{\partial^2 w}{\partial t^2}+f_3(x,\theta,t)-P_z\right]\qquad\text{(c)}$$

It can be seen that neglecting the above mentioned terms is equivalent to neglecting the term $\frac{Q_\theta}{r}$ in equation (5.1-1b) and the terms $\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial x}$ in the expressions for the change in curvature and the twist of the middle of the shell.

The simplified equations become

$$\frac{\partial^2 u}{\partial x^2} + \frac{(1-\nu)}{2r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{(1+\nu)}{2r}\frac{\partial^2 v}{\partial x \partial \theta} - \frac{\nu}{r}\frac{\partial w}{\partial x} = \frac{(1-\nu^2)}{E}\left[\rho\left(\frac{\partial^2 u}{\partial t^2}\right)\right] + F_1(x, \theta, t) \qquad (a)$$

$$\frac{(1+\nu)}{2r}\frac{\partial^2 u}{\partial x \partial \theta} + \frac{(1-\nu)}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} - \frac{1}{r^2}\frac{\partial w}{\partial \theta} = \frac{(1-\nu^2)}{E}\left[\rho\left(\frac{\partial^2 v}{\partial t^2}\right)\right] + F_2(x, \theta, t) \qquad (b) \quad (5.5\text{-}2)$$

$$\frac{\nu}{r}\frac{\partial u}{\partial x} + \frac{1}{r^2}\frac{\partial v}{\partial \theta} - \frac{1}{r^2} w - \frac{h^2}{12}\nabla^4 w = \frac{(1-\nu^2)}{E}\left[\rho\left(\frac{\partial^2 w}{\partial t^2}\right)\right] + F_3(x, \theta, t) \qquad (c)$$

where

$$F_1(x, \theta, t) = \frac{1-\nu^2}{Eh}(f_1 - P_1) \qquad (a)$$

$$F_2(x, \theta, t) = \frac{1-\nu^2}{Eh}(f_2 - P_2) \qquad (b) \quad (5.5\text{-}3)$$

$$F_3(x, \theta, t) = \frac{1-\nu^2}{Eh}(f_3 - P_3) \qquad (c)$$

5.6 Transformation to Dimensionless Equations

In order to present the equations of motion in dimensionless form the following parameters will be used

$$\xi = \frac{x}{L}, \qquad \tau = \omega_1 t, \qquad \lambda = L/r$$
$$\bar{u} = u/L, \qquad \bar{v} = v/L, \qquad \bar{w} = w/L \tag{5.6-1}$$
$$\sigma = h/L, \qquad \mu = \frac{\rho L^2 \omega_1^{\,2}}{E},$$

The equations of motion become

$$\frac{\partial^2 \bar{u}}{\partial \xi^2} + \frac{(1-\nu)}{2}\lambda^2 \frac{\partial^2 \bar{u}}{\partial \theta^2} + \frac{(1+\nu)}{2}\lambda \frac{\partial^2 \bar{v}}{\partial \xi \partial \theta} - \nu\lambda \frac{\partial \bar{w}}{\partial \xi} = \frac{(1-\nu^2)}{E} \rho\omega_1^2 L^2 \frac{\partial^2 \bar{u}}{\partial \tau^2}$$
$$+ G_1(\xi, \theta, \tau) \qquad \text{(a)}$$

$$\frac{1+\nu}{2}\lambda \frac{\partial^2 \bar{u}}{\partial \xi \partial \theta} + \frac{1-\nu}{2} \frac{\partial^2 \bar{v}}{\partial \xi^2} + \lambda^2 \frac{\partial^2 \bar{v}}{\partial \theta^2} - \lambda^2 \frac{\partial \bar{w}}{\partial \theta} = \frac{(1-\nu^2)}{E} \rho\omega_1^2 L^2 \frac{\partial^2 v}{\partial \tau^2}$$
$$+ G_2(\xi, \theta, \tau) \qquad \text{(b)} \tag{5.6-2}$$

$$\nu\lambda \frac{\partial \bar{u}}{\partial \xi} + \lambda^2 \frac{\partial \bar{v}}{\partial \theta} - \lambda^2 \bar{w} - \frac{\sigma^2}{12} \bar{\nabla}^4 \bar{w} = \frac{(1-\nu^2)}{E} \rho\omega_1^2 L^2 \frac{\partial^2 \bar{w}}{\partial \tau^2} + G_3(\xi, \theta, \tau) \qquad \text{(c)}$$

where

$$G_1(\xi, \theta, \tau) = LF_1(\xi, \theta, \tau) \qquad \text{(a)}$$
$$G_2(\xi, \theta, \tau) = LF_2(\xi, \theta, \tau) \qquad \text{(b)} \tag{5.6-3}$$
$$G_3(\xi, \theta, \tau) = LF_3(\xi, \theta, \tau) \qquad \text{(c)}$$

and

$$\bar{\nabla}^2 = \frac{\partial^2}{\partial \xi^2} + \lambda^2 \frac{\partial^2}{\partial \theta^2} \tag{5.6-4}$$

5.7 Intermediate and Final Forcing Functions

5.7.1 Inertial Forces

Using the dimensionless coordinates

$$X_o = \frac{X}{L}, \quad Y_o = \frac{Y}{L}, \quad \tau = \omega_1 t \tag{5.7-1}$$

the inertial forces in equations (5.3-3) can be written in dimensionless form as follows:

$$f_1(\xi, \theta, \tau) = \rho h L \omega_1^2 (\ddot{X}_o \cos\psi + \ddot{Y}_o \sin\psi - \frac{r}{L}\ddot{\psi} \cos\theta - \dot{\psi}^2 \xi) \quad (a)$$

$$f_2(\xi, \theta, \tau) = \rho h L \omega_1^2 (\ddot{X}_o \sin\psi - \ddot{Y}_o \cos\psi + \frac{r}{L}\dot{\psi}^2 \cos\theta - \ddot{\psi}\xi) \sin\theta \quad (b)$$

$$f_3(\xi, \theta, \tau) = \rho h L \omega_1^2 (\ddot{X}_o \sin\psi - \ddot{Y}_o \cos\psi + \frac{r}{L}\dot{\psi}^2 \cos\theta - \ddot{\psi}\xi) \cos\theta \quad (c)$$

(5.7-2)

5.7.2 Distributed Surface Forces

Since only the gimbaled thrust T(t) is considered in the analysis the distributed forces P_1, P_2, and P_3 in equations (5.5-3) are written

$$P_1(\theta, t) = T(t) \cos K\psi - \rho g h \cos\psi \quad (a)$$

$$P_2(\theta, t) = -\left[T(t) \sin K\psi + \rho g h \sin\psi\right] \sin\theta \quad (b)$$

$$P_3(\theta, t) = -\left[T(t) \sin K\psi + \rho g h \sin\psi\right] \cos\theta \quad (c)$$

(5.7-3)

In terms of dimensionless time parameter $\tau = \omega_1 t$ the equations are written

$$P_1(\theta, \tau) = T_o(1 - \gamma\cos\bar{\Omega}\tau)\cos K\psi - \rho g h \cos\psi \qquad \text{(a)}$$

$$P_2(\theta, \tau) = -\left[T_o(1 - \gamma\cos\bar{\Omega}\tau)\sin K\psi + \rho g h \sin\psi\right]\sin\theta \qquad \text{(b)} \qquad (5.7\text{-}4)$$

$$P_3(\theta, \tau) = -\left[T_o(1 - \gamma\cos\bar{\Omega}\tau)\sin K\psi + \rho g h \sin\psi\right]\cos\theta \qquad \text{(c)}$$

where

$$\bar{\Omega} = \frac{\Omega}{\omega_1} \qquad (5.7\text{-}5)$$

5.7.3 <u>Final Forcing Terms</u>

Substituting equations (5.7-2) and (5.7-4) into equations (5.6-3) the final forcing terms in dimensionless form are written

$$G_1(\xi, \theta, \tau) = (1-\nu^2)\mu\left[\frac{\bar{T}_o}{2}(1-\gamma\cos\bar{\Omega}\tau)\cos K\psi - \frac{\ddot{\psi}}{\lambda}\cos\theta - \dot{\psi}^2\xi\right] \qquad \text{(a)}$$

$$G_2(\xi, \theta, \tau) = (1-\nu^2)\mu\left[-\frac{\bar{T}_o}{2}(1-\gamma\cos\bar{\Omega}\tau)\sin K\psi + \frac{\dot{\psi}^2}{\lambda}\cos\theta - \ddot{\psi}^2\xi\right]\sin\theta \qquad \text{(b)}$$

$$(5.7\text{-}6)$$

$$G_3(\xi, \theta, \tau) = (1-\nu^2)\mu\left[-\frac{\bar{T}_o}{2}(1-\gamma\cos\bar{\Omega}\tau)\sin K\psi + \frac{\dot{\psi}^2}{\lambda}\cos\theta - \ddot{\psi}^2\xi\right]\cos\theta \qquad \text{(c)}$$

where

$$\mu = \frac{\rho L^2 \omega_1^2}{E} \qquad (5.7\text{-}7)$$

and

$$\bar{T}_o = \frac{T_o}{\rho h L^2 \omega_1^2} \qquad (5.7\text{-}8)$$

6.0 GALERKIN METHOD

6.1 General

The method of solution of equations (5.6-2) will be the Galerkin procedure used earlier in Technical Memo #38, where the advantages of the method were listed in CONCLUSIONS, p. 46 of that memo.

6.2 Approximating Forms Used in the Galerkin Procedure

Rewriting the equations of motion (5.6-2)

$$\frac{\partial^2 \bar{u}}{\partial \xi^2} + \frac{(1-\nu)}{2} \lambda^2 \frac{\partial^2 \bar{u}}{\partial \theta^2} + \frac{(1+\nu)}{2} \lambda \frac{\partial^2 \bar{v}}{\partial \xi \partial \theta} - \nu\lambda \frac{\partial \bar{w}}{\partial \xi}$$

$$= (1-\nu^2)\mu \frac{\partial^2 \bar{u}}{\partial \tau^2} + G_1(\xi, \theta, \tau) \qquad \text{(a)}$$

$$\frac{(1+\nu)}{2} \lambda \frac{\partial^2 \bar{u}}{\partial \xi \partial \theta} + \frac{(1-\nu)}{2} \frac{\partial^2 \bar{v}}{\partial \xi^2} + \lambda^2 \frac{\partial^2 \bar{v}}{\partial \theta^2} - \lambda^2 \frac{\partial \bar{w}}{\partial \theta}$$

$$= (1-\nu^2)\mu \frac{\partial^2 v}{\partial \tau^2} + G_2(\xi, \theta, \tau) \qquad \text{(b)} \qquad (6.2\text{-}1)$$

$$\nu\lambda \frac{\partial \bar{u}}{\partial \xi} + \lambda^2 \frac{\partial \bar{v}}{\partial \theta} - \lambda^2 \bar{w} - \frac{\sigma^2}{12} \bar{\nabla}^4 \bar{w} = (1-\nu^2)\mu \frac{\partial^2 \bar{w}}{\partial \tau^2}$$

$$+ G_3(\xi, \theta, \tau) \qquad \text{(c)}$$

The Galerkin procedure requires the use of assumed approximating functions for $\bar{u}$, $\bar{v}$, and $\bar{w}$ in the above equations. In this analysis we chose the approximating forms

$$\tilde{u}(\xi, \theta, \tau) = \sum_{m=0}^{M} \sum_{n=0}^{N} U_{mn}(\tau) \; f_{mn}(\xi, \theta) + u^*(\xi, \tau) \qquad \text{(a)}$$

$$\bar{v}(\xi, \theta, \tau) = \sum_{m=0}^{M} \sum_{n=0}^{N} V_{mn}(\tau) \quad g_{mn}(\xi, \theta) + v^{*}(\xi, \theta, \tau) \qquad (b) \qquad (6.2\text{-}2)$$

$$\bar{w}(\xi, \theta, \tau) = \sum_{m=0}^{M} \sum_{n=0}^{N} W_{mn}(\tau) \quad h_{mn}(\xi, \theta) + w^{*}(\xi, \theta, \tau) \qquad (c)$$

where

$$u^{*}(\xi, \tau) = f^{*}(\xi, \tau) = - \frac{1-\nu^{2}}{Eh} \frac{T_{o}}{2}(1-\gamma\cos\bar{\Omega}\tau) \cos k\psi \ (\xi - \frac{\xi^{2}}{2}) \qquad (a)$$

$$v^{*}(\xi, \theta, \tau) = g^{*}(\xi, \tau) \sin \theta \qquad (b) \ (6.2\text{-}3)$$

$$w^{*}(\xi, \theta, \tau) = h^{*}(\xi, \tau) \cos \theta \qquad (c)$$

and

$$f_{mn}(\xi, \theta) = \cos \frac{m\pi}{2} (\xi+1) \cos n\theta \qquad (a)$$

$$g_{mn}(\xi, \theta) = \sin \frac{m\pi}{2} (\xi+1) \sin n\theta \qquad (b) \ (6.2\text{-}4)$$

$$h_{mn}(\xi, \theta) = \sin \frac{m\pi}{2} (\xi+1) \cos n\theta \qquad (c)$$

It can be seen that the functions $f^{*}(\xi, \tau)$, $g^{*}(\xi, \tau)$, $h^{*}(\xi, \tau)$, $f_{mn}(\xi, \theta)$, $g_{mn}(\xi, \theta)$ and $h_{mn}(\xi, \theta)$ satisfies the boundary conditions at $\xi = -1$ and $\xi = +1$. Additionally, as shown in APPENDIX A, the assumed forms for $\bar{u}$, $\bar{v}$ and $\bar{w}$ satisfies the stress resultant-displacement relationship which renders the required edge loading.

$$(N_{\xi})_{\xi=-1} = -T_{o}(1 - \gamma\cos\bar{\Omega}\tau) \cos k\psi \qquad (a)$$

$$(6.2\text{-}5)$$

$$(N_{\xi})_{\xi=+1} = 0 \qquad (b)$$

The significance of the indices m and n in $f_{mn}(\xi, \theta)$, $g_{mn}(\xi, \theta)$, and $h_{mn}(\xi, \theta)$ can be more easily understood by referring to the vibration shapes of Figures 3(a) and 3(b) and the nodal arrangement of Figure 4.

6.3 Resulting Equations After Use of the Galerkin Method

Substituting the approximating functions $\tilde{u}$, $\tilde{v}$, and $\tilde{w}$ and their required partial derivatives into equations (6.2-1), multiplying the first, second, and third resulting equations by the orthogonal functions $f_{jk}(\xi, \theta)$, $g_{jk}(\xi, \theta)$ and $h_{jk}(\xi, \theta)$, respectively, and forming double integrals, we have the following results:

$$\sum_{m=0}^{M} \sum_{n=0}^{N} \Bigg\{ \left[(1-\nu^2)\mu \int_{-1}^{1}\int_{0}^{2\pi} f_{mn}(\xi, \theta)\, f_{jk}(\xi, \theta)\, d\xi\, d\theta \right] \ddot{U}_{mn}(\tau)$$
$$- \left[\int_{-1}^{1}\int_{0}^{2\pi} \left(\frac{\partial^2 f_{mn}(\xi, \theta)}{\partial \xi^2} + \frac{(1-\nu)}{2} \lambda^2 \frac{\partial^2 f_{mn}(\xi, \theta)}{\partial \theta} \right) f_{jk}(\xi, \theta)\, d\xi\, d\theta \right] U_{mn}(\tau)$$
$$- \left[\frac{(1+\nu)}{2} \lambda \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial^2 g_{mn}(\xi, \theta)}{\partial \xi \partial \theta} f_{jk}(\xi, \theta)\, d\xi\, d\theta \right] V_{mn}(\tau)$$
$$+ \left[\nu\lambda \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial h_{mn}(\xi, \theta)}{\partial \xi} f_{jk}(\xi, \theta)\, d\xi\, d\theta \right] W_{mn}(\tau) \Bigg\}$$
$$= -\int_{-1}^{1}\int_{0}^{2\pi} G_1(\xi, \theta, \tau)\, f_{jk}(\xi, \theta)\, d\xi\, d\theta + \int\int \frac{\partial^2 f^*(\xi, \tau)}{\partial \xi^2} f_{jk}(\xi, \theta)\, d\xi\, d\theta$$
$$+ \int_{-1}^{1}\int_{0}^{2\pi} \frac{(1+\nu)\lambda}{2} \frac{\partial^2 \left[g^*(\xi, \tau) \sin\theta \right]}{\partial \xi \partial \theta} f_{jk}(\xi, \theta)\, d\xi\, d\theta$$
$$+ \nu\lambda \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial \left[h^*(\xi, \tau) \cos\theta \right]}{\partial \xi} f_{jk}(\xi, \theta)\, d\xi\, d\theta$$

$$j = 1, 2, \ldots, M_1$$

$$k = 1, 2, \ldots, N_1 \qquad (6.3\text{-}1)$$

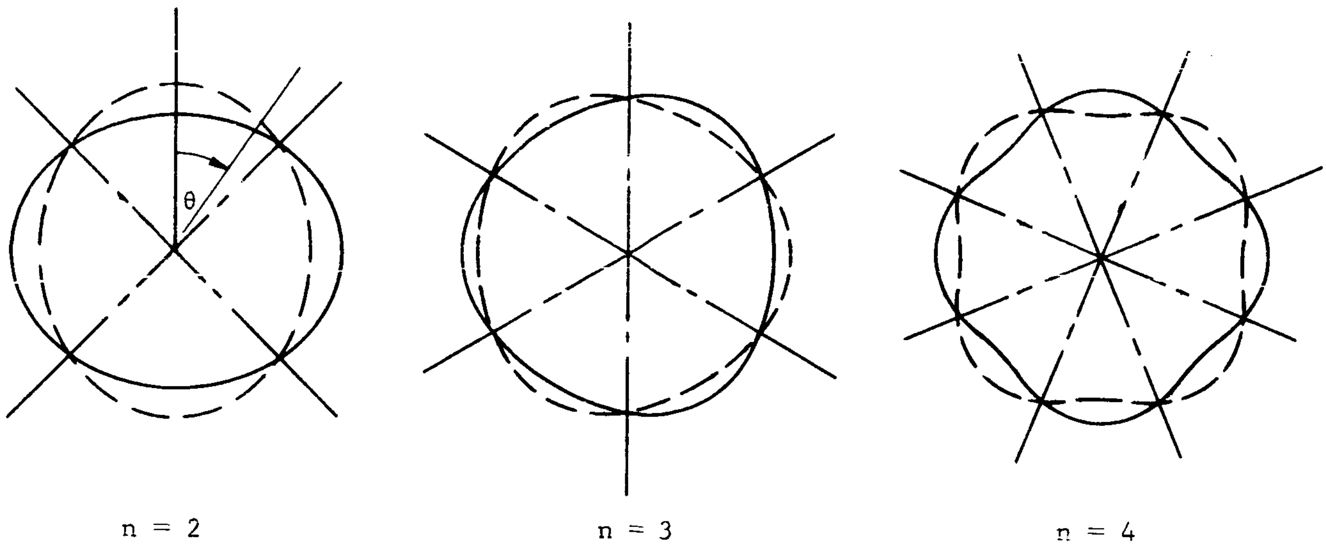


Figure 3(a) CIRCUMFERENTIAL VIBRATION SHAPES FOR THE FREE-FREE, THIN-WALLED CYLINDER

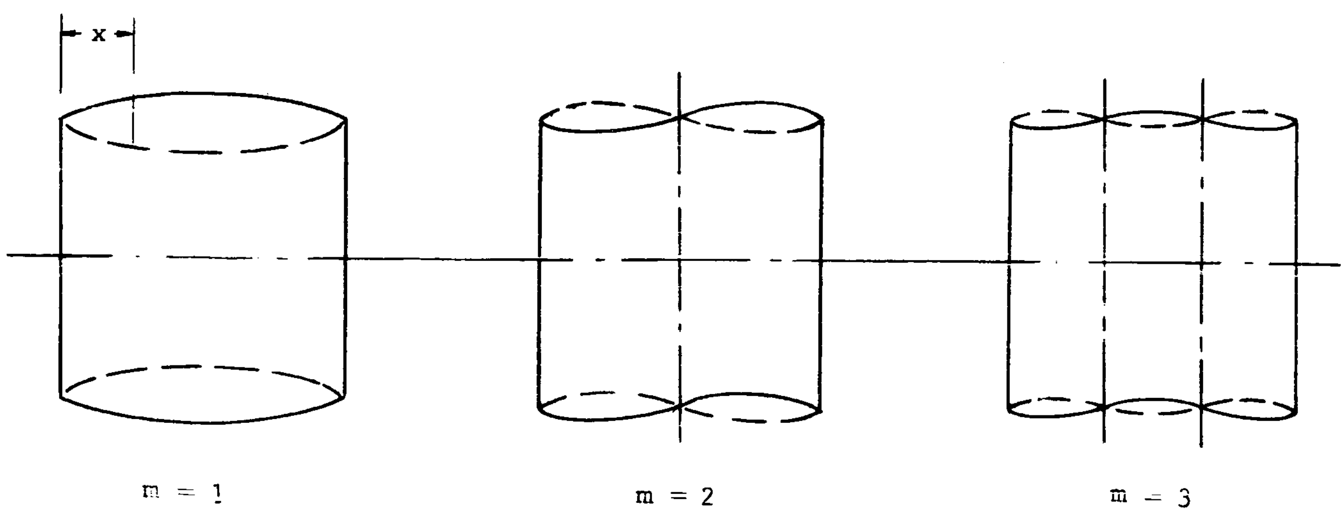


Figure 3(b) AXIAL VIBRATION SHAPES FOR THE FREE-FREE, THIN-WALLED CYLINDER

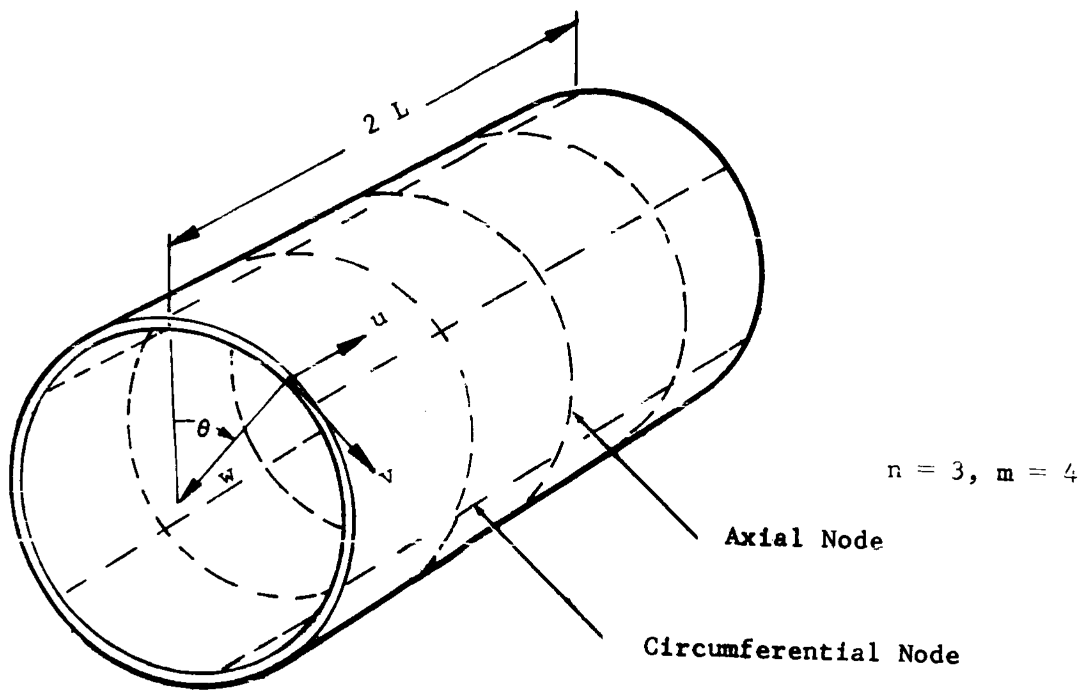


Figure 4 NODAL ARRANGEMENT OF A THIN-WALLED CYLINDER FOR THE CASE OF m = 4, n = 3

$$\sum_{m=0}^{M}\sum_{n=0}^{N}\Bigg\{-\left[\frac{(1+\nu)\lambda}{2}\int_{-1}^{1}\int_{0}^{2\pi}\frac{\partial^2 f_{mn}(\xi,\theta)}{\partial\xi\partial\theta}\; g_{jk}(\xi,\theta)\; d\xi d\theta\right]U_{mn}(\tau)$$

$$+\left[(1-\nu^2)\mu\int_{-1}^{1}\int_{0}^{2\pi} g_{mn}(\xi,\theta)\, g_{jk}(\xi,\theta)\, d\xi d\theta\right]\ddot{V}_{mn}(\tau)$$

$$-\left(\int_{-1}^{1}\int_{0}^{2\pi}\left[\frac{(1-\nu)}{2}\;\frac{\partial^2 g_{mn}(\xi,\theta)}{\partial\xi^2}+\lambda^2\;\frac{\partial^2 g_{mn}(\xi,\theta)}{\partial\theta^2}\right] g_{jk}(\xi,\theta)\, d\xi d\theta\right)V_{mn}(\tau)$$

$$+\left[\lambda^2\int_{-1}^{1}\int_{0}^{2\pi}\frac{h_{mn}(\xi,\theta)}{\partial\theta}\; g_{jk}(\xi,\theta)\, d\xi d\theta\right]W_{mn}(\tau)\Bigg\}$$

$$=-\int_{-1}^{1}\int_{0}^{2\pi} G_2(\xi,\theta,\tau)\; g_{jk}(\xi,\theta)\, d\xi d\theta\; -\frac{1+\nu}{2}\,\lambda\int_{-1}^{1}\int_{0}^{2\pi}\frac{\partial^2\left[f^*(\xi,\tau)\right]}{\partial\xi\partial\theta}\, g_{jk}(\xi,\theta)\, d\xi d\theta$$

$$-\frac{(1-\nu)}{2}\int_{-1}^{1}\int_{0}^{2\pi}\frac{\partial^2\left[g^*(\xi,\tau)\sin\theta\right]}{\partial\xi^2}\; g_{jk}(\xi,\theta)\, d\xi d\theta$$

$$-\lambda^2\int_{-1}^{1}\int_{0}^{2\pi}\frac{\partial^2\left[g^*(\xi,\tau)\sin\theta\right]}{\partial\theta^2}\; g_{jk}(\xi,\theta)\, d\xi d\theta$$

$$+\lambda^2\int_{-1}^{1}\int_{0}^{2\pi}\frac{\partial\left[h^*(\xi,\tau)\cos\theta\right]}{\partial\theta}\, g_{jk}(\xi,\theta)\, d\xi d\theta$$

$$j = 1, 2, \ldots, M_1$$

$$k = 1, 2, \ldots, N_1 \qquad (6.3\text{-}2)$$

$$
\begin{aligned}
\sum_{m=0}^{M}\sum_{n=0}^{N}\Bigg\{ & -\left[\nu\lambda \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial f_{mn}(\xi,\theta)}{\partial \xi} h_{jk}(\xi,\theta)\, d\xi d\theta \right] \ddot{U}_{mn}(\tau) \\
& -\left[\lambda^2 \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial g_{mn}(\xi,\theta)}{\partial \theta} h_{jk}(\xi,\theta)\, d\xi d\theta \right] V_{mn}(\tau) \\
& +\left[(1-\nu^2)\mu \int_{-1}^{1}\int_{0}^{2\pi} h_{mn}(\xi,\theta)\, h_{jk}(\xi,\theta)\, d\xi d\theta \right] \ddot{W}_{mn}(\tau) \\
& +\left(\int_{-1}^{1}\int_{0}^{2\pi}\left[\lambda^2 h_{mn}(\xi,\theta) + \frac{\sigma^2}{12}\nabla^4 h_{mn}(\xi,\theta)\right] h_{jk}(\xi,\theta)\, d\xi d\theta \right) W_{mn}(\tau)\Bigg\} \\
= & -\int_{-1}^{1}\int_{0}^{2\pi} G_3(\xi,\theta)\, h_{jk}(\xi,\theta)\, d\xi d\theta - \nu\lambda \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial f^*(\xi,\tau)}{\partial \xi} h_{jk}(\xi,\theta)\, d\xi d\theta \\
& -\lambda^2 \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial\left[g^*(\xi,\tau)\sin\theta\right]}{\partial\theta} h_{jk}(\xi,\theta)\, d\xi d\theta \\
& \qquad +\lambda^2 \int_{-1}^{1}\int_{0}^{2\pi} h^*(\xi,\tau)\cos\theta\, h_{jk}(\xi,\theta)\, d\xi d\theta \\
& +\frac{\sigma^2}{12}\int_{-1}^{1}\int_{0}^{2\pi} \nabla^4\left[h^*_{jk}(\xi,\tau)\cos\theta\right] h_{jk}(\xi,\theta)\, d\xi d\theta
\end{aligned}
$$

$$
j = 0, 1, \ldots, M_1
$$

$$
k = 0, 1, \ldots, N_1 \qquad (6.3\text{-}3)
$$

6.4 Remaining Equations After Consideration of the Indices j and k

6.4.1 For $j = m = 0,\ k = n = 0$

$$(1-\nu^2)\mu\ 4\pi\ \ddot{U}_{00}(\tau) = -\int_{-1}^{1}\int_{0}^{2\pi} G_1(\xi,\ \theta,\ \tau)\ d\xi\, d\theta$$

$$-\int_{-1}^{1}\int_{0}^{2\pi}\frac{\partial^2\left[f^*(\xi,\ \tau)\right]}{\partial\xi^2}\ d\xi\, d\theta\ -\int_{-1}^{1}\int_{0}^{2\pi}\frac{(1+\nu)}{2}\ \lambda\ \frac{\partial^2\left[g^*(\xi,\ \tau)\ \sin\theta\right]}{\partial\xi\partial\theta}\ d\xi d\theta$$

$$+\ \nu\lambda\int_{-1}^{1}\int_{0}^{2\pi}\frac{\partial\left[h^*(\xi,\ \tau)\ \cos\theta\right]}{\partial\xi}\ d\xi\, d\theta \qquad (6.4\text{-}1)$$

6.4.2 For $j = m = 0,\ k = n = 1,\ 2,\ \ldots\ ,\ N_1$

$$(1-\nu^2)\mu\ (2\pi)\ \ \ddot{U}_{0k}(\tau) + \frac{(1-\nu)}{2}\ \lambda^2 k^2\ \ (2\pi)\ U_{0k}(\tau) =$$

$$-\int_{-1}^{1}\int_{0}^{2\pi} G_1(\xi,\ \theta,\ \tau)\ \cos k\theta\ \ d\xi d\theta + \int_{-1}^{1}\int_{0}^{2\pi}\frac{\partial^2 f^*(\xi,\ \tau)}{\partial\xi^2}\ \cos k\theta\ d\xi d\theta$$

$$+\frac{(1+\nu)\lambda}{2}\int_{-1}^{1}\int_{0}^{2\pi}\frac{\partial^2\left[g^*(\xi,\ \tau)\ \sin\theta\right]}{\partial\xi\partial\theta}\ \cos k\theta\ d\xi d\theta$$

$$-\ \nu\lambda\int_{-1}^{1}\int_{0}^{2\pi}\frac{\partial\left[h^*(\xi,\ \tau)\ \cos\theta\right]}{\partial\xi}\ \cos k\theta\ d\xi d\theta \qquad (6.4\text{-}2)$$

