NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Report No. 32-698

The Analysis of Time Multiplexing Systems for Partial Success

Donall G. Bourke

JET PROPULSION LABORATORY
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Donall G. Bourke

Reed B. Harker, *Manager*
Spacecraft Telemetry and Command

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CONTENTS

TABLES

FIGURES

ABSTRACT

34416

This Report is an attempt to improve analysis methods for time multiplexing systems under partial-success criteria. Results of the analysis are in the form of probability distribution for the number of surviving measurements in a multiplexer configuration. This form of result allows such criteria to be applied as the expected number of measurements, the expected value of the weighted measurements, or the probability that the number of surviving weighted measurements will be greater than some preassigned value.

The analysis for multiplexer configuration is based upon first modeling the basic multiplexer deck. A representative solid-state deck used by the Jet Propulsion Laboratory is used as an example. After the probability distribution of surviving measurements is obtained, its general form is used to extend the analysis to general and more complex multiplexer configurations of more than one deck. Some numerical results are presented and areas for further investigations are discussed.

Author

I. INTRODUCTION

To increase the reliability of the data handling systems of future space vehicles, reliability techniques and system organization concepts must be extended. If, for any generalized system, a reliability criterion of complete success is specified, the design problem for that system becomes one of minimization of nonredundant system elements to accomplish a given functional requirement. After this minimization has been accomplished, the only recourses available for increasing reliability are in the component quality, circuit design, and redundancy disciplines. In a data handling system, however, a reliability criterion of complete success is not necessarily the most realistic, since there are many combinations of internal failures which, while resulting in some information loss, could not necessarily be classified as a complete system failure. The design problem for a system under a partial success criterion is generally to insure that if failures occur, the effect of the failures should be minimized. Implicit in most present day design processes is the analysis task.

A function basic to a data handling system is time multiplexing, and it is in association with this function that partial success criteria assume a great deal of meaning. Analysis methods of time multiplexing systems to measure their ability to produce partial successes have, at best, been time consuming to the point of generating negligible inputs to the design process. This Report is an attempt to improve the analysis methods for time multiplexing systems by increasing the realism and significance of the final results and also by decreasing the time required to obtain these results. While this Report is theoretical in nature and lacks empirical support from tests of actual multiplexer configurations, the results are thought to be of sufficient value to warrant dissemination without this empirical evidence.

The analysis for multiplexer configurations is based upon modeling the basic element of the system, the multiplexer deck. This model is based upon a solid-state deck and switch arrangement used by the Jet Propulsion Laboratory in multiplexing systems for engineering data. The probability distribution of the number of surviving data points on the deck is derived in two forms, each form having its own particular usefulness for subsequent analysis in the report. The first form of the probability distribution indicates the probability of, as one example, i data points surviving; the second form indicates the probability of i data points surviving as a function of the location of the points on the deck. This Report then utilizes the deck probability distributions to obtain expressions for the probability distributions of the number of surviving data points for two general multiplexer configurations of more than one deck. A significant reduction in the total number of outcomes which need to be explicitly elaborated is obtained by accounting for a small number of primary failure combinations and performing convolution operations with various deck probability distributions for each primary failure combination. The convolution operations are also simplified by transforming the deck probability distribution into generating functions and consequently transforming the convolution operation into multiplication of polynomials. The technique of analysis for general multiplexer configurations is applicable to any type of basic deck, provided that the deck output can be described in the form of a probability distribution for the number of surviving data points.

The final sections of this Report treat the analysis problem for more complex multiplexing systems, present some numerical results obtained from a computer program of a basic deck model, and outline areas for further study.

The analysis methods presented must rely on high-speed computer capability to achieve results. If this capability can be assumed, then the analysis methods presented will provide an effective tool for the reliability evaluation of relatively complex time multiplexing systems.

II. DERIVATIONS FOR THE BASIC DECK MODEL

A. Introduction

Figure 1 is a functional representation of a multiplexer deck used as the basic model for this analysis. There is a total of N stages which sequentially route data to the deck output. Each stage consists of a solid-state switch S_j, driven by a memory element M_j, with j ranging from 1 to N. The stages are turned on sequentially when clock pulses are applied to the input. Whenever the last stage N, is turned on, the next clock pulse will turn on stage zero through a feedback mechanism, and the sequential cycle repeats. Stage zero is a memory element M_0, which will deliver a sync indication to another point in the data system of which the deck is a member. This element, along with the feedback, clock conditioning, and any signal conditioning circuits common to all data points on the deck comprise the series (in a reliability sense) portion of the deck.

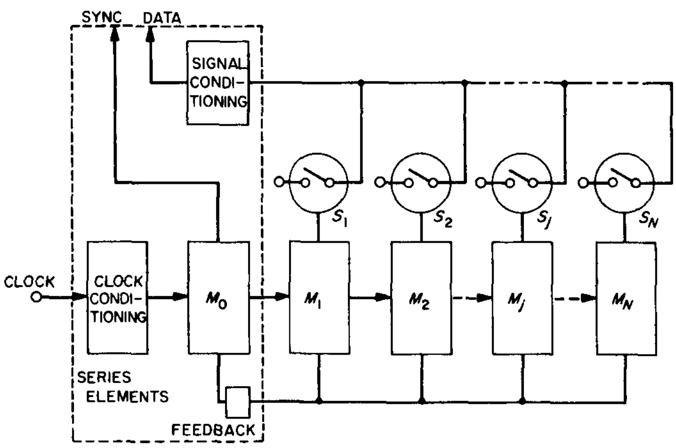

Fig. 1. Basic multiplexer deck model

B. Definitions and Assumptions

Let

$s \stackrel{\Delta}{=}$ probability of a switch being successful

$s_o \triangleq$ probability of a switch failing in the "open" mode such that the data point is isolated from the bus regardless of input drive signal and the degree of isolation is high enough such that other successful data points on the bus are not considered as having failed

$s_c \triangleq$ probability of a switch failing in the "closed" mode such that the degree of isolation of the data point is low enough to cause failure of all other data points on the bus

$m \triangleq$ probability of a memory element being successful

$m_o \triangleq$ probability of a memory element failing in the "open" mode such that it passes no drive signal to the next memory element and indicates an "open" signal to its attendant switch

$m_c \triangleq$ probability of a memory element failing in the "closed" mode such that it indicates a "close" signal to its attendant switch and passes no signal or a "close" signal to the next memory element.

It is assumed that

$$s + s_o + s_c = 1$$

$$m + m_o + m_c = 1$$

consequently s_o, s_c, m_o, and m_c could be of the form,

$[1 - s] \times$ [conditional probability of "open" (or "closed") mode given failure]

and

$[1 - m] \times$ [conditional probability of "open" (or "closed") mode given failure]

It will also be assumed that the conditional probabilities above are not functions of time or state of the element although $[1 - s]$ and $[1 - m]$ will necessarily have time arguments. The conditional probabilities above will not be used explicitly in the remainder of this Report but will be implicit in the terms s_o, s_c, m_o, and m_c; they were used here only to state an assumption more clearly.