6.4.3 For $j = m = 1,\ 2,\ \ldots\ ,\ M_1,\ k = n = 0$

$$\left((1-\nu^2)\mu\int_{-1}^{1}\int_{0}^{2\pi}\left[\cos\frac{j\pi}{2}\ (\xi+1)\right]^2\ \ d\xi d\theta\right)\ddot{U}_{j0}(\tau)$$

$$+\left(\left(\frac{j\pi}{2}\right)^2\int_{-1}^{1}\int_{0}^{2\pi}\left[\cos\frac{j\pi}{2}\ (\xi+1)\right]^2\ \ d\xi d\theta\right)U_{j0}(\tau)$$

$$+\left(\nu\lambda\ \left(\frac{j\pi}{2}\right)\int_{-1}^{1}\int_{0}^{2\pi}\left[\cos\frac{j\pi}{2}\ (\xi+1)\right]^2\ \ d\xi d\theta\right)W_{j0}(\tau)$$

$$
\begin{aligned}
= & -\int_{-1}^{1}\int_{0}^{2\pi} G_1(\xi,\ \theta,\ \tau)\cos\frac{j\pi}{2}(\xi+1)\ d\xi\, d\theta \\
& -\int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial^2 f^*(\xi,\ \tau)}{\partial\xi^2}\cos\frac{j\pi}{2}(\xi+1)\ d\xi\, d\theta \\
& -\frac{(1+\nu)}{2}\lambda\int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial^2\left[g^*(\xi,\ \tau)\sin\theta\right]}{\partial\xi\,\partial\theta}\cos\frac{j\pi}{2}(\xi+1)\ d\xi\, d\theta \\
& +\nu\lambda\int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial\left[h^*(\xi,\ \tau)\cos\theta\right]}{\partial\xi}\cos\frac{j\pi}{2}(\xi+1)\ d\xi\, d\theta
\end{aligned}
\qquad (6.4\text{-}3)
$$

$$
\begin{aligned}
& \left(\nu\lambda\left(\frac{j\pi}{2}\right)\int_{-1}^{1}\int_{0}^{2\pi}\left[\sin\frac{j\pi}{2}(\xi+1)\right]^2 d\xi\, d\theta\right) U_{j0}(\tau) \\
& +\left((1-\nu^2)\mu\int_{-1}^{1}\int_{0}^{2\pi}\left[\sin\frac{j\pi}{2}(\xi+1)\right]^2 d\xi\, d\theta\right)\ddot{W}_{j0}(\tau) \\
& +\left(\lambda^2\int_{-1}^{1}\int_{0}^{2\pi}\left[\sin\frac{j\pi}{2}(\xi+1)\right]^2 d\xi\, d\theta\right) W_{j0}(\tau) \\
& +\left(\frac{\sigma^2}{12}\left(\frac{j\pi}{2}\right)^4\int_{-1}^{1}\int_{0}^{2\pi}\left[\sin\frac{j\pi}{2}(\xi+1)\right]^2 d\xi\, d\theta\right) W_{j0}(\tau) \\
= & -\int_{-1}^{1}\int_{0}^{2\pi} G_3(\xi,\ \theta,\ \tau)\sin\frac{j\pi}{2}(\xi+1)\ d\xi\, d\theta \\
& -\nu\lambda\int_{-1}^{1}\int_{0}^{2\pi}\frac{\partial\left[f^*(\xi,\ \tau)\right]}{\partial\xi}\sin\frac{j\pi}{2}(\xi+1)\ d\xi\, d\theta \\
& -\lambda^2\int_{-1}^{1}\int_{0}^{2\pi}\frac{\partial\left[g^*(\xi,\ \tau)\sin\theta\right]}{\partial\theta}\sin\frac{j\pi}{2}(\xi+1)\ d\xi\, d\theta \\
& +\lambda^2\int_{-1}^{1}\int_{0}^{2\pi} h^*(\xi,\ \tau)\cos\theta\sin\frac{j\pi}{2}(\xi+1)\ d\xi\, d\theta \\
& +\frac{\sigma^2}{12}\int_{-1}^{1}\int_{0}^{2\pi}\nabla^4\left[h^*(\xi,\ \tau)\cos\theta\right]\sin\frac{j\pi}{2}(\xi+1)\ d\xi\, d\theta
\end{aligned}
\qquad (6.4\text{-}4)
$$

6.4.4 <u>For $j = m = 1, 2, \ldots, M_1$; $k = n = 1, 2, \ldots, N_1$</u>

$$(1-\nu^2)\mu\pi\,\ddot{U}_{jk}(\tau) + \left(\frac{j^2\pi^2+2(1-\nu)\lambda^2k^2}{4}\right)\pi\, U_{jk}(\tau) - \frac{(1+\nu)\lambda}{4}\, jk\pi^2\, V_{jk}(\tau)$$

$$+ \frac{\nu\lambda j\pi^2}{2}\, W_{jk}(\tau) = - \int_{-1}^{1}\int_{0}^{2\pi} G_1(\xi, \theta, \tau)\cos\frac{j\pi}{2}(\xi+1)\cos k\theta \;d\xi\, d\theta$$

$$- \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial^2[f^*(\xi, \tau)]}{\partial\xi^2}\cos\frac{j\pi}{2}(\xi+1)\cos k\theta \;d\xi\, d\theta$$

$$- \frac{(1+\nu)\lambda}{2}\int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial^2[g^*(\xi, \tau)\sin\theta]}{\partial\xi\partial\theta}\cos\frac{j\pi}{2}(\xi+1)\cos k\theta \;d\xi\, d\theta$$

$$+ \nu\lambda\int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial[h^*(\xi, \tau)\cos\theta]}{\partial\xi}\cos\frac{j\pi}{2}(\xi+1)\cos k\theta \;d\xi\, d\theta \qquad (6.4\text{-}5)$$

Rearranging (dividing thru by $(1-\nu^2)\,\mu\pi$)

$$\ddot{U}_{jk}(\tau) + \frac{j^2\pi^2 + 2(1-\nu)\lambda^2k^2}{4(1-\nu^2)\mu}\, U_{jk}(\tau) - \frac{jk\;\pi\;\lambda}{4(1-\nu)\mu}\, V_{jk}(\tau)$$

$$+ \frac{\nu\lambda j\pi}{2(1-\nu^2)\mu}\, W_{jk}(\tau) = - \frac{1}{\mu\pi(1-\nu^2)}\left[\int_{-1}^{1}\int_{0}^{2\pi} G_1(\xi, \theta, \tau)\cos\frac{j\pi}{2}(\xi+1)\cos k\theta \;d\xi\, d\theta\right.$$

$$+ \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial^2[f^*(\xi, \tau)]}{\partial\xi^2}\cos\frac{j\pi}{2}(\xi+1)\cos k\theta \;d\xi\, d\theta$$

$$+ \frac{(1+\nu)\lambda}{2}\int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial^2[g^*(\xi, \tau)\sin\theta]}{\partial\xi\partial\theta}\cos\frac{j\pi}{2}(\xi+1)\cos k\theta \;d\xi\, d\theta$$

$$\left.- \nu\lambda\int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial[h^*(\xi, \tau)\cos\theta]}{\partial\xi}\cos\frac{j\pi}{2}(\xi+1)\cos k\theta \;d\xi\, d\theta\right] = \left[Q_{jk}(\tau)\right]_1 \qquad (6.4\text{-}6)$$

$$- \frac{(1+\nu)}{2} \frac{jk\ \pi^2}{2} U_{jk}(\tau) + (1-\nu^2)\mu\pi\ \ddot{V}_{jk}(\tau) + \left[\frac{(1-\nu)j^2\pi^2}{8} + \lambda^2 k^2\right]\pi\ V_{jk}(\tau)$$

$$- \lambda^2 k\pi\ W_{jk}(\tau) = - \int_{-1}^{1}\int_{0}^{2\pi} G_2(\xi,\ \theta,\ \tau) \sin \frac{j\pi}{2} (\xi+1) \sin k\theta\ d\xi d\theta$$

$$- \frac{1+\nu}{2} \lambda \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial^2 f^*(\xi,\tau)}{\partial\xi\partial\theta} \sin \frac{j\pi}{2} (\xi+1) \sin k\theta\ d\xi d\theta$$

$$- \frac{(1-\nu)}{2} \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial^2[g^*(\xi,\ \tau) \sin\ \theta]}{\partial\xi^2} \sin \frac{j\pi}{2} (\xi+1) \sin k\theta\ d\xi d\theta$$

$$- \lambda^2 \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial^2[g^*(\xi,\ \tau) \sin\ \theta]}{\partial\theta^2} \sin \frac{j\pi}{2} (\xi+1) \sin k\theta\ d\xi d\theta$$

$$+ \lambda^2 \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial[h^*(\xi,\ \tau) \cos\ \theta]}{\partial\theta} \sin \frac{j\pi}{2} (\xi+1) \sin k\theta\ d\xi d\theta \qquad (6.4\text{-}7)$$

Rearranging (dividing thru by $(1-\nu^2)\mu\pi$)

$$- \frac{jk\lambda\pi}{4(1-\nu)\mu} U_{jk}(\tau) + \ddot{V}_{jk}(\tau) + \frac{(1-\nu)\ j^2\pi^2 + 8\ \lambda^2 k^2}{8(1-\nu^2)\ \mu} V_{jk}(\tau)$$

$$- \frac{\lambda^2 k}{(1-\nu^2)\mu} W_{jk}(\tau) = - \frac{1}{(1-\nu^2)\mu\pi} \Bigg[\int_{-1}^{1}\int_{0}^{2\pi} G_2(\xi,\ \theta,\ \tau) \sin \frac{j\pi}{2} (\xi+1) \sin k\theta\ d\xi d\theta$$

$$+ \frac{1+\nu}{2} \lambda \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial^2 f^*(\xi,\ \tau)}{\partial\xi\partial\theta} \sin \frac{j\pi}{2} (\xi+1) \sin k\theta\ d\xi d\theta$$

$$+ \frac{(1-\nu)}{2} \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial^2[g^*(\xi,\ \tau) \sin\ \theta]}{\partial\xi^2} \sin \frac{j\pi}{2} (\xi+1) \sin k\theta\ d\xi d\theta$$

$$+ \lambda^2 \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial^2[g^*(\xi,\ \tau) \sin\ \theta]}{\partial\theta^2} \sin \frac{j\pi}{2} (\xi+1) \sin k\theta\ d\xi d\theta$$

$$- \lambda^2 \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial[h^*(\xi,\ \tau) \cos\ \theta]}{\partial\theta} \sin \frac{j\pi}{2} (\xi+1) \sin k\theta\ d\xi d\theta \Bigg] = \left[Q_{jk}(\tau)\right]_2 \qquad (6.4\text{-}8)$$

$$\nu\lambda \frac{j\pi^2}{2} U_{jk}(\tau) - \lambda^2 k\pi \quad V_{jk}(\tau) + (1-\nu^2)\mu\pi \ddot{W}_{jk}(\tau)$$

$$+ \left\{\lambda^2 + \frac{\sigma^2}{12}\left[\left(\frac{j\pi}{2}\right)^4 + 2\lambda^2 k^2 \left(\frac{j\pi}{2}\right)^2 + \lambda^4 k^4\right]\right\}\pi \ W_{jk}(\tau)$$

$$= - \int_{-1}^{1}\int_{0}^{2\pi} G_3(\xi, \ \theta, \ \tau) \sin \frac{j\pi}{2} (\xi+1) \cos k\theta \ \ d\xi d\theta$$

$$- \nu\lambda \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial\left[f^*(\xi, \ \tau)\right]}{\partial\xi} \sin \frac{j\pi}{2} (\xi+1) \cos k\theta \ d\xi d\theta$$

$$- \lambda^2 \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial\left[g^*(\xi, \ \tau) \sin \theta\right]}{\partial\theta} \sin \frac{j\pi}{2} (\xi+1) \cos k\theta \ d\xi d\theta$$

$$+ \lambda^2 \int_{-1}^{1}\int_{0}^{2\pi} \left[h^*(\xi, \ \tau) \cos\theta\right] \sin \frac{j\pi}{2} (\xi+1) \cos k\theta \ d\xi d\theta$$

$$+ \frac{\sigma^2}{12} \int_{-1}^{1}\int_{0}^{2\pi} \overline{\nabla}^4 \left[h^*(\xi, \ \tau) \cos\theta\right] \sin \frac{j\pi}{2}(\xi+1) \cos k\theta \quad d\xi d\theta \qquad (6.4\text{-}9)$$

Rearranging (dividing thru by $(1-\nu^2)\mu \ \pi$)

$$\frac{j\pi\nu\lambda}{(1-\nu^2)\mu} U_{jk}(\tau) - \frac{k\lambda^2}{(1-\nu^2)\mu} V_{jk}(\tau) + \ddot{W}_{jk}(\tau)$$

$$+ \frac{12\lambda^2 + \sigma^2\left(\frac{j^2\pi^2}{4} + k^2\lambda^2\right)^2}{12(1-\nu^2)\mu} W_{jk}(\tau)$$

$$= - \frac{1}{(1-\nu^2)\mu\pi}\left[\int_{-1}^{1}\int_{0}^{2\pi} G_3(\xi, \ \theta, \ \tau) \sin \frac{j\pi}{2} (\xi+1) \cos k\theta \quad d\xi d\theta\right.$$

$$+ \nu\lambda \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial f^*(\xi, \ \tau)}{\partial\xi} \sin \frac{j\pi}{2} (\xi+1) \cos k\theta \quad d\xi d\theta$$

$$+ \lambda^2 \int_{-1}^{1} \int_{0}^{2\pi} \frac{\partial\left[g^*(\xi, \tau) \sin\theta\right]}{\partial\theta} \sin\frac{j\pi}{2}(\xi+1) \cos k\theta \; d\xi d\theta$$

$$- \lambda^2 \int_{-1}^{1} \int_{0}^{2\pi} \left[h^*(\xi, \tau) \cos\theta\right] \sin\frac{j\pi}{2}(\xi+1) \cos k\theta \; d\xi d\theta$$

$$\left. - \frac{\sigma^2}{12} \int_{-1}^{1} \int_{0}^{2\pi} \bar{\nabla}^4\left[h^*(\xi, \tau) \cos\theta\right] \sin\frac{j\pi}{2}(\xi+1) \cos k\theta \; d\xi d\theta \right] = \left[Q_{jk}(\tau)\right]_3 \quad (6.4\text{-}10)$$

6.4.5 Simplification and Summary of the Equations

$$\ddot{U}_{00}(\tau) = - \frac{1}{4\pi(1-\nu^2)\mu} \left[\int_{-1}^{1} \int_{0}^{2\pi} G_1(\xi, \theta, \tau) \, d\xi d\theta + \int_{-1}^{1} \int_{0}^{2\pi} \frac{\partial^2\left[f^*(\xi, \tau)\right]}{\partial\xi^2} d\xi d\theta \right.$$

$$+ \frac{(1+\nu)\lambda}{2} \int_{-1}^{1} \int_{0}^{2\pi} \frac{\partial^2\left[g^*(\xi, \tau) \sin\theta\right]}{\partial\xi\partial\theta} d\xi d\theta$$

$$\left. - \nu\lambda \int_{-1}^{1} \int_{0}^{2\pi} \frac{\partial\left[h^*(\xi, \tau) \cos\theta\right]}{\partial\xi} d\xi d\theta \right] = P_{00}(\tau) \quad (6.4\text{-}11)$$

$j = 0$

$k = 0$

$$\ddot{U}_{0k}(\tau) + \frac{k^2\lambda^2}{2(1+\nu)\mu} U_{0k}(\tau) = - \frac{1}{2\pi(1-\nu^2)\mu} \left[\int_{-1}^{1} \int_{0}^{2\pi} G_1(\xi, \theta, \tau) \cos k\theta \, d\xi d\theta \right.$$

$$+ \int_{-1}^{1} \int_{0}^{2\pi} \frac{\partial^2\left[f^*(\xi, \tau)\right]}{\partial\xi^2} \cos k\theta \; d\xi d\theta$$

$$+ \frac{(1+\nu)\lambda}{2} \int_{-1}^{1} \int_{0}^{2\pi} \frac{\partial^2\left[g^*(\xi, \tau) \sin\theta\right]}{\partial\xi\partial\theta} \cos k\theta \; d\xi d\theta$$

$$\left. - \nu\lambda \int_{-1}^{1} \int_{0}^{2\pi} \frac{\partial\left[h^*(\xi, \tau) \cos\theta\right]}{\partial\xi} \cos k\theta \; d\xi d\theta \right] = P_{0k}(\tau) \quad (6.4\text{-}12)$$

$j = 0$

$k = 1, 2, \ldots, N_1$

$$\ddot{U}_{j0}(\tau) + \frac{j^2\pi^2}{4(1-\nu^2)\mu} U_{j0}(\tau) + \nu\lambda \frac{j\pi}{2(1-\nu^2)\mu} W_{j0}(\tau) = - \frac{1}{2\pi(1-\nu^2)\mu}$$

$$\Bigg[\int_{-1}^{1}\int_{0}^{2\pi} G_1(\xi, \theta, \tau) \cos \frac{j\pi}{2} (\xi+1)\, d\xi d\theta$$

$$+ \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial^2 \left[f^*(\xi, \tau)\right]}{\partial \xi^2} \cos \frac{j\pi}{2} (\xi+1)\, d\xi d\theta$$

$$+ \frac{(1+\nu)\lambda}{2} \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial^2 \left[g^*(\xi, \tau) \sin \theta\right]}{\partial \xi \partial \theta} \cos \frac{j\pi}{2} (\xi+1)\, d\xi d\theta$$

$$- \nu\lambda \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial \left[h^*(\xi, \tau) \cos \theta\right]}{\partial \xi} \cos \frac{j\pi}{2} (\xi+1)\, d\xi d\theta \Bigg] = \Big[P_{j0}(\tau)\Big]_1 \qquad (6.4\text{-}13)$$

$$j = 1, 2, \ldots, M_1$$

$$k = 0$$

$$\frac{\nu\lambda}{2(1-\nu^2)\mu} U_{j0}(\tau) + \ddot{W}_{j0}(\tau) + \left[\frac{\lambda^2}{(1-\nu^2)\mu} + \frac{\sigma^2 j^4\pi^4}{192(1-\nu^2)\mu}\right] W_{j0}(\tau)$$

$$= - \frac{1}{2\pi(1-\nu^2)\mu} \frac{1}{\int_{-1}^{1}\int_{0}^{2\pi} \left[\sin \frac{j\pi}{2} (\xi+1)\right]^2 d\xi d\theta} \Bigg[G_3(\xi, \theta, \tau) \sin \frac{j\pi}{2} (\xi+1)\, d\xi d\theta$$

$$+ \nu\lambda \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial \left[f^*(\xi, \tau)\right]}{\partial \xi} \sin \frac{j\pi}{2} (\xi+1)\, d\xi d\theta$$

$$+ \lambda^2 \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial \left[g^*(\xi, \tau) \sin \theta\right]}{\partial \theta} \sin \frac{j\pi}{2} (\xi+1)\, d\xi d\theta$$

$$- \lambda^2 \int_{-1}^{1}\int_{0}^{2\pi} h^*(\xi, \tau) \cos \theta \sin \frac{j\pi}{2} (\xi+1)\, d\xi d\theta$$

$$- \frac{\sigma^2}{12} \int_{-1}^{1} \int_{0}^{2\pi} \bar{\nabla}^4 \left[h^*(\xi, \tau) \cos\theta\right] \sin \frac{j\pi}{2} (\xi+1) \, d\xi d\theta \Bigg] = \left[P_{j0}(\tau)\right]_2 \qquad (6.4\text{-}14)$$

Finally, the equations may be written

$$\ddot{U}_{jk}(\tau) + \frac{j^2\pi^2 + 2(1-\nu)\lambda^2k^2}{4(1-\nu^2)\mu} U_{jk}(\tau) - \frac{jk\lambda\pi}{4(1-\nu)\mu} V_{jk}(\tau)$$

$$+ \frac{jk\lambda\pi}{2(1-\nu^2)\mu} W_{jk}(\tau) = \left[Q_{jk}(\tau)\right]_1 \qquad (6.4\text{-}15)$$

$$- \frac{jk\lambda\pi}{4(1-\nu)\mu} U_{jk}(\tau) + \ddot{V}_{jk}(\tau) + \frac{(1-\nu)j^2\pi^2 + 8\lambda^2k^2}{8(1-\nu^2)\mu} V_{jk}(\tau)$$

$$- \frac{k\lambda^2}{(1-\nu^2)\mu} W_{jk}(\tau) = \left[Q_{jk}(\tau)\right]_2 \qquad (6.4\text{-}16)$$

$$\frac{j\nu\lambda\pi}{(1-\nu^2)\mu} U_{jk}(\tau) - \frac{k\lambda^2}{(1-\nu^2)} V_{jk}(\tau) + \ddot{W}_{jk}(\tau)$$

$$+ \frac{12\lambda^2 + \sigma^2\left(\frac{j^2\pi^2}{4} + k^2\lambda^2\right)^2}{12(1-\nu^2)\mu} W_{jk}(\tau) = \left[Q_{jk}(\tau)\right]_3 \qquad (6.4\text{-}17)$$

6.4.6 Initial Conditions

The initial conditions used in this analysis will be as follows for $\tau = 0$

$$\bar{u}(\xi, \theta, 0) = \sum_{m=0}^{M} \sum_{n=0}^{N} U_{mn} \cos \frac{m\pi}{2} (\xi+1) \cos n\theta + u^*(\xi, \theta, 0) \qquad (a)$$

$$\overline{v}(\xi,\ \theta,\ 0) = \sum_{m=0}^{M} \sum_{n=0}^{N} v_{mn} \sin \frac{m\pi}{2} (\xi+1) \sin n\theta + v^{*}(\xi,\ \theta,\ 0) \qquad \text{(b)} \quad (6.4\text{-}18)$$

$$\overline{w}(\xi,\ \theta,\ 0) = \sum_{m=0}^{M} \sum_{n=0}^{N} w_{mn} \sin \frac{m\pi}{2} (\xi+1) \cos n\theta + w^{*}(\xi,\ \theta,\ 0) \qquad \text{(c)}$$

where

$$u^{*}(\xi,\ 0) = -\frac{1-\nu^2}{Eh} \frac{T_o}{2} (1-\gamma) \cos k\psi \left(\xi - \frac{\xi^2}{2}\right) \qquad \text{(a)}$$

$$v^{*}(\xi,\ \theta,\ 0) = g^{*}(\xi,\ 0)_n \sin\theta \qquad \text{(b)} \quad (6.4\text{-}19)$$

$$w^{*}(\xi,\ \theta,\ 0) = g^{*}(\xi,\ 0)_n \cos\theta \qquad \text{(c)}$$

and

$$U_{mn}(0) = u_{mn} \qquad \text{(a)}$$

$$V_{mn}(0) = v_{mn} \qquad \text{(b)} \quad (6.4\text{-}20)$$

$$W_{mn}(0) = w_{mn} \qquad \text{(c)}$$

Secondly, the initial conditions

$$\frac{\partial\overline{u}}{\partial\tau} (\xi,\ \theta,\ 0) = 0 \qquad \text{(a)}$$

$$\frac{\partial\overline{v}}{\partial\tau} (\xi,\ \theta,\ 0) = 0 \qquad \text{(b)} \quad (6.4\text{-}21)$$

$$\frac{\partial\overline{w}}{\partial\tau} (\xi,\ \theta,\ 0) = 0 \qquad \text{(c)}$$

will be imposed.

Prior to solving equations (6.4-11) thru (6.4-17) it will be necessary to determine expressions for $\psi(\tau)$, the rigid-body angle of rotation about the center of mass, and for $g^{*}(\xi,\ \tau)$, the end displacement of the free-free cylinder.

7.0 DETERMINATION OF THE RIGID BODY ANGLE OF ROTATION

7.1 Solution of the Matheau Equation

Rewriting equation (5.2-1c)

$$\frac{d^2\psi}{dt^2} + \frac{2\pi r\ L\ T_o}{I}(1 - \gamma\cos\Omega t)\sin K\psi = 0 \qquad (7.1\text{-}1)$$

and restricting ψ such that $\sin K_\psi \approx K\psi$ renders

$$\frac{d^2\psi}{dt^2} + \frac{2\pi\ r\ L\ T_o K}{I}(1 - \Upsilon\cos\Omega t)\psi = 0 \qquad (7.1\text{-}2)$$

where by changing to a dimensionless time $\tau_1 = \frac{\Omega}{2}t$, and introducing the dimensionless parameters

$$a = \frac{8\pi\ r\ L\ T_o K}{I\Omega^2} \qquad q = \frac{\Upsilon a}{2} \qquad (7.1\text{-}3)$$

the equation may be written

$$\frac{d^2\psi}{d\tau_1^2} + (a - 2q\cos 2\tau_1)\psi = 0 \qquad (7.1\text{-}4)$$

Equation (7.1-4) is a particular case of a linear second order differential equation with periodic coefficients considered as the canonical form of the Matheau equation having different solutions according to the values of the parameters a and q. The theory for the solution of equation (7.1-4) is given in McLachlan (ref. 5), Tisserand (ref. 6), and Whittaker (ref. 7).

For cylindrical shells approximating the size of a large booster, and considering the magnitude of the thrust, T_o, that they encounter, proper range restrictions on the parameters a and q will be

$$0<a<1 \qquad 0<q \leq 0.05 \tag{7.1-5}$$

The solution of equation (7.1-4) takes different forms according to the values of a and q. A stability for the Matheau function in the (a, q) plane is shown on page 40 of McLachlan (ref. 5).

Considering the range restrictions of relations (7.1-5) the stable solution of equation (7.1-4) for q small and positive can be written

$$\psi(\tau_1) = e^{i\beta\tau_1} \sum_{r=-\infty}^{\infty} C_{2r} e^{i2r\tau_1} \tag{7.1-6}$$

Substituting equation (7.1-6) into equation (7.1-4) and equating the coefficient of $e^{i2r\tau_1}$ to zero for $r = -\infty$ to $+\infty$ yields the recurrence relation

$$\left[a - (2r + \beta)^2\right] C_{2r} - q\,(C_{2r+2} + C_{2r-2}) = 0 \tag{7.1-7}$$

Equation (7.1-7) is a linear difference equation and it can be shown (see pp. 37 and 90 of McLachlan, ref. 5) that

$$\left|\frac{C_{2r+2}}{C_{2r}}\right| \simeq \frac{q}{4(r+1)} \rightarrow 0 \quad \text{as } r \rightarrow +\infty \tag{7.1-8}$$

assuring the convergence of the series in equation (7.1-6). Dividing equation (7.1-7) by $\left[(2r + \beta)^2 - a\right]$ renders

$$C_{2r} + \frac{q}{\left[(2r + \beta)^2 - a\right]} (C_{2r+2} + C_{2r-2}) = 0 \tag{7.1-9}$$

which can be written as

$$C_{2r} + \xi_{2r}(C_{2r+2} + C_{2r-2}) = 0 \qquad (7.1\text{-}10)$$

With r= ... -2, -1, 0, 1, 2, ... we obtain the system of linear equations

$$\begin{array}{llllllllll}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot = 0 \\
\cdot + 0 + & \xi_{-4}C_{-6} + & C_{-4} & + \xi_{-4}C_{-2} + & 0 & + & 0 + & 0 + & 0 & + \cdot = 0 \\
\cdot + 0 + & 0 & + \xi_{-2}C_{-4} + & C_{-2} & + \xi_{-2}C_0 + & 0 & + & 0 + & 0 & + \cdot = 0 \\
\cdot + 0 + & 0 & + 0 & + \xi_0C_{-2} + & C_0 & + \xi_0C_2 + & 0 & + & 0 & + \cdot = 0 \\
\cdot + 0 + & 0 & + 0 & + 0 & + \xi_2C_0 + & C_2 & + \xi_2C_4 + & 0 & + & \cdot = 0 \\
\cdot + 0 + & 0 & + 0 & + 0 & + 0 & + \xi_4C_2 + & C_4 & + \xi_4C_6 & + & \cdot = 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot = 0
\end{array} \qquad (7.1\text{-}11)$$

For equation (7.1-6) to be a solution of the Matheau equation (7.1-4), equations (7.1-11) must be a consistent system of equations, i.e., they must be satisfied simultaneously, thus the determinant of their coefficient must vanish. The infinite determinant is written

$$\Delta(\beta) = \begin{vmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \xi_{-4} & 1 & \xi_{-4} & 0 & 0 & 0 & 0 & \cdot \\
\cdot & 0 & \xi_{-2} & 1 & \xi_{-2} & 0 & 0 & 0 & \cdot \\
\cdot & 0 & 0 & \xi_0 & 1 & \xi_0 & 0 & 0 & \cdot \\
\cdot & 0 & 0 & 0 & \xi_2 & 1 & \xi_2 & 0 & \cdot \\
\cdot & 0 & 0 & 0 & 0 & \xi_4 & 1 & \xi_4 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{vmatrix} = 0 \qquad (7.1\text{-}12)$$

When expanded this constitutes an equation in β.

Some of the properties of $\Delta(\beta)$ are as follows:

1. $\Delta(-\beta) = \Delta(\beta)$, even function of β
2. $\Delta(\beta-2) = \Delta(\beta) = \Delta(\beta+2)$, periodic in 2 (7.1-13)
3. Only singularities of $\Delta(\beta)$ are simple poles which occur when $\left[(2r + \beta)^2 - a\right] = 0$, i.e., $\beta = a^{\frac{1}{2}} - 2r$ and $\beta = -(a^{\frac{1}{2}} + 2r)$.

Note that the function

$$\chi(\beta) = \frac{1}{\cos \beta\pi - \cos \pi\, a^{\frac{1}{2}}} \tag{7.1-14}$$

has simple poles at the same values of β (the assumption is made that $a \neq 4r^2$, so $\beta = 0$ is not a pole) while its period is that of $\Delta(\beta)$. From the above statement it follows that the function

$$\psi(\beta) = \Delta(\beta) - C\,\chi(\beta) \tag{7.1-15}$$

will have no singularities if C is suitably chosen, so by Liouville's theorem it must be a constant.

To determine C we proceed as follows: When $\beta \to \infty$ all of the ξ in the infinite determinant tend toward zero, and the diagonal elements alone remain. Thus

$$\lim_{\beta \to \infty} \Delta(\beta) = 1 \tag{7.1-16}$$

$$\lim_{\beta \to \infty} \chi(\beta) = 0 \tag{7.1-17}$$

and therefore

$$\psi(\beta) = 1 \tag{7.1-18}$$

Then

$$C\chi(\beta) = \Delta(\beta) - 1 \tag{7.1-19}$$

or

$$C = \frac{\Delta(\beta) - 1}{\chi(\beta)} \tag{7.1-20}$$

When $\beta = 0$,

$$\chi(0) = \frac{1}{(1 - \cos \pi a^{\frac{1}{2}})} \tag{7.1-21}$$

while the value of $\Delta(\beta)$ is $\Delta(0)$, and

$$\xi_r = \frac{q}{(4r^2 - a)} \quad , \quad a \neq 4r^2 \tag{7.1-22}$$

so equation (7.1-20) is written

$$C = \left[\Delta(0) - 1\right]\left[1 - \cos \pi a^{\frac{1}{2}}\right] \tag{7.1-23}$$

The preceding analysis has been based upon Whittaker (ref. 13).

The determinantal equation is satisfied if β is such that $\Delta(\beta) = 0$. Substituting this for $\Delta(\beta)$ in equation (7.1-15) and equating to equation (7.1-20) renders

$$\frac{1}{\chi(\beta)} = \left[1 - \Delta(0)\right]\left[1 - \cos \pi a^{\frac{1}{2}}\right] \tag{7.1-24}$$

Substituting equation (7.1-14) into the above equation results in

$$\cos \beta\pi = \cos \pi a^{\frac{1}{2}} + \left[1 - \Delta(0)\right] \left[1 - \cos \pi a^{\frac{1}{2}}\right] \tag{7.1-25}$$

$$= 1 - \Delta(0) \left[1 - \cos \pi a^{\frac{1}{2}}\right] \tag{7.1-26}$$

or

$$\sin^2 \frac{\beta\pi}{2} = \Delta(0) \sin^2 \frac{\pi a^{\frac{1}{2}}}{2} \qquad (a \neq 4r^2) \tag{7.1-27}$$

The values of β which satisfy equation (7.1-27) also satisfy the infinite determinant, equations (7.1-12). Hence β is determined if $\Delta(0)$ can be evaluated. If q is small, see Tisserand (ref. 6), $\Delta(0)$ is approximated as

$$\Delta(0) \simeq 1 - \frac{\pi q^2}{4a^{\frac{1}{2}}(a - 1)} \cot \pi a^{\frac{1}{2}} \tag{7.1-28}$$

so that equation (7.1-27) may be written

$$\cos \beta\pi = \cos \pi a^{\frac{1}{2}} + \frac{\pi q^2}{4a^{\frac{1}{2}}(a - 1)} \sin \pi a^{\frac{1}{2}} \tag{7.1-29}$$

Various numerical examples have indicated that only three terms of the series in equation (7.1-6) are sufficient to render an adequate solution of equation (7.1-4) for the restricted ranges of a and q in equation (7.1-5). Thus C_4 and all higher order coefficients and C_{-4} and all lower order coefficients are considered as zero. The linear difference equation [equation (7.1-7)] yields for $r = 1$ the following:

$$C_2 = \frac{q}{a - (2 + \beta)^2} \left(C_0 + \cancelto{0}{C_4}\right) \tag{7.1-30}$$

and for $r = -1$

$$C_{-2} = \frac{q}{a - (\beta - 2)^2} \left(C_0 + \cancelto{0}{C_{-4}}\right) \tag{7.1-31}$$

For $r = 0$, equation (7.1-7) provides a check, we should have

$$(a - \beta^2) = q^2 \left[\frac{1}{a - (2 + \beta)^2} + \frac{1}{a - (\beta - 2)^2} \right] \tag{7.1-32}$$

How well equation (7.1-32) is satisfied is a measure of the adequacy of the solution. Since the solutions of the Matheau equation (equation 7.1-4) are arbitrary to a constant multiplier, an assigned value of $C_0 = 1$ results in

$$C_{-2} = \frac{q}{a - (\beta - 2)^2}, \quad C_2 = \frac{q}{a - (2 + \beta)^2} \tag{7.1-33}$$

Referring to equation (7.1-6) and restricting the series to three terms, the general solution of equation (7.1-4) is written

$$\begin{aligned} \psi(\tau_1) = {} & a_1 \left[C_{-2} \cos (2-\beta)\tau_1 + \cos \beta\tau_1 + C_2 \cos (2+\beta)\tau_1 \right] \\ & + a_2 \left[-C_{-2} \sin (2-\beta)\tau_1 + \sin \beta\tau_1 + C_2 \sin (2+\beta)\tau_1 \right] \end{aligned} \tag{7.1-34}$$

Since $\tau_1 = \frac{\Omega}{2}t = \frac{\Omega}{2\omega_1} \tau$, equation (7.1-35) can be written for time τ

$$\begin{aligned} \psi(\tau) = {} & a_1 \left[C_{-2} \cos (2-\beta)\Omega^*\tau + \cos \beta\Omega^*\tau + C_2 \cos (2+\beta)\Omega^*\tau \right] \\ & + a_2 \left[-C_{-2} \sin (2-\beta)\Omega^*\tau + \sin \beta\Omega^*\tau + C_2 \sin (2+\beta)\Omega^*\tau \right] \end{aligned} \tag{7.1-35}$$

where

$$\Omega^* = \frac{\Omega}{2\omega_1} \tag{7.1-36}$$

7.2 Expression for the Angle of Rotation

The expression for the angle of rotation $\psi(\tau)$ is written

$$\psi(\tau) = a_1\left[C_{-2} \cos (2-\beta) \frac{\Omega}{2\omega_1} \tau + \cos \beta\frac{\Omega}{2\omega_1} \tau + C_2 \cos (2+\beta) \frac{\Omega}{2\omega_1} \tau\right]$$
$$+ a_2\left[-C_{-2} \sin (2-\beta) \frac{\Omega}{2\omega_1} \tau + \sin \beta\frac{\Omega}{2\omega_1}\tau + C_2 \sin (2+\beta) \frac{\Omega}{2\omega_1} \tau\right] \tag{7.2-1}$$

$$\psi_{\tau=0} = \psi_0 = a_1(1 + C_2 + C_{-2}) \tag{7.2-2}$$

$$\left.\frac{d\psi}{d\tau}\right|_{\tau=0} = \frac{\Omega}{2\omega_1} a_2\left[\beta + (2+\beta) C_2 - (2-\beta) C_{-2}\right] = \bar{\omega}_0 \tag{7.2-3}$$

For initial conditions

$$\psi(0) = \psi_o \qquad\qquad \left.\frac{d\psi}{d\tau}\right|_{\tau=0} = 0 \tag{7.2-4}$$

Equation (7.2-1) can be written

$$\psi(\tau) = \sum_{i=1}^{3} A_i \cos \Omega_i\tau \tag{7.2-5}$$

where

$$A_1 = a_1 C_{-2} = A_2C_{-2} \qquad (a)$$

$$A_2 = a_1 = \frac{\psi_o}{1 + C_2 + C_{-2}} \qquad (b) \tag{7.2-6}$$

$$A_3 = a_1C_2 = A_2C_2 \qquad (c)$$

and

$$\Omega_1 = (2-\beta) \frac{\Omega}{2\omega_1} \qquad (a) \tag{7.2-7}$$

$$\Omega_2 = \beta \frac{\Omega}{2\omega_1} \qquad \text{(b)}$$

$$\Omega_3 = (2+\beta)\, \frac{\Omega}{2\omega_1} \qquad \text{(c)} \qquad (7.2\text{-}7)$$

7.3 Results

The important results of Section 7 are summarized below.

1. The general solution of the Matheau equation [equation (7.1-4)] is given by equation (7.1-6).
2. The a's in equation (7.1-35) are determined from initial conditions by equation (7.2-2) and (7.2-3).
3. The C's in equations (7.2-2) and (7.2-3) are determined by equations (7.1-33).
4. The value of the infinite determinant of equation (7.1-12) when $\beta = 0$ is determined from the equation (7.1-28).
5. The stability parameter β is determined from either equation (7.1-27) or equation (7.1-29).

8.0 END DISPLACEMENTS

8.1 Method of Determination

The transverse displacement (restricted to the x-y plane) at the ends of the cylindrical shell will be determined as follows:

A. The free-free cylinder will be treated as a free-free beam (see reference 8) of equal mass and length.

B. The transverse displacement equation of a previous analysis will be used since solutions are fully detailed.

8.2 Application of Beam Theory For the Free-Free Beam

The generalized displacement equation is written

$$g^{*}(\xi_1 \tau) = q_B \xi + \sum_{n=1}^{N} q_n(\tau) \phi_n(\xi) \tag{8.2-1}$$

where

q_B - rotation of the undeflected beam (cylinder) axis about the center of mass.

$q_n(\tau)$ - generalized coordinate associated with $\phi_n(\xi)$.

$\phi_n(\xi)$ - nth vibration mode shape of a free-free beam (cylinder). See Figure 5 for mode shapes.

Since the beam under consideration will be a free-free beam, $\phi_n(\xi)$ is written

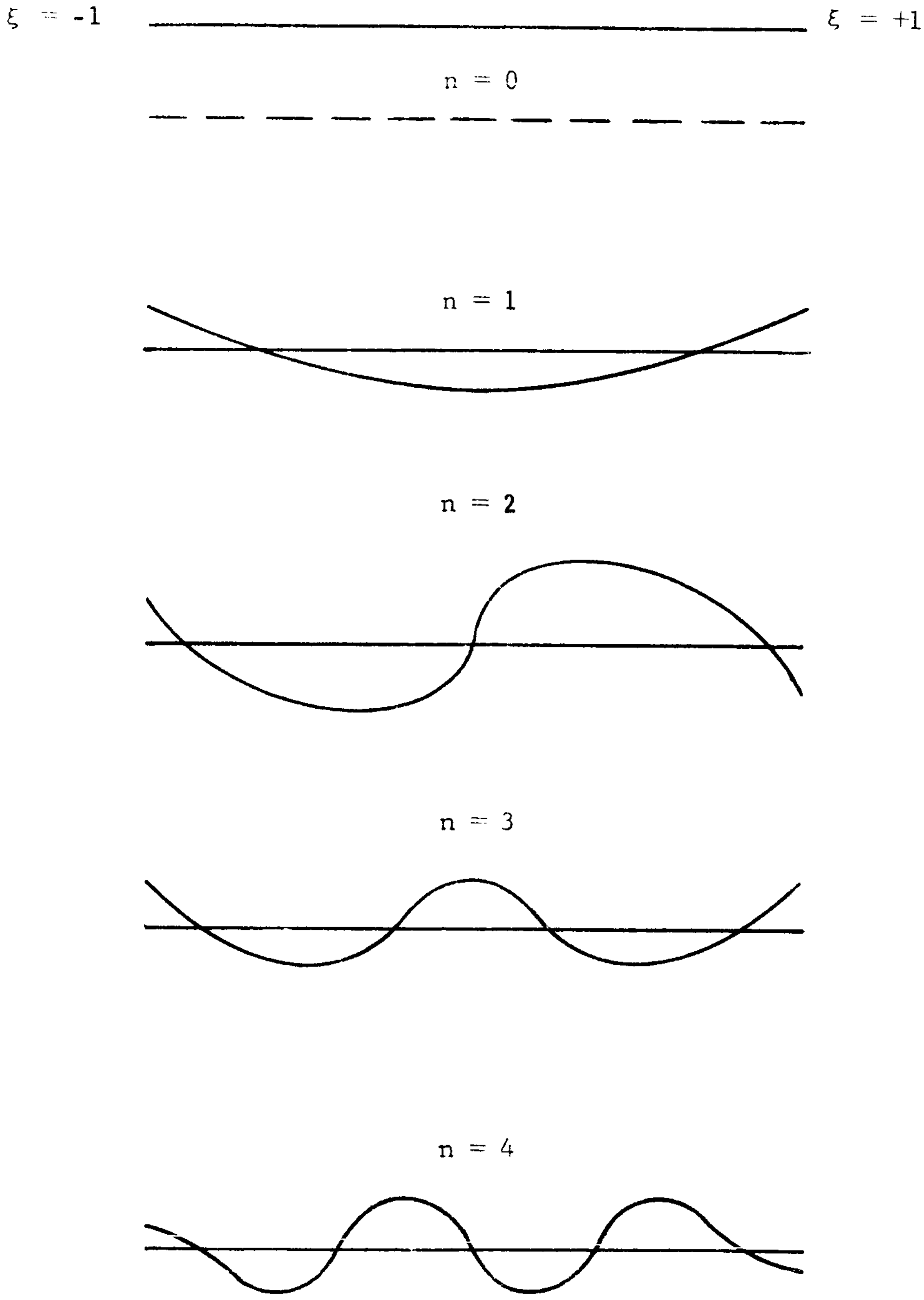


Figure 5 SINGLE SPAN BEAM HAVING FREE-FREE END CONDITIONS SHOWING THE SHAPES OF THE FIRST FEW NATURAL MODES OF VIBRATION IN FLEXURE

$$\phi_n(\xi) = \cosh \frac{\lambda n}{2}(\xi+1) + \cos \frac{\lambda n}{2}(\xi+1) - \alpha_n\Big[\sinh \frac{\lambda n}{2}(\xi+1)$$

$$+ \sin \frac{\lambda n}{2}(\xi+1)\Big] \qquad (8.2\text{-}2)$$

Properties of this function $\phi_n(\xi)$ are listed in Timoshenko (ref. 9) and Young (ref. 10) so there is little need for further discussion except that $\phi_n(\xi)$ satisfies the differential equation.

$$\frac{d^4\phi_n}{d\xi^4} = \lambda_n^4\,\phi_n \qquad (8.2\text{-}3)$$

where

$$\lambda_n^{\;4} = \omega_n^{\;2}\,\frac{m(2L)^4}{EI} \qquad (8.2\text{-}4)$$

and

ω_n - lateral bending frequency of the nth mode of the free-free beam

m - mass per unit length of beam

2L - length of the uniform beam, (the beam length of 2L was chosen to conform to the cylinder length)

EI - bending stiffness of the uniform beam.

8.3 End Displacements and Derivatives

The spatial and time derivatives of the displacement equations are as follows:

$$g^*(\xi, \tau) = q_B(\tau)\xi + \sum_{n=1}^{N} q_n(\tau)\;\phi_n(\xi) \qquad (8.3\text{-}1)$$

$$\ddot{g}^*(\xi, \tau) = \ddot{q}_B(\tau)\xi + \sum_{n=1}^{N} \ddot{q}_n(\tau)\;\phi_n(\xi) \qquad (8.3\text{-}2)$$

$$g'^{*}(\xi,\tau) = q_B(\tau) + \sum_{n=1}^{N} q_n(\tau)\left(\frac{\lambda n}{2}\right)\phi_n'(\xi) \qquad (8.3\text{-}3)$$

$$g''^{*}(\xi,\tau) = \sum_{n=1}^{N} q_n(\tau)\left(\frac{\lambda n}{2}\right)^2 \phi_n''(\xi) \qquad (8.3\text{-}4)$$

$$g'''^{*}(\xi,\tau) = \sum_{n=1}^{N} q_n(\tau)\left(\frac{\lambda n}{2}\right)^3 \phi_n'''(\xi) \qquad (8.3\text{-}5)$$

$$g''''^{*}(\xi,\tau) = \sum_{n=1}^{N} q_n(\tau)\left(\frac{\lambda n}{2}\right)^4 \phi_n''''(\xi)$$

$$= \sum_{n=1}^{N} q_n(\tau)\left(\frac{\lambda n}{2}\right)^4 \phi_n(\xi) \qquad (8.3\text{-}6)$$

The above terms will be substituted into equations, as need, in subsequent evaluation of terms.

9.0 SOLUTION OF THE ORDINARY DIFFERENTIAL EQUATIONS

9.1 Laplace Transformation of the Equations

Equations (6.4-13) and (6.4-14) are rewritten

$$\ddot{U}_{j0}(\tau) + \frac{j^2\pi^2}{4(1-\nu^2)\mu} U_{j0}(\tau) + \frac{j\pi\nu\lambda}{2(1-\nu^2)\mu} W_{j0}(\tau) = \left[P_{j0}(\tau)\right]_1 \tag{9.1-1}$$

$$\frac{j\pi\nu\lambda}{2(1-\nu^2)\mu} U_{j0}(\tau) + \ddot{W}_{j0}(\tau) + \frac{192\lambda^2 + j^4\pi^4\sigma^2}{192(1-\nu^2)\mu} W_{j0}(\tau) = \left[P_{j0}(\tau)\right]_2 \tag{9.1-2}$$

Transforming the above equations

$$(s^2 + c_{11}^{j0})\,\overline{U}_{j0}(\tau) + c_{12}^{j0}\,\overline{W}_{j0}(\tau) = \left[\overline{P}_{j0}(\tau)\right]_1 + s u_{j0} + \cancelto{0}{\dot{u}_{j0}} \tag{9.1-3}$$

$$c_{21}^{j0}\,\overline{U}_{j0}(\tau) + (s^2 + c_{22}^{j0})\,\overline{W}_{j0}(\tau) = \left[\overline{P}_{j0}(\tau)\right]_2 + s w_{j0} + \cancelto{0}{\dot{w}_{j0}} \tag{9.1-4}$$

where

$$\left.\begin{aligned}
c_{11}^{j0} &= \frac{j^2\pi^2}{4(1-\nu^2)\mu} && \text{(a)}\\
c_{12}^{j0} &= \frac{j\pi\nu\lambda}{2(1-\nu^2)\mu} && \text{(b)}\\
c_{21}^{j0} &= c_{12}^{j0} && \text{(c)}\\
c_{22}^{j0} &= \frac{192\lambda^2 + j^4\pi^4\sigma^2}{192(1-\nu^2)\mu} && \text{(d)}
\end{aligned}\right\} \tag{9.1-5}$$

Solving for $\overline{U}_{j0}(\tau)$ and $W_{j0}(\tau)$ we have

$$\overline{U}_{j0}(\tau) = \frac{\left[\overline{P}_{j0}(\tau)\right]_1 \; s^2+c_{22}^{j0} \; - c_{12}^{j0}\left[\overline{P}_{j0}(\tau)\right]_2 + u_{j0} \; s\left(s^2+c_{22}^{j0}\right) - s w_{j0} \; c_{12}^{j0}}{\left[\left(s^2+\omega_{j0}^1\right)^2\right]\left[\left(s^2+\omega_{j0}^2\right)^2\right]} \tag{9.1-6}$$

$$\overline{W}_{j0}(\tau) = \frac{\left[\overline{P}_{j0}(\tau)\right]_2 \left(s^2+c_{11}^{j0}\right) - c_{21}^{j0}\left[\overline{P}_{j0}(\tau)\right]_1 + w_{j0} \; s\left(s^2+c_{11}^{j0}\right) - s u_{j0} \; c_{21}^{j0}}{\left[\left(s^2+\omega_{j0}^1\right)^2\right]\left[\left(s^2+\omega_{j0}^2\right)^2\right]} \tag{9.1-7}$$

where

$$\Delta_{j0}(s) = \begin{vmatrix} s^2+c_{11}^{j0} & c_{12}^{j0} \\ c_{21}^{j0} & s^2+c_{22}^{j0} \end{vmatrix} = \left[\left(s^2+\omega_{j0}^1\right)^2\right]\left[\left(s^2+\omega_{j0}^2\right)^2\right] \tag{9.1-8}$$

Equations (6.4-15) thru (6.4-17) are rewritten

$$\ddot{U}_{jk}(\tau) + \frac{j^2\pi^2 + 2(1-\nu)\,\lambda^2 k^2}{4(1-\nu^2)\mu}\,U_{jk}(\tau) - \frac{jk\pi\lambda}{4(1-\nu)\mu}\,V_{jk}(\tau)$$

$$+ \frac{\nu\lambda j\pi}{2(1-\nu^2)\mu}\,W_{jk}(\tau) = \left[Q_{jk}(\tau)\right]_1 \tag{9.1-9}$$

$$- \frac{jk\lambda\pi}{4(1-\nu^2)\mu}\,U_{jk}(\tau) + \ddot{V}_{jk}(\tau) + \frac{(1-\nu)\,j^2\pi^2 + 8\lambda^2k^2}{8(1-\nu^2)\mu}\,V_{jk}(\tau)$$

$$- \frac{\lambda^2 k}{(1-\nu^2)\mu}\,W_{jk}(\tau) = \left[Q_{jk}(\tau)\right]_2 \tag{9.1-10}$$

$$\frac{j\pi\nu\lambda}{(1-\nu^2)\mu} U_{jk}(\tau) - \frac{k\lambda^2}{(1-\nu^2)\mu} V_{jk}(\tau) + \ddot{W}_{jk}(\tau)$$

$$+ \frac{12\lambda^2 + \sigma^2\left(\frac{j^2\pi^2}{4} + k^2\lambda^2\right)^2}{12(1-\nu^2)\mu} W_{jk}(\tau) = \left[Q_{jk}(\tau)\right]_3 \qquad (9.1\text{-}11)$$

Transforming the above equations we have

$$\left(s^2 + c_{11}^{jk}\right) \overline{U}_{jk}(\tau) + c_{12}^{jk} \overline{V}_{jk}(\tau) + c_{13}^{jk} \overline{W}_{jk}(\tau) = \left[\overline{Q}_{jk}(\tau)\right]_1 + su_{jk} \qquad (9.1\text{-}12)$$

$$c_{21}^{jk} \overline{U}_{jk}(\tau) + \left(s^2 + c_{22}^{jk}\right) \overline{V}_{jk}(\tau) + c_{23}^{jk} \overline{W}_{jk}(\tau) = \left[\overline{Q}_{jk}(\tau)\right]_2 + sv_{jk} \qquad (9.1\text{-}13)$$

$$c_{31}^{jk} \overline{U}_{jk}(\tau) + c_{32}^{jk} \overline{V}_{jk}(\tau) + \left(s^2 + c_{33}^{jk}\right) \overline{W}_{jk}(\tau) = \left[\overline{Q}_{jk}(\tau)\right]_3 + sw_{jk} \qquad (9.1\text{-}14)$$

where $\dot{u}_{jk}$, $\dot{v}_{jk}$, and $\dot{w}_{jk}$ are assumed to be zero and

$$c_{11}^{jk} = \frac{j^2\pi^2 + 2(1-\nu)\, k^2\lambda^2}{4(1-\nu^2)\mu} \qquad (a)$$

$$c_{12}^{jk} = -\frac{jk\lambda\pi}{4(1-\nu)\mu} \qquad (b)$$

$$c_{13}^{jk} = \frac{j\nu\lambda\pi}{2(1-\nu^2)\mu} \qquad (c)$$

$$c_{21}^{jk} = c_{12}^{jk} \qquad (d)$$

$$c_{22}^{jk} = \frac{j^2\pi^2(1-\nu) + 8k^2\lambda^2}{8(1-\nu^2)\mu} \qquad (e) \qquad (9.1\text{-}15)$$

$$c_{23}^{jk} = - \frac{k\lambda^2}{(1-\nu^2)\mu} \tag{f}$$

$$c_{31}^{jk} = c_{13}^{jk} \tag{g}$$

$$c_{32}^{jk} = c_{23}^{jk} \tag{h}$$

$$c_{33}^{jk} = \frac{12\lambda^2 + \sigma^2 \left(\frac{j^2\pi^2}{4} + k^2\lambda^2\right)^2}{12(1-\nu^2)\mu} \tag{i}$$

Defining

$$\Delta_{jk}(s) = \begin{vmatrix} s^2 + c_{11}^{jk} & c_{12}^{jk} & c_{13}^{jk} \\ c_{21}^{jk} & s^2 + c_{22}^{jk} & c_{23}^{jk} \\ c_{31}^{jk} & c_{32}^{jk} & s^2 + c_{33}^{jk} \end{vmatrix}$$

$$= \left[\left(s^2 + \omega_{jk}^1\right)^2\right] \left[\left(s^2 + \omega_{jk}^2\right)^2\right] \left[\left(s^2 + \omega_{jk}^3\right)^2\right] \tag{9.1-16}$$

$$\Delta_1(s) = \begin{vmatrix} \left[\overline{Q}_{jk}(\tau)\right]_1 + su_{jk} & c_{12}^{jk} & c_{13}^{jk} \\ \left[\overline{Q}_{jk}(\tau)\right]_2 + sv_{jk} & s^2 + c_{22}^{jk} & c_{23}^{jk} \\ \left[\overline{Q}_{jk}(\tau)\right]_3 + sw_{jk} & c_{32}^{jk} & s^2 + c_{33}^{jk} \end{vmatrix} \tag{9.1-17}$$

$$\Delta_2(s) = \begin{vmatrix} s^2 + c_{11}^{jk} & \left[\bar{Q}_{jk}(\tau)\right]_1 + su_{jk} & c_{13}^{jk} \\ c_{21}^{jk} & \left[\bar{Q}_{jk}(\tau)\right]_2 + sv_{jk} & c_{23}^{jk} \\ c_{31}^{jk} & \left[\bar{Q}_{jk}(\tau)\right]_3 + sw_{jk} & s^2 + c_{33}^{jk} \end{vmatrix} \qquad (9.1\text{-}18)$$

$$\Delta_3(s) = \begin{vmatrix} s^2 + c_{11}^{jk} & c_{11}^{jk} & \left[\bar{Q}_{jk}(\tau)\right]_1 + su_{jk} \\ c_{21}^{jk} & s^2 + c_{22}^{jk} & \left[\bar{Q}_{jk}(\tau)\right]_2 + sv_{jk} \\ c_{31}^{jk} & c_{32}^{jk} & \left[\bar{Q}_{jk}(\tau)\right]_3 + sw_{jk} \end{vmatrix} \qquad (9.1\text{-}19)$$

The equations for the transformed displacements $\bar{U}_{jk}(\tau)$, $\bar{V}_{jk}(\tau)$, and $\bar{W}_{jk}(\tau)$ can be written

$$\begin{aligned} \bar{U}_{jk}(\tau) = \frac{\Delta_1(s)}{\Delta jk(s)} = \Bigg\{ & \left\{\left[\bar{Q}_{jk}(\tau)\right]_1 + su_{jk}\right\} \left[\left(s^2 + c_{22}^{jk}\right)\left(s^2 + c_{33}^{jk}\right) - \left(c_{32}^{jk}\right)^2\right] \\ & + \left\{\left[\bar{Q}_{jk}(\tau)\right]_2 + sv_{jk}\right\} \left[c_{32}^{jk} \; c_{13}^{jk} - \left(s^2 + c_{33}^{jk}\right) c_{12}^{jk}\right] \\ & + \left\{\left[\bar{Q}_{jk}(\tau)\right]_3 + sw_{jk}\right\} \left[c_{12}^{jk} \; c_{23}^{jk} - \left(s^2 + c_{22}^{jk}\right) c_{13}^{jk}\right] \Bigg\} \Big/ \Delta_{jk}(s) \end{aligned} \qquad (9.1\text{-}20)$$

$$\overline{V}_{jk}(\tau) = \frac{\Delta_2(s)}{\Delta_{jk}(s)} = \left\{\left\{\left[\overline{Q}_{jk}(\tau)\right]_1 + su_{jk}\right\}\left[c_{31}^{jk}\, c_{23}^{jk} - \left(s^2 + c_{33}^{jk}\right) c_{21}^{jk}\right]\right.$$

$$+ \left\{\left[\overline{Q}_{jk}(\tau)\right]_2 + sv_{jk}\right\}\left[\left(s^2 + c_{11}^{jk}\right)\left(s^2 + c_{33}^{jk}\right) - \left(c_{31}^{jk}\right)^2\right] \qquad (9.1\text{-}21)$$

$$\left. + \left\{\left[\overline{Q}_{jk}(\tau)\right]_3 + sw_{jk}\right\}\left[c_{21}^{jk}\, c_{13}^{jk} - \left(s^2 + c_{11}^{jk}\right) c_{23}^{jk}\right]\right\} \Big/ \Delta_{jk}(s)$$

$$\overline{W}_{jk}(\tau) = \frac{\Delta_3(s)}{\Delta_{jk}(s)} = \left\{\left\{\left[\overline{Q}_{jk}(\tau)\right]_1 + su_{jk}\right\}\left[c_{21}^{jk}\, c_{32}^{jk} - \left(s^2 + c_{22}^{jk}\right) c_{31}^{jk}\right]\right.$$

$$+ \left\{\left[\overline{Q}_{jk}(\tau)\right]_2 + sv_{jk}\right\}\left[c_{31}^{jk}\, c_{12}^{jk} - \left(s^2 + c_{11}^{jk}\right) c_{32}^{jk}\right] \qquad (9.1\text{-}22)$$

$$\left. + \left\{\left[\overline{Q}_{jk}(\tau)\right]_3 + sw_{jk}\right\}\left[\left(s^2 + c_{11}^{jk}\right)\left(s^2 + c_{22}^{jk}\right) - \left(c_{21}^{jk}\right)^2\right]\right\} \Big/ \Delta_{jk}(s)$$

Equation (6.4-11) is rewritten

$$\ddot{U}_{00}(\tau) = -\frac{1}{4\pi(1-\nu^2)}\left[\int_{-1}^{1}\int_{0}^{2\pi} G_1(\xi, \theta, \tau)\, d\xi d\theta + \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial^2\left[f^*(\xi, \tau)\right]}{\partial\xi^2}\, d\xi d\theta\right.$$

$$+ \frac{(1+\nu)\lambda}{2}\int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial^2\left[g^*(\xi, \tau)\sin\theta\right]}{\partial\xi\partial\theta}\, d\xi d\theta$$

$$\left. - \nu\lambda\int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial\left[h^*(\xi, \tau)\ \cos\theta\right]}{\partial\xi}\, d\xi d\theta\right] = P_{00}(\tau) \qquad (9.1\text{-}23)$$

Substituting $G_1(\xi, \theta, \tau)$, $P_{00}(\tau)$ is evaluated

$$P_{00}(\tau) = - \frac{1}{4\pi(1-\nu^2)} \left\{ \int_{-1}^{1} \int_{0}^{2\pi} (1-\nu^2)\mu \left[\frac{\bar{T}_o}{2} (1 - \gamma\cos \bar{\Omega}\tau) \cos K\psi - \frac{\ddot{\psi}}{\lambda} \cos\theta \right.\right.$$

$$\left.\left. - \dot{\psi}^2 \xi \right] d\xi\, d\theta + 2\pi \frac{(1-\nu^2)}{Eh} T_o (1-\gamma\cos \bar{\Omega}\tau) \cos K\psi \right\} \qquad (9.1\text{-}24)$$

which reduces to

$$P_{00}(\tau) = - \frac{\bar{T}_o}{2} (1-\gamma \cos \bar{\Omega}\tau) \cos K\psi + \frac{\dot{\psi}^2}{2(1-\nu^2)} \left[\frac{\xi^2}{2} \Big|_{-1}^{1} \right]$$

$$- \frac{\bar{T}_o}{2\ Eh} (1-\gamma \cos \bar{\Omega}\tau) \cdot \cos K\psi \qquad (9.1\text{-}25)$$

Equation (9.1-23) can now be written

$$\bar{U}_{00}(\tau) = - \frac{\bar{T}_o}{2} (1-\gamma\cos \bar{\Omega}\tau) \cos K\psi - \frac{T_o}{2\ Eh} (1-\gamma\cos \bar{\Omega}\tau) \cos K\psi$$

$$= - \frac{1}{2} \left(\bar{T}_o + \frac{T_o}{Eh} \right) (1-\gamma\cos \bar{\Omega}\tau) \cos K\psi \qquad (9.1\text{-}26)$$

Rewriting equation (6.4-12)

$$\ddot{U}_{0k}(\tau) + \frac{k^2\lambda^2}{2(1+\nu)\mu} U_{0k}(\tau) = - \frac{1}{2\pi(1-\nu^2)\mu} \int_{-1}^{1} \int_{0}^{2\pi} G_1(\xi, \theta, \tau) \cos k\theta \; d\xi\, d\theta$$

$$+ \frac{1}{2\pi(1-\nu^2)\mu} \int_{-1}^{1} \int_{0}^{2\pi} \frac{\partial^2 [f^*(\xi, \tau)]}{\partial \xi^2} \cos k\theta \; d\xi\, d\theta$$

$$+\frac{\lambda}{4\pi(1-\nu)\mu}\int_{-1}^{1}\int_{0}^{2\pi}\frac{\partial^2\left[g^*(\xi,\tau)\sin\theta\right]}{\partial\xi\partial\theta}\cos k\theta\ d\xi d\theta$$

$$-\frac{\nu\lambda}{2\pi(1-\nu^2)\mu}\int_{-1}^{1}\int_{0}^{2\pi}\frac{\partial\left[h^*(\xi,\tau)\cos\theta\right]}{\partial\xi}\cos k\theta\ d\xi d\theta = P_{0k}(\tau) \quad (9.1\text{-}27)$$

Substituting $G_1(\xi, \theta, \tau)$, we can write $P_{0k}(\tau)$

$$P_{0k}(\tau) = -\frac{1}{2\pi(1-\nu^2)\mu}\int_{-1}^{1}\int_{0}^{2\pi}(1-\nu^2)\mu\left[\frac{\overline{T}_o}{2}(1-\gamma\cos\overline{\Omega}\tau)\cos K\psi - \frac{\ddot{\psi}}{\lambda}\cos\theta\right.$$

$$\left. - \dot{\psi}^2\xi\right]\cos k\theta\ d\xi d\theta$$

$$+\frac{\lambda}{4\pi(1-\nu)\mu}\int_{-1}^{1}\int_{0}^{2\pi}\frac{\partial g^*(\xi,\tau)}{\partial\xi}\cos\theta\cos k\theta\ d\xi d\theta$$

$$-\frac{\nu\lambda}{2\pi(1-\nu^2)\mu}\int_{-1}^{1}\int_{0}^{2\pi}\frac{\partial h^*(\xi,\tau)}{\partial\xi}\cos\theta\cos k\theta\ d\xi d\theta \quad (9.1\text{-}28)$$

$$=\frac{1}{\pi}\int_{-1}^{1}\frac{\ddot{\psi}}{\lambda}\cos\theta\cos k\theta\ d\theta + \frac{\lambda}{4\pi(1-\nu)\mu}\int_{-1}^{1}\int_{0}^{2\pi}\frac{\partial g^*(\xi,\tau)}{\partial\xi}\cos\theta\cos k\theta\ d\xi d\theta$$

$$-\frac{\nu\lambda}{2\pi(1-\nu^2)\mu}\int_{-1}^{1}\int_{0}^{2\pi}\frac{\partial h^*(\xi,\tau)}{\partial\xi}\cos\theta\cos k\theta\ d\xi d\theta$$

$$=\delta_{1k}\left[\frac{\ddot{\psi}}{\lambda}+\frac{\lambda}{4(1-\nu)\mu}\int_{-1}^{1}\frac{\partial g^*(\xi,\tau)}{\partial\xi}d\xi - \frac{\nu\lambda}{2(1-\nu^2)\mu}\int_{-1}^{1}\frac{\partial h^*(\xi,\tau)}{\partial\xi}d\xi\right]$$

$$(9.1\text{-}29)$$

Since $g^*(\xi, \tau) = h^*(\xi, \tau)$ the expression for $P_{0k}(\tau)$ is written

$$P_{0k}(\tau) = \delta_{1k}\left[\frac{\ddot{\psi}}{\lambda}+\frac{\lambda}{4(1+\nu)\mu}\int_{-1}^{1}\frac{\partial g^*(\xi,\tau)}{\partial\xi}d\xi\right] \quad (9.1\text{-}30)$$

which, upon substitution of $g^*(\xi, \tau)$, reduces to

$$P_{0k}(\tau) = \Delta_{1k}\left\{\frac{\ddot{\psi}}{\lambda} + \frac{\lambda}{4(1+\nu)\mu}\left[2q_B(\tau) + \sum_{n=1}^{3} q_n(\tau)\frac{\lambda n}{2}\int \phi_n{}'(\xi)\, d\xi\right]\right\} \quad (9.1\text{-}31)$$

Substituting for $\psi(\tau)$ and evaluating, the equation will be

$$P_{0k}(\tau) = -\frac{\delta_{1k}}{\lambda}\sum_{i=1}^{I} A_i\Omega_i{}^2 \cos\Omega_i\tau + \frac{\lambda\delta_{1k}}{4(1+\nu)\mu}\left[2\, q_B(\tau) + \sum_{n=1}^{N}\phi_n(\xi)\Big|_{-1}^{1} q_n(\tau)\right] \quad (9.1\text{-}32)$$

Equation (9.1-27) can now be written

$$\ddot{U}_{0k}(\tau) + (\omega_{0k})^2 U_{0k}(\tau) = \frac{\delta_{1k}}{\lambda}\sum_{i=1}^{I} A_i\Omega_i{}^2 \cos\Omega_i\tau + \frac{\lambda\delta_{1k}}{4(1+\nu)\mu}\left[2\, q_B(\tau) + \sum_{n=1}^{N}\phi_n(\xi)\Big|_{-1}^{1} q_n(\tau)\right] \quad (9.1\text{-}33)$$