There now remains to be considered what constitutes failure of the deck, or failure of a single data point. First, if any memory element in the sample space fails in the "closed" or "open" mode, the full sequencing capability of the deck will be lost. If a memory element fails in the "open" mode and all preceding memory elements are successful, then the deck will short count through the internal feedback mechanism. It is felt that such a failure will not cause complete deck failure if the sync memory element M_0 is still good, because data processing on the ground could possibly determine the data points surviving in a short counted sequence. If any memory element fails in a "closed" mode, the deck will short circuit through the internal feedback mechanism and the entire sequencing function will be stopped. In this case, although there may be a single data point surviving, the absence of sync identification may cause some ambiguity as to which point was surviving. Therefore, this case is treated as a complete failure of the deck. Loss of memory element M_0 either in the "open" or "closed" state is also considered a failure of all data points on the deck. Second, if a switch coupled to a failed "closed" memory element is good or bad, or if any switch fails in the "closed" state regardless of the memory element states, then the deck is considered to have failed. Thus, any partial survival of data points with unambiguous identification must be on a sample space conditioned by the event that no switch or memory element "closed" failures have occurred, and that the series elements mentioned previously are successful.

A few more assumptions should be stated at this point:

1. Although loading conditions, duty cycles, etc., will change as failures occur, it is assumed that there is no dependence among the probabilities of the individual elements, i.e., they will change only as a function of time.

2. If a switch or memory element fails in a certain mode, it is assumed that it will remain in this mode over the period of time considered.

Essentially, the probability distribution of data points surviving will be derived assuming mutual independence among the deck elements in a probability sense, but will take into account the functional dependence existing among these elements.

C. Analysis

As mentioned before, any distribution of data point partial success must be conditional on the event that no switch or memory element failures in the "closed" mode have occurred and the event that the deck series elements are successful. To arrive at this conditional distribution, the first notation is that any combination of successes, "open" failures and "closed" failures for the switches, is given by terms of a trinomial distribution denoted by S. Thus,

$$S(X_s, X_o, X_c) = \frac{N!}{X_s!\,X_o!\,X_c!}(s)^{X_s}(s_o)^{X_o}(s_c)^{X_c}$$

is the probability that in N trials (N switches) there are X_s successes, X_o "open" failures, and X_c "closed" failures, where

$$X_s + X_o + X_c = N$$

Similarly, for those N memory elements which drive the switches, any combination of memory element successes and failures is given by terms of a trinomial distribution denoted by M. Therefore:

$$M(Y_s, Y_o, Y_c) = \frac{N!}{Y_s!\,Y_o!\,Y_c!}(m)^{Y_s}(m_o)^{Y_o}(m_c)^{Y_c}$$

where Y_s is the number of memory element successes, Y_o is the number of "open" failures, and Y_c is the number of "closed" failures and, of course,

$$Y_s + Y_o + Y_c = N$$

Let

$P\{E_s\} \triangleq$ probability of events E_s where E_s is the success of all series elements in the deck, i.e., memory element M_0, the feedback circuits, the clock conditioning, and any signal conditioning common to all data points on the deck.

Note that the probability distribution of any surviving data points on the deck will then be of the form

$$\begin{aligned} P\{E_s, X_c = 0, Y_c = 0\}\, S(X_s, X_o | X_c = 0)\, M(Y_s, Y_o | Y_c = 0) \\ = P\{E_s\}\, P\{X_c = 0\}\, P\{Y_c = 0\} \\ \times S(X_s, X_o | X_c = 0)\, M(Y_s, Y_o | Y_c = 0) \end{aligned} \tag{1}$$

because of the mutual independence in a probability sense of the elements in the deck. With the assumption of independence between S and M, we can take each trinomial distribution independently and condition it with the events $X_c = 0$, and $Y_c = 0$. The processes are identical for both, so let us take the switch distribution. Now

$$S(X_s, X_o | X_c = 0) = \frac{S(X_s, X_o, X_c = 0)}{S(X_c = 0)} \tag{2}$$

The term in the numerator of Eq. (2) is

$$\begin{aligned} S(X_s, X_o, X_c = 0) &= \frac{N!}{X_s!\,(N - X_s!)}(s)^{X_s}(s_o)^{N-X_s} \\ &= \binom{N}{X_s}(s)^{X_s}(s_o)^{N-X_s} \end{aligned} \tag{3}$$

where

$\binom{N}{X_s}$ is the standard notation for a binomial coefficient.

Note, however, that Eq. (3) is not a term from the binomial distribution since

$$s + s_o \neq 1$$

The denominator in Eq. (2) is

$$S(X_c = 0) = \sum_{X_s=0}^{N}\binom{N}{X_s}(s)^{X_s}(s_o)^{N-X_s} = (s + s_o)^N$$

Equation (2) then becomes

$$S(X_s, X_o | X_c = 0) = \frac{\binom{N}{X_s}(s)^{X_s}(s_o)^{N-X_s}}{(s + s_o)^N}$$

Using the same method to condition the memory element distribution, final terms in the distribution of data point partial success in Statement (1) will be of the form

$$\begin{aligned} &P\{E_s\}\,[(s + s_o)(m + m_o)]^N \\ &\quad\times\left[\frac{\binom{N}{X_s}(s)^{X_s}(s_o)^{N-X_s}\binom{N}{Y_s}(m)^{Y_s}(m_o)^{N-Y_s}}{[(s + s_o)(m + m_o)]^N}\right] \\ &= P\{E_s\}\left[\binom{N}{X_s}(s)^{X_s}(s_o)^{N-X_s}\binom{N}{Y_s}(m)^{Y_s}(m_o)^{N-Y_s}\right] \end{aligned} \tag{4}$$

It is the terms in the square brackets on the right side of Eq. (4) which require further rearrangement for our purposes, since with the form shown in Statement (4), specific arguments of X_s and Y_s do not yield unique values of the number of data points surviving. Let

$P_N D\{i\} = P_N D\{N - j\} \triangleq$ The probability of $N - j$ data points surviving on a deck of length N.

The notation $N - j$ is temporarily used here for greater ease in explanation while deriving subsequent expressions. Figure 2 shows the end of the deck previously

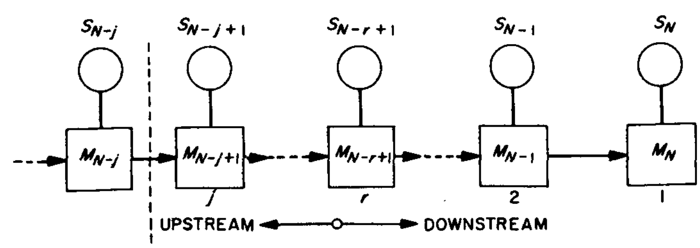

Fig. 2. Basic deck re-labeled

shown in Fig. 1, with some re-labeling necessary for our explanation.

Assume that the general term for $P_N D\{N-j\}$ points surviving is being evaluated. In this term, the maximum number of memory element "open" failures which can be tolerated is j and all these possible failures must occur in elements M_{N-j+1} to M_N. Thus the subterms in $P_N D\{N-j\}$ have factors of $(m)^{N-r}(m_o)^r$ where $r = 1, 2, \cdots, j$. There are $\binom{j}{r}$ different ways in which r "open" failures may occur in j memory elements. If a failure occurs in the $N-j+1$ memory element, it will fail all data points downstream from $N-j$ and will leave no possibilities for any switch "open" failures upstream from $N-j+1$. If an "open" failure occurs in, say, element M_{N-1}, it will fail all data points downstream from $N-2$, but leaves a possible $j-2$ "open" switch failures which may be tolerated upstream from $N-1$. Of course, "open" switch failures may be tolerated downstream from, and in, any "open" failed memory element in the j elements of the deck. Generalizing, it can be stated that for j possible data point losses and r possible memory element losses there are, in the last $r + k' \leqq j$ elements,

$\binom{r+k'-1}{r}$ possible arrangements of memory element "open" failures,

$(r+k')$ data points downstream which have induced failures but can tolerate switch successes or switch "open" failures,

$j-(r+k')$ other data points upstream which can tolerate "open" switch failures of which there are

$$\binom{N-(r+k')}{j-(r+k')}$$

possible arrangements.