Transforming, equation (9.1-33) is written

$$\bar{U}_{0k}(\tau)\left[s^2 + \left(\omega_{0k}\right)^2\right] = su_{0k} - \frac{\delta_{1k}}{\lambda}\mathcal{L}\left[\sum_{i=1}^{E} A_i\Omega_i{}^2 \cos\Omega_1\tau\right] + \frac{\lambda\delta_{1k}}{4(1+\nu)\mu}\mathcal{L}\left[2\, q_B(\tau) + \sum_{n=1}^{N}\phi_n(\xi)\Big|_{-1}^{1} q_n(\tau)\right] \quad (9.1\text{-}34)$$

and the solution becomes

$$U_{0k}(\tau) = u_{0k}\cos\omega_{ok}\tau$$

$$- \frac{\delta_{1k}}{\lambda \omega_{0k}} \sum_{i=1}^{I} A_i \, \Omega_i^2 \int_0^{\tau} \cos \Omega_i(\eta) \sin \omega_{0k}(\tau-\eta) \, d\eta$$

$$+ \frac{\lambda \delta_{1k}}{2(1+\nu)\mu \, \omega_{0k}} \int_0^{\tau} q_B(\eta) \sin \omega_{0k} (\tau-\eta) \, d\eta$$

$$+ \frac{\lambda \delta_{1k}}{4(1+\nu)\mu \, \omega_{0k}} \sum_{n=1}^{N} \phi_n(\xi) \Big|_{-1}^{1} \int_0^{\tau} q_n(\eta) \sin \omega_{0k}(\tau-\eta) \, d\eta \qquad (9.1\text{-}35)$$

Evaluating the first integral and $\phi(\xi)$ at the limits ± 1, $U_{0k}(\tau)$ is written

$$U_{0k}(\tau) = u_{0k} \cos \omega_{0k}\tau + \frac{\delta_{1k}}{\lambda} \sum_{i=1}^{I} A_i \frac{\Omega_i^2}{\Omega_i^2 - (\omega_{0k})^2} (\cos \Omega_i \tau - \cos \omega_{0k}\tau)$$

$$+ \frac{\lambda \nu}{2(1+\nu)\mu \, \omega_{0k}} \left\{ \int_0^{\tau} q_B(\eta) \sin \omega_{0k}(\tau-\eta) \, d\eta \right.$$

$$\left. + \sum_{n=1}^{N} \left[1 + (-1)^n\right] \int_0^{\tau} q_n(\eta) \sin \omega_{0k}(\tau-\eta) \, d\eta \right\} \qquad (9.1\text{-}36)$$

Substituting for the convolution integrals from INTEGRALS, $U_{0k}(\tau)$ is written

$$U_{0k}(\tau) = U_{0k} \cos \omega_{0k}\tau + \frac{\delta_{1k}}{\lambda} \sum_{i=1}^{I} \frac{A_i \Omega_i^2}{\Omega_i^2 - (\omega_{0k})^2} (\cos \Omega_i \tau - \cos \omega_{ok}\tau)$$

$$+ \frac{\lambda \delta_{1k}}{2(1+\nu)\mu \, \omega} \sum_{s=-S}^{S} \frac{c_B^{(s)}}{(\alpha_n+s)\bar{\Omega}^2 - (\omega_{0k})^2} \left\{ \cos \omega_{0k}\tau \right.$$

$$+ i \frac{\left(\alpha_r + s\right)}{\omega_{0k}} \bar{\Omega} \sin \omega_{0k}\tau - e^{i(\alpha_r + s)\bar{\Omega}\tau} \Big\}$$

$$+ \sum_{s=-S}^{+S} \sum_{n=1}^{N} \left[1 + (-1)^n\right] \frac{c_n(s)}{\left[(\alpha_r + s)\bar{\Omega}\right]^2 - (\omega_{0k})^2} \Big\{ \cos \omega_{0k}\tau$$

$$+ i \frac{(\alpha_r + s)}{\omega_{0k}} \bar{\Omega} \sin \omega_{0k}\tau - e^{i(\alpha_r + s)\bar{\Omega}\tau} \Big\} \qquad (9.1\text{-}37)$$

and in final form, the solution is written

$$U_{0k}(\tau) = u_{0k} \cos \omega_{0k}\tau + \frac{\delta_{1k}}{\lambda} \sum_{i=1}^{I} \frac{A_i \Omega_i^2}{\Omega_i^2 - (\omega_{0k})^2} \left(\cos \Omega_i\tau - \cos \omega_{0k}\tau\right)$$

$$+ \frac{\lambda \delta_{1k}}{2(1+\nu)\mu} \sum_{s=-S}^{+S} \sum_{n=1}^{N} \frac{c_B(s) + \left[1 + (-1)^n\right] c_n(s)}{\left[(\alpha_r + s)\bar{\Omega}\right]^2 - (\omega_{0k})^2}$$

$$\Big\{ \cos \omega_{0k}\tau + i \frac{(\alpha_r + s)\bar{\Omega}}{\omega_{0k}} \sin \omega_{0k}\tau - e^{i(\alpha_r + s)\bar{\Omega}\tau} \Big\} \qquad (9.1\text{-}38)$$

$k = 1, 2, \ldots, N_1$

$i = 1, 2, \ldots I$

$r = 1, 2, \ldots, R$

$s = -S$ to $+S$

Evaluation of $\left[P_{j0}(\tau)\right]_1$ in equation (9.1-1)

$$P_{j0}(\tau)_1 = -\frac{1}{2\pi(1-\nu^2)\mu} \int_{-1}^{1}\int_{0}^{2\pi} G_1(\xi, \theta, \tau) \cos \frac{j\pi}{2} (\xi + 1)\, d\xi d\theta$$

$$- \frac{1}{2(1-\nu^2)\mu} \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial^2 [f^*(\xi, \tau)]}{\partial \xi^2} \cos \frac{j\pi}{2} (\xi + 1)\, d\xi d\theta$$

$$- \frac{\lambda}{4\pi(1-\nu)\mu} \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial^2[g^*(\xi,\tau)\sin\theta]}{\partial\xi\partial\theta} \cos\frac{j\pi}{2}(\xi+1)\, d\theta$$

$$+ \frac{\nu\lambda}{2\pi(1-\nu^2)\mu} \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial[h^*(\xi,\tau)\cos\theta]}{\partial\xi} \cos\frac{j\pi}{2}(\xi+1)\, d\xi d\theta \quad (9.1\text{-}39)$$

Substituting for $G_1(\xi, \theta, \tau)$, $\left[P_{j0}(\tau)\right]$ is written

$$P_{j0}(\tau)_1 = - \frac{1}{2\pi(1-\nu^2)\mu} \int_{-1}^{1}\int_{0}^{2\pi} (1-\nu^2)\mu \left[\frac{\overline{T}}{2}(1-\gamma\cos\overline{\Omega}\tau)\cos K\psi - \frac{\ddot{\psi}}{\lambda}\cos\theta - \dot{\psi}^2\xi\right]\cos\frac{j\pi}{2}(\xi+1)\, d\xi d\theta$$

$$- \frac{1}{2\pi(1-\nu^2)\mu} \int_{-1}^{1}\left[(2\pi)\frac{(1-\nu^2)}{Eh}\frac{\overline{T}_o}{2}(1-\gamma\cos\overline{\Omega}\tau)\cos K\psi\right]\cos\frac{j\pi}{2}(\xi+1)\, d\xi$$

$$- \frac{\lambda}{4\pi(1-\nu)\mu} \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial g^*(\xi,\tau)}{\partial\xi}\cos\theta\cos\frac{j\pi}{2}(\xi+1)\, d\xi d\theta$$

$$+ \frac{\nu\lambda}{2\pi(1-\nu^2)\mu} \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial h^*(\xi,\tau)}{\partial\xi}\cos\theta\cos\frac{j\pi}{2}(\xi+1)\, d\xi d\theta \quad (9.1\text{-}40)$$

which reduces to

$$\left[P_{j0}(\tau)\right]_1 = \dot{\psi}^2(\tau)\int_{-1}^{1} \xi\cos\frac{j\pi}{2}(\xi+1)\, d\xi$$

$$= \dot{\psi}^2(\tau)\ \frac{2}{j\pi}^2\ \left[(-1)^j - 1\right] \quad (9.1\text{-}41)$$

Substituting $\dot{\psi}^2(\tau)$, the expression for $P_{j0}(\tau)_1$ becomes

$$\left[P_{j0}(\tau)\right]_1 = \left(\frac{2}{j\pi}\right)^2 \left[(-1)^j - 1\right] \left\{ \sum_{i=1}^{I} \sum_{\substack{m=1\\ i\neq 1}}^{M} A_i A_m \Omega_i \Omega_m \sin\Omega_i\tau \sin\Omega_m\tau \right.$$

$$+ \sum_{i=m=1}^{I} A_i^2 \, \Omega_i^2 \sin^2 \Omega_i \tau \Bigg\} \qquad (9.1\text{-}42)$$

From equation (9.1-2), $\left[P_{j0}(\tau)\right]_2$ is written

$$\left[P_{j0}(\tau)\right]_2 = - \frac{1}{2\pi(1-\nu^2)\mu} \int_{-1}^{1}\int_{0}^{2\pi} G_3(\xi, \theta, \tau) \sin \frac{j\pi}{2} (\xi+1) \, d\xi d\theta$$

$$- \frac{\nu\lambda}{2\pi(1-\nu^2)\mu} \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial f^*(\xi, \tau)}{\partial \xi} \sin \frac{j\pi}{2} (\xi+1) \, d\xi d\theta$$

$$- \frac{\lambda^2}{2\pi(1-\nu^2)\mu} \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial\left[g^*(\xi, \tau) \sin\theta\right]}{\partial \theta} \sin \frac{j\pi}{2} (\xi+1) \, d\xi d\theta$$

$$+ \frac{\lambda^2}{2\pi(1-\nu^2)\mu} \int_{-1}^{1}\int_{0}^{2\pi} h^*(\xi, \tau) \cos\theta \sin \frac{j\pi}{2} (\xi+1) \, d\xi d\theta$$

$$+ \frac{\sigma^2}{24\pi(1-\nu^2)\mu} \int_{-1}^{1}\int_{0}^{2\pi} \overline{\nabla}^4 \left[h^*(\xi, \tau) \cos\theta\right] \sin \frac{j\pi}{2} (\xi+1) \, d\xi d\theta \qquad (9.1\text{-}43)$$

Substituting for $G_3(\xi, \theta, \tau)$, the expression becomes

$$\left[P_{j0}(\tau)\right]_2 = - \frac{1}{2\pi(1-\nu^2)\mu} \int_{-1}^{1}\int_{0}^{2\pi} (1-\nu^2)\mu \left[- \frac{\overline{T}_o}{2} (1-\gamma \cos \overline{\Omega}\tau) \sin \psi \right.$$

$$\left. + \frac{\dot{\psi}^2}{\lambda} \cos\theta - \ddot{\psi} \, \xi \right] \cos\theta \sin \frac{j\pi}{2} (\xi+1) \, d\xi d\theta$$

$$- \frac{\nu\lambda}{2 \, (1-\nu^2)\mu} \left[(2\pi) \frac{1-\nu^2}{Eh} \frac{T_o}{2} (1-\gamma\cos \overline{\Omega}\tau) \cos \psi \int_{-1}^{1} (1-\xi) \sin \frac{j\pi}{2} (\xi+1) \, d\xi \right]$$

$$- \frac{\lambda^2}{2\pi(1-\nu^2)\mu} \int_{-1}^{1}\int_{0}^{2\pi} g^*(\xi, \tau) \cos\theta \sin \frac{j\pi}{2} (\xi+1)\, d\xi d\theta$$

$$+ \frac{\lambda^2}{2\pi(1-\nu^2)\mu} \int_{-1}^{1}\int_{0}^{2\pi} h^*(\xi, \tau) \cos\theta \sin \frac{j\pi}{2} (\xi+1)\, d\xi d\theta$$

$$+ \frac{\sigma^2}{24\pi(1-\nu^2)\mu} \int_{-1}^{1}\int_{0}^{2\pi} \overline{\nabla}^4\left[h^*(\xi, \tau) \cos\theta\right] \sin \frac{j\pi}{2} (\xi+1)\, d\xi d\theta \qquad (9.1\text{-}44)$$

Rearranging

$$\left[P_{j0}(\tau)\right]_2 = - \frac{1}{2\pi} \frac{\dot{\psi}}{\lambda} \int_{-1}^{1}\int_{0}^{2\pi} \cos^2\theta \sin \frac{j\pi}{2} (\xi+1)\, d\xi d\theta$$

$$+ \frac{\nu\lambda}{\mu E h} \frac{T_o}{2} (1-\gamma\cos \overline{\Omega}\tau) \cos K\psi \frac{2}{j\pi} \left[1 + (-1)^j\right]$$

$$+ \frac{\sigma^2}{24\pi(1-\nu^2)\mu} \int_{-1}^{1}\int_{0}^{2\pi} \overline{\nabla}^4\left[h^*(\xi, \tau) \cos\theta\right] \sin \frac{j\pi}{2} (\xi+1)\, d\xi d\theta \qquad (9.1\text{-}45)$$

Inserting for $\overline{\nabla}^4\left[h^*(\xi, \tau) \cos\theta\right]$ the equation becomes

$$\left[P_{j0}(\tau)\right]_2 = - \frac{\dot{\psi}^2}{2\lambda} \int_{-1}^{1} \sin \frac{j\pi}{2} (\xi+1)\, d\xi$$

$$+ \frac{\nu\lambda}{\mu E h} \frac{T_o}{2} (1-\gamma\cos \overline{\Omega}\tau) \cos K\psi \frac{2}{j\pi} \left[1 - (-1)^j\right]$$

$$- \frac{\nu\lambda}{\mu E h} \frac{T_o}{2} (1-\gamma\cos \overline{\Omega}\tau) \cos K\psi \frac{2}{j\pi} \left[1 + (-1)^j\right]$$

$$+ \frac{\sigma^2}{24(1-\nu^2)\mu} \int_{-1}^{1}\int_{0}^{2\pi} \left\{ \frac{\partial^4}{\partial\xi^4}\left[h^*(\xi, \tau) \cos\theta\right] + 2\lambda^2 \frac{\partial^4}{\partial\xi^2\partial\theta^2}\left[h^*(\xi, \tau) \cos\theta\right]\right.$$

$$+ \lambda^4 \frac{\partial^4}{\partial\theta^4} [h^*(\xi, \tau) \cos\theta] \Big\} \sin \frac{j\pi}{2} (\xi+1)\, d\xi d\theta \qquad (9.1\text{-}46)$$

which reduces to

$$\left[P_{j0}(\tau)\right]_2 = \frac{\dot{\psi}^2(\tau)}{j\pi\lambda} \left[(-1)^j - 1\right] + \frac{2\nu\lambda}{j\pi\mu Eh} T_o(1-\gamma\cos\bar{\Omega}\tau) \qquad (9.1\text{-}47)$$

or in terms of $\left[P_{j0}(\tau)\right]_1$

$$\left[P_{j0}(\tau)\right]_2 = \frac{j\pi}{4\lambda} \left[P_{j0}(\tau)\right]_1 + \frac{2\nu\lambda}{j\pi\mu Eh} T_o(1-\gamma\cos\bar{\Omega}\tau) \qquad (9.1\text{-}48)$$

Evaluation of $\left[Q_{jk}(\tau)\right]_1$ from equation (9.1-9)

$$\left[Q_{jk}(\tau)\right]_1 = -\frac{1}{\pi(1-\nu^2)\mu} \int_{-1}^{1}\int_{0}^{2\pi} G_1(\xi, \theta, \tau) \cos \frac{j\pi}{2} (\xi+1) \cos k\theta \, d\xi d\theta$$

$$- \frac{1}{\pi(1-\nu^2)\mu} \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial^2 f^*(\xi, \tau)}{\partial\xi^2} \cos \frac{j\pi}{2} (\xi+1) \cos k\theta \, d\xi d\theta$$

$$- \frac{\lambda}{2\pi(1-\nu)\mu} \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial^2 [g^*(\xi, \tau) \sin]}{\partial\xi\partial\theta} \cos \frac{j\pi}{2} (\xi+1) \cos k\theta \, d\xi d\theta$$

$$+ \frac{\nu\lambda}{\pi(1-\nu^2)\mu} \int_{-1}^{1}\int_{0}^{2\pi} \frac{\partial\, h^*(\xi, \tau)\cos\theta}{\partial\xi} \cos \frac{j\pi}{2} (\xi+1) \cos k\theta \, d\xi d\theta \qquad (9.1\text{-}49)$$

Substituting for $G_1(\xi, \theta, \tau)$, equation (9.1-49) becomes

$$\left[Q_{jk}(\tau)\right]_1 = -\frac{1}{\pi(1-\nu^2)\mu} \int_{-1}^{1}\int_{0}^{2\pi} (1-\nu^2)\mu \left[\frac{\bar{T}_o}{2} (1-\gamma\cos\bar{\Omega}\tau) \cos K\psi\right.$$

$$\left. - \frac{\ddot{\alpha}}{\lambda} \cos\theta - \dot{\alpha}^2 \xi \right] \cos \frac{j\pi}{2} (\xi+1) \cos k\theta \, d\xi d\theta$$

$$+\frac{1}{\pi(1-\nu^2)\mu}\left[\frac{1-\nu^2}{Eh}\ \frac{T_o}{2}\ (1-\gamma\cos\bar{\Omega}\tau)\cos K\psi\int_{-1}^{1}\cos\frac{j\pi}{2}(\xi+1)\cos k\theta\, d\xi\, d\theta\right.$$

$$-\frac{\lambda}{2\pi(1-\nu)\mu}\int_{-1}^{1}\int_{0}^{2\pi}\frac{\partial g^*(\xi,\tau)}{\partial\xi}\cos\frac{j\pi}{2}(\xi+1)\cos\theta\cos k\theta\, d\xi\, d\theta$$

$$+\frac{\nu\lambda}{\pi(1-\nu^2)\mu}\int_{-1}^{1}\int_{0}^{2\pi}\frac{\partial h^*(\xi,\tau)}{\partial\xi}\cos\frac{j\pi}{2}(\xi+1)\cos\theta\cos k\theta\, d\xi\, d\theta \qquad (9.1\text{-}50)$$

which reduces to

$$\left[Q_{jk}(\tau)\right]_1 = -\frac{\lambda\,\delta_{1k}}{2(1-\nu)\mu}\int_{-1}^{1}\frac{\partial g^*(\xi,\tau)}{\partial\xi}\cos\frac{j\pi}{2}(\xi+1)\, d\xi$$

$$+\frac{\nu\lambda\,\delta_{1k}}{(1-\nu^2)\mu}\int_{-1}^{1}\frac{\partial h^*(\xi,\tau)}{\partial\xi}\cos\frac{j\pi}{2}(\xi+1)\, d\xi \qquad (9.1\text{-}51)$$

$$\left[Q_{jk}(\tau)\right]_1 = \frac{\lambda\,\delta_{1k}}{2(1+\nu)\mu}\int_{-1}^{1}\frac{\partial g^*(\xi,\tau)}{\partial\xi}\cos\frac{j\pi}{2}(\xi+1)\, d\xi \qquad (9.1\text{-}52)$$

Substituting for $g^*(\xi,\tau)$

$$\left[Q_{jk}(\tau)\right]_1 = \frac{\lambda\,\delta_{1k}}{2(1+\nu)\mu}\int\left[q_B(\tau)+\sum_{n=1}^{N}\frac{\lambda n}{2}\,\phi_n'(\xi)\,q_n(\tau)\right]\cos\frac{j\pi}{2}(\xi+1)\, d\xi$$

$$(9.1\text{-}53)$$

Evaluating

$$\left[Q_{jk}(\tau)\right]_1 = \frac{\lambda\,\delta_{1k}}{2(1+\nu)\mu}\,q_B(\tau)\int\cos\frac{j\pi}{2}(\xi+1)\, d\xi$$

$$\frac{\lambda\,\delta_{1k}}{2(1+\nu)\mu}\sum_{n=1}^{N}\frac{\lambda n}{2}\,q_n(\tau)\int\phi_n'(\xi)\cos\frac{j\pi}{2}(\xi+1)\, d\xi \qquad (9.1\text{-}54)$$

Finally

$$\left[Q_{jk}(\tau)\right]_1 = \frac{\delta_{1k}\,\lambda}{2(1+\nu)\mu} \sum_{n=1}^{N} \frac{\lambda_n}{2} z_{jn}\, q_n(\tau) \qquad (9.1\text{-}55)$$

Evaluating $Q_{jk}(\tau)_2$ from equation (9.1-10)

$$\left[Q_{jk}(\tau)\right]_2 = -\frac{1}{(1-\nu^2)\mu}\int_{-1}^{1}\int_{0}^{2\pi} G_2(\xi,\ \theta,\ \tau)\sin\frac{j\pi}{2}(\xi+1)\sin k\theta\, d\xi d\theta$$

$$-\frac{\lambda}{2(1-\nu)\pi\mu}\int_{-1}^{1}\int_{0}^{2\pi}\frac{\partial^2 f^*(\xi,\ \tau)}{\partial\xi\partial\theta}\sin\frac{j\pi}{2}(\xi+1)\sin k\theta\, d\xi d\theta$$

$$-\frac{1}{2\pi(1+\nu)\mu}\int_{-1}^{1}\int_{0}^{2\pi}\frac{\partial^2[g^*(\xi,\ \tau)\sin\theta]}{\partial\xi^2}\sin\frac{j\pi}{2}(\xi+1)\sin k\theta\, d\xi d\theta$$

$$-\frac{\lambda^2}{\pi(1-\nu^2)\mu}\int_{-1}^{1}\int_{0}^{2\pi}\frac{\partial^2[g^*(\xi,\ \tau)\sin\theta]}{\partial\theta^2}\sin\frac{j\pi}{2}(\xi+1)\sin k\theta\, d\xi d\theta$$

$$+\frac{\lambda^2}{\pi(1-\nu^2)\mu}\int_{-1}^{1}\int_{0}^{2\pi}\frac{\partial[h^*(\xi,\ \tau)\cos\theta]}{\partial\theta}\sin\frac{j\pi}{2}(\xi+1)\sin k\theta\, d\xi d\theta$$

$$(9.1\text{-}56)$$

Substituting for $G_2(\xi,\ \theta,\ \tau)$, equation (9.1-56) becomes

$$= -\frac{1}{\pi(1-\nu^2)\mu}\int_{-1}^{1}\int_{0}^{2\pi}\left[(1-\nu^2)\mu\left[-\frac{\bar{T}_o}{2}(1-\gamma\cos\bar{\Omega}\tau)\sin K\psi\right.\right.$$

$$\left.\left.+\frac{\dot{\psi}}{\lambda}\cos\theta - \ddot{\psi}\xi\right]\sin\theta\right]\sin\frac{j\pi}{2}(\xi+1)\sin k\theta\, d\xi d\theta$$

$$-\frac{\delta_{1k}}{2(1+\nu)\mu}\int \frac{\partial^2 g^*(\xi,\ \tau)}{\partial\xi^2}\sin\frac{j\pi}{2}(\xi+1)\, d\xi$$

$$+ \frac{\lambda^2 \delta_{1k}}{(1-\nu^2)\mu} \int_{-1}^{1} g^*(\xi, \tau) \sin \frac{j\pi}{2} (\xi+1)\, d\xi\, d\theta$$

$$- \frac{\lambda^2 \delta_{1k}}{(1-\nu^2)\mu} \int_{-1}^{1} h^*(\xi, \tau) \sin \frac{j\pi}{2} (\xi+1)\, d\xi\, d\theta \qquad (9.1\text{-}57)$$

which reduces to

$$Q_{jk}(\tau)_2 = \delta_{1k} \left\{ \int_{-1}^{1} \left[\frac{\overline{T}_o}{2} (1-\gamma\cos\overline{\Omega}\tau) \sin K\psi + \ddot{\psi}\xi \right] \sin \frac{j\pi}{2} (\xi+1)\, d\xi \right\}$$

$$- \frac{\dot{\psi}^2}{\lambda} \int_{-1}^{1}\int_{0}^{2\pi} \sin \frac{j\pi}{2} (\xi+1) \cos\theta \sin\theta \sin k\theta\, d\xi\, d\theta$$

$$- \frac{\delta_{1k}}{2(1+\nu)\mu} \int_{-1}^{1} \frac{\partial^2 g^*(\xi, \tau)}{\partial \xi^2} \sin \frac{j\pi}{2} (\xi+1)\, d\xi \qquad (9.1\text{-}58)$$

Substituting $g^*(\xi, \tau)$, equation (9.1-58) becomes

$$\left[Q_{jk}(\tau)\right]_2 = \delta_{1k} \frac{T_o}{2} (1-\gamma\cos\overline{\Omega}\tau\, K\psi(\tau)) \int \sin \frac{j\pi}{2} (\xi+1)\, d\xi$$

$$+ \delta_{1k}\ddot{\psi}(\tau) \int_{-1}^{1} \xi \sin \frac{j\pi}{2} (\xi+1)\, d\xi$$

$$- \frac{\dot{\psi}^2(\tau)}{\lambda\pi} \int_{-1}^{1}\int_{0}^{2\pi} \cos\theta \sin\theta \sin k\theta \sin \frac{j\pi}{2} (\xi+1)\, d\xi\, d\theta \qquad (9.1\text{-}59)$$

$$- \frac{\delta_{1k}}{2(1+\nu)\mu} \int_{-1}^{1} g''^*(\xi, \tau) \sin \frac{j\pi}{2} (\xi+1)\, d\xi$$

which reduces to

$$\left[Q_{jk}(\tau)\right]_2 = \delta_{1k} \frac{T_o}{2} (1- \gamma\cos \bar{\Omega}\tau) \frac{2K}{j\pi} [1 - (-1)^j] \psi(\tau)$$

$$- \delta_{1k} \frac{2}{j\pi} \left[1 + (-1)^j\right] \ddot{\psi}(\tau)$$

$$- \frac{\delta_{2k}}{j\lambda} \left[1 - (-1)^j\right] \dot{\psi}^2(\tau)$$

$$- \frac{\delta_{1k}}{2(1+\nu)\mu} \sum_{n=1}^{N} \left(\frac{\lambda n}{2}\right)^2 q_n(\tau) \int_{-1}^{1} \phi_n''(\xi) \sin \frac{j\pi}{2} (\xi+1)\, d\xi \qquad (9.1\text{-}60)$$

Finally, the resulting equation is

$$\left[Q_{jk}(\tau)\right]_2 = \delta_{1k} \frac{T_o}{2} (1- \gamma\cos \bar{\Omega}\tau) \frac{2K}{j\pi} \left[1 - (-1)^j\right] \psi(\tau)$$

$$-\delta_{1k} \frac{2}{j\pi} \left[1 + (-1)^j\right] \ddot{\psi}(\tau)$$

$$- \frac{\delta_{2k}}{j\lambda} \left[1 - (-1)^j \right] \dot{\psi}^2(\tau)$$

$$- \frac{\delta_{1k}}{2(1+\nu)\mu} \sum_{n=1}^{N} \left(\frac{\lambda n}{2}\right)^2 x_{jn}\, q_n(\tau) \qquad (9.1\text{-}61)$$

Substituting for ψ, $\dot{\psi}^2$ and $\ddot{\psi}$ equation (9.1-61) becomes

$$\left[Q_{jk}(\tau)\right]_2 = \delta_{1k} \frac{T_o}{2} (1-\gamma\cos \bar{\Omega}\tau) \frac{2K}{j\pi} \left[1 - (-1)^j\right] \sum_{i=1}^{I} A_i \cos \Omega_i\tau$$

$$+ \delta_{1k} \frac{2}{j\pi} \left[1 + (-1)^j\right] \sum_{i=1}^{I} A_i \Omega_i^2 \cos \Omega_i \tau$$

$$- \frac{\delta_{2k}}{j\lambda} \left[1 - (-1)^j\right] \sum_{i=1}^{I} \sum_{p=11}^{P} A_i A_p \Omega_i \Omega_p \sin \Omega_i \tau \sin \Omega_p \tau$$

$$- \frac{\delta_{2k}}{j\lambda} \left[1 - (-1)^j\right] \sum_{i=p=1}^{I} A_i^2 \Omega_i^2 \sin^2 \Omega_i \tau$$

$$- \frac{\delta_{1k}}{2(1+\nu)\mu} \sum_{n=1}^{N} \left(\frac{\lambda n}{2}\right)^2 \chi_{jn} q_n(\tau) \tag{9.1-62}$$

Evaluation of $Q_{jk}(\tau)_3$ from equation (9.1-11)

$$\left[Q_{jk}(\tau)_3\right] = - \frac{1}{(1-\nu^2)\mu} \int_{-1}^{1} \int_{0}^{2\pi} G_3(\xi, \theta, \tau) \sin \frac{j\pi}{2} (\xi+1) \cos k\theta \, d\xi d\theta$$

$$- \frac{\nu\lambda}{\pi(1-\nu^2)\mu} \int_{-1}^{1} \int_{0}^{2\pi} \frac{\partial f^*(\xi, \tau)}{\partial \xi} \sin \frac{j\pi}{2} (\xi+1) \cos k\theta \, d\xi d\theta$$

$$- \frac{\lambda^2}{\pi(1-\nu^2)\mu} \int_{-1}^{1} \int_{0}^{2\pi} \frac{\partial[g^*(\xi, \tau) \sin\theta]}{\partial \theta} \sin \frac{j\pi}{2} (\xi+1) \cos k\theta \, d\xi d\theta$$

$$+ \frac{\lambda^2}{\pi(1-\nu^2)\mu} \int_{-1}^{1} \int_{0}^{2\pi} [h^*(\xi, \tau) \cos] \sin \frac{j\pi}{2} (\xi+1) \cos k\theta \, d\xi d\theta$$

$$+ \frac{\sigma^2}{12\pi(1-\nu^2)\mu} \int_{-1}^{1} \int_{0}^{2\pi} \bar{\nabla}^4 [h^*(\xi, \tau)\cos\theta] \sin \frac{j\pi}{2} (\xi+1) \cos k\theta \, d\xi d\theta \tag{9.1-63}$$

Substituting for $G_3(\xi, \theta, \tau)$, equation (9.1-63) becomes

$$= - \frac{1}{\pi(1-\nu^2)\mu} \int_{-1}^{1}\int_{0}^{2\pi} (1-\nu^2)\mu \left[- \frac{\bar{T}_o}{2} (1-\gamma\cos \bar{\Omega}\tau) \sin K\psi + \frac{\dot{\psi}^2}{\lambda} \cos \theta \right.$$

$$\left. - \ddot{\psi}\xi \right] \cos\theta \sin \frac{j\pi}{2} (\xi+1) \cos k\theta \, d\xi d\theta$$

$$- \frac{\nu\lambda}{\pi(1-\nu^2)\mu} \left[\frac{1-\nu^2}{Eh} \frac{T_o}{2} (1-\gamma\cos \bar{\Omega}\tau)\cos K\psi \int_{-1}^{1}\int_{0}^{2\pi} (1-\xi)\sin \frac{j\pi}{2} (\xi+1)\cos k\theta \, d\xi d\theta \right.$$

$$- \frac{\lambda^2 \delta_{1k}}{(1-\nu^2)\mu} \int_{-1}^{1} g^*(\xi, \tau) \sin \frac{j\pi}{2} (\xi+1) \, d\xi$$

$$+ \frac{\lambda^2 \delta_{1k}}{(1-\nu^2)\mu} \int_{-1}^{1} h^*(\xi, \tau) \sin \frac{j\pi}{2} (\xi+1) \, d\xi$$

$$+ \frac{\sigma^2}{12\pi(1-\nu^2)\mu} \int_{-1}^{1}\int_{0}^{2\pi} \left\{ \frac{\partial^4}{\partial\xi^4} \left[h^*(\xi, \tau)\cos\theta\right] + 2\lambda^2 \frac{\partial^4}{\partial\xi^2 \partial\theta^2} \left[h^*(\xi, \tau)\cos \theta\right] \right.$$

$$\left. + \lambda^4 \frac{\partial^4}{\partial\theta^4} \left[h^*(\xi, \tau) \cos\theta\right] \right\} \sin \frac{j\pi}{2} (\xi+1) \cos k\theta \, d\xi d\theta \qquad (9.1\text{-}64)$$

rearranging, equation (9.1-64) becomes

$$\left[Q_{jk}(\tau)\right]_3 = \delta_{1k} \left\{ \int_{-1}^{1} \left[\frac{\bar{T}_o}{2}(1-\gamma\cos \bar{\Omega}\tau) \sin K\psi + \ddot{\psi}\xi \right] \sin \frac{j\pi}{2} (\xi+1) \, d\xi \right\}$$

$$- \frac{\dot{\psi}^2}{\lambda} \int_{-1}^{1}\int_{0}^{2\pi} \sin \frac{j\pi}{2} (\xi+1) \cos\theta \cos\theta \cos k\theta \, d\xi d\theta$$

$$- \frac{\lambda^2 \delta_{1k}}{(1-\nu^2)\mu} \int_{-1}^{1} g^*(\xi, \tau) \sin \frac{j\pi}{2} (\xi+1)\, d\xi$$

$$+ \frac{\lambda^2 \delta_{1k}}{(1-\nu^2)\mu} \int_{-1}^{1} h^*(\xi, \tau) \sin \frac{j\pi}{2} (\xi+1)\, d\xi$$

$$+ \frac{\sigma^2}{12(1-\nu^2)\mu} \left\{ \int_{-1}^{1} \left[\frac{\partial^4}{\partial \xi^4} h^*(\xi, \tau) - 2\lambda^2 \frac{\partial^2}{\partial \xi^2} h^*(\xi, \tau) \right.\right.$$

$$\left.\left. + \lambda^4 h^*(\xi, \tau) \right] \sin \frac{j\pi}{2} (\xi+1)\, d\xi \right\} \qquad (9.1\text{-}65)$$

which reduces to

$$\left[Q_{jk}(\tau)\right]_3 = \delta_{1k} \frac{\overline{T}_o}{2} (1-\gamma\cos \overline{\Omega}\tau) \frac{2K}{j\pi} \left[1 - (-1)^j\right] \psi(\tau)$$

$$- \delta_{1k} \frac{2}{j\pi} \left[1 + (-1)^j\right] \ddot{\psi}(\tau)$$

$$+ \delta_{1k} \frac{\sigma^2}{12(1-\nu^2)\mu} \left\{ \int_{-1}^{1} \left[g''''^*(\xi, \tau) - 2\lambda^2 g''^*(\xi, \tau) \right.\right.$$

$$\left.\left. + \lambda^4 g^*(\xi, \tau) \right] \sin \frac{j\pi}{2} (\xi+1)\, d\xi \right\}$$

$$- \frac{\delta_{2k}}{j\lambda} \left[1 - (-1)^j\right] \dot{\psi}^2(\tau) \qquad (9.1\text{-}66)$$