With the above relationships

$$P_N D\{N-j\} = P\{E_s\}\left[m^N \binom{N}{j} s^{N-j}(s_o)^j + s^{N-j}\sum_{r=1}^{j} m^{N-r}(m_o)^r \sum_{k'=0}^{j-r}\binom{r+k'-1}{r-1} \times \binom{N-(r+k')}{j-(r+k')}(s+s_o)^{r+k'}(s_o)^{j-(r+k')}\right]$$

For $j = 0, 1, 2, \cdots, N-1$ and after a change of variables, namely, $k = r + k'$ and $i = N - j$

$$P_N D\{i\} = P\{E_s\}\left[m^N \binom{N}{N-i} s^i (s_o)^{N-i} + s^i \sum_{r=1}^{N-i} m^{N-r}(m_o)^r \sum_{k=r}^{N-i}\binom{k-1}{r-1} \times \binom{N-k}{N-i-k}(s+s_o)^k (s_o)^{N-i-k}\right] \tag{5}$$

for $i = 1, 2, \cdots, N$.

For $i = 0$ consider failure of the deck due to any elements failing in the "close" mode and also due to some combinations of "open" failures, given by Eq. (5) with $i = 0$. Therefore

$$P_N D\{0\} = 1 - [(s+s_o)(m+m_o)]^N P\{E_s\} + P\{E_s\} \times \left[(ms_o)^N + \sum_{r=1}^{N} m^{N-r}(m_o)^r \sum_{k=r}^{N}\binom{k-1}{r-1}(s+s_o)^k(s_o)^{N-k}\right] \tag{6}$$

The verification that Eq. (6) and (7) are terms of a probability distribution, or simply that

$$\sum_{i=0}^{N} P_N D\{i\} = 1$$

is given in Appendix A.

The next probability distribution which will be needed is the probability distribution of a multiplexer deck for arguments of

$i \triangleq$ number of data points surviving,

$x_i \triangleq$ the greatest index of one of the i data points surviving. Index here is the number of the data point which indicates its place in the multiplexer deck. Since x_i is, by definition, the greatest index of the i points surviving, there cannot be any points surviving on the deck with indices greater than x_i.

Figure 3 illustrates the model to be used here and the method of indexing. It is essentially a simpler representation of Fig. 1, and all assumptions stated for the previous derivation apply to this derivation. Many methods used

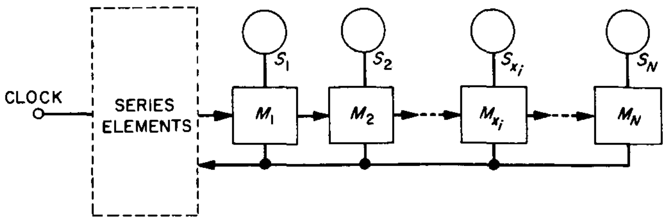

Fig. 3. Method of deck indexing

to derive the expression for $P_N D\{i\}$ can also be applied here. In fact, defining

$P_N D\{i, x_i\} \triangleq$ the probability of exactly i data point successes with index x_i,

this derivation can be entered at the point of stating that

$$P_N D\{0, x_0\} = P_N D\{0\}$$

where x_0 is merely a notation and has no meaning in terms of a greatest index. For $i = 1, 2, \cdots, N$ consider what is the probability of exactly i successes for a fixed index x_i and fixed indices

$$x_{i-1}, x_{i-2}, \cdots, x_2, x_1$$

where

$$x_i > x_{i-1} > x_{i-2} > \cdots > x_2 > x_1 \geqq 1$$

The probability is

$[P\{E_s\} s^i m^{x_i} (s_o)^{x_i - i}] \times$ [probability of 0 successes in the remaining $N - x_i$ elements of the deck given no "closed" failures may occur]

This probability is independent of the value of the indexes $x_1, x_2, \cdots, x_{i-1}$. Now, the probability of exactly i successes for a fixed value of x_i for all values of $x_1, x_2, \cdots, x_i$ is

$\left[P\{E_s\}\binom{x_i - 1}{i - 1} s^i m^{x_i} (s_o)^{x_i - i}\right] \times$ [probability of 0 successes in the remaining $N - x_i$ elements of the deck given no "closed" failures may occur]

Since there are

$$\binom{x_i - 1}{i - 1}$$

ways that the remaining $i - 1$ successes can be distributed in the $x_i - 1$ switches of index less than x_i. Also, the probability of 0 successes in the remaining $N - x_i$ elements of the deck given that no "closed" failures occur, can be borrowed from the previous derivation, more specifically, from the second term on the right side of Eq. (6) with $P\{E_s\} = 1$ and $N - x_i$ substituted for N. Therefore

$$P_N D\{i, x_i\} = P\{E_s\}\binom{x_i - 1}{i - 1} s^i m^{x_i} (s_o)^{x_i - i} \times \left[(ms_o)^{N - x_i} + \sum_{r=1}^{N - x_i} (m)^{N - x_i - r} (m_o)^r \times \sum_{k=r}^{N - x_i} \binom{k - 1}{r - 1} (s + s_o)^k (s_o)^{N - x_i - k}\right] \tag{7}$$

for $i = 1, 2, \cdots, N$, $x_i = i, i + 1, \cdots, N$ and, at the risk of being repetitive,

$$P_N D\{0, x_0\} = P_N D\{0\} \tag{8}$$

The verification that Eq. (7) and (8) are terms of probability distribution is given in Appendix B.

III. ANALYSIS OF MULTIPLEXER CONFIGURATIONS

A. Introduction

In this section, the derivations of the probability for the number of surviving points for the basic deck model will be applied to general multiplexer configurations. The first analysis example is rather simple, and consequently will find somewhat limited application. The example does serve as a good vehicle for illustrating the general method of approach to the multiplexer analysis problem. From this first example, the progression is to a more general analysis method which is extremely flexible in application. The last example is a method of approach to a very complex and interdependent multiplexer configuration which is similar to the type used on the *Mariner C* spacecraft.

B. Analysis of a Simple Multiplexer Configuration with Two Levels of Commutation

Though a relatively simple configuration is used in this first analysis, it will serve as a vehicle to typify an application of the single deck analysis to configurations of more than one deck. After the initial derivation for the probability of data points serving in the basic multiplexer configuration, some variations in the basic configuration will be made for cases where the derivation is still applicable. In Fig. 4 a multiplexer is shown with N subdecks, each of length N_{SD}, the outputs of which are routed to the N channels of the main deck, D_0. This main deck has an extension deck, D_{N+1}, of arbitrary length N_E, the output of which is connected to the main deck data bus. Note that D_{N+1} needs no sync memory element. A counter divides the basic clock and its output is routed to the subdecks in parallel. Each subdeck has its own sync memory element which is routed to some point after the output of the multiplexer. Each subdeck does not necessarily have to have this synchronization indication, if the counter is provided with more memory and the capability to reset the subdecks to some initial state and the counter indication is used instead of subdeck indications. As far as the analysis is concerned it matters little which of the two schemes is assumed.

The primary assumption for this analysis is that the probability distribution of surviving points for the subdecks are the same. Note that this allows only limited application of the analysis to actual configurations. Some flexibility in application is allowed by the fact that the series elements of each deck do not necessarily have to be of the same function, although they must have the same probability of success.

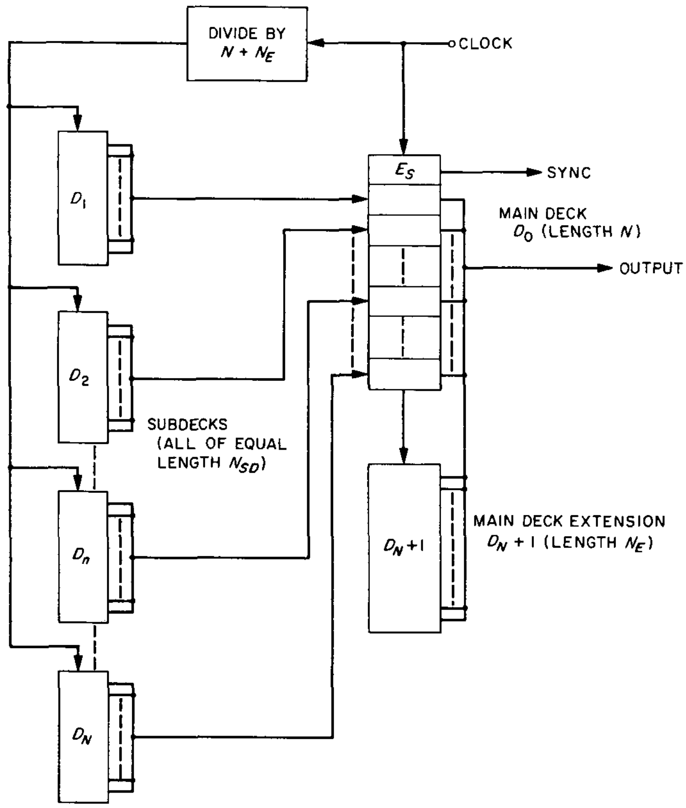

Fig. 4. Simple multiplexer configuration

Let

$P_{N(N_{SD})+N_E} M_4 \{i\} \triangleq$ The probability of i data points succeeding (with sync) in the multiplexer of Fig. 4, where the maximum number of data points in the multiplexer is $N(N_{SD}) + N_E$

$P\{C\} \triangleq$ The probability of success of the counter

$P\{\overline{C}\} \triangleq$ The probability of failure of the counter

and

$P_{N_j} D_j \{i\} \triangleq$ The probability of exactly i data points serving on deck j where the total number of data points in the deck is N_j. If there is little doubt as to what N_j is when a particular deck is specified, it will be dropped. For example

$$P_N D_0 \{i\} = P D_0 \{i\}$$

Begin the analysis by first noting that the main deck output and the output of the extension are connected. Thus if any "closed" failures occur either in the main deck or its extension, the entire multiplexer has failed. Therefore, the probability of zero points surviving in the distribution $P\,M_4\,\{i\}$ will contain the term

$$1 - [(s + s_o)(m + m_o)]^{N+N_E} P\{E_s\}$$

in addition to others. The important point here is that the event of the multiplexer failed because of "closed" failures in the main deck, or its extension is accounted for by this term, and henceforth when dealing with these two decks it is necessary to speak of terms from their conditional distribution given that no "closed" failures occur, times the probability that the event by which they are conditioned has occurred.

Let

$$P_{N_E} D_{N+1}\{i, X_c = Y_c = 0\} = [(s + s_o)(m + m_o)]^{N_E} \times P_{N_E} D_{N+1}\{i | X_c = Y_c = 0\}$$

$$= m^{N_E} \binom{N_E}{i} s^i (s_o)^{N_E - i} + s^i \sum_{r=1}^{N_E - i} m^{N_E - r} (m_o)^r \sum_{k=r}^{N_E - i} \binom{k-1}{r-1} \times \binom{N_E - k}{N_E - i - k} (s + s_o)^k (s_o)^{N_E - i - k}$$

for $i = 0, 1, 2, \cdots, N_E$. To continue the analysis with a clearer understanding, separate the distribution $P_N D_0\{i\}$ into two parts. Let

$$P_N D_0\{i\}_1 \triangleq P\{E_s\}(ms_o)^N$$

and

$$P_N D_0\{i\}_2 \triangleq P\{E_s\} s^i \sum_{r=1}^{N} m^{N-r} (m_o)^r \cdots \sum_{k=r}^{N} \binom{k-1}{r-1} \binom{N-k}{N-i-k} (s + s_o)^k (s_o)^{N-i-k}$$

These expressions are those parts of $P_N D_0\{i\}$ in which an output clock is delivered, and is not delivered to the main deck extension, respectively. At this point, it is necessary to define some notation associated with generating functions. Appendix C presents a short discussion of generating functions and their usage in this Report. Let

$${}_{N_j}D_j(S) = \sum_{i=0}^{N_j} {}_{N_j}D_j\{i\} S^i \triangleq$$ The generating function for the probability distribution of multiplexer deck, D_j. As before, the subscript N_j will be dropped whenever possible.

$${}_{N_{SD}}D(S) = \sum_{i=0}^{N_{SD}} {}_{N_{SD}}D\{i\} S^i \triangleq$$ The generating function for the probability distribution for any subdeck. Since the subdeck probability distributions are equal, so are their generating functions.

Also let

$$C_j(S) \triangleq \sum_{k=0}^{jN_{SD}} C_{j-k} S^k \triangleq [{}_{N_{SD}}D(S)]^j$$

be the generating function of the j-fold convolution of the subdeck probability distributions, and let

$$B_j(S) \triangleq \sum_{k=0}^{jN_{SD}+N_E} B_{j-k} S^k \triangleq [{}_{N_{SD}}D(S)]^j \, {}_{N_E}D_{N+1}(S)$$

be the generating function of the $j + 1$-fold convolution of j subdeck probability distributions with the probability distribution $P_{N_E} D_{N+1}\{i, X_c = Y_c = 0\}$. Note that a single subscript to the right of the letters B and C above represents a number of convolutions, whereas when used with the letter D it represents a deck number.

For convenience the coefficients

$$C_{0-k}(s) = 1$$

will be defined for all k. With this last definition note that

$$B_0(S) = {}_{N_E}D_{N+1}(S)$$

and, therefore

$$B_{0-k} = P_{N_E} D_{N+1}\{k, X_c = Y_c = 0\}$$

Now it is possible to derive the terms for the probability distribution

$$P_{N(N_{SD})+N_E} M_4\{i\}$$

The term $P\,M_4\,\{0\}$ is derived by brute force enumeration, and can be conveniently listed in seven mutually exclusive outcomes. The first outcome is

$$1 - [(s + s_o)(m + m_o)]^{N+N_E} P\{E_s\}$$

as before. The second is

$$PD_0\{0\}_1 B_{0-0}$$

which includes the counter-failed or not-failed, all probability states of the subdecks (which sum to 1), the main deck delivering on output to its extension, and both the main deck and its extension yielding no data point successes. The third outcome is

$$P\{\bar{C}\}\sum_{j=1}^{N} PD_0\{j\}_1 B_{0-0}$$

which includes the counter in a failed state, all states of the main deck yielding a finite number of data points, and the extension yielding zero data points. The fourth is