Substituting ψ, $\ddot{\psi}$ and $g^*(\xi, \tau)$, equation (9.1-66) becomes

$$\left[Q_{jk}(\tau)\right]_3 = \delta_{1k} \frac{\overline{T}_o}{2} (1-\gamma\cos\overline{\Omega}\tau) \frac{2K}{j\pi} \left[(1 - (-1)^j\right] \sum_{i=1}^{I} A_i \cos \Omega_i\tau$$

$$+ \delta_{1k} \left(\frac{2}{j\pi}\right) \left[1 + (-1)^j\right] \sum_{i=1}^{I} A_i\Omega_i^2 \cos \Omega_i\tau$$

$$+ \delta_{1k} \frac{\sigma^2}{12(1-\nu^2)\mu} \sum_{n=1}^{N} \left(\frac{\lambda n}{2}\right)^4 q_n(\tau) \int_{-1}^{1} \phi''''_n(\xi) \sin \frac{j\pi}{2} (\xi+1)\, d\xi$$

$$+ \delta_{1k} \frac{\sigma^2 \; 2\lambda^2}{12(1-\nu^2)\mu} \sum_{n=1}^{N} \left(\frac{\lambda n}{2}\right)^2 q_n(\tau) \int_{-1}^{1} \phi''_n(\xi) \sin \frac{j\pi}{2} (\xi+1)\, d\xi$$

$$+ \delta_{1k} \frac{\sigma^2\lambda^4}{12(1-\nu^2)\mu} \left\{ q_B(\tau) \int_{-1}^{1} \xi \sin \frac{j\pi}{2} (\xi+1)\, d\xi \right.$$

$$\left. + \sum_{n=1}^{N} q_n(\tau) \int_{-1}^{1} \phi_n(\xi) \sin \frac{j\pi}{2} (\xi+1)\, d\xi \right\}$$

$$- \frac{\delta_{2k}}{j\lambda} \left[1 - (-1)^j\right] \sum_{i=1}^{I} \sum_{\substack{p=1 \\ i\neq p}}^{P} A_iA_p\, \Omega_i\Omega_p \sin \Omega_i\tau \sin \Omega_p\tau$$

$$- \frac{\delta_{2k}}{j\lambda} \left[1 - (-1)^j\right] \sum_{i=p=1}^{I} A_i^2\Omega_i^2 \sin^2\Omega_i\tau \qquad (9.1\text{-}67)$$

which reduces to

$$\left[Q_{jk}(\tau)\right]_3 = \delta_{1k} \frac{2}{j\pi} \sum_{i=1}^{I} \left\{ \frac{T_o}{2} (1-\gamma \cos \bar{\Omega}\tau)\, K \left[1 - (-1)^j\right] + \left[1 + (-1)^j\right] \Omega_i^2 \;\; A_i \cos \Omega_i \tau \right\}$$

$$- \frac{\delta_{2k}}{j\lambda} \left[1 - (-1)^j\right] \sum_{i=p=1}^{I} A_i^2 \Omega_i^2 \;\; \sin^2 \Omega_i \tau$$

$$- \frac{\delta_{2k}}{j\lambda} \left[1 - (-1)^j\right] \sum_{i=1}^{I} \sum_{p=1}^{P} A_i A_p \;\; \Omega_i \Omega_p \;\; \sin \Omega_i \tau \sin \Omega_p \tau$$

$$+ \delta_{1k} \frac{\sigma^2}{12(1-\nu^2)\mu} \left[\sum_{n=1}^{N} \left\{ s_{jn} \left[\left(\frac{\lambda n}{2}\right)^4 + \lambda^4 \right] - 2\lambda^2 \;\; \frac{\lambda n}{2}^{\;2} \;\; x_{jn} \right\} q_n(\tau) + \lambda^4 \; m_{jo} \; q_B(\tau) \right] \quad (9.1\text{-}68)$$

9.2 Inverse Transforms of the Equations

Performing inverse transform techniques on equations (9.1-6) and (9.1-7) we have

$$U_{j0}(\tau) = \frac{\left\{ u_{j0} \left[c_{22}^{j0} - \left[\left(\omega_{j0}^1 \right)^2 \right] - c_{12}^{j0} \; w_{j0} \right\} }{\left[\left(\omega_{j0}^2 \right)^2 - \left(\omega_{j0}^1 \right)^2 \right]} \cos \; \omega_{j0}^1 \tau$$

$$- \frac{\left\{u_{j0} \left[c_{22}^{j0} - \left(\omega_{j0}^{2}\right)^{2}\right] - c_{12}^{j0} w_{j0}\right\}}{\left[\left(\omega_{j0}^{2}\right)^{2} - \left(\omega_{j0}^{1}\right)^{2}\right]} \cos \omega_{j0}^{2} \tau$$

$$+ \frac{c_{22}^{j0} - \left(\omega_{j0}^{1}\right)^{2}}{\omega_{j0}^{1}\left[\left(\omega_{j0}^{2}\right)^{2} - \left(\omega_{j0}^{1}\right)\right]} \int_{0}^{\tau} \left[P_{j0}(\eta)\right]_{1} \sin \omega_{j0}^{1} (\tau-\eta)\, d\eta$$

$$- \frac{c_{22}^{j0} - \left(\omega_{j0}^{2}\right)^{2}}{\omega_{j0}^{2}\left[\left(\omega_{j0}^{2}\right)^{2} - \left(\omega_{j0}^{1}\right)^{2}\right]} \int_{0}^{\tau} \left[P_{j0}(\eta)\right]_{1} \sin \omega_{j0}^{2} (\tau-\eta)\, d\eta$$

$$- \frac{c_{12}^{j0}}{\omega_{j0}^{1}\left[\left(\omega_{j0}^{2}\right)^{2} - \left(\omega_{j0}^{1}\right)^{2}\right]} \int_{0}^{\tau} \left[P_{j0}(\eta)\right]_{2} \sin \omega_{j0}^{1} (\tau-\eta)\, d\eta$$

$$+ \frac{c_{12}^{j0}}{\omega_{j0}^{2}\left[\left(\omega_{j0}^{2}\right)^{2} - \left(\omega_{j0}^{1}\right)^{2}\right]} \int_{0}^{\tau} \left[P_{j0}(\eta)\right]_{2} \sin \omega_{j0}^{2} (\tau-\eta)\, d\eta \qquad (9.2\text{-}1)$$

$$j = 1, 2, \ldots, M_1$$

$$k = 0$$

$$W_{j0}(\tau) = \frac{\left\{w_{j0} \left[c_{11}^{j0} - \left(\omega_{j0}^{1}\right)^{2}\right] - c_{21}^{j0} u_{j0}\right\}}{\left[\left(\omega_{j0}^{2}\right) - \left(\omega_{j0}^{1}\right)^{2}\right]} \cos \omega_{j0}^{1} \tau$$

$$- \frac{\left\{w_{j0} \left[c_{11}^{j0} - \left(\omega_{j0}^{2}\right)^{2}\right] - c_{21}^{j0} u_{j0}\right\}}{\left[\left(\omega_{j0}^{2}\right)^{2} - \left(\omega_{j0}^{1}\right)^{2}\right]} \cos \omega_{j0}^{2} \tau$$

$$+ \frac{c_{11}^{j0} - \left(\omega_{j0}^{1}\right)^{2}}{\omega_{j0}^{1}\left[\left(\omega_{j0}^{2}\right)^{2} - \left(\omega_{j0}^{1}\right)^{2}\right]} \int_{0}^{\tau} \left[P_{j0}(\eta)\right]_{2} \sin \omega_{j0}^{1} (\tau-\eta)\, d\eta$$

$$- \frac{c_{11}^{j0} - \left(\omega_{j0}^{2}\right)^2}{\omega_{j0}^{2}\left[\left(\omega_{j0}^{2}\right)^2 - \left(\omega_{j0}^{1}\right)^2\right]} \int_0^{\tau} \left[P_{j0}(\eta)\right]_2 \sin \omega_{j0}^{2} (\tau-\eta)\, d\eta$$

$$- \frac{c_{21}^{j0}}{\omega_{j0}^{1}\left[\left(\omega_{j0}^{2}\right)^2 - \left(\omega_{j0}^{1}\right)^2\right]} \int_0^{\tau} \left[P_{j0}(\eta)\right]_1 \sin \omega_{j0}^{1}(\tau-\eta)\, d\eta$$

$$+ \frac{c_{21}^{j0}}{\omega_{j0}^{2}\left[\left(\omega_{j0}^{2}\right)^2 - \left(\omega_{j0}^{1}\right)^2\right]} \int_0^{\tau} \left[P_{j0}(\eta)\right]_1 \sin \omega_{j0}^{2} (\tau-\eta)\, d\eta \qquad (9.2\text{-}2)$$

$$j = 1,\ 2,\ \ldots,\ M_1$$
$$k = 0$$

Performing inverse transform techniques on equations (9.1-20) thru (9.1-22) we have

$$U_{jk}(\tau) = \mathcal{L}^{-1}\left\{\bar{U}_{jk}(\tau)\right\} = \sum_{m=1}^{3} \frac{1}{D_{jk}^{m}} \left[\left\langle u_{jk}\left\{\left[c_{22}^{jk} - \left(\omega_{jk}^{m}\right)^2\right]\left[c_{33}^{jk} - \left(\omega_{jk}^{m}\right)^2\right] - \left(c_{23}^{jk}\right)^2\right\}\right.\right.$$

$$- v_{jk}\left\{c_{12}^{jk}\left[c_{33}^{jk} - \left(\omega_{jk}^{m}\right)^2\right] - c_{13}^{jk}\, c_{23}^{jk}\right\}$$

$$\left. - w_{jk}\left\{c_{13}^{jk}\left[c_{22}^{jk} - \left(\omega_{jk}^{m}\right)^2\right] - c_{12}^{jk}\, c_{23}^{jk}\right\}\right\rangle \cos \omega_{jk}^{m}\tau$$

$$+\left\{\frac{\left[c_{22}^{jk} - \left(\omega_{jk}^{m}\right)^2\right]\left[c_{33}^{jk} - \left(\omega_{jk}^{m}\right)^2\right] - \left(c_{32}^{jk}\right)^2}{\omega_{jk}^{m}}\right\}\int_0^{\tau}\left[Q_{jk}(\eta)\right]_1 \sin \omega_{jk}^{m} (\tau-\eta)\, d\eta$$

$$
- \left\{ \frac{c_{12}^{jk}\left[c_{33}^{jk} - \left(\omega_{jk}^{m}\right)^2\right] - c_{13}^{jk}\, c_{23}^{jk}}{\omega_{jk}^{m}} \right\} \int_0^{\tau} \left[Q_{jk}(\eta)\right]_2 \sin \omega_{jk}^{m} (\tau - \eta)\, d\eta
$$

$$
- \left\{ \frac{c_{13}^{jk}\left[c_{22}^{jk} - \left(\omega_{jk}^{m}\right)^2\right] - c_{12}^{jk}\, c_{23}^{jk}}{\omega_{jk}^{m}} \right\} \int_0^{\tau} \left[Q_{jk}(\eta)\right]_3 \sin \omega_{jk}^{m} (\tau - \eta)\, d\eta \Bigg] \quad (9.2\text{-}3)
$$

$$
j = 1, 2, \ldots, M_1
$$

$$
k = 1, 2, \ldots, N_1
$$

$$
V_{jk}(\tau) = \mathcal{L}^{-1} \left\{ \overline{V}_{jk}(\tau) \right\}
$$

$$
= \sum_{m=1}^{3} \frac{1}{D_{jk}^{m}} \Bigg[\Bigg\langle - u_{jk} \left\{ c_{12}^{jk} \left[c_{33}^{jk} - \left(\omega_{jk}^{m}\right)^2 \right] - c_{13}^{jk}\, c_{23}^{jk} \right\}
$$

$$
+ v_{jk} \left\{ \left[{}^{jk}_{11} - \left(\omega_{jk}^{m}\right)^2 \right] \left[c_{33}^{jk} - \left(\omega_{jk}^{m}\right)^2 \right] - \left(c_{13}^{jk}\right)^2 \right\}
$$

$$
- w_{jk} \left\{ c_{23}^{jk} \left[c_{11}^{jk} - \left(\omega_{jk}^{m}\right)^2 \right] - c_{12}^{jk}\, c_{13}^{jk} \right\} \Bigg\rangle \cos \omega_{jk}^{m} \tau
$$

$$
- \left\{ \frac{c_{12}^{jk} \left[c_{33}^{jk} - \left(\omega_{jk}^{m}\right)^2 \right] - c_{13}^{jk}\, c_{23}^{jk}}{\omega_{jk}^{m}} \right\} \int_0^{\tau} \left[Q_{jk}(\eta)\right]_1 \sin \omega_{jk}^{m} (\tau - \eta)\, d\eta
$$

$$
+ \left\{ \frac{\left[c_{11}^{jk} - \left(\omega_{jk}^{m}\right)^2 \right] \left[c_{33}^{jk} - \left(\omega_{jk}^{m}\right)^2 \right] - c_{13}^{jk\,2}}{\omega_{jk}^{m}} \right\} \int_0^{\tau} \left[Q_{jk}(\eta)\right]_2 \sin \omega_{jk}^{m} (\tau - \eta)\, d\eta
$$

$$
- \left\{ \frac{c_{23}^{jk} \left[c_{11}^{jk} - \left(\omega_{jk}^{m}\right)^2 \right] - c_{12}^{jk}\, c_{13}^{jk}}{\omega_{jk}^{m}} \right\} \int_0^{\tau} \left[Q_{jk}(\eta)\right]_3 \sin \omega_{jk}^{m} (\tau - \eta)\, d\eta \Bigg] \quad (9.2\text{-}4)
$$

$$
j = 1, 2, \ldots, M_1
$$

$$
k = 1, 2, \ldots, N_1
$$

$$W_{jk}(\tau) = \mathcal{L}^{-1}\left\{\overline{W}_{jk}(\tau)\right\}$$

$$= \sum_{m=1}^{3} \frac{1}{D_{jk}^{m}} \left[\left\langle - u_{jk} \left\{ c_{13}^{jk} \left[c_{22}^{jk} - \left(\omega_{jk}^{m} \right)^2 \right] - c_{12}^{jk} c_{23}^{jk} \right\} \right.\right.$$

$$- v_{jk} \left\{ c_{23}^{jk} \left[c_{11}^{jk} - \left(\omega_{jk}^{m} \right)^2 \right] - c_{12}^{jk} c_{13}^{jk} \right\}$$

$$\left. + w_{jk} \left\{ \left[c_{11}^{jk} - \left(\omega_{jk}^{m} \right)^2 \right] \left[c_{22}^{jk} - \left(\omega_{jk}^{m} \right)^2 \right] - \left(c_{12}^{jk} \right)^2 \right\} \right\rangle \cos \omega_{jk}^{m} \tau$$

$$- \left\{ \frac{c_{13}^{jk} \left[c_{22}^{jk} - \left(\omega_{jk}^{m} \right)^2 \right] - c_{12}^{jk} c_{23}^{jk}}{\omega_{jk}^{m}} \right\} \int_0^{\tau} \left[Q_{jk}(\eta) \right]_1 \sin \omega_{jk}^{m} (\tau - \eta) \, d\eta$$

$$- \left\{ \frac{c_{23}^{jk} \left[c_{11}^{jk} - \left(\omega_{jk}^{m} \right)^2 \right] - c_{12}^{jk} c_{13}^{jk}}{\omega_{jk}^{m}} \right\} \int_0^{\tau} \left[Q_{jk}(\eta) \right]_2 \sin \omega_{jk}^{m} (\tau - \eta) \, d\eta$$

$$+ \left\{ \frac{\left[c_{11}^{jk} - \left(\omega_{jk}^{m} \right)^2 \right] \left[c_{22}^{jk} - \left(\omega_{jk}^{m} \right)^2 \right] - \left(c_{12}^{jk} \right)^2}{\omega_{jk}^{m}} \right\} \int_0^{\tau} \left[Q_{jk}(\eta) \right]_3 \sin \omega_{jk}^{m} (\tau - \eta) \, d\eta$$

(9.2-5)

$j = 1, 2, \ldots, M_1$

$k = 1, 2, \ldots, N_1$

where

$$D_{jk}^{m} = \prod_{\substack{i=1 \\ i \neq m}}^{3} \left[\left(\omega_{jk}^{i}\right)^2 - \left(\omega_{jk}^{m}\right)^2 \right] \tag{9.2-6}$$

9.3 Evaluation of the Convolution Integrals

The evaluation of the convolution integrals appearing in equations (9.2-1) thru (9.2-6) will be presented in this section.

9.3.1 Substitution of Forcing Functions

By substituting the forcing functions $\left[P_{j0}(\tau)\right]_1$, $\left[P_{j0}(\tau)\right]_2$, $\left[Q_{jk}(\tau)\right]_1$, $\left[Q_{jk}(\tau)\right]_2$, and $\left[Q_{jk}(\tau)\right]_3$ and using η as a dummy variable the integrals in equations (9.2-1) thru (9.2-6) are evaluated as follows:

$$I_1 = \int_0^{\tau} \left[P_{j0}(\eta)_1\right] \sin \omega_{j0}^{1} (\tau-\eta)\, d\eta \tag{9.3-1}$$

$$I_1 = \left(\frac{2}{j\pi}\right)^2 \left[(-1)^j - 1\right] \sum_{i=1}^{I} \sum_{p=1}^{P} A_i A_p\ \Omega_i \Omega_p \int_0^{\tau} \sin \Omega_i(\eta) \sin \Omega_p(\eta) \sin \omega_{j0}^{1}(\tau-\eta)\, d\eta$$

$$+ \left(\frac{2}{j\pi}\right)^2 \left[(-1)^j - 1\right] \sum_{i=p=1}^{I} A_i^{\,2} \Omega_i^{\,2} \int_0^{\tau} \sin^2 \Omega_i(\eta) \sin \omega_{j0}^{1}(\tau-\eta)\, d\eta \tag{9.3-2}$$

$$I_2 = \int_0^{\tau} \left[P_{j0}(\eta)\right]_1 \sin \omega_{j0}^{2} (\tau-\eta)\, d\eta \tag{9.3-3}$$

$$I_2 = \left(\frac{2}{j\pi}\right)^2 \left[(-1)^j - 1\right] \sum_{\substack{i=1 \\ i \neq m}}^{I} \sum_{m=1}^{M} A_i A_p \Omega_i \Omega_p \int_0^{\tau} \sin \Omega_i(\eta) \sin \Omega_p(\eta) \sin \omega_{j0}^{2}(\tau-\eta)\, d\eta$$

$$+ \left(\frac{2}{j\pi}\right)^2 \left[(-1)^j - 1\right] \sum_{i=1}^{I} A_i^{\,2} \Omega_i^{\,2} \int_0^{\tau} \sin^2 \Omega_i(\eta) \sin \omega_{j0}^{2}(\tau-\eta)\, d\eta \tag{9.3-4}$$

$$I_3 = \int_0^\tau \left[P_{j0}(\eta)\right]_2 \sin \omega^1_{j0} (\tau-\eta)\, d\eta \qquad (9.3\text{-}5)$$

$$I_3 = \int_0^\tau \left\{\frac{j\pi}{4\lambda}\left[P_{j0}(\eta)\right]_1 + \frac{2\nu\lambda}{j\pi\mu Eh} T_o \left[1-\gamma\cos \bar{\Omega}(\eta)\right]\right\} \sin \omega^1_{j0} (\tau-\eta)\, d\eta \qquad (9.3\text{-}6)$$

$$I_3 = \frac{j\pi}{4\lambda}\left(\frac{2}{j\pi}\right)^2\left[(-1)^j - 1\right]\sum_{i=1}^{I}\sum_{p=1}^{P} A_i A_p \Omega_i \Omega_p \int_0^\tau \sin \Omega_i(\eta) \sin\Omega_p(\eta) \sin\omega^1_{j0}(\tau-\eta)\, d\eta$$

$$+ \frac{j\pi}{4\lambda}\left(\frac{2}{j\pi}\right)^2\left[(-1)^j - 1\right]\sum_{i=1}^{I} A_i^{\,2}\Omega_i^{\,2} \int_0^\tau \sin^2\Omega_i(\eta) \sin \omega^1_{j0}(\tau-\eta)\, d\eta$$

$$+ \frac{2\nu\lambda\, T_o}{j\pi\mu Eh}\int_0^\tau \sin \omega^1_{j0} (\tau-\eta)\, d\eta - \frac{2\nu\lambda\, T_o\gamma}{j\pi\mu\, Eh}\int_0^\tau \cos \bar{\Omega}(\eta) \sin \omega^1_{j0} (\tau-\eta)\, d\eta$$

(9.3-7)

$$I_4 = \int_0^\tau \left[P_{j0}(\eta)\right]_2 \sin \omega^2_{j0} (\tau-\eta)\, d\eta \qquad (9.3\text{-}8)$$

$$I_4 = \frac{j\pi}{4\lambda}\left(\frac{2}{j\pi}\right)^2\left[(-1)^j - 1\right]\sum_{\substack{i=1\\ i\neq p}}^{I}\sum_{p=1}^{P} A_i A_p\, \Omega_i \Omega_p \int_0^\tau \sin\Omega_i(\eta) \sin\Omega_p(\eta) \sin \omega^2_{j0}(\tau-\eta)\, d\eta$$

$$+ \frac{j\pi}{4\lambda}\left(\frac{2}{j\pi}\right)^2\left[(-1)^i - 1\right]\sum_{i=p=1}^{I} A_i^{\,2}\Omega_i^{\,2} \int_0^\tau \sin^2 \Omega_i(\eta) \sin \omega^2_{j0} (\tau-\eta)\, d\eta$$

$$+ \frac{2\nu\lambda T_o}{j\pi\mu Eh}\int_0^\tau \sin \omega^2_{j0} (\tau-\eta)\, d\eta - \frac{2\nu\lambda T_o\gamma}{j\pi\mu Eh}\int_0^\tau \cos \bar{\Omega}\eta \sin \omega^2_{j0} (\tau-\eta)\, d\eta \qquad (9.3\text{-}9)$$

$$J_1 = \int_0^\tau \left[Q_{jk}(\eta)\right]_1 \sin \omega^m_{jk} (\tau-\eta)\, d\eta \qquad (9.3\text{-}10)$$

$$J_1 = \int_0^\tau \left\{ \frac{\lambda\delta 1k}{2(1+\nu)\mu} \sum_{n=1}^{N} \frac{\lambda n}{2} z_{jn} q_n(\eta) \right\} \sin \omega_{jk}^m (\tau-\eta)\, d\eta \qquad (9.3\text{-}11)$$

$$J_1 = \frac{\lambda\delta 1k}{2(1+\nu)\mu} \sum_{m=1}^{M} \sum_{n=1}^{N} \left(\frac{\lambda n}{2}\right) z_{jn} \int_0^\tau q_n(\eta) \sin \omega_{jk}^m (\tau-\eta)\, d\eta \qquad (9.3\text{-}12)$$

$$J_2 = \int_0^\tau \left[Q_{jk}(\eta)\right]_2 \sin \omega_{jk}^m (\tau-\eta)\, d\eta \qquad (9.3\text{-}13)$$

$$\begin{aligned}
J_2 = {} & \delta_{1k} \left(\frac{2}{j\pi}\right) \sum_{i=1}^{I} \sum_{m=1}^{M} \left\{ \frac{K\,T_o}{2} \left[1 - (-1)^j\right] \right. \\
& \left. + \left[1 + (-1)^j\right] \Omega_i^{\,2} \right\} A_i \int_0^\tau \cos\Omega_i(\eta) \sin \omega_{jk}^m (\tau-\eta)\, d\eta \\
& - \delta_{1k} \left(\frac{2}{j\pi}\right) \sum_{i=1}^{I} \sum_{m=1}^{M} \frac{K\,T_o}{2} \left[1 - (-1)^j\right] A_i\, \gamma \\
& \int_0^\tau \cos\bar{\Omega}(\eta) \cos\Omega_i(\eta) \sin \omega_{jk}^m (\tau-\eta)\, d\eta \\
& - \delta_{2k} \left[1 - (-1)^j\right] \sum_{\substack{i=1 \\ i\neq p}}^{I} \sum_{p=1}^{P} \sum_{m=1}^{M} A_i A_p \Omega_i \Omega_p \int_0^\tau \sin\Omega_i(\eta) \sin\Omega_p(\eta) \sin\omega_{jk}^m(\tau-\eta) d\eta \\
& - \delta_{2k} \left[1 - (-1)^j\right] \sum_{i=p=1}^{I} \sum_{m=1}^{M} A_i^{\,2} \Omega_i^{\,2} \int_0^\tau \sin^2\Omega_i(\eta) \sin\omega_{jk}^m (\tau-\eta)\, d\eta \\
& - \frac{\delta 1k}{2(1+\nu)\mu} \sum_{m=1}^{M} \sum_{n=1}^{N} \left(\frac{\lambda n}{2}\right)^2 x_{jn} \int_0^\tau q_n(\eta) \sin \omega_{jk}^m (\tau-\eta)\, d\eta
\end{aligned} \qquad (9.3\text{-}14)$$

$$J_3 = \int_0^\tau \left[Q_{jk}(\eta)\right]_3 \sin \omega_{jk}^m (\tau-\eta)\, d\eta \tag{9.3-15}$$

$$J_3 = \delta_{1k} \left(\frac{2}{j\pi}\right) \sum_{i=1}^{I} \sum_{m=1}^{M} \left\{ \frac{K\,T_o}{2} \left[1 - (-1)^j\right] \right.$$

$$\left. + \left[1 + (-1)^j\right] \Omega_i^{\,2} \right\} A_i \int_0^\tau \cos\Omega_i(\eta) \sin\omega_{jk}^m (\tau-\eta)\, d\eta$$

$$- \delta_{1k} \left(\frac{2}{j\pi}\right) \sum_{i=1}^{I} \sum_{m=1}^{M} \frac{K\,T_o}{2} \left[1 - (-1)^j\right] A_i \gamma$$

$$\int_0^\tau \cos\bar{\Omega}(\eta) \cos\Omega_i(\eta) \sin \omega_{jk}^m (\tau-\eta)\, d\eta$$

$$- \delta_{2k} \left[1 - (-1)^j\right] \sum_{\substack{i=1 \\ i \neq p}}^{I} \sum_{p=1}^{P} \sum_{m=1}^{M} A_i A_p \Omega_i \Omega_p \quad \sin\Omega_i(\eta) \sin\Omega_p(\eta) \sin\omega_{jk}^m (\tau-\eta) d\eta$$

$$- \delta_{2k} \left[1 - (-1)^j\right] \sum_{i=p=1}^{I} \sum_{m=1}^{M} A_i^{\,2} \Omega_i^{\,2} \int_0^\tau \sin^2\Omega_i(\eta) \sin\omega_{jk}^m (\tau-\eta)\, d\eta$$

$$+ \delta_{1k} \frac{\sigma^2}{12(1-\nu^2)\mu} \sum_{m=1}^{M} \sum_{n=1}^{N} \left\{ s_{jn} \left[\left(\frac{\lambda n}{2}\right)^4 + \lambda^4 \right] \right.$$

$$\left. - 2\lambda^2 \frac{\lambda n}{2}^2 x_{jn} \right\} \int_0^\tau q_n(\eta) \sin \omega_{jk}^m (\tau-\eta)\, d\eta$$

$$+ \delta_{1k} \frac{\sigma^2}{12(1-\nu^2)\mu} \sum_{m=1}^{M} \lambda^4\, m_{j0} \int_0^\tau q_B(\eta) \sin \omega_{jk}^m (\tau-\eta)\, d\eta \tag{9.3-16}$$

9.3.2 Evaluation of Remaining Integrals

The remaining integrals of equations (9.3-1) thru (9.3-16) have been evaluated and are listed below.

$$\int_0^\tau \sin \Omega_i(\eta) \sin \Omega_p(\eta) \sin \omega_{j0}^1 (\tau-\eta)\, d\eta$$

$$= \frac{\omega_{j0}^1}{2} \left\{ \frac{\cos(\Omega_i+\Omega_p)\tau - \cos \omega_{j0}^1\tau}{(\Omega_i+\Omega_p)^2 - (\omega_{j0}^1)^2} - \frac{\cos(\Omega_i-\Omega_p)\tau - \cos \omega_{j0}^1\tau}{(\Omega_i-\Omega_p)^2 - (\omega_{j0}^1)^2} \right\} \quad (9.3\text{-}17)$$

$$\int_0^\tau \sin \Omega_i(\eta) \sin\Omega_p(\eta) \sin \omega_{jk}^m (\tau-\eta)\, d\eta$$

$$= \frac{\omega_{j0}^2}{2} \left\{ \frac{\cos(\Omega_i+\Omega_p)\tau - \cos \omega_{j0}^2\tau}{(\Omega_i+\Omega_p)^2 - (\omega_{j0}^2)^2} - \frac{\cos(\Omega_i-\Omega_p)\tau - \cos \omega_{j0}^2\tau}{(\Omega_i-\Omega_p)^2 - (\omega_{j0}^2)^2} \right\} \quad (9.3\text{-}18)$$

$$\int_0^\tau \sin \Omega_i(\eta) \sin\Omega_p(\eta) \sin \omega_{jk}^m (\tau-\eta)$$

$$= \frac{\omega_{jk}^m}{2} \left\{ \frac{\cos(\Omega_i+\Omega_p)\tau - \cos \omega_{jk}^m}{(\Omega_i+\Omega_p)^2 - (\omega_{jk}^m)^2} - \frac{\cos(\Omega_i-\Omega_p)\tau - \cos \omega_{jk}^m\tau}{(\Omega_i-\Omega_p)^2 - (\omega_{jk}^m)^2} \right\} \quad (9.3\text{-}19)$$

$$\int_0^\tau \sin^2\Omega_i(\eta) \sin \omega_{j0}^1 (\tau-\eta)\, d\eta$$

$$= \frac{1}{2\omega_{j0}^1} (1-\cos \omega_{j0}^1\tau)$$

$$+ \frac{\omega_{j0}^1}{\left[2\ (2\Omega_i)^2 - (\omega_{j0}^1)^2\right]} (\cos 2\Omega_i\tau - \cos \omega_{j0}^1\tau) \quad (9.3\text{-}20)$$

$$\int_0^\tau \sin^2\Omega_i(\eta)\,\sin\omega_{j0}^2(\tau-\eta)\,d\eta$$

$$= \frac{1}{2\omega_{j0}^2}\left(1 - \cos\omega_{j0}^2\tau\right)$$

$$+ \frac{\omega_{j0}^2}{2\left[(2\Omega_i)^2 - (\omega_{j0}^2)^2\right]}\left(\cos 2\Omega_i\tau - \cos\omega_{j0}^2\tau\right) \qquad (9.3\text{-}21)$$

$$\int_0^\tau \sin^2\Omega_i(\eta)\,\sin\omega_{jk}^m(\tau-\eta)\,d\eta$$

$$= \frac{1}{2\omega_{jk}^m}\left(1 - \cos\omega_{jk}^m\tau\right)$$

$$+ \frac{\omega_{jk}^m}{2\left[(2\Omega_i)^2 - (\omega_{jk}^m)^2\right]}\left(\cos 2\Omega_i\tau - \cos\omega_{jk}^m\tau\right) \qquad (9.3\text{-}22)$$

$$\int_0^\tau \cos\bar{\Omega}(\eta)\,\sin\omega_{j0}^1(\tau-\eta)\,d\eta$$

$$= -\frac{\omega_{j0}^1}{\left[\bar{\Omega}^2 - (\omega_{j0}^1)^2\right]}\left(\cos\bar{\Omega}\tau - \cos\omega_{j0}^1\tau\right) \qquad (9.3\text{-}23)$$

$$\int_0^\tau \cos\bar{\Omega}\eta\,\sin\omega_{j0}^2(\tau-\eta)\,d\eta$$

$$= -\frac{\omega_{j0}^2}{\left[\bar{\Omega}^2 - (\omega_{j0}^2)^2\right]}\left(\cos\bar{\Omega}\tau - \cos\omega_{j0}^2\tau\right) \qquad (9.3\text{-}24)$$

$$\int_0^{\tau} \sin\, \omega_{j0}^{1}\, (\tau-\eta)\, d\eta$$

$$= \frac{1}{2\omega_{j0}^{1}} \cos\omega_{j0}^{1}\, (\tau-\eta) \Big|_{0}^{\tau} = \frac{1}{2\omega_{j0}^{1}} (1 - \cos\omega_{j0}^{1}\tau) \qquad (9.3\text{-}25)$$

$$\int_0^{\tau} \sin\omega_{j0}^{2}\, (\tau-\eta)\, d\eta = \frac{1}{2\omega_{j0}^{2}} (1 - \cos\omega_{j0}^{2}\tau) \qquad (9.3\text{-}26)$$

$$\int_0^{\tau} \cos\bar{\Omega}\eta\, \cos\Omega_i\eta\, \sin\omega_{jk}^{m}\, (\tau-\eta)\, d\eta$$

$$= -\frac{\omega_{jk}^{m}}{\left[2\, (\bar{\Omega}+\Omega_i)^2 - (\omega_{jk}^{m})^2\right]} \left[\cos(\bar{\Omega}+\Omega_i)\tau - \cos\omega_{jk}^{m}\tau\right]$$

$$-\frac{\omega_{jk}^{m}}{\left[2\, (\bar{\Omega}-\Omega_i)^2 - (\omega_{jk}^{m})^2\right]} \left[\cos(\bar{\Omega}-\Omega_i)\tau - \cos\omega_{jk}^{m}\tau\right] \qquad (9.3\text{-}27)$$

$$\int_0^{\tau} \cos\Omega_i\eta\, \sin\omega_{jk}^{m}\, (\tau-\eta)\, d\eta$$

$$= -\frac{\omega_{jk}^{m}}{\left[\Omega_i^{2} - (\omega_{jk}^{m})^2\right]} \left[(\cos\Omega_i\tau - \cos\omega_{jk}^{m}\tau)\right] \qquad (9.3\text{-}28)$$