$$P\{C\}\sum_{j=1}^{N} PD_0\{j\}_1 B_{j-0}$$

which includes the counter in a successful state, all states of the main deck yielding $j > 0$ data points, and for each j, the zero term of the j-fold convolution of the subdeck distribution with the extension distribution. The fifth outcome is

$$PD_0\{0\}_2\,[(s+s_o)(m+m_o)]^{N_E}$$

which includes the counter-failed or not-failed, all states of subdecks, the main deck yielding zero data points and delivering no clock to the main deck extension, and all states of main deck extension $P_{N_E} D_{N+1}\{i, X_c = Y_c = 0\}$, which sum to

$$[(s+s_o)(m+m_o)]^{N_E}$$

The last two outcomes are

$$P\{\bar{C}\}\sum_{j=1}^{N} PD_0\{j\}_2\,[(s+s_o)(m+m_o)]^{N_E}$$

and

$$P\{C\}\sum_{j=1}^{N} PD_0\{j\}_2\, C_{j-0}\,[(s+s_o)(m+m_o)]^{N_E}$$

Using the fact that $P\{C\} + P\{\bar{C}\} = 1$ and combining terms

$$\begin{aligned} PM_4\{0\} = 1 &- [(s+s_o)(m+m_o)]^{N+N_E} P\{E_s\} \\ &+ \sum_{j=0}^{N} PD_0\{j\}_1 (P\{\bar{C}\} B_{0-0} + P\{C\} B_{j-0}) \\ &+ [(s+s_o)(m+m_o)]^{N_E} \\ &\times \sum_{j=0}^{N} PD_0\{j\}_2 (P\{\bar{C}\} + P\{C\} C_{j-0}) \end{aligned} \tag{9}$$

For $PM_4\{i\}$ and $i > 0$, consider two ranges of i. First, if

$$1 \leqq i \leqq N_E$$

the counter can be allowed to fail; second, if

$$N_E < i \leqq N(N_{SD}) + N_E$$

then the counter cannot be allowed to fail, since if it does, there is only a maximum of N_E points which can survive.

Therefore, for $1 \leqq i \leqq N_E$,

$$PM_4\{i\} = P\{C\}\left[\sum_{k=\left\lfloor\frac{i+N_E+N_{SD}+1}{N_E+N_{SD}}\right\rfloor}^{N} PD_0\{k\}_1 B_{k-i} + \sum_{r=\left\lfloor\frac{i+N_{SD}-1}{N_{SD}}\right\rfloor}^{N} PD_0\{r\}_2\, C_{r-i}\,[(s+s_o)(m+m_o)]^{N_E}\right] + P\{\bar{C}\}\left[\sum_{t=\left\lfloor\frac{i+N_E+N_{SD}-1}{N_E+N_{SD}}\right\rfloor}^{N} PD_0\{t\}_1 B_{0-i}\right] \tag{10}$$

where $\lfloor \cdots \rfloor$ denotes "the integer part of $\cdots$." For

$$N_E < i \leqq N(N_{SD}) + N_E$$

$$PM_4\{i\} = P\{C\}\left[\sum_{k=\left\lfloor\frac{i+N_E+N_{SD}+1}{N_E+N_{SD}}\right\rfloor}^{N} PD_0\{k\}_1 B_{k-i} + \sum_{r=\left\lfloor\frac{i+N_{SD}-1}{N_{SD}}\right\rfloor}^{N} PD_0\{r\}_2\, C_{r-i}\,[(s+s_o)(m+m_o)]^{N_E}\right] \tag{11}$$

Verification that the $PM_4\{i\}$ are terms of a probability distribution is available but is omitted, because it is not considered to be particularly edifying.

Two useful variations of the configuration in Fig. 4 can be made and not affect the basic form of the analysis. The first variation considers the main deck extension to be of length zero. In this case the counter becomes a series element for any data point survival. Therefore, all that is needed to alter the basic derivation is to:

1. Set all $N_E = 0$

2. Set $P\{\bar{C}\} = 0$

3. Set all $B_{0-k} = 1$ for all k

4. Replace the expression in the $PM_4\{0\}$ term

$$1 - [(s + s_o)(m + m_o)]^{N+N_E} P\{E_s\}$$

with

$$1 - [(s + s_o)(m + m_o)]^N P\{C\} P\{E_s\}$$

For a main deck extension of zero length after collecting terms

$$PM_4\{0\} = 1 - [(s + s_o)(m + m_o)]^N P\{C\} P\{E_s\}$$

$$+ P\{C\}\left[\sum_{j=0}^{N} PD_0\{j\} C_{j-0}\right]$$

and for $1 \leq i \leq N(N_{SD})$

$$PM_4\{i\} = P\{C\}\left[\sum^{N} PD_0\{k\} C_{j-i}\right]$$

$$k = \left\lceil \frac{i + N_{SD} - 1}{N_{SD}} \right\rceil$$

The second variation considers the possibility of using the main deck or the main deck extension to clock the sequencing of the subdecks, instead of the counter. To alter the basic derivation for this variation, merely set $P\{\bar{C}\} = 0$, and $P\{C\} = 1$ in Eq. (9) and (11) and extend the range of i associated with Eq. (11) to

$$1 \leq i \leq N(N_{SD}) + N_E$$

Equation (10) is neglected for this variation. The final form of the probability is straightforward and will not be presented here.

The two variations discussed here for the multiplexer configuration in Fig. 4 are about the only ones which can be easily accommodated by this analysis. For example, how is a configuration analyzed in which the subdecks do not have the same probability distributions? This case would occur quite frequently in practice if signal conditioning associated with subdecks has different success probabilities, or if the multiplexer had more than two levels of commutation, where subdecks themselves have signals from lower commutation levels fed through them. Also, how could the configuration of Fig. 4 be analyzed if the outputs of the subdecks did not enter the first N channels of the main deck, but were dispersed into the inputs of the main deck and its extension with single points between subdeck inputs? How can transducer reliability be introduced into single point inputs if desired? In answer to these questions, a more generalized analysis method is presented in the next section.

IV. GENERAL ANALYSIS FOR A COMPLEX MULTIPLEXER CONFIGURATION WITH TWO LEVELS OF COMMUTATION

Figure 5 is a generalized multiplexer configuration. The main deck, D_0, receives pulses from the clock source and sequentially connects the output of N subdecks to the output. The main deck has a sync element which is considered to be part of its series elements of D_0. The subdecks $D_1, D_2, \cdots, D_N$ are considered to have probability distributions of surviving points

$$P_{N_1}D_1, P_{N_2}D_2, \cdots, P_{N_N}D_N$$

respectively, which are not necessarily equal with respect to the maximum number of surviving points possible, or with respect to the probability terms, even if the maximum number of surviving points are equal. The subdecks are clocked by a counter, which has a probability of success $P\{C\}$. It should be pointed out at this time *that the analysis of a generalized subdeck may also account for any single source inputs to the main deck.* Whether the analyst chooses to make single source inputs