9.3.3 Summary of Evaluated Convolution Integrals

Substituting the results of equations (9.3-17) thru (9.3-28) into equations (9.3-1) thru (9.3-16) the convolution integrals are evaluated. A summary of the evaluated convolution integrals appear below.

$$I_1 = \int_0^{\tau} \left[P_{j0}(\eta)_1\right] \sin \omega_{j0}^1 (\tau-\eta)\, d\eta$$

$$= \left(\frac{2}{j\pi}\right)^2 \left[(-1)^j - 1\right] \sum_{i=1}^{I} \sum_{\substack{p=1 \\ i \neq p}}^{P} A_i A_p \Omega_i \Omega_p \left[\frac{\omega_{j0}^1}{2} \left\{ \frac{\cos(\Omega_i+\Omega_p)\tau - \cos \omega_{j0}^1 \tau}{(\Omega_i+\Omega_p)^2 - (\omega_{j0}^1)^2} \right.\right.$$

$$\left.\left. - \frac{\cos(\Omega_i-\Omega_p)\tau - \cos \omega_{j0}^1 \tau}{(\Omega_i-\Omega_p)^2 - (\omega_{j0}^1)^2} \right\} \right]$$

$$+ \left(\frac{2}{j\pi}\right)^2 \left[(-1)^j - 1\right] \sum_{i=p=1}^{I} \left[A_i^2 \Omega_i^2 \; \frac{1}{2\omega_{j0}^1} (1-\cos \omega_{j0}^1 \tau) \right.$$

$$\left. + \frac{\omega_{j0}^1}{2\left[(2\Omega_i)^2 - (\omega_{j0}^1)^2\right]} (\cos 2\Omega_i \tau - \cos \omega_{j0}^1 \tau) \right] \qquad (9.3\text{-}29)$$

$$I_2 = \int_0^{\tau} \left[P_{j0}(\eta)\right]_1 \sin \omega_{j0}^2 (\tau-\eta)\, d\eta$$

$$= \left(\frac{2}{j\pi}\right)^2 \left[(-1)^j - 1\right] \sum_{i=1}^{I} \sum_{\substack{p=1 \\ i \neq p}}^{P} A_i A_p \Omega_i \Omega_p \left[\frac{\omega_{j0}^2}{2} \left\{ \frac{\cos(\Omega_i+\Omega_p)\tau - \cos \omega_{j0}^2 \tau}{(\Omega_i+\Omega_p)^2 - (\omega_{j0}^2)^2} \right.\right.$$

$$\left.\left. - \frac{\cos(\Omega_i-\Omega_p)\tau - \cos \omega_{j0}^2 \tau}{(\Omega_i-\Omega_p)^2 - (\omega_{j0}^2)^2} \right\} \right]$$

$$+ \left(\frac{2}{j\pi}\right)^2 \left[(-1)^j - 1\right] \sum_{i=p=1}^{I} A_i^2 \Omega_i^2 \left[\frac{1}{2\omega_{j0}^2} (1-\cos \omega_{j0}^2 \tau) \right.$$

$$+\frac{\omega_{j0}^{2}}{\left[2\,(2\Omega_i)^2-(\omega_{j0}^{2})^2\right]}\left(\cos 2\Omega_i\tau-\cos\omega_{j0}^{2}\tau\right)\Bigg] \qquad (9.3\text{-}30)$$

$$I_3=\int_0^{\tau}\left[P_{j0}(\eta)\right]_2\ \sin\omega_{j0}^{1}(\tau-\eta)$$

$$=\frac{j\pi}{4\lambda}\left(\frac{2}{j\pi}\right)^2\left[(-1)^j-1\right]\sum_{\substack{i=1\\ i\neq p}}^{I}\sum_{p=1}^{P}A_iA_p\Omega_i\Omega_p\left[\frac{\omega_{j0}^{1}}{2}\left\{\frac{\cos(\Omega_i+\Omega_p)\tau-\cos\omega_{j0}^{1}\tau}{(\Omega_i+\Omega_p)^2-(\omega_{j0}^{1})^2}\right.\right.$$

$$\left.\left.-\frac{\cos(\Omega_i-\Omega_p)\tau-\cos\omega_{j0}^{1}\tau}{(\Omega_i-\Omega_p)^2-(\omega_{j0}^{1})^2}\right\}\right]$$

$$+\frac{j\pi}{4\lambda}\left(\frac{2}{j\pi}\right)^2\left[(-1)^j-1\right]\sum_{i=p=1}^{I}A_i^{\,2}\Omega_i^{\,2}\left[\frac{1}{2\,\omega_{j0}^{1}}\left(1-\cos\omega_{j0}^{1}\tau\right)\right.$$

$$\left.+\frac{\omega_{j0}^{1}}{\left[2\,(2\Omega_i)^2-(\omega_{j0}^{1})^2\right]}\left(\cos 2\Omega_i\tau-\cos\omega_{j0}^{1}\tau\right)\right]$$

$$+\frac{2\nu\lambda\,T_0}{j\pi\mu\,Eh}\left[\frac{1}{2\,\omega_{j0}^{1}}\left(1-\cos\omega_{j0}^{1}\tau\right)\right]$$

$$-\frac{2\nu\lambda\,T_o\gamma}{j\pi\mu\,Eh}\left[-\frac{\omega_{j0}^{1}}{\left[(\bar{\Omega}^2-(\omega_{j0}^{1})^2\right]}\left(\cos\bar{\Omega}\tau-\cos\omega_{j0}^{1}\tau\right)\right] \qquad (9.3\text{-}31)$$

$$I_4=\int_0^{\tau}\left[P_{j0}(\eta)\right]_1\ \sin\omega_{j0}^{2}(\tau-\eta)\,d\eta$$

$$= \frac{j\pi}{4\lambda}\left(\frac{2}{j\pi}\right)^2 \left[(-1)^j - 1\right] \sum_{i=1}^{I}\sum_{p=1}^{P} A_i A_p \Omega_i \Omega_p \left[\frac{\omega_{j0}^1}{2}\left\{\frac{\cos(\Omega_i+\Omega_p)\tau - \cos\omega_{j0}^2\tau}{(\Omega_i+\Omega_p)^2 - (\omega_{j0}^2)^2}\right.\right.$$

$$\left.\left. - \frac{\cos(\Omega_i+\Omega_p)\tau - \cos\omega_{j0}^2\tau}{(\Omega_i-\Omega_p)^2 - (\omega_{j0}^2)^2}\right\}\right] + \frac{j\pi}{4\lambda}\left(\frac{2}{j\pi}\right)^2\left[(-1)^j - 1\right]\sum_{i=p=1}^{I} A_i^2 \Omega_i^2 \left[\frac{1}{2\omega_{j0}^2}(1-\cos\omega_{j0}^2\tau)\right.$$

$$\left. + \frac{\omega_{j0}^2}{2\left[(2\Omega_i)^2 - (\omega_{j0}^2)^2\right]}(\cos 2\Omega_i - \cos\omega_{j0}^2\tau)\right] + \frac{2\nu\lambda T_o}{j\pi\mu Eh}\left[\frac{1}{2\omega_{j0}^2}(1 - \cos\omega_{j0}^2\tau)\right]$$

$$- \frac{2\nu\lambda T_o\gamma}{j\pi\mu Eh}\left[-\frac{\omega_{j0}^2}{\left[(\bar{\Omega}^2 - (\omega_{j0}^2)^2\right]}(\cos\bar{\Omega}\tau - \cos\omega_{j0}^2\tau)\right] \qquad (9.3\text{-}32)$$

$$J_1 = \int_0^\tau \left[Q_{jk}(\eta)\right]_1 \sin\omega_{jk}^m(\tau-\eta)\,d\eta$$

$$= \frac{\lambda\delta 1k}{2(1+\nu)\mu}\sum_{m=1}^{M}\sum_{n=1}^{N}\frac{\lambda n}{2} z_{jn}\int_0^\tau q_n(\eta)\sin\omega_{jk}^m(\tau-\eta)\,d\eta \qquad (9.3\text{-}33)$$

$$J_2 = \int_0^\tau Q_{jk}(\eta)_2 \sin\omega_{jk}^m(\tau-\eta)\,d\eta$$

$$= \delta_{1k}\frac{2}{j\pi}\sum_{i=1}^{I}\sum_{m=1}^{M}\left\{\frac{K\,T_o}{2}\left[1-(-1)^j\right] + \left[1+(-1)^j\right]\Omega_i^2\right\} A_i$$

$$\cdot\left\{-\frac{\omega_{jk}^m}{\left[\Omega_i^2 - (\omega_{jk}^m)^2\right]}(\cos\Omega_i\tau - \cos\omega_{jk}^m\tau)\right\}$$

$$- \delta_{1k} \left(\frac{2}{j\pi}\right) \sum_{i=1}^{I} \sum_{m=1}^{M} \frac{K\,T_o}{2} \left[1 - (-1)^j\right] A_i \gamma$$

$$\cdot \left\{ - \frac{\omega_{jk}^{m}}{2\left[(\bar{\Omega}+\Omega_i)^2 - (\omega_{jk}^{m})^2\right]} \left[\cos(\bar{\Omega}+\Omega_i)\tau - \cos\omega_{jk}^{m}\tau\right] \right.$$

$$\left. - \frac{\omega_{jk}^{m}}{2\left[(\bar{\Omega}-\Omega_i)^2 - (\omega_{jk}^{m})^2\right]} \left[\cos(\bar{\Omega}-\Omega_i)\tau - \cos\omega_{jk}^{m}\tau\right] \right\}$$

$$- \delta_{2k} \left[1 - (-1)^j\right] \sum_{\substack{i=1 \\ i \neq p}}^{I} \sum_{p=1}^{P} \sum_{m=1}^{M} A_i A_p \Omega_i \Omega_p$$

$$\frac{\omega_{jk}^{m}}{2} \left\{ \frac{\cos(\Omega_i+\Omega_p)\tau - \cos\omega_{jk}^{m}\tau}{(\Omega_i+\Omega_p)^2 - (\omega_{jk}^{m})^2} - \frac{\cos(\Omega_i-\Omega_p)\tau - \cos\omega_{jk}^{m}\tau}{(\Omega_i-\Omega_p)^2 - (\omega_{jk}^{m})^2} \right\}$$

$$- \delta_{2k} \left[1 - (-1)^j\right] \sum_{i=p=1}^{I} \sum_{m=1}^{M} \frac{A_i^2 \Omega_i^2}{2} \left\{ \frac{1 - \cos\omega_{jk}^{m}\tau}{\omega_{jk}^{m}} + \frac{\omega_{jk}^{m}(\cos 2\Omega_i\tau - \cos\omega_{jk}^{m}\tau)}{\left[(2\Omega_i)^2 - (\omega_{jk}^{m})^2\right]} \right\}$$

$$- \frac{\delta_{1k}}{2(1+\nu)\mu} \sum_{m=1}^{M} \sum_{n=1}^{N} \left(\frac{\lambda n}{2}\right)^2 X_{jn} \int_0^{\tau} q_n(\eta) \sin \omega_{jk}^{m} (\tau-\eta)\, d\eta \qquad (9.3\text{-}34)$$

$$J_3 = \int_0^{\tau} \left[Q_{jk}(\eta)\right]_3 \sin \omega_{jk}^{m} (\tau-\eta)\, d\eta$$

$$= \delta_{1k} \left(\frac{2}{j\pi}\right) \sum_{i=1}^{I} \sum_{m=1}^{M} \left\{ \frac{K\,T_o}{2} \left[1 - (-1)^j\right] + \left[1 + (-1)^j\right] \Omega_i^2 \right\} A_i$$

$$\cdot \left\{ - \frac{\omega_{jk}^{m}}{\left[\Omega_i^2 - (\omega_{jk}^{m})^2\right]} (\cos\Omega_i\tau - \cos\omega_{jk}^{m}\tau) \right\}$$

$$- \delta_{1k} \left(\frac{2}{j\pi}\right) \sum_{i=1}^{I} \sum_{m=1}^{M} \frac{K\,T_o}{2} \left[1 - (-1)^j\right] A_i \gamma$$

$$\left\{ - \frac{\omega_{jk}^m}{\left[2\ \bar{\Omega}+\Omega_i{}^2 - (\ {}^m_{jk})^2\right]} \left[\cos(\bar{\Omega}+\Omega_i)\tau - \cos\omega_{jk}^m \tau\right] \right.$$

$$\left. - \frac{\omega_{jk}^m}{\left[2\ (\bar{\Omega}-\Omega_i)^2 - (\omega_{jk}^m)^2\right]} \left[\cos(\bar{\Omega}-\Omega_i)\ - \cos\omega_{jk}^m \tau\right] \right\}$$

$$- \delta_{2k} \left[1 - (-1)^j\right] \sum_{\substack{i=1 \\ i \neq p}}^{I} \sum_{p=1}^{P} \sum_{m=1}^{M} A_i A_p \Omega_i \Omega_p$$

$$\frac{\omega_{jk}^m}{2} \left\{ \frac{\cos(\Omega_i+\Omega_p)\tau - \cos\omega_{jk}^m \tau}{(\Omega_i+\Omega_p)^2 - (\omega_{jk}^m)^2} - \frac{\cos(\Omega_i-\Omega_p)\tau - \cos\omega_{jk}^m \tau}{(\Omega_i-\Omega_p)^2 - (\omega_{jk}^m)^2} \right\}$$

$$- \delta_{2k} \left[1 - (-1)^j\right] \sum_{i=p=1}^{I} \sum_{m=1}^{M} \frac{A_i{}^2 \Omega_i{}^2}{2} \left\{ \frac{1-\cos\omega_{jk}^m \tau}{\omega_{jk}^m} + \frac{\omega_{jk}^m (\cos 2\Omega_i \tau - \cos\omega_{jk}^m \tau)}{(2\Omega_i)^2 - (\omega_{jk}^m)^2} \right\}$$

$$+\delta_{1k} \frac{\sigma^2}{12(1-\nu^2)\mu} \sum_{m=1}^{M} \sum_{n=1}^{N} \left\{ s_{jn} \left[\left(\frac{\lambda n}{2}\right)^4 + \lambda^4\right] - 2\lambda^2 \left(\frac{\lambda n}{2}\right)^2 x_{jn} \right\} \int_0 q_n(\eta) \sin\omega_{jk}^m (\tau-\eta)\ d\eta$$

$$+ \delta_{1k} \frac{\sigma^2}{12(1-\nu^2)\mu} \sum_{m=1}^{M} \lambda^4 m_{j0} \int_0^{\tau} q_B(\eta)\ \sin\ \omega_{jk}^m\ (\tau-\eta)\ d\eta \qquad (9.3\text{-}35)$$

In terms of J_2 , J_3 can be written

$$J_3 = J_2 + \frac{\delta_{1k}}{2(1+\nu)\mu} \sum_{m=1}^{M} \sum_{n=1}^{N} \left(\frac{\lambda n}{2}\right)^2 x_{jn} \int_0^{\tau} q_n(\eta)\ \sin\ \omega_{jk}^m\ (\tau-\eta)\ d\eta$$

$$+ \delta_{1k} \frac{\sigma^2}{12(1-\nu^2)\mu} \left[\sum_{m=1}^{M} \sum_{n=1}^{N} \left\{ \varepsilon_{jn} \left[\left(\frac{\lambda n}{2}\right)^4 + \lambda^4 \right] - 2\lambda^2 \left(\frac{\lambda n}{2}\right)^2 x_{jn} \right\} \right.$$

$$\left. \int_0^\tau q_n(\eta) \sin\omega_{jk}^m (\tau-\eta)\, d\eta + \sum_{m=1}^{M} \lambda^4 m_{j0} \int_0^\tau q_B(\eta) \sin\omega_{jk}^m (\tau-\eta)\, d\eta \right]$$

(9.3-36)

9.4 Evaluated Displacement Equations

Substituting the evaluated convolution integrals of equations (9.3-29) thru (9.3-36) into equations (9.2-1) thru (912-5) the displacement equations are evaluated.

From equation (9.2-1)

$$U_{j0}(\tau) = \frac{\left\{ U_{j0} \left[c_{22}^{j0} - \left(\omega_{j0}^1\right)^2 \right] - c_{12}^{j0}\, w_{j0} \right\}}{\left[\left(\omega_{j0}^2\right)^2 - \left(\omega_{j0}^1\right)^2 \right]} \cos \omega_{j0}^1 \tau$$

$$- \frac{\left\{ U_{j0} \left[c_{22}^{j0} - \left(\omega_{j0}^2\right)^2 \right] - c_{12}^{j0}\, w_{j0} \right\}}{\left[\left(\omega_{j0}^2\right)^2 - \left(\omega_{j0}^1\right)^2 \right]} \cos \omega_{j0}^2 \tau$$

$$+ \frac{c_{22}^{j0} - \omega_{j0}^{1\ 2}}{\left[\left(\omega_{j0}^2\right)^2 - \left(\omega_{j0}^1\right)^2 \right]} \left\{ \left(\frac{2}{j\pi}\right)^2 \left[(-1)^j - 1\right] \left[\sum_{\substack{i=1 \\ i \neq p}}^{I} \sum_{p=1}^{P} A_i A_p \Omega_i \Omega_p \frac{1}{2} \left\{ \frac{\cos(\Omega_i+\Omega_p)\tau - \cos\omega_{j0}^1 \tau}{(\Omega_i+\Omega_p)^2 - \left(\omega_{j0}^1\right)^2} \right. \right. \right.$$

$$\left. - \frac{\cos(\Omega_i-\Omega_p)\tau - \cos\omega_{j0}^1 \tau}{(\Omega_i-\Omega_p)^2 - \left(\omega_{j0}^1\right)^2} \right\} + \sum_{i=p=1}^{I} A_i^2 \Omega_i^2 \left\{ \frac{1}{2\left(\omega_{j0}^1\right)^2} (1 - \cos \omega_{j0}^1 \tau) \right.$$

$$+ \frac{1}{2\left[(2\Omega_i)^2 - \left(\omega_{j0}^1\right)^2\right]} (\cos 2\Omega_i\tau - \cos \omega_{j0}^1\tau)\Big\}\Big]\Big]\Big\}$$

$$- \frac{c_{22}^{j0} - \left(\omega_{j0}^2\right)^2}{\left[\left(\omega_{j0}^2\right)^2 - \left(\omega_{j0}^1\right)^2\right]} \left\{\left(\frac{2}{j\pi}\right)^2 \left[(-1)^j - 1\right]\left[\sum_{i=1}^{I} \sum_{\substack{p=1 \\ i\neq p}}^{P} A_i A_p \Omega_i \Omega_p \; \frac{1}{2}\left\{\frac{\cos(\Omega_i+\Omega_p)\tau - \cos\omega_{j0}^2\tau}{(\Omega_i+\Omega_p)^2 - \left(\omega_{j0}^2\right)^2}\right.\right.\right.$$

$$\left.- \frac{\cos(\Omega_i-\Omega_p)\tau - \cos\omega_{j0}^2\tau}{(\Omega_i-\Omega_p)^2 - \left(\omega_{j0}^2\right)^2}\right\} + \sum_{i=p=1}^{I} A_i^2 \Omega_i^2 \left\{\frac{1}{2(\omega_{j0}^2)^2} (1 - \cos\omega_{j0}^2\tau)\right.$$

$$\left.\left.\left.+ \frac{1}{2\left[(2\Omega_i)^2 - (\omega_{j0}^2)^2\right]} (\cos 2\Omega_i\tau - \cos \omega_{j0}^2\tau)\right\}\right]\right]\Big\}$$

$$- \frac{c_{12}^{j0}}{\left[\left(\omega_{j0}^2\right)^2 - \left(\omega_{j0}^1\right)^2\right]} \left\{\frac{j\pi}{4\lambda}\left(\frac{2}{j\pi}\right)^2\left[(-1)^j - 1\right]\left[\sum_{i=1}^{I} \sum_{\substack{p=1 \\ i\neq p}}^{P} A_i A_p \Omega_i \Omega_p \; \frac{1}{2}\left\{\frac{\cos(\Omega_i+\Omega_p)\tau - \cos\omega_{j0}^1\tau}{(\Omega_i+\Omega_p)^2 - \left(\omega_{j0}^1\right)^2}\right.\right.\right.$$

$$\left.- \frac{\cos(\Omega_i-\Omega_p)\tau - \cos\omega_{j0}^1\tau}{(\Omega_i-\Omega_p)^2 - \left(\omega_{j0}^1\right)^2}\right\} + \sum_{i=p=1}^{I} A_i^2 \Omega_i^2 \left\{\frac{1}{2\left(\omega_{j0}^1\right)^2} (1 - \cos \omega_{j0}^1\tau)\right.$$

$$\left.\left.+ \frac{1}{2\left[(2\Omega_i)^2 - \left(\omega_{j0}^1\right)^2\right]} (\cos 2\Omega_i\tau - \cos\omega_{j0}^1\tau)\right\}\right] + \frac{2\nu\lambda T_o}{j\pi\mu Eh}\left[\frac{1}{2(\omega_{j0}^1)^2} (1 - \cos\omega_{j0}^1\tau)\right.$$

$$\left.\left.- \frac{2\nu\lambda T_o\gamma}{j\pi\mu Eh}\left[- \frac{1}{\bar{\Omega}^2 - \left(\omega_{j0}^1\right)^2} (\cos\bar{\Omega}\tau - \cos\omega_{j0}^1\tau)\right]\right]\right\}$$

$$+ \frac{c_{12}^{j0}}{\left[(\omega_{j0}^2)^2 - (\omega_{j0}^1)^2\right]}\left\{\frac{j\pi}{4\lambda}\left(\frac{2}{j\pi}\right)^2 \left[(-1)^j - 1\right]\left[\sum_{i=1}^{I} \sum_{\substack{p=1 \\ i\neq p}}^{P} A_i A_p \Omega_i \Omega_p \; \frac{1}{2}\left\{\frac{\cos(\Omega_i+\Omega_p)\tau - \cos\omega_{j0}^2\tau}{(\Omega_i+\Omega_p)^2 - (\omega_{j0}^2)^2}\right.\right.\right.$$

$$
- \frac{\cos(\Omega_i-\Omega_p)\tau \ \cos\omega_{j0}^2\tau}{(\Omega_i-\Omega_p)^2 - (\omega_{j0}^2)^2}\Bigg\} + \sum_{i=p=1}^{I} A_i^2\Omega_i^2 \Bigg\{\frac{1}{2(\omega_{j0}^2)^2}(1-\cos\omega_{j0}^2\tau)
$$

$$
+ \frac{1}{2\left[(2\Omega_i)^2-(\omega_{j0}^2)^2\right]}(\cos 2\Omega_i\tau - \cos\omega_{j0}^2\tau)\Bigg\}\Bigg] + \frac{2\nu\lambda T_o}{j\pi\mu Eh}\left[\frac{1}{2(\omega_{j0}^2)^2}(1-\cos\omega_{j0}^2\tau)\right.
$$

$$
\left. - \frac{2\nu\lambda T_o\gamma}{j\pi\mu Eh}\left[-\frac{1}{\bar{\Omega}^2-(\omega_{j0}^2)^2}(\cos\bar{\Omega}\tau - \cos\omega_{j0}^2\tau)\right]\right]\Bigg\} \tag{9.4-1}
$$

Rewriting equation (9.4-1) we have

$$
U_{j0}(\tau) = \sum_{m=1}^{2}\frac{(-1)^m}{\left(\omega_{j0}^1\right)^2 - \left(\omega_{j0}^2\right)^2}\Bigg(\Bigg\{U_{j0}\left[c_{22}^{j0} - \left(\omega_{j0}^m\right)^2\right] - c_{12}^{j0}W_{j0}\Bigg\}\cos\omega_{j0}^m\tau
$$

$$
+ \left[c_{22}^{j0} - \left(\omega_{j0}^m\right)^2\right]\Bigg[\left(\frac{2}{j\pi}\right)^2\left[(-1)^j-1\right]\sum_{\substack{i=1\\ i\neq p}}^{I}\sum_{p=1}^{P}A_iA_p\frac{\Omega_i\Omega_p}{2}\Bigg\{\frac{\cos(\Omega_i+\Omega_p)\tau-\cos\omega_{j0}\tau}{(\Omega_i+\Omega_p)^2 - (\omega_{j0}^m)^2}
$$

$$
- \frac{\cos(\Omega_i-\Omega_p)\tau-\cos\omega_{j0}^m\tau}{(\Omega_i-\Omega_p)^2 - (\omega_{j0}^m)^2}\Bigg\} + \sum_{i=p=1}^{I}\frac{A_i^2\Omega_i^2}{2}\Bigg\{\frac{1-\cos\omega_{j0}^m\tau}{(\omega_{j0}^m)^2}
$$

$$
+ \frac{\cos 2\Omega_i\tau - \cos\omega_{j0}^m\tau}{(2\Omega_i)^2 - (\omega_{j0}^m)^2}\Bigg\}\Bigg]\Bigg\}
$$

$$
- c_{12}^{j0}\left\{\frac{j\pi}{4\lambda}\left(\frac{2}{j\pi}\right)^2\left[(-1)^j - 1\right]\right.\Bigg[\sum_{\substack{i=1\\ i\neq p}}^{I}\sum_{p=1}^{P}\frac{A_iA_p\Omega_i\Omega_p}{2}\Bigg\{\frac{\cos(\Omega_i+\Omega_p)\tau-\cos\omega_{j0}^m\tau}{(\Omega_i+\Omega_p)^2 - (\omega_{j0}^m)^2}
$$

$$
- \frac{\cos(\Omega_i-\Omega_p)\tau-\cos\omega_{j0}^m\tau}{(\Omega_i-\Omega_p)^2 - (\omega_{j0}^m)^2}\Bigg\} + \sum_{i=p=1}^{I}\frac{A_i^2\Omega_i^2}{2}\Bigg\{\frac{(1-\cos\omega_{j0}^m\tau}{(\omega_{j0}^m)^2} + \frac{\cos 2\Omega_i\tau-\cos\omega_{j0}^m\tau}{(2\Omega_i)^2 - (\omega_{j0}^m)^2}\Bigg\}\Bigg]
$$

$$+\frac{2\nu\lambda T_o}{j\pi\mu Eh}\left\{\frac{1-\cos\omega_{j0}^{m}\tau}{2(\omega_{j0}^{m})^2}+\frac{\gamma(\cos\bar{\Omega}\tau-\cos\omega_{j0}^{m}\tau)}{\bar{\Omega}^2-(\omega_{j0}^{m})^2}\right\}\Bigg]\Bigg)\Bigg) \qquad (9.4\text{-}2)$$

which reduces to

$$U_{j0}(\tau) = \sum_{m=1}^{2}\frac{(-1)^m}{(\omega_{j0}^{1})^2-(\omega_{j0}^{2})^2}\Bigg(\Bigg(\left\{U_{j0}\left[c_{22}^{j0}-(\omega_{j0}^{m})^2\right]-c_{12}^{j0}\right\}\cos\omega_{j0}^{m}\tau$$

$$+\left[c_{22}^{j0}-\frac{j\pi}{4\lambda}c_{12}^{j0}-(\omega_{j0}^{m})^2\right]\Bigg\{\Bigg[\left(\frac{2}{j\pi}\right)^2\left[(-1)^j-1\right]\Bigg[\sum_{i=1}^{I}\sum_{\substack{p=1\\ i\neq p}}^{P}\frac{A_iA_p\Omega_i\Omega_p}{2}$$

$$\cdot\left\{\frac{\cos(\Omega_i+\Omega_p)\tau-\cos\omega_{j0}^{m}\tau}{(\Omega_i+\Omega_p)^2-(\omega_{j0}^{m})^2}-\frac{\cos(\Omega_i-\Omega_p)\tau-\cos\omega_{j0}^{m}\tau}{(\Omega_i-\Omega_p)^2-(\omega_{j0}^{m})^2}\right\}$$

$$+\sum_{i=p=1}^{I}\frac{A_i^{2}\Omega_i^{2}}{2}\left\{\frac{1-\cos\omega_{j0}^{m}\tau}{\omega_{j0}^{m\,2}}+\frac{\cos 2\Omega_i\tau-\cos\omega_{j0}^{m}\tau}{(2\Omega_i)^2-(\omega_{j0}^{m})^2}\right\}\Bigg]\Bigg]\Bigg\}$$

$$-c_{12}^{j0}\frac{2\nu\lambda T_o}{j\pi\mu Eh}\left\{\frac{1-\cos\omega_{j0}^{m}\tau}{2(\omega_{j0}^{m})^2}+\frac{\gamma(\cos\bar{\Omega}\tau-\cos\omega_{j0}^{m}\tau)}{\bar{\Omega}^2-(\omega_{j0}^{m})^2}\right\}\Bigg)$$

(9.4-3)

$i=1,\ 2,\ 3$

$j=1,\ 2,\ \ldots,\ M_1$

$k=0$

$m=1,\ 2$

$p=1,\ 2,\ 3$

From equation (9.2-2), the displacement equation $W_{j0}(\tau)$ is written

$$W_{j0}(\tau) = \sum_{m=1}^{2} \frac{(-1)^m}{(\omega_{j0}^1)^2 - (\omega_{j0}^2)^2} \Bigg(\left\{ w_{j0}\left[c_{11}^{j0} - (\omega_{j0}^m)^2\right] - c_{21}^{j0} u_{j0} \right\} \cos\omega_{j0}^m\tau$$

$$+ \left[c_{11}^{j0} - (\omega_{j0}^m)^2\right] \left\{ \left[\frac{j\pi}{4\lambda}\left(\frac{2}{j\pi}\right)^2 \left[(-1)^j - 1\right] \left[\sum_{i=1}^{I} \sum_{p=1}^{P} \frac{A_i A_p \Omega_i \Omega_p}{2} \right.\right.\right.$$

$$\cdot \left\{ \frac{\cos(\Omega_i+\Omega_p)\tau - \cos\omega_{j0}^m\tau}{(\Omega_i+\Omega_p)^2 - (\omega_{j0}^m)^2} - \frac{\cos(\Omega_i-\Omega_p)\tau - \cos\omega_{j0}^m\tau}{(\Omega_i-\Omega_p)^2 - (\omega_{j0}^m)^2} \right\}$$

$$+ \sum_{i=p=1}^{I} \frac{A_i^2 \Omega_i^2}{2} \left\{ \frac{1-\cos\omega_{j0}^m\tau}{(\omega_{j0}^m)^2} + \frac{\cos 2\Omega_i\tau - \cos\omega_{j0}^m\tau}{(2\Omega_i)^2 - (\omega_{j0}^m)^2} \right\} \Bigg]$$

$$+ \frac{2\nu\lambda}{j\pi\mu} \frac{T_o}{Eh} \left\{ \frac{1-\cos\omega_{j0}^m\tau}{2(\omega_{j0}^m)^2} + \frac{\gamma(\cos\bar{\Omega}\tau - \cos\omega_{j0}^m\tau)}{\bar{\Omega}^2 - (\omega_{j0}^m)^2} \right\} \Bigg] \Bigg\}$$

$$- c_{21}^{j0} \left\{ \left[\left(\frac{2}{j\pi}\right)^2 \left[(-1)^j - 1\right] \left[\sum_{i=1}^{I} \sum_{\substack{p=1 \\ i \neq p}}^{P} \frac{A_i A_p \Omega_i \Omega_p}{2} \right.\right.\right.$$

$$\cdot \left\{ \frac{\cos(\Omega_i+\Omega_p)\tau - \cos\omega_{j0}^m\tau}{(\Omega_i+\Omega_p)^2 - (\omega_{j0}^m)^2} - \frac{\cos(\Omega_i-\Omega_p)\tau - \cos\omega_{j0}^m\tau}{(\Omega_i-\Omega_p)^2 - (\omega_{j0}^m)^2} \right\}$$

$$+ \sum_{i=p=1}^{I} \frac{A_i^2 \Omega_i^2}{2} \left\{ \frac{1-\cos\omega_{j0}^m}{(\omega_{j0}^m)^2} + \frac{\cos 2\Omega_i\tau - \cos\omega_{j0}^m\tau}{(2\Omega_i)^2 - (\omega_{j0}^m)^2} \right\} \Bigg] \Bigg] \Bigg\} \Bigg) \qquad (9.4\text{-}4)$$

which reduces to

$$W_{j0}(\tau) = \sum_{m=1}^{2} \frac{(-1)^m}{(\omega_{j0}^1)^2-(\omega_{j0}^2)^2} \left(\left\{ w_{j0}\left[c_{11}^{j0} - (\omega_{j0}^m)^2\right] - c_{21}^{j0}\, U_{j0} \right\} \cos\omega_{j0}^m \tau \right.$$

$$+ \left\{ \left[c_{11}^{j0} - (\omega_{j0}^m)^2\right] \frac{j\pi}{4\lambda} - c_{12}^{j0} \right\} \left\{ \left(\frac{2}{j\pi}\right)^2 \left[(-1)^j - 1\right] \left[\sum_{\substack{i=1 \\ i \neq p}}^{I} \sum_{p=1}^{P} \frac{A_i A_p \Omega_i \Omega_p}{2} \right. \right.$$

$$\cdot \left\{ \frac{\cos(\Omega_i+\Omega_p)\tau - \cos\omega_{j0}^m \tau}{(\Omega_i+\Omega_p)^2 - (\omega_{j0}^m)^2} - \frac{\cos(\Omega_i-\Omega_p)\tau - \cos\omega_{j0}^m \tau}{(\Omega_i-\Omega_p)^2 - (\omega_{j0}^m)^2} \right\}$$

$$\left. \left. + \sum_{i=p=1}^{I} \frac{A_i^2\, \Omega_i^2}{2} \left\{ \frac{1 - \cos\, \omega_{j0}^m \tau}{(\omega_{j0}^m)^2} + \frac{\cos\, 2\Omega_i \tau - \cos\, \omega_{j0}^m \tau}{(2\Omega_i)^2 - (\omega_{j0}^m)^2} \right\} \right] \right\}$$

$$\left. + \left[c_{11}^{j0} - (\omega_{j0}^m)^2\right] \frac{2\nu\lambda T_o}{j\pi\mu E h} \left\{ \frac{1 - \cos\omega_{j0}^m \tau}{(\omega_{j0}^m)^2} + \frac{\gamma(\cos\bar{\Omega}\tau - \cos\omega_{j0}^m \tau)}{\bar{\Omega}^2 - (\omega_{j0}^m)^2} \right\} \right) \qquad (9.4\text{-}5)$$

$i = 1, 2, 3$

$j = 1, 2, \ldots, M_1$

$k = 0$

$m = 1, 2$

$p = 1, 2, 3$

From equation (9.2-3) the displacement $U_{jk}(\tau)$ becomes

$$U_{jk}(\tau) = \sum_{m=1}^{3} \frac{1}{D_{jk}^m} \left(\left\langle u_{jk} \left\{ \left[c_{22}^{jk} - (\omega_{jk}^m)^2\right] \left[c_{33}^{jk} - (\omega_{jk}^m)^2\right] - (c_{23}^{jk})^2 \right\} \right. \right.$$

$$- v_{jk} \left\{ c_{12}^{jk} \left[c_{33}^{jk} - (\omega_{jk}^m)^2\right] - c_{13}^{jk}\, c_{23}^{jk} \right\}$$

$$\left. - w_{jk} \left\{ c_{13}^{jk} \left[c_{22}^{jk} - (\omega_{jk}^m)^2\right] - c_{12}^{jk}\, c_{23}^{jk} \right\} \right\rangle \cos\omega_{jk}^m \tau$$

$$+ \frac{A_{jk}}{\omega^m_{jk}} \frac{\lambda\delta_{1k}}{2(1+\nu)\mu} \sum_{n=1}^{N} \left(\frac{\lambda n}{2}\right) z_{jn} \int_0^{\tau} q_n(\eta) \sin\omega^m_{jk} (\tau-\eta)\, d\eta$$

$$- \left(B_{jk} + c_{jk} \right) \left\{ \left[\delta_{1k} \left(\frac{2}{j\pi}\right) \sum_{i=1}^{I} A_i \left[\frac{\gamma\, K\, T_o}{2} \left[1 - (-1)^j\right] \right.\right.\right.$$

$$\cdot \left\{ \frac{1}{2\left[(\bar{\Omega}+\Omega_i)^2 - (\omega^m_{jk})^2\right]} \left[\cos(\bar{\Omega}+\Omega_i)\tau - \cos\omega^m_{jk}\tau\right] + \frac{1}{2\ (\bar{\Omega}-\Omega_i)^2 - (\omega^m_{jk})^2} \right.$$

$$\left. \cdot \left[\cos(\bar{\Omega}-\Omega_i)\tau - \cos\omega^m_{jk}\tau \right] \right\}$$

$$- \left\{ \frac{K\ T_o}{2} \left[1 - (-1)^j\right] + \left[1 + (-1)^j\right] \Omega_i^{\,2} \right\}$$

$$\left.\left. \cdot \left\{ \frac{1}{\Omega_i^{\,2} - (\omega^m_{jk})^2} \left[\cos\ \Omega_i \tau - \cos\ \omega^m_{jk}\tau\right] \right\} \right] \right]$$

$$- \delta_{2k} \left[1 - (-1)^j\right] \left[\sum_{i=1}^{I} \sum_{\substack{p=1 \\ i\neq p}}^{P} \frac{A_i A_p \Omega_i \Omega_p}{2} \right.$$

$$\left\{ \frac{\cos(\Omega_i+\Omega_p)\tau - \cos\omega^m_{jk}\tau}{(\Omega_i+\Omega_p)^2 - (\omega^m_{jk})^2} - \frac{\cos(\Omega_i-\Omega_p)\tau - \cos\omega^m_{jk}\tau}{(\Omega_i-\Omega_p)^2 - (\omega^m_{jk})^2} \right\}$$

$$\left. + \sum_{i=p=1}^{I} \frac{A_i^{\,2}\Omega_i^{\,2}}{2} \left\{ \frac{1-\cos\omega^m_{jk}\tau}{(\omega^m_{jk})^2} + \frac{(\cos 2\Omega_i\tau - \cos\omega^m_{jk}\tau)}{(2\Omega_i)^2 - (\omega^m_{jk})^2} \right\} \right]$$

$$\left. - \frac{\delta_{1k}}{\omega^m_{jk}\ 2(1+\nu)\mu} \sum_{n=1}^{N} \left(\frac{\lambda n}{2}\right)^2 x_{jn} \int_0^{\tau} q_n(\eta) \sin\omega^m_{jk} (\tau-\eta)\, d\eta \right\} \right]$$

$$
- \frac{c_{jk}}{\omega_{jk}^{m}} \, \delta. \left\{ \left[\frac{\delta_{1k}}{2(1+\nu)\mu} \frac{1}{} \sum_{n=1}^{N} \left(\frac{\lambda n}{2}\right)^{2} x_{jn} \int_{0}^{\tau} q_n(\eta) \sin\omega_{jk}^{m} (\tau-\eta)\, d\eta \right.\right.
$$

$$
+ \frac{\delta_{1k}}{12(1-\nu^{2})\mu} \left[\sum_{n=1}^{N} \left\{ s_{jn} \left[\left(\frac{\lambda n}{2}\right)^{4} + \lambda^{4} \right] - 2\lambda^{2} \left(\frac{\lambda n}{2}\right)^{2} x_{jn} \right\} \right.
$$

$$
\left.\left.\left. \cdot \int_{0}^{\tau} q_n(\eta) \sin\omega_{jk}^{m} (\tau-\eta)\, d\eta + \lambda^{4} m_{j0} \int_{0}^{\tau} q_B(\eta) \sin\omega_{jk}^{m} (\tau-\eta)\, d\eta \right] \right\} \right)
$$

(9.4-6)

$i = 1, 2, 3$

$j = 1, 2, \ldots, M_1$

$k = 1, 2, \ldots, N_1$

$m = 1, 2, 3$

$n = 1, 2, \ldots, N$

$p = 1, 2, 3$

Evaluating the remaining integrals of equation (9.4-6) we can now write

$$
U_{jk}(\tau) = \sum_{m=1}^{3} \frac{1}{D_{jk}^{m}} \left(\left\{ A_{jk} u_{jk} - B_{jk} v_{jk} - c_{jk} w_{jk} \right\} \cos\omega_{jk}^{m} \tau \right.
$$

$$
+ \frac{A_{jk} \, \lambda \, \delta_{1k}}{2(1+\nu)\mu \, \omega_{jk}^{m}} \sum_{s=-S}^{+S} \sum_{n=1}^{N} \lambda \left(\frac{n}{2}\right) z_{jn} \frac{c_n^{(s)}}{\left[(\alpha_r+s)\bar{\Omega}\right]^{2} - (\omega_{jk}^{m})^{2}}
$$

$$
\left\{ \cos\omega_{jk}^{m} \tau + \frac{i(\alpha_r+s)}{\omega_{jk}^{m}} \bar{\Omega} \sin\omega_{jk}^{m} \tau - e^{i(\alpha_r+s)\bar{\Omega}\tau} \right\}
$$

$$- (B_{jk}+C_{jk})\left\{\left[\delta_{1k}\left(\frac{2}{j\pi}\right)\sum_{i=1}^{I} A_i\left[\frac{\gamma K\ T_o}{2}\left[1-(-1)^j\right]\left\{\frac{\cos(\bar{\Omega}+\Omega_i)\tau-\cos\omega_{jk}^{m}\tau}{2\left[(\bar{\Omega}+\Omega_i)^2-(\omega_{jk}^{m})^2\right]}\right.\right.\right.\right.$$

$$\left. + \frac{\cos(\bar{\Omega}-\Omega_i)\tau-\cos\omega_{jk}^{m}\tau}{2\left[(\bar{\Omega}-\Omega_i)^2-(\omega_{jk}^{m})^2\right]}\right\}$$

$$\left. - \left\{\frac{K\ T_o}{2}\left[1-(-1)^j\right]+\left[1+(-1)^j\right]\Omega_i^{\,2}\right\}\frac{\cos\Omega_i\tau-\cos\omega_{jk}^{m}\tau}{\Omega_i^{\,2}-(\omega_{jk}^{m})^2}\right]$$

$$- \delta_{2k}\left[1-(-1)^j\right]\left[\sum_{i=1}^{I}\sum_{\substack{p=1\\ i\neq p}}^{P}\frac{A_iA_p\ \Omega_i\Omega_p}{2}\right.$$

$$\cdot\left\{\frac{\cos(\Omega_i+\Omega_p)\tau-\cos\omega_{jk}^{m}\tau}{(\Omega_i+\Omega_p)^2-(\omega_{jk}^{m})^2}-\frac{\cos(\Omega_i-\Omega_p)\tau-\cos\omega_{jk}^{m}\tau}{(\Omega_i-\Omega_p)^2-(\omega_{jk}^{m})^2}\right\}$$

$$+ \sum_{i=p=1}^{I}\frac{A_i^{\,2}\Omega_i^{\,2}}{2}\left\{\frac{1-\cos\omega_{jk}^{m}\tau}{(\omega_{jk}^{m})^2}+\frac{\cos 2\Omega_i\tau-\cos\omega_{jk}^{m}\tau}{(2\Omega_i)^2-(\omega_{jk}^{m})^2}\right\}$$

$$- \frac{\delta_{1k}}{2(1+\nu)\mu}\sum_{s=-S}^{+S}\sum_{n=1}^{N}\left(\frac{\lambda n}{2}\right)^2 x_{jn}\frac{C_n^{(s)}}{\left[(\alpha_r+s)\bar{\Omega}\right]^2-(\omega_{jk}^{m})^2}$$

$$\left.\left.\left\{\cos\omega_{jk}^{m}\tau+\frac{i(\alpha_r+s)}{\omega_{jk}^{m}}\bar{\Omega}\sin\omega_{jk}^{m}\tau-e^{i(\alpha_r+s)\bar{\Omega}\tau}\right\}\right\}\right]$$

$$- c_{jk}\ \delta_{1k}\left\{\left[\frac{1}{2(1+\nu)\mu}\sum_{s=-S}^{+S}\sum_{n=1}^{N}\left(\frac{\lambda n}{2}\right)^2 x_{jn}\frac{c_n^{(s)}}{\left[(\alpha_r+s)\bar{\Omega}\right]^2-(\omega_{jk}^{m})^2}\right.\right.$$

$$\cdot \left\{ \cos\omega_{jk}^{m}\tau + i\,\frac{(\alpha_r+s)}{\omega_{jk}^{m}}\,\bar{\Omega}\sin\omega_{jk}^{m}\tau - e^{i(\alpha_r+s)\bar{\Omega}\tau} \right\}$$

$$+ \frac{1}{12(1-\nu^2)\mu}\left[\sum_{s=-S}^{+S} \sum_{n=1}^{N} \left\{ \varepsilon_{jn}\left[\left(\frac{\lambda n}{2}\right)^4 + \lambda^4\right] - 2\lambda^2 \left(\frac{\lambda n}{2}\right)^2 x_{jn} \right\} \right.$$

$$\frac{C_n^{(s)}}{\left[(\alpha_r+s)\bar{\Omega}\right]^2-(\omega_{jk}^{m})^2}\left\{ \cos\omega_{jk}^{m}\tau + i\,\frac{(\alpha_r+s)}{\omega_{jk}^{m}}\,\bar{\Omega}\sin\omega_{jk}^{m}\tau - \omega_{jk}^{m}\, e^{i(\alpha_r+s)\bar{\Omega}\tau} \right\}$$

$$+ \lambda^4 m_{j0} \sum_{s=-S}^{+S} \frac{C_B^{(s)}}{\left[(\alpha_r+s)\bar{\Omega}\right]^2-(\omega_{jk}^{m})^2}$$

$$\left. \cdot \left\{ \cos\omega_{jk}^{m}\tau + i\,\frac{(\alpha_r+s)}{\omega_{jk}^{m}}\,\bar{\Omega}\sin\omega_{jk}^{m}\tau - e^{i(\alpha_r+s)\bar{\Omega}\tau} \right\} \right] \Bigg) \qquad (9.4\text{-}7)$$

$i = 1, 2, 3$

$j = 1, 2, \ldots, M_1$

$k = 1, 2, \ldots, N_1$

$m = 1, 2, 3$

$n = 1, 2, \ldots, N$

$p = 1, 2, 3$

$r = 1, 2, \ldots, R$

$s = -S$ to $+S$

From equation (9.2-4) the displacement $V_{jk}(\tau)$ becomes

$$V_{jk}(\tau) = \sum_{m=1}^{3} \frac{1}{D_{jk}^{m}} \Bigg(\left\{ - u_{jk} D_{jk} + v_{jk} E_{jk} + w_{jk} F_{jk} \right\} \cos\omega_{jk}^{m}\tau$$

$$- \frac{D_{jk}}{\omega_{jk}^{m}} \frac{\lambda\ \delta_{1k}}{2(1+\nu)\mu} \sum_{n=1}^{N} \left(\frac{\lambda n}{2}\right) z_{jn} \int_0^{\tau} q_n(\eta) \sin\omega_{jk}^{m} (\tau-\eta)\, d\eta$$

$$+ \left(E_{jk} - F_{jk}\right) \left\{ \left[\delta_{1k} \left(\frac{2}{j\pi}\right) \sum_{i=1}^{I} A_i \left[\frac{\gamma K\ T_o}{2} \left[1 - (-1)^j\right] \right.\right.\right.$$

$$\cdot \left\{ \frac{1}{2\left[(\bar{\Omega}+\Omega_i)^2 - (\omega_{jk}^{m})^2\right]} \left[\cos(\bar{\Omega}+\Omega_i)\tau - \cos\omega_{jk}^{m}\tau\right] \right.$$

$$\left. + \frac{\omega_{jk}^{m}}{2\left[(\bar{\Omega}-\Omega_i)^2 - (\omega_{jk}^{m})^2\right]} \left[\cos(\bar{\Omega}-\Omega_i)\tau - \cos\omega_{jk}^{m}\tau\right] \right.$$

$$- \left\{ \frac{K\ T_o}{2} \left[1 - (-1)^j\right] + \left[1 + (-1)^j\right] \Omega_i^{\,2} \right\}$$

$$\cdot \left\{ \frac{1}{\Omega_i^{\,2} - (\omega_{jk}^{m})^2} \left[\cos\ \Omega_i\tau - \cos\omega_{jk}^{m}\tau\right] \right.$$

$$- \delta_{2k} \left[1 - (-1)^j\right] \left[\sum_{i=1}^{I} \sum_{\substack{p=1 \\ i\neq p}}^{P} \frac{A_i A_p\ \Omega_i \Omega_p}{2} \right.$$

$$\cdot \left\{ \frac{\cos(\Omega_i+\Omega_p)\tau - \cos\omega_{jk}^{m}\tau}{(\Omega_i+\Omega_p)^2 - (\omega_{jk}^{m})^2} - \frac{\cos(\Omega_i-\Omega_p)\tau - \cos\omega_{jk}^{m}\tau}{(\Omega_i-\Omega_p)^2 - (\omega_{jk}^{m})^2} \right\}$$

$$\left. + \sum_{i=p=1}^{I} \frac{A_i^{\,2}\ \Omega_i^{\,2}}{2} \left\{ \frac{1 - \cos\omega_{jk}^{m}\tau}{(\omega_{jk}^{m})^2} + \frac{\cos\ 2\Omega_i\tau - \cos\omega_{jk}^{m}\tau}{(2\Omega_i)^2 - (\omega_{jk}^{m})^2} \right\} \right]$$

$$\left. - \frac{\delta_{1k}}{\omega_{jk}^{m}\ 2(1+\nu)\mu} \sum_{n=1}^{N} \left(\frac{\lambda n}{2}\right)^2 x_{jn} \int_0^{\tau} q_n(\eta) \sin\omega_{jk}^{m} (\tau-\eta)\, d\eta \right\} \right]$$

$$- \frac{F_{jk}}{\omega_{jk}^{m}} \left\{ \delta_{1k} \frac{1}{2(1+\nu)\mu} \sum_{n=1}^{N} \left(\frac{\lambda n}{2}\right)^2 x_{jn} \int_0^{\tau} q_n(\eta) \sin \omega_{jk}^{m} (\tau-\eta)\, d\eta \right.$$

$$+ \frac{1}{12(1-\nu^2)\mu} \left[\sum_{n=1}^{N} \left\{ s_{jn} \left[\left(\frac{\lambda n}{2}\right)^4 + \lambda^4 \right] - 2\lambda^2 \left(\frac{\lambda n}{2}\right)^2 x_{jn} \right\} \right.$$

$$\left. \left. \cdot \int_0^{\tau} q_n(\eta) \sin\omega_{jk}^{m}(\tau-\eta)\, d\eta + \lambda^4 m_{j0} \int_0^{\tau} q_B(\eta) \sin\omega_{jk}^{m} (\tau-\eta)\, d\eta \right\} \right] \Bigg) \qquad (9.4\text{-}8)$$

Evaluating the remaining integrals of equation (9.4-8) we have

$$V_{jk}(\tau) = \sum_{m=1}^{3} \frac{1}{D_{jk}^{m}} \Bigg(\left\{ - D_{jk} u_{jk} + E_{jk} v_{jk} - F_{jk} w_{jk} \right\} \cos \omega_{jk}^{m} \tau$$

$$- \frac{D_{jk}\, \lambda\, \delta_{1k}}{2(1+\nu)\mu\, \omega_{jk}^{m}} \sum_{s=-S}^{+S} \sum_{n=1}^{N} \left(\frac{\lambda n}{2}\right) z_{jn} \frac{C_n^{(s)}}{\left[(\alpha_r+s)\bar{\Omega}\right]^2 - (\omega_{jk}^{m})^2}$$

$$\cdot \left\{ \omega_{jk}^{m} \cos\omega_{jk}^{m}\tau + i(\alpha_r+s)\bar{\Omega} \sin\omega_{jk}^{m}\tau - \omega_{jk}^{m} e^{i(\alpha_r+s)\bar{\Omega}\tau} \right\}$$

$$+ (E_{jk}-F_{jk}) \left\{ \delta_{1k} \left(\frac{2}{j\pi}\right) \sum_{i=1}^{I} A_i \left[\frac{\gamma K T_o}{2} \left[1-(-1)^j\right] \left\{ \frac{\cos(\bar{\Omega}+\Omega_i)\tau - \cos\omega_{jk}^{m}\tau}{2\left[(\bar{\Omega}+\Omega_i)^2 - (\omega_{jk}^{m})^2\right]} \right. \right. \right.$$

$$\left. + \frac{\cos(\bar{\Omega}-\Omega_i)\tau - \cos\omega_{jk}^{m}\tau}{2\left[(\bar{\Omega}-\Omega_i)^2 - (\omega_{jk}^{m})^2\right]} \right\}$$

$$\left. - \left\{ \frac{K\, T_o}{2} \left[1 - (-1)^j\right] + \left[1 + (-1)^j\right] \Omega_i^2 \right\} \frac{\cos\Omega_i \tau - \cos\omega_{jk}^{m}\tau}{\Omega_i^2 - (\omega_{jk}^{m})^2} \right]$$

$$- \delta_{2k} \left[1 - (-1)^j\right] \left[\sum_{\substack{i=1 \\ i \neq p}}^{I} \sum_{p=1}^{P} \frac{A_i A_p \Omega_i \Omega_p}{2} \right.$$

$$\cdot \left\{ \frac{\cos(\Omega_i + \Omega_p)\tau - \cos\omega_{jk}^m \tau}{(\Omega_i + \Omega_p)^2 - (\omega_{jk}^m)^2} - \frac{\cos(\Omega_i + \Omega_p)\tau - \cos\omega_{jk}^m \tau}{(\Omega_i - \Omega_p)^2 - (\omega_{jk}^m)^2} \right\}$$

$$\left. + \sum_{i=p=1}^{I} \frac{A_i^2 \Omega_i^2}{2} \left\{ \frac{1 - \cos\omega_{jk}^m \tau}{(\omega_{jk}^m)} + \frac{\cos 2\Omega_i \tau - \cos\omega_{jk}^m \tau}{(2\Omega_i)^2 - (\omega_{jk}^m)^2} \right\} \right]$$

$$- \frac{\delta_{1k}}{2(1+\nu)\mu} \sum_{s=-S}^{+S} \sum_{n=1}^{N} \left(\frac{\lambda n}{2}\right)^2 x_{jn} \frac{C_n^{(s)}}{\left[(\alpha_r + s)\bar{\Omega}^2 - (\omega_{jk}^m)^2\right]}$$

$$\left. \cdot \left\{ \cos\omega_{jk}^m \tau + i \frac{(\alpha_r + s)}{\omega_{jk}^m} \bar{\Omega} \sin\omega_{jk}^m \tau - e^{i(\alpha_r + s)\bar{\Omega}\tau} \right\} \right]$$

$$- F_{jk} \, \delta_{1k} \left\{ \left[\frac{1}{2(1+\nu)\mu} \sum_{s=-S}^{+S} \sum_{n=1}^{N} \left(\frac{\lambda n}{2}\right)^2 x_{jn} \frac{C_n^{(s)}}{\left[(\alpha_r + s)\bar{\Omega}\right]^2 - (\omega_{jk}^m)^2} \right. \right.$$

$$\cdot \left\{ \cos\omega_{jk}^m \tau + i \frac{(\alpha_r + s)}{\omega_{jk}^m} \bar{\Omega} \sin\omega_{jk}^m \tau - e^{i(\alpha_r + s)\bar{\Omega}\tau} \right\}$$

$$+ \frac{1}{12(1-\nu^2)\mu} \left[\sum_{s=-S}^{+S} \sum_{n=1}^{N} s_{jn} \left[\left(\frac{\lambda n}{2}\right)^4 + \lambda^4 \right] - 2\lambda^2 \left(\frac{\lambda n}{2}\right)^2 x_{jn} \right\}$$

$$\frac{C_n^{(s)}}{\left[(\alpha_r + s)\bar{\Omega}\right]^2 - (\omega_{jk}^m)^2} \left\{ \cos\omega_{jk}^m \tau + i \frac{(\alpha_r + s)}{\omega_{jk}^m} \bar{\Omega} \sin\omega_{jk}^m \tau - e^{i(\alpha_r + s)\bar{\Omega}\tau} \right\}$$

$$+ \lambda^4 m_{j0} \sum_{s=-S}^{+S} \frac{C_B^{(s)}}{\left[(\alpha+s)\bar{\Omega}\right]^2 - (\omega_{jk}^m)^2}$$

$$\cdot \left\{ \cos\omega_{jk}^m \tau + i \frac{(\alpha_r + s)}{{}_{jk}^{m}} \bar{\Omega} \sin\omega_{jk}^m \tau - e^{i(\alpha_r+s)\bar{\Omega}\tau} \right\} \Bigg] \Bigg\} \Bigg) \qquad (9.4\text{-}9)$$

$i=1, 2, 3$

$j = 1, 2, \ldots, M_1$

$k = 1, 2, \ldots, N_1$

$m = 1, 2, 3$

$n = 1, 2, \ldots, N$

$p = 1, 2, 3$

$r = 1, 2, \ldots, R$

$s = -S$ to $+S$

From equation (9.2-5) the displacement $W_{jk}(\tau)$ becomes

$$W_{jk}(\tau) = \sum_{m=1}^{3} \frac{1}{D_{jk}^m} \left(\left\{ - u_{jk} G_{jk} - v_{jk} H_{jk} + w_{jk} I_{jk} \right\} \cos\omega_{jk}^m \tau \right.$$

$$- \frac{G_{jk}}{\omega_{jk}^m} \frac{\lambda \delta_{1k}}{2(1+\nu)\mu} \sum_{n=1}^{N} \left(\frac{\lambda n}{2}\right) z_{jn} \int_0^\tau q_n(\eta) \sin\omega_{jk}^m (\tau - \eta)\, d\eta$$

$$- \left(H_{jk} - I_{jk} \right) \left\{ \left[\delta_{1k} \left(\frac{2}{j\pi}\right) \sum_{i=1}^{I} A_i \left[\frac{\gamma K T_o}{2} \left[1 - (-1)^j \right] \right. \right. \right.$$

$$\cdot \left\{ \frac{1}{2\left[(\bar{\Omega}+\Omega_i)^2 - (\omega_{jk}^m)^2 \right]} \left[\cos(\bar{\Omega}+\Omega_i)\tau - \cos\omega_{jk}^m \tau \right] \right.$$

$$+ \frac{1}{2\left[(\bar{\Omega}-\Omega_i)^2-(\omega_{jk}^m)^2\right]} \left[\cos(\bar{\Omega}-\Omega_i)\tau - \cos\omega_{jk}^m\tau\right]$$

$$- \left\{\frac{K\ T_o}{2} \left[1 - (-1)^j\right] + \left[1 + (-1)^j\right] \Omega_i^{\ 2}\right\}$$

$$\cdot\left\{\frac{1}{\Omega_i^{\ 2}-(\omega_{jk}^m)^2} \left[\cos\Omega_i\tau - \cos\omega_{jk}^m\tau\right]\right\}\Bigg]$$

$$- \delta_{2k}\left[1 - (-1)^j\right]\Bigg[\sum_{i=1}^{I}\sum_{\substack{p=1\\ i\neq p}}^{P} \frac{A_i A_p\ \Omega_i\Omega_p}{2}$$

$$\cdot\left\{\frac{\cos(\Omega_i+\Omega_p)\tau-\cos\omega_{jk}^m\tau}{(\Omega_i+\Omega_p)^2 - (\omega_{jk}^m)^2} - \frac{\cos(\Omega_i-\Omega_p)\tau-\cos\omega_{jk}^m\tau}{(\Omega_i-\Omega_p)^2 - (\omega_{jk}^m)^2}\right\}$$

$$+ \sum_{i=p=1}^{I} \frac{A_i^{\ 2}\Omega_i^{\ 2}}{2} \left\{\frac{1 - \cos\omega_{jk}^m\tau}{(\omega_{jk}^m)^2} + \frac{\cos\ 2\Omega_i\tau-\cos\omega_{jk}^m\tau}{(2\Omega_i)^2 - (\omega_{jk}^m)^2}\right\}\Bigg]$$

$$- \frac{\delta_{1k}}{\omega_{jk}^m\ 2(1+\nu)\mu} \sum_{n=1}^{N} \left(\frac{\lambda n}{2}\right)^2 x_{jn} \int_0^\tau q_n(\eta)\ \sin\omega_{jk}^m(\tau-\eta)\ d\eta\Bigg\}\Bigg]$$

$$+ \frac{I_{jk}}{\omega_{jk}^m}\ \delta_{1k}\Bigg\{\Bigg[\frac{1}{2(1+\nu)\mu}\sum_{n=1}^{N}\left(\frac{\lambda n}{2}\right)^2 x_{jn} \int_0^\tau q_n(\eta)\ \sin\omega_{jk}^m(\tau-\eta)\ d\eta$$

$$+ \frac{1}{12(1-\nu^2)\mu}\left[\sum_{n=1}^{N}\left\{s_{jn}\left[\left(\frac{\lambda n}{2}\right)^4 + \lambda^4\right] - 2\lambda^2\left(\frac{\lambda n}{2}\right)^2 x_{jn}\right\}\right.$$

$$\cdot \int_0^{\tau} q_n(\eta) \sin\omega_{jk}^m (\tau-\eta)\, d\eta + \lambda^4 m_{j0} \int_0^{\tau} q_B(\eta) \sin\omega_{jk}^m (\tau-\eta)\, d\eta \Big]\Big\}\Big) \tag{9.4-10}$$

Evaluating the remaining integrals of equation (9.4-10) we have

$$W_{jk}(\tau) = \sum_{m=1}^{3} \frac{1}{D_{jk}^m} \Bigg(\Big\{ - G_{jk} u_{jk} - H_{jk} v_{jk} + I_{jk} w_{jk} \Big\} \cos\omega_{jk}^m \tau$$

$$- \frac{G_{jk} \lambda\, \delta_{1k}}{2(1+\nu)\mu} \sum_{s=-S}^{+S} \sum_{n=1}^{N} \left(\frac{\lambda n}{2}\right) z_{jn} \frac{C_n^{(s)}}{\left[(\alpha_r+s)\bar{\Omega}\right]^2 - (\omega_{jk}^m)^2}$$

$$\cdot \left\{ \cos\omega_{jk}^m \tau + i \frac{(\alpha_r+s)}{\omega_{jk}^m} \bar{\Omega} \sin\omega_{jk}^m \tau - e^{i(\alpha_r+s)\bar{\Omega}\tau} \right\}$$

$$- (H_{jk} - I_{jk}) \left\{ \left[\delta_{1k} \left(\frac{2}{j\pi}\right) \sum_{i=1}^{I} A_i \left[\frac{\gamma K T_o}{2} \left[1 - (-1)^j\right] \left\{ \frac{\cos(\bar{\Omega}+\Omega_i)\tau - \cos\omega_{jk}^m \tau}{2\left[(\bar{\Omega}+\Omega_i)^2 - (\omega_{jk}^m)^2\right]} \right. \right. \right. \right.$$

$$\left. + \frac{\cos(\bar{\Omega}-\Omega_i)\tau - \cos\omega_{jk}^m \tau}{2\left[(\bar{\Omega}-\Omega_i)^2 - (\omega_{jk}^m)^2\right]} \right\}$$

$$- \left\{ \frac{K T_o}{2} \left[1 - (-1)^j\right] + \left[1 + (-1)^j\right] \Omega_i^2 \right\} \frac{\cos\Omega_i \tau - \cos\omega_{jk}^m \tau}{\Omega_i^2 - (\omega_{jk}^m)^2} \Bigg]$$

$$- \delta_{2k} \left[1 - (-1)^j\right] \Bigg[\sum_{i=1}^{I} \sum_{\substack{p=1 \\ i \neq p}}^{P} \frac{A_i A_p \Omega_i \Omega_p}{2}$$

$$\cdot \left\{ \frac{\cos(\Omega_i+\Omega_p)\tau - \cos\omega_{jk}^{m}\tau}{(\Omega_i+\Omega_p)^2 - (\omega_{jk}^{m})^2} - \frac{\cos(\Omega_i-\Omega_p)\tau - \cos\omega_{jk}^{m}\tau}{(\Omega_i-\Omega_p)^2 - (\omega_{jk}^{m})^2} \right\}$$

$$+ \sum_{i=p=1}^{I} \frac{A_i^{\,2}\Omega_i^{\,2}}{2} \left\{ \frac{1-\cos\omega_{jk}^{m}\tau}{(\omega_{jk}^{m})^2} + \frac{\cos 2\Omega_i\tau - \cos\omega_{jk}^{m}\tau}{(2\Omega_i)^2 - (\omega_{jk}^{m})^2} \right\}$$

$$- \frac{\delta_{1k}}{2(1+\nu)\mu} \sum_{s=-S}^{+S} \sum_{n=1}^{N} \left(\frac{\lambda n}{2}\right)^2 x_{jn} \frac{C_n^{(s)}}{\left[(\alpha_r+s)\bar{\Omega}\right]^2 - (\omega_{jk}^{m})^2}$$

$$\cdot \left\{ \cos\omega_{jk}^{m}\tau + \frac{i(\alpha_r+s)}{\omega_{jk}^{m}}\bar{\Omega}\sin\omega_{jk}^{m}\tau - e^{i(\alpha_r+s)\bar{\Omega}\tau} \right\} \Bigg] \Bigg\}$$

$$+ I_{jk}\,\delta_{1k} \left\{ \left[\frac{i}{2(1+\nu)\mu} \sum_{s=-S}^{+S} \sum_{n=1}^{N} \frac{\lambda n}{2}^2 x_{jn} \frac{C_n^{(s)}}{\left[(\alpha+s)\bar{\Omega}\right]^2 - (\omega_{jk}^{m})^2} \right.\right.$$

$$\cdot \left\{ \cos\omega_{jk}^{m}\tau + \frac{i(\alpha_r+s)}{\omega_{jk}^{m}}\bar{\Omega}\sin\omega_{jk}^{m}\tau - e^{i(\alpha_r+s)\bar{\Omega}\tau} \right\}$$

$$+ \frac{1}{12(1-\nu^2)\mu} \left[\sum_{s=-S}^{+S} \sum_{n=1}^{N} \left\{ s_{jn} \left[\left(\frac{\lambda n}{2}\right)^4 + \lambda^4 \right] - 2\lambda^2 \left(\frac{\lambda n}{2}\right)^2 x_{jn} \right\} \right.$$

$$\cdot \frac{C_n^{(s)}}{(\alpha_r+s)\bar{\Omega}^{\,2} - (\omega_{jk}^{m})^2} \left\{ \cos\omega_{jk}^{m}\tau + \frac{i(\alpha_r+s)}{\omega_{jk}^{m}}\bar{\Omega}\sin\omega_{jk}^{m}\tau - e^{i(\alpha_r+s)\bar{\Omega}\tau} \right\}$$

$$+ \lambda^4 m_{j0} \sum_{s=-S}^{+S} \frac{C_B^{(s)}}{\left[(\alpha_r+s)\Omega\right]^2 - (\omega_{jk}^{m})^2}$$

$$\cdot \left\{ \cos \omega_{jk}^{m} \tau + \frac{i(\alpha_r + s)}{\omega_{jk}^{m}} \bar{\Omega} \sin \omega_{jk}^{m} \tau - e^{i(\alpha_r + s)\bar{\Omega}\tau} \right\} \Bigg] \Bigg\} \Bigg) \qquad (9.4\text{-}11)$$

$i = 1, 2, 3$

$j = 1, 2, \ldots, M$

$k = 1, 2, \ldots, N_1$

$m = 1, 2, 3$

$n = 1, 2, \ldots N$

$p = 1, 2, 3$

$r = 1, 2, \ldots, R$

$s = -S$ to $+S$

10.0 STABILITY ANALYSIS

10.1 Inspection of the Displacement Equations

Inspection of the displacement equations shows that certain denominator frequency factors, if equated to zero, would render the displacements unbounded. The solutions of these factors will produce the unstable values of the thrust frequency, Ω.