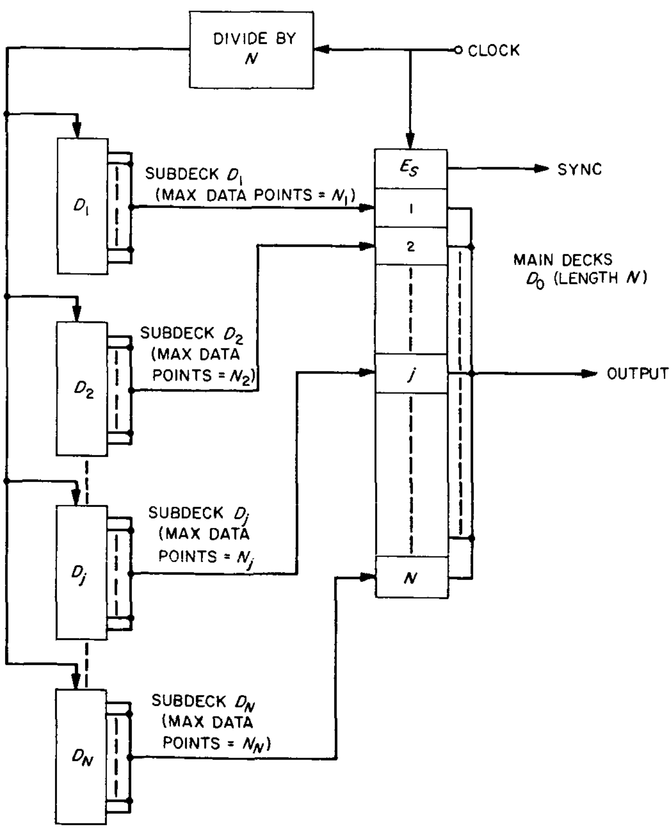

Fig. 5. Generalized multiplexer

probabilistic, by taking into account any signal conditioning equipment or transducers associated with the source, or deterministic, by neglecting such equipment, is at his own discretion. For example, with a probabilistic single source input, the probability distribution of the number of surviving points will be of the form $p(0) + p(1)$ with generating function $p(0) + p(1)S$. If the input is deterministic, $p(1)$ becomes 1 and the generating function becomes S. Single source inputs, of course, do not depend on the counter C, for success. The same consideration for establishing sync on the subdecks discussed in the preceding subsection also apply here.

In approaching this analysis, it should be stressed that the final output of the analysis procedure is a probability distribution of surviving points for the multiplexer. Thus, a procedure is set up whereby a one-dimensional array of numbers (the probability distribution) is generated. Since computer operations are in mind for any particular analysis task, matrix notation is used to define the operations necessary to generate the final output array. Many different matrices which will have to be employed to set up the analysis. If the operations were programmed on a computer, many matrix operations explicitly outlined here would lose their identity. The procedure outlined here also attempts to define iterative operations for building up the final output array such that the use of memory in computer operations could be kept to a tolerable minimum.

First in the analysis procedure is the methodical task of setting down definitions. Let

$P_{N_j}D_j\{i\} = P_{D_j}\{i\} \triangleq$ The probability of i data points surviving on subdeck D_j, $j = 1, 2, \cdots, N$; $i = 0, 1, \cdots, N_j$, where N_j is the maximum number of points the subdeck can deliver. Note that N_j will be dropped if no confusion will arise from doing so.

$\max_j (N_j) \triangleq N_{max}(1) \triangleq$ The maximum number of data points which can be delivered by any of the single subdecks.

$\underset{j \neq k}{max}(N_j + N_k) \triangleq N_{max}(2) \triangleq$ The maximum number of data points which can be delivered by two subdecks, over all combinations of two subdecks. Note that,

$$N_{max}(2) = \underset{j}{max}(N_j) + \underset{j \neq k}{max}(N_k)$$

From the above definitions $M_{max}(3), N_{max}(4), \cdots, N_{max}(N)$ should be clear. Also it should be evident that,

$N_{max}(N) = \sum_{\text{All } j} (N_j) \triangleq N_{max} \triangleq$ The maximum number of data points which the multiplexer can deliver.

Also let,

$|P_{N_{max}} M_5|_F \triangleq$ The final output matrix of dimension $(N_{max}+1) \times 1$. The number in the rth row will contain the probability of $r-1$ data points surviving in the multiplexer of Fig. 5.

$|P_{N_{max}} M_5|_I \triangleq$ The output matrix of dimension $(N_{max}+1) \times 1$ in the initial state. The term in the first row for this matrix is $P_N D_0 \{0, x_o\} = P_N D_0 \{0\}$ which will be recognized as the term for the probability of zero data points from the main deck. All the remaining elements are zero.

$$|APD_0| \triangleq \left| \begin{matrix} (x_i, i) = \\ \dfrac{P_N D_0 \{i, x_i\}}{\dbinom{x_i - 1}{i - 1}} \end{matrix} \right|$$

$\triangleq$ The auxiliary matrix of the main deck probability distribution where the element in the x_ith row and ith column is as indicated above. The matrix has dimensions of $N \times N$ and has elements of zero above the diagonal since $P_N D_0 \{i, x_i\}$ is not defined for $i > x_i$. The sum of the rows and columns of this matrix do not sum to 1.

$$|PD_j|_1 \triangleq \left| \begin{matrix} (i, j) = \\ PD_j (i-1) \end{matrix} \right|_1$$

$\triangleq$ The initial matrix of all subdeck probability distributions where the element in the ith row and jth column is the probability of $i-1$ data points surviving on subdeck D_j. This matrix will have dimension $[N_{max}(1) + 1] \times N$. The columns of this matrix will sum to 1.

$$|SPD_j|_1 \triangleq \left| \begin{matrix} (i, j) = \\ SPD_j \{i-1\} \end{matrix} \right|_1$$

$\triangleq$ The initial matrix of the single point subdecks, where the element in the ith row and jth column is the probability of $i-1$ points surviving on subdeck (single point) D_j for all D_j which have single point outputs. Otherwise the elements in the jth column will be zero. This matrix is considered to have a dimension of $[N_{max}(1) + 1] \times N$ for convenience purposes only. Note that if the analyst decided to make all single point inputs to the main deck deterministic this matrix would contain all 1's in the second row and zeros elsewhere.

$|F_i| \triangleq$ An $N \times 1$ matrix with a 1 in the ith row and zeros in the remaining rows. This matrix will be used to "filter" out terms from $|APD_0|$, hence F is used as a notation.

$|D_{[N_{max}+1] \times M}| \triangleq$ An $(N_{max}+1) \times M$ matrix with 1's in the ith row and jth column if $i = j$, and zeros elsewhere. For example,

$$|D_{5 \times 3}| = \left| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right|$$

This matrix will be used for dimensioning purposes to keep the matrix operations consistent.

There are enough definitions to start the procedure of building up the initial output matrix $|P_{N_{max}}M_5|_I$ to the final matrix of terms by successive addition of mutually exclusive terms. About the only way to make the explanation of the operations reasonably lucid is to explain the first step of the process, show its relation to the whole process and then generalize to succeeding steps. Now the form of the operations needed to build up the final output matrix will be as follows:

$$\begin{aligned}|P_{N_{max}}M_5|_F = |P_{N_{max}}M_5|_I + |D_{[N_{max}+1]\times[N_{max}(1)+1]}| \\ \times (P\{C\}|PD_j|_1|APD_0|\,|F_1| \\ + P\{\bar{C}\}|SPD_j|_1|APD_0|\,|F_1|)\end{aligned}$$

$+$ Second step $+$ third step $+ \cdots$ etc.

Viewing the first step in more detail shows the operation of

$$|APD_0|\,|F_1| = \left|\begin{matrix} PD_0\{1,1\} \\ PD_0\{1,2\} \\ PD_0\{1,3\} \\ \cdot \\ \cdot \\ \cdot \\ PD_0\{1,N\} \end{matrix}\right|$$

since

$$\binom{x_i - 1}{0} = 1$$

Here an $N \times 1$ matrix was produced whose elements are the probabilities of single data point successes for indices of $x_i = 1, 2, \cdots, N$ on the main deck. For each one of these indices and for a single main deck data point success, there is the possibility of $0, 1, \cdots, N_j$ successes from subdeck D_j where $j = x_i$. Consequently the operation of

expresses these possibilities in an $[N_{max}(1) + 1] \times 1$ matrix.This matrix must be multiplied by $P\{C\}$ because the counter must be successful for these possibilities to occur, and then it must be dimensioned for addition to the column matrix $|P_{N_{max}}M_5|$. The latter part of the first step will be self explanatory. It accounts for the counter in a failed state and successes for single data point sources into the main deck, which do not depend on clocking from the counter.

Operations in the second step are

$$\begin{aligned}|D_{[N_{max}+1]\times[N_{max}(2)+1]}|(P\{C\}|PD_j|_2\,|APD_0|\,|F_2| \\ + P\{\bar{C}\}\,|SPD_j|_2\,|APD_0|\,|F_2|)\end{aligned}$$

where

$$|PD_j|_2 \text{ and } |SPD_j|_2$$

have yet to be defined. First note that the operation of

$$|APD_0|\,|F_2| = \left|\begin{matrix} 0 \\ \dfrac{PD_0\{2,2\}}{\binom{1}{1}} \\ \dfrac{PD_0\{2,3\}}{\binom{2}{1}} \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{PD_0\{2,N\}}{\binom{N-1}{1}} \end{matrix}\right|$$

produces an $N \times 1$ matrix whose elements are the probability of any particular two main deck data point successes for a given index x_i, where x_i ranges from 2 to N. Now each term of two successes with index x_i, on the main deck creates the possibility of surviving points

$$|PD_j|\,|APD_0|\,|F_1| = \left|\begin{matrix} PD_1\{0\}PD_0\{1,1\} + PD_2\{0\}PD_0\{1,2\} + \cdots + PD_N\{0\}PD_0\{1,N\} \\ PD_1\{1\}PD_0\{1,1\} + PD_2\{1\}PD_0\{1,2\} + \cdots + PD_N\{1\}PD_0\{1,N\} \\ PD_1\{2\}PD_0\{1,1\} + \cdots \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ \cdot \\ PD_1\{N\}PD_0\{1,1\} + \cdots \qquad + PD_N\{N_N\}PD_0\{1,N\} \end{matrix}\right|$$

from combinations of two subdecks. One of the subdecks will be D_{x_i} and the other subdeck number may range from 1 to $x_i - 1$. Thus the combinations will be $D_{x_i}D_1$, $D_{x_i}D_2$, $\cdots$, $D_{x_i}D_{x_{i-1}}$. For any two subdecks, the probability distribution of the number of surviving points is given by convolution of their respective probability distributions, assuming that the decks operate independently. This assumption can be considered valid, since we have already accounted for the dependencies which exist between subdecks, the counter and main deck. Obviously the operation

$$|APD_0|\ |F_k|$$

will produce a column matrix whose elements are the probability of any particular k main deck data point successes for a given value of x_i, where x_i will range from k to N. Each term will give rise to

$$\binom{x_i - 1}{i - 1} = \binom{x_i - 1}{k - 1}$$

distinct k-fold convolutions of the probability distribution of the subdecks of which one is always PD_{x_i}.

Consider the matrix

$$|PD_j|_2$$

and how it is formulated from

$$|PD_j|_1$$

After explaining this process and consequently the remainder of step 2, it is possible to generalize to the process of producing

$$|PD_j|_{k+1} \text{ from } |PD_j|_k$$

and to any arbitrary step $k + 1$. First $|PD_j|_1$ is right multiplied by an $N \times N$ matrix with zeros on and below the main diagonal and 1's elsewhere. For example if $|PD_j|_1$ were

$$\begin{vmatrix} PD_1\{0\} & PD_2\{0\} & PD_3\{0\} \\ PD_1\{1\} & PD_2\{1\} & PD_3\{1\} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ PD_1\{4\} & PD_2\{4\} & PD_3\{4\} \end{vmatrix}$$

right multiplication by

$$\begin{vmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix}$$

would yield

$$\begin{vmatrix} 0 & PD_1\{0\} & PD_1\{0\} + PD_2\{0\} \\ 0 & PD_1\{1\} & PD_1\{1\} + PD_2\{1\} \\ 0 & \cdot & \cdot \\ & \cdot & \cdot \\ 0 & \cdot & \cdot \\ 0 & PD_1\{4\} & PD_1\{4\} + PD_2\{4\} \end{vmatrix}$$

This operation has the effect of shifting all columns in $|PD_j|$ once to the right and adding all elements in the same row and in columns, say 1 through $c - 1$, to the element in column c. The result is an $[N_{max}(1) + 1] \times N$ matrix which will be called $|PD_jSUM|_1$. Now the element in the rth row mth column of $|PD_j|_2$ is the coefficient a_{r-1} in the polynomial in S, say $a_0 + a_1S + a_2S^2 + \cdots$ generated by

$$\left((|PD_jSUM|_1\ |F_m|)^T \begin{vmatrix} 1 \\ S \\ S^2 \\ \cdot \\ \cdot \\ S^{N_{max}(1)} \end{vmatrix} \right) \left((|PD_j|_1 |F_m|)^T \begin{vmatrix} 1 \\ S \\ S^2 \\ \cdot \\ \cdot \\ S^{N_{max}(1)} \end{vmatrix} \right)$$

and $|PD_j|_2$ will be an $[N_{max}(2) + 1] \times N$ matrix. The generation of $|PD_j|_{k+1}$ from $|PD_j|_k$ can now be generalized. Let

$|SUM_k| \triangleq$ An $N \times N$ matrix with zeros on and below the main diagonal, zeros on the $k - 1$ diagonals immediately above the main diagonal, and 1's elsewhere.

$|S_k| \triangleq$ An $[N_{max}(k) + 1] \times 1$ matrix whose element in the rth row is equal to S^{r-1}.

For the general operation, $|PD_j|_{k+1}$ is an

$$[N_{max}(k + 1) + 1] \times N$$

matrix whose element in the rth row and mth column is the coefficient a_{r-1} in the polynomial

$$a_o + a_1S + a_2S^2 + a_3S^3 + \cdots =$$

$$((|PD_j|_k\, |SUM_k|\, |F_m|)^T\, |S_k|)\, ((|PD_j|_1\, |F_m|)^T\, |S_k|)$$

Note that in an iterative procedure, after $|PD_j|_{k+1}$ has been derived, $|PD_j|_k$ (for $k \neq 1$) need not be returned in memory. It should be noted finally that the same procedure above is applied in producing

$$|SPD_j|_{k+1} \text{ from } |SPD_j|_k$$

Finally, the process of building up the final output probability distribution matrix $|P_{N_{max}}M_5|_F$ can be generalized to

$$|P_{N_{max}}M_5|_F = |P_{N_{max}}M_5|_I + \sum_{k=1}^{N} |D_{[N_{max}+1] \times [N_{max}(k)+1]}|$$

$$(P\{C\}\, |PD_j|_k\, |APD_0|\, |F_k|$$

$$+\ P\{\bar{C}\}\, |SPD_j|_k\, |APD_0|\, |F_k|)$$

There is no general proof that $|P_{N_{max}}M_5|_F$ will contain a probability distribution. There is a proof for two *particular* cases, but they are omitted in this Report.