10.2 Stability Equations

From $U_{0k}(\tau)$

$$\Omega_i = \omega_{0k} \qquad i = 1, 2, 3 \tag{10.2-1}$$

where as defined earlier (Matheau equation consideration)

$$\Omega_1 = (1 - \frac{\beta}{2})\frac{\Omega}{\omega_1} = (2 - \beta)\frac{\Omega}{2\omega_1} \qquad \text{(a)}$$

$$\Omega_2 = \frac{\beta\Omega}{2\omega_1} \qquad \text{(b)} \qquad (10.2\text{-}2)$$

$$\Omega_3 = (1 + \frac{\beta}{2})\frac{\Omega}{\omega_1} = (2 + \beta)\frac{\Omega}{2\omega_1} \qquad \text{(c)}$$

If we define

$$\varepsilon_i = \begin{cases} -\frac{1}{2} & \text{if } i = 1 \\ \frac{1}{2} & \text{if } i = 2 \\ \frac{1}{2} & \text{if } i = 3 \end{cases} \tag{10.2-3}$$

the eqs. (10.2-2) can be expressed as

$$\Omega_i = \left[(i-2)^2 + \varepsilon_i \beta\right] \frac{\Omega}{\omega_1} \tag{10.2-4}$$

Substituting for values of i the resulting equations

$$i = 1, \quad \Omega_1 = (1 - \beta/2)\frac{\Omega}{\omega_1} \qquad \text{(a)}$$

$$i = 2, \quad \Omega_2 = \frac{\beta\Omega}{2\omega_1} \qquad \text{(b)} \qquad (10.2\text{-}5)$$

$$i = 3, \quad \Omega_3 = (1 + \beta/2)\frac{\Omega}{\omega_1} \qquad \text{(c)}$$

check with equations (10.2-2). Using β and ε_i, we can now present the stability equations with a brief heading to indicate their source.

From $U_{0k}(\tau)$

$$\Omega_i = \left[(i-2)^2 + \varepsilon_i \beta\right] \frac{\Omega}{\omega_1} = \omega_{0k} \tag{10.2-6}$$

$i = 1, 2, 3$

$k = 1, 2, \ldots, N_1$

From $U_{0k}(\tau)$

$$\left[(\alpha_r + s)\bar{\Omega}\right]^2 - (\omega_{0k})^2 = 0 \tag{10.2-7}$$

$$\bar{\Omega} = \frac{\omega_{0k}}{|(\alpha_r + s)|} = \frac{\Omega}{\omega_1} \tag{10.2-8}$$

$k = 1, 2, \ldots, N_1$

$r = 1, 2, \ldots, R$

$s = -S$ to $+S$

From $U_{j0}(\tau)$ and $W_{j0}(\tau)$

$$\bar{\Omega}^2 - (\omega^m_{j0})^2 = 0 \tag{10.2-9}$$

$$\bar{\Omega} = \frac{\Omega}{\omega_1} = \omega^m_{j0} \tag{10.2-10}$$

$j = 1, 2, \ldots, M_1$

$m = 1, 2$

From $U_{j0}(\tau)$ and $W_{j0}(\tau)$

$$(2\Omega_i)^2 - (\omega^m_{j0})^2 = 0 \tag{10.2-11}$$

$$2\Omega_i = \omega^m_{j0} \tag{10.2-12}$$

$$2\left[(i-2)^2 + \varepsilon_i\beta\right]\frac{\Omega}{\omega_1} = \omega^m_{j0} \tag{10.2-13}$$

$i = 1, 2, 3$

$j = 1, 2, \ldots, M_1$

$m = 1, 2$

From $U_{j0}(\tau)$ and $W_{j0}(\tau)$

$$(\Omega_i + \Omega_p)^2 - (\omega^m_{j0})^2 = 0 \qquad (10.2\text{-}13)$$

$$\Omega_i + \Omega_p = \omega^m_{j0} \tag{10.2-14}$$

$$\left|(i - 2)^2 + (p - 2)^2 + (\varepsilon_i + \varepsilon_p)\beta\right|\frac{\Omega}{\omega_1} = \omega^m_{j0} \tag{10.2-15}$$

$i = 1, 2, 3$

$j = 1, 2, \ldots, M_1$

$m = 1, 2$

$p = 1, 2, 3$

From $U_{j0}(\tau)$ and $W_{j0}(\tau)$

$$(\Omega_i - \Omega_p)^2 - (\omega_{j0}^m)^2 = 0 \qquad (10.2\text{-}16)$$

$$(\Omega_i - \Omega_p) = \omega_{j0}^m \qquad (10.2\text{-}17)$$

$$\left|(i - 2)^2 - (p - 2)^2 + (\varepsilon_i - \varepsilon_p)\beta\right| \frac{\Omega}{\omega_1} = \omega_{j0}^m \qquad (10.2\text{-}18)$$

$i = 1, 2, 3$

$j = 1, 2, \ldots, M_1$

$m = 1, 2$

$p = 1, 2, 3$

From $U_{jk}(\tau)$, $V_{jk}(\tau)$, and $W_{jk}(\tau)$

$$\Omega_i^2 - (\omega_{jk}^m)^2 = 0 \qquad (10.2\text{-}19)$$

$$\Omega_i = \omega_{jk}^m \qquad (10.2\text{-}20)$$

$$\left[(i - 2)^2 + \varepsilon_i\beta\right] \frac{\Omega}{\omega_1} = \omega_{jk}^m \qquad (10.2\text{-}21)$$

$i = 1, 2, 3$

$j = 1, 2, \ldots, M_1$

$k = 1, 2, \ldots, N_1$

$m = 1, 2, 3$

From $U_{jk}(\tau)$, $V_{jk}(\tau)$, and $W_{jk}(\tau)$

$$(\bar{\Omega} + \Omega_i)^2 - (\omega_{jk}^m)^2 = 0 \qquad (10.2\text{-}22)$$

$$(\bar{\Omega} + \Omega_i) = \omega_{jk}^{m} \tag{10.2-23}$$

$$\left|(i - 2)^2 + \varepsilon_i\beta + 1\right| \frac{\Omega}{\omega_1} = \omega_{jk}^{m} \tag{10.2-24}$$

$i = 1, 2, 3$

$j = 1, 2, \ldots, M_1$

$k = 1, 2, \ldots, N_1$

$m = 1, 2, 3$

From $U_{jk}(\tau)$, $V_{jk}(\tau)$, and $W_{jk}(\tau)$

$$(\bar{\Omega} - \Omega_i)^2 - (\omega_{jk}^{m})^2 = 0 \tag{10.2-25}$$

$$(\bar{\Omega} - \Omega_i) = \omega_{jk}^{m} \tag{10.2-26}$$

$$\left|(i - 2)^2 + \varepsilon_i\beta - 1\right| \frac{\Omega}{\omega_1} = \omega_{jk}^{m} \tag{10.2-27}$$

$i = 1, 2, 3$

$j = 1, 2, \ldots, M_1$

$k = 1, 2, \ldots, N_1$

$m = 1, 2, 3$

From $U_{jk}(\tau)$, $V_{jk}(\tau)$, and $W_{jk}(\tau)$

$$(\Omega_i + \Omega_p)^2 - (\omega_{jk}^{m})^2 = 0 \tag{10.2-28}$$

$$(\Omega_i + \Omega_p) = \omega_{jk}^{m} \tag{10.2-29}$$

$$\left|(i - 2)^2 + (p - 2)^2 + (\varepsilon_i + \varepsilon_p)\beta\right| \frac{\Omega}{\omega_1} = \omega_{jk}^{m} \tag{10.2-30}$$

$i = 1, 2, 3$

$j = 1, 2, \ldots, M_1$

$k = 1, 2, \ldots, N_1$

$m = 1, 2, 3$

$p = 1, 2, 3$

From $U_{jk}(\tau)$, $V_{jk}(\tau)$, and $W_{jk}(\tau)$

$$(\Omega_i - \Omega_p)^2 - (\omega^m_{jk})^2 = 0 \tag{10.2-31}$$

$$(\Omega_i - \Omega_p) = \omega^m_{jk} \tag{10.2-32}$$

$$\left|(i - 2)^2 - (p - 2)^2 + (\varepsilon_i - \varepsilon_p)\beta\right| \frac{\Omega}{\omega_1} = \omega^m_{jk} \tag{10.2-33}$$

$i = 1, 2, 3$

$j = 1, 2, \ldots, M_1$

$k = 1, 2, \ldots, N_1$

$m = 1, 2, 3$

$p = 1, 2, 3$

From $U_{jk}(\tau)$, $V_{jk}(\tau)$, and $W_{jk}(\tau)$

$$(2\Omega_i)^2 - (\omega^m_{jk})^2 = 0 \tag{10.2-34}$$

$$2\Omega_i = \omega^m_{jk} \tag{10.2-35}$$

$$2\left[(i - 2)^2 + \varepsilon_i\beta\right] \frac{\Omega}{\omega_1} = \omega^m_{jk} \tag{10.2-36}$$

$i = 1, 2, 3$

$j = 1, 2, \ldots, M_1$

$k = 1, 2, \ldots, N_1$

$m = 1, 2, 3$

$p = 1, 2, 3$

From $U_{jk}(\tau)$, $V_{jk}(\tau)$, and $W_{jk}(\tau)$

$$\left[(\alpha_r + s)\bar{\Omega}\right]^2 - (\omega_{jk}^m)^2 = 0 \tag{10.2-37}$$

$$\bar{\Omega} = \frac{\omega_{jk}^m}{|(\alpha_r + s)|} = \frac{\Omega}{\omega_1} \tag{10.2-38}$$

$j = 1, 2, \ldots, M_1$

$k = 1, 2, \ldots, N_1$

$m = 1, 2, 3$

$r = 1, 2, \ldots, R$

$s = -S$ to $+S$

10.3 Solution of the Stability Equation

10.3.1 Computer Methods

Reference to the indicial variation in each of the stability equations shows that the use of hand-made calculations is virtually prohibitive, so computer solutions must be made available. The indicial range is of particular importance to the programmer since it determines the number of unstable values of the thrust frequency Ω which effectively governs the size of the computer operations, something very important to consider when working with computers of limited memory space.

10.3.2 Equations to be Solved

The stability equations that must be solved and the number of unstable values that each equation will render for values of M_1, N_1, R, and S, are summarized in tabular form for the convenience of the programmer. As in the previous stability analysis of Tech. Memo. No. 38, values of

$M_1 = N_1 = 10$ will be used leaving only R and S to vary in equations (10.2-8) and (10.2-38).

Stability Equation	Number of Unstable Values of Thrust Frequency Ω
1. Eq. (10.2-6)	30
2. Eq. (10.2-10)	20
3. Eq. (10.2-13)	60
4. Eq. (10.2-15)	180
5. Eq. (10.2-18)	180
6. Eq. (10.2-21)	900
7. Eq. (10.2-24)	900
8. Eq. (10.2-27)	900
9. Eq. (10.2-30)	2700
10. Eq. (10.2-33)	2700
11. Eq. (10.2-36)	2700

12. Eq. (10.2- 8): The number of unstable values of the thrust frequency Ω for values of R and S are given below

		S									
		1	2	3	4	5	6	7	8	9	10
R	1	20	40	60	80	100	120	140	160	180	200
	2	40	80	120	160	200	240	280	320	360	400
	3	60	120	180	240	300	360	420	480	540	600
	4	80	160	240	320	400	480	560	640	720	800
	5	100	200	300	400	500	600	700	800	900	1000
	6	120	240	360	480	600	720	840	960	1080	1200
	7	140	280	420	560	700	840	980	1120	1260	1400
	8	160	320	480	640	800	960	1120	1280	1440	1600
	9	180	360	540	720	900	1080	1260	1440	1620	1800
	10	200	400	600	800	1000	1200	1400	1600	1800	2000

13. Eq. (10.2-38) : The number of unstable values of the thrust frequency Ω for values of R and S are given below.

R		S: 1	2	3	4	5	6	7	8	9	10
	1	600	1200	1800	2400	3000	3600	4200	4800	5400	6000
	2	1200	2400	3600	4800	6000	7200	8400	9600	10,800	12,000
	3	1800	3600	5400	7200	9000	10,800	12,600	14,400	16,200	18,000
	4	2400	4800	7200	9600	12,000	14,400	16,800	19,200	21,600	24,000
R	5	3000	6000	9000	12,000	15,000	18,000	21,000	24,000	27,000	30,000
	6	3600	7200	10,800	14,400	18,000	21,600	25,200	28,800	32,400	36,000
	7	4200	8400	12,600	16,800	21,000	25,200	29,400	33,600	37,800	42,000
	8	4800	9600	14,400	19,200	24,000	28,800	33,600	38,400	44,200	48,000
	9	5400	10,800	16,200	21,600	27,000	32,400	37,800	43,200	48,600	54,000
	10	6000	12,000	18,000	24,000	30,000	36,000	42,000	48,000	54,000	60,000

10.3.3 Analysis of the Stability Equations

With the exception of Eqs. (10.2-8) and (10.2-38), the stability equations (10.2-6) thru (10.2-38) contain the stability parameter β

$$\beta = \frac{2}{\pi}\left(\sin^{-1}\left\{[\Delta(0)]^{\frac{1}{2}} \sin\frac{\pi a^{\frac{1}{2}}}{2}\right\}\right) \tag{10.2-39}$$

where

$$\Delta(0) = 1 + \frac{\pi a^{3/2}\gamma^2}{16(1-a)} \cot\frac{\pi a^{\frac{1}{2}}}{2} \tag{10.2-40}$$

$$a = \frac{8\pi r L T_o K}{I\Omega^2} \tag{10.2-41}$$

which shows that β is a function of Ω by virtue of equations (10.2-40) and (10.2-41).

The stability equations (10.2-8) and (10.2-38) contain the parameter α which is determined from beam theory presented in Tech. Memo. No. 35. A FORTRAN IV Computer Program is being prepared that will render solutions to the stability equations (10.2-6) thru (10.2-38) for a given thin-walled cylinder. This computer program will be made available as an appendix to this report or as a separate publication.

11.0 CONCLUSIONS

11.1 General Comment

This report is a continuation in the analysis of the dynamic structural behavior of a large rocket booster during powered flight. The thin cylindrical model used in this consideration was subjected to the same time-varying, gimbaled thrust as that used in NSL Tech. Memo. No. 38.

11.2 Natural Frequencies

The natural frequencies of the free vibration of a free-free, thin-walled circular cylindrical shell were obtained. Since the homogenous portion of the transformed equations (9.1-3), (9.1-4), and equations (9.1-12) thru (9.1-14) are identical to those of Tech. Memo. No. 38, the natural frequencies for the cylinders will be identical.

11.3 Unstable Values of the Thrust Frequency, Ω.

Referring to 10.0 STABILITY ANALYSIS of this report, and comparing the stability equations of Hill (ref. 1), one can see that the present analysis has rendered two additional stability equations, (10.2-8) and (10.2-38). This was to be expected since consideration of the end displacements $q^*(\xi, \tau)$ of the cylindrical model used in this analysis have added nonhomogeneous terms to the forcing terms $P_{0k}(\tau)$, $\left[P_{j0}(\tau)\right]_1$, $\left[P_{j0}(\tau)\right]_2$, $\left[Q_{jk}(\tau)\right]_1$, $\left[Q_{jk}(\tau)\right]_2$, and $\left[Q_{jk}(\tau)\right]_3$.

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APPENDIX A. STRESS RESULTANT - DISPLACEMENT RELATIONS

A-I Nondimensionalizing the Equation

The stress resultant equation is written

$$N_x = \frac{Eh}{1-\nu^2}\left[\frac{\partial u}{\partial x} + \frac{\nu}{r}\left(\frac{\partial v}{\partial \theta} - w\right)\right] \tag{A-1}$$

Substituting the relations into equation (A-1)

$$\bar{u} = \frac{u}{L}, \qquad \bar{v} = \frac{v}{L}, \qquad \bar{w} = \frac{w}{L} \tag{A-2}$$

and performing the required derivatives results in the following dimensionless expression

$$N_\xi = \frac{Eh}{1-\nu^2}\left[\frac{\partial \bar{u}}{\partial \xi} + \frac{\nu}{r}\left(L\frac{\partial \bar{v}}{\partial \theta} - L\bar{w}\right)\right] \tag{A-3}$$

A-II Edge Conditions

Substituting the chosen approximating forms for $\bar{u}$, $\bar{v}$, and $\bar{w}$ (see Section 6.2) into equation (A-3), we have

$$\begin{aligned} N_\xi = \frac{Eh}{1-\nu^2} \Bigg\{ \Bigg[& \left[-\frac{m\pi}{2}\sin\frac{m\pi}{2}(\xi+1)\cos n\theta\right]U_{mn}(\tau) \\ & - \frac{1-\nu^2}{Eh}\frac{T_o}{2}(1-\gamma\cos\bar{\Omega}\tau)\cos K\psi(1-\xi) \\ & + \frac{\nu L}{r}\left\{\left[\sin\frac{m\pi}{2}(\xi+1)\, n\cos n\theta\right]V_{mn}(\tau) + g^*(\xi,\tau)\cos\theta\right. \\ & \left. - \left[\sin\frac{m\pi}{2}(\xi+1)\cos n\theta\right]W_{mn}(\tau) + h^*(\xi,\tau)\cos\theta\right\}\Bigg]\end{aligned} \tag{A-4}$$

At $\xi = -1$

$$N_{\xi} = \frac{Eh}{1-\nu^2}\left\{-\frac{(1-\nu^2)}{Eh}\frac{T_o}{2}(1-\gamma\cos\bar{\Omega}\tau)\cos K\psi \quad (2) + \frac{L\nu}{r}\left[g^*(-1,\tau)\cos\theta - h^*(-1,\tau)\cos\theta\right]\right\} \tag{A-5}$$

$$= -T_o(1-\gamma\cos\bar{\Omega}\tau)\cos K\psi \tag{A-6}$$

At $\xi = +1$

$$N_{\xi} = \frac{Eh}{1-\nu^2}L\frac{\nu}{r}\left[g^*(+1,\tau)\cos\theta - h^*(+1,\tau)\cos\theta\right] = 0 \tag{A-7}$$

APPENDIX B. REPRESENTATION OF THE ϕ-FUNCTIONS

B-I ϕ-Functions for the Free-Free Beam

The ϕ-functions and their derivatives for the free-free beam are summarized below.

$$\phi_n(\xi) = \cosh \frac{\lambda_n}{2}(\xi+1) + \cos \frac{\lambda_n}{2}(\xi+1) - \alpha_n \left[\sinh \frac{\lambda_n}{2}(\xi+1) + \sin \frac{\lambda_n}{2}(\xi+1)\right] \tag{B-1}$$

$$\frac{d\phi_n}{d\xi} = \frac{\lambda_n}{2}\left\{\sinh \frac{\lambda_n}{2}(\xi+1) - \sin \frac{\lambda_n}{2}(\xi+1) - \alpha_n \left[\cosh \frac{\lambda_n}{2}(\xi+1) + \cos \frac{\lambda_n}{2}(\xi+1)\right]\right\} \tag{B-2}$$

$$\frac{d^2\phi_n}{d\xi^2} = \frac{\lambda_n^2}{4}\left\{\cosh \frac{\lambda_n}{2}(\xi+1) - \cos \frac{\lambda_n}{2}(\xi+1) - \alpha_n \left[\sinh \frac{\lambda_n}{2}(\xi+1) - \sin \frac{\lambda_n}{2}(\xi+1)\right]\right\} \tag{B-3}$$

$$\frac{d^3\phi}{d\xi^3} = \frac{\lambda_n^3}{8}\left\{\sinh \frac{\lambda_n}{2}(\xi+1) + \sin \frac{\lambda_n}{2}(\xi+1) - \alpha_n \left[\cosh \frac{\lambda_n}{2}(\xi+1) - \cos \frac{\lambda_n}{2}(\xi+1)\right]\right\} \tag{B-4}$$

$$\frac{d^4\phi_n}{d\xi^4} = \frac{\lambda_n^4}{16}\left\{\cosh \frac{\lambda_n}{2}(\xi+1) + \cos \frac{\lambda_n}{2}(\xi+1) - \alpha_n \left[\sinh \frac{\lambda_n}{2}(\xi+1) + \sin \frac{\lambda_n}{2}(\xi+1)\right]\right\} \tag{B-5}$$

B-II Evaluation at the Ends

$$\sum_{n=1}^{N} \phi_n\begin{pmatrix}+1\\-1\end{pmatrix} = \sum_{n=1}^{N}\left[\cosh \lambda_n + \cos \lambda_n - \alpha_n(\sinh \lambda_n + \sin \lambda_n) - 2\right]$$

$$= \sum_{n=1}^{3} 2\left[1 + (-1)^n\right] \tag{B-6}$$

$$\sum_{n=1} \phi_n'\begin{pmatrix}+1\\-1\end{pmatrix} = \sum_{n=1}^{N}\left[\sinh \lambda_n - \sin \lambda_n - \alpha_n(\cosh \lambda_n + \cos \lambda_n) + 2\alpha_n\right]$$

$$= \begin{cases} = 2\ (-1.96500) & n = 1 \\ = 0 & n = 2 \\ = 2\ (-1.99993) & n = 3 \\ = 0 & n = 4 \\ = 2\ (-2.00000) & n = 5 \end{cases} \tag{B-7}$$

$$\sum_{n=1}^{N} \phi_n'' \begin{pmatrix}+1\\-1\end{pmatrix} = \sum_{n=1}^{N} \left[\cosh \lambda_n - \cos \lambda_n - \alpha_n (\sinh \lambda_n - \sin \lambda_n)\right]$$

$$= 0 \quad \text{for all integer values } n \tag{B-8}$$

$$\sum_{n=1}^{N} \phi_n''' \begin{pmatrix}+1\\-1\end{pmatrix} = \sum_{n=1}^{N} \left[\sinh \lambda_n + \sin \lambda_n - \alpha_n (\cosh \lambda_n - \cos \lambda_n)\right]$$

$$= 0 \quad \text{for all integer values of } n \tag{B-9}$$

$$\sum_{n=1}^{N} \phi_n'''' \begin{pmatrix}+1\\-1\end{pmatrix} = \sum_{n=1}^{N} \left[\cosh \lambda_n + \cos \lambda_n - \alpha_n (\sinh \lambda_n + \sin \lambda_n) - 2\right]$$

$$= \sum_{n=1}^{N} 2\left[1 + (-1)^n\right] \tag{B-10}$$

Observe that

$$\frac{d^4 \phi_n}{d\xi^4} = \frac{\lambda_n^4}{16} \phi_n \tag{B-11}$$

and

$$\phi_n'''' = \phi_n \tag{B-12}$$

APPENDIX C. EVALUATED COEFFICIENT INTEGRALS

The evaluated coefficient integrals appearing as Equations (3.0-17) through (3.0-23) of Section 3.0 - LIST OF TERMS AND INTEGRALS APPEARING IN THE REPORT, are presented as follows:

$$m_{j0} = \int_{-1}^{+1} \xi \sin \frac{j\pi}{2}(\xi+1)\, d\xi$$

$$= -\frac{2}{j\pi}\left[1 + (-1)^j\right] \qquad \text{(C-1)}$$

$$r_{j0} = \int_{-1}^{+1} \sin \frac{j\pi}{2}(\xi+1)\, d\xi$$

$$= \frac{2}{j\pi}\left[1 - (-1)^j\right] \qquad \text{(C-2)}$$

$$s_{jn} = \sum_{n=1}^{3} \left\{ \int_{-1}^{+1} \phi_n(\xi) \sin \frac{j\pi}{2}(\xi+1)\, d\xi \right\}$$

$$= 2(j\pi)^3 \sum_{n=1}^{3} \frac{\left[1 - (-1)^j\right]}{\left[(j\pi)^4 - \lambda_n^{\,4}\right]} \left\{ \cosh \lambda_n + \cos \lambda_n - 2 - \alpha_n \left[\sinh \lambda_n + \sin \lambda_n\right] \right\} \qquad \text{(C-3)}$$

$$t_{jn} = \sum_{n=1}^{3} \left\{ \int_{-1}^{+1} \phi_n'(\xi) \sin \frac{j\pi}{2}(\xi+1)\, d\xi \right\}$$

$$= 2(j\pi)^3 \sum_{n=1}^{3} \frac{\left[1 - (-1)^j\right]}{\left[(j\pi)^4 - \lambda_n^{\,4}\right]} \left\{ \sinh \lambda_n - \sin \lambda_n - \alpha_n \left[\cosh \lambda_n + \cos \lambda_n - 2\right] \right\} \qquad \text{(C-4)}$$

$$x_{jn} = \sum_{n=1}^{3} \left\{ \int_{-1}^{+1} \phi_n''(\xi) \sin \frac{j\pi}{2} (\xi+1)\, d\xi \right\}$$

$$= -2(j\pi) \sum_{n=1}^{3} \lambda_n^2 \frac{\left[1 - (-1)^j\right]}{\left[(j\pi)^4 - \lambda_n^4\right]} \left\{ \cosh \lambda_n - \cos \lambda_n - 2 \right.$$

$$\left. - \alpha_n \left[\sinh \lambda_n + \sin \lambda_n\right] \right\} = - \sum_{n=1}^{3} \frac{\lambda_n^2}{(j\pi)^2} s_{jn} \qquad \text{(C-5)}$$

$$y_{jn} = \sum_{n=1}^{3} \left\{ \int_{-1}^{+1} \phi_n'''(\xi) \sin \frac{j\pi}{2} (\xi+1)\, d\xi \right\}$$

$$= -2(j\pi) \sum_{n=1}^{3} \lambda_n^2 \frac{\left[1 - (-1)^j\right]}{\left[(j\pi)^4 - \lambda_n^4\right]} \left\{ \sinh \lambda_n - \sin \lambda_n \right.$$

$$\left. - \alpha_n \left[\cosh \lambda_n + \cos \lambda_n - 2\right] \right\} = - \sum_{n=1}^{3} \frac{\lambda_n^2}{(j\pi)^2} t_{jn} \qquad \text{(C-6)}$$

$$z_{jn} = \sum_{n=1}^{3} \left\{ \int_{-1}^{+1} \phi_n'(\xi) \cos \frac{j\pi}{2} (\xi+1)\, d\xi \right\}$$

$$= 2 \sum_{n=1}^{3} \lambda_n^3 \frac{\left[1 - (-1)^j\right]}{\left[(j\pi)^4 - \lambda_n^4\right]} \left\{ \cosh \lambda_n + \cos \lambda_n - 2 \right.$$

$$\left. - \alpha_n \left[\sinh \lambda_n + \sin \lambda_n\right] \right\} = \sum_{n=1}^{3} \frac{\lambda_n^3}{(j\pi)^3} s_{jn} \qquad \text{(C-7)}$$

where

$$j = 1, 2, \ldots, M \qquad \text{(C-8)}$$

$$\lambda_n^4 = \omega_n^2 \frac{m\,(2L)^4}{EI} \qquad \text{(C-9)}$$

The characteristic number λ_n for the free-free beam for integer values of n has been calculated in Young, (ref. 10).

$$\begin{aligned} \lambda_1^4 &= 500.5639 && \text{(a)} \\ \lambda_2^4 &= 3{,}803.5390 && \text{(b)} \qquad \text{(C-10)} \\ \lambda_3^4 &= 14{,}617.6299 && \text{(c)} \end{aligned}$$

The values of α_n are as follows

$$\begin{aligned} \alpha_1 &= 0.9825 && \text{(a)} \\ \alpha_2 &= 1.0007 && \text{(b)} \qquad \text{(C-11)} \\ \alpha_3 &= 0.9999 && \text{(c)} \end{aligned}$$

Having the values of λ_n available, we can now determine the natural circular frequency of the n^{th} mode of vibration of the free-free beam (cylinder of equal mass)

$$\omega_n = \left(\frac{\lambda_n}{2L}\right)^2 \sqrt{\frac{EI}{m}} \qquad \text{(C-12)}$$

where

E - modulus of elasticity of beam (cylinder) material, lb/in^2.

I - moment of inertia of a cross-section of the beam (cylinder) about its neutral axis, in^4.

m - mass per unit length of the beam (cylinder), $lb\text{-}sec^2/in^2$.