Variations in the general configuration of Fig. 5 can be introduced without nullifying the applicability of the procedure. For example, if the subdecks derive clocking from the main deck instead of the counter, this can be easily accounted for by methods similar to those demonstrated in the latter part of the preceding subsection. The form of matrix operations however is not altered; only particular elements in certain matrices. Multiplexer configurations which have more than two levels can easily be accommodated by this procedure by considering the subdecks to be analogous to main decks for the next lowest level of commutation.

V. ANALYSIS APPROACH TO A MULTIPLEXER CONFIGURATION WITH DEPENDENT CLOCKING FUNCTIONS

A. Introduction

Figure 6 is a diagram of a multiplexer in which clocking functions are performed redundantly by the decks themselves. The figure is not general, but can easily be made so. The main clock enters the highest data level in the multiplexer and clocks the decks in parallel. The deck clock outputs are routed to synchronizers, which are essentially ac coupled *OR* gates. The output of the synchronizers clocks the next lowest level and performs a resetting function on the decks which input to it in order to maintain proper mutual phasing of these decks. It is evident that the deck lengths on any data level are all equal, since the synchronizer scheme would be useless otherwise. This fact, however, does not tend to simplify the analysis to any great degree since the individual probability distribution of decks on the same level may differ because of signal conditioning equipment differences or because of a different number of sub-deck inputs to the decks. External clocking is supplied to the isolator deck and the highest data level through a counter. The counter maintains correct phasing between the isolator deck and the decks in the highest data level.

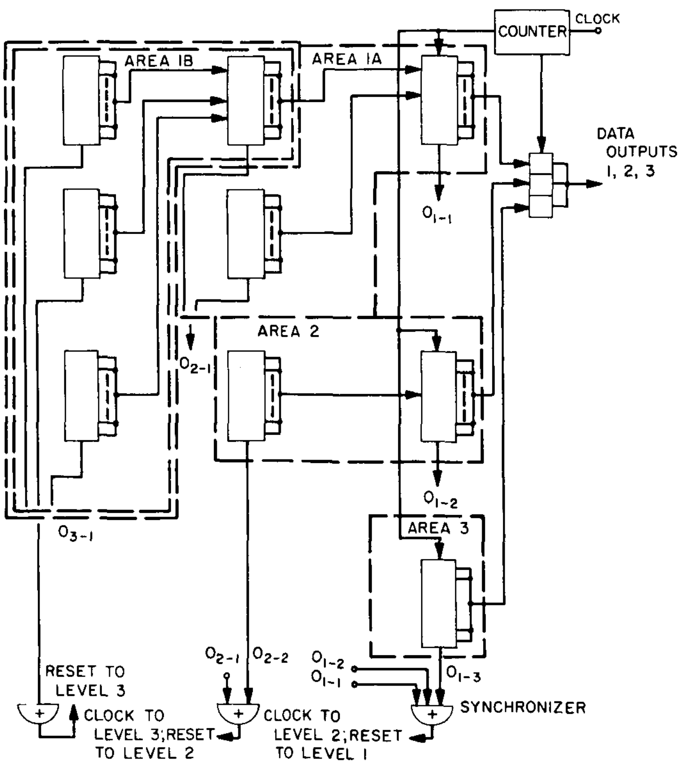

Fig. 6. Multiplexer with redundant clocking functions

The main emphasis in this subsection will be on the approach to the analysis. There are no answers which can be put neatly in closed form. The approach outlined here will be sketchy, but its purpose will be served if the impression is left that for any specific configuration, there will always be some fringe outcome enumeration which has to be done. A specific example of this was noted when a counter was postulated in the last section, and by doing this the final closed form for the answer was made somewhat less aesthetic in a mathematical sense.

B. Approach to the Analysis

In the multiplexer under scrutiny it was noted that the presence of synchronizers eliminates the need for most of the individual deck synchronization elements ;however, there is a price to be paid. If any synchronizer fails, clocking is lost to all lower data levels, and synchronization is lost to the decks which input to the failed synchronizer. Each deck has its own resetting and short counting feature, but this does not guarantee that all decks will stay in synchronization after a synchronizer failure. True, if the data is distinguishable over the ensemble of data points of the decks, and redundant with respect to time histories, then it is quite possible that ground data processing could eventually decommutate the data. Whether the analyst decides to consider this mode a success is up to his own discretion. A synchronizer failure is defined here as a failure to all lower data levels and to its own level. Also, if a synchronizer does not fail, then the requirement for clocking to this next lower level is that at least one deck in the synchronizer's level have all its memory elements succeed. This condition will also insure that data points on decks, which may be short counting, can be decommutated, provided that ground data processing is sensitive to this kind of condition on the spacecraft.

Figure 6 is divided into three areas. The basis for dividing the multiplexer like this is as follows: All decks which eventually output to, for example, DATA OUTPUT J, are included in the area j. Temporarily the isolator deck, the counter, and the synchronizers will be disregarded. The multiplexer configurations in the respective areas can be analyzed independently of each other, but at the same time, a bookkeeping system can be set up for the probability distributions as a function of the clocking outputs to the synchronizer, i.e., the 0_{k-j} shown in the figure. These probability distributions are in essence joint probability distributions of the number of surviving points and number of clocking outputs which are successful. A clocking output 0_{k-j} from any area j is defined as being successful if at least one of the decks in level k in area j is successful in delivering an output. These joint probability distributions will be of the form

$$p\{i, 0_{1-j}, 0_{2-j}, \cdots, 0_{L-j}\}$$

where L is the maximum number of levels in area j and the 0_{k-j} are defined to assume value 0 otherwise. How is this accomplished? From the figure it can be seen that areas 2 and 3 are simple and can readily be accommodated using the general analysis scheme in the previous subsection with minor modifications for bookkeeping. Area 1 is somewhat more difficult. However, this area can be broken down into Area 1A, and Area 1B can be analyzed separately up to the output of the data deck which feeds into Area 1A, and then the two separate analyses can be combined.

The question may be asked at this point how these joint probability distributions in any particular area (or subarea) are derived. Both probability distributions of the individual decks, $PD_j\{i\}$ and $PD_j\{i, x_i\}$ can easily be separated into those terms in which the deck clock output is successful, and those terms in which a clock output is not successful. The terms will be mutually exclusive.

Once this is done a bookkeeping system can be set up in applying succeeding general analysis procedures, as outlined in the preceding section, to build up arrays of the joint probability distributions for each condition of the outputs in each area. Once these joint distributions for each area are produced, appropriate distributions may be combined by convolution operations subject to failure modes postulated in the synchronizers, counter, the isolator deck, and the outputs of each area. To outline this approach in further detail would only result in an elaborate amount of unnecessary definitions and detailed operations.

VI. SOME RESULTS AND AREAS FOR FURTHER STUDY

The probability distribution for the number of surviving points on a multiplexer deck $PD\{i\}$ has been programmed on a 1620 computer for entering arguments of deck length, absolute failure probabilities, and conditional failure probabilities for switches and memory elements. Table 1 presents the results of two runs. Run times from execution to final output were 5 min for a deck of length 10, and 24 min for a deck of length 20. Run time increases exponentially as a function of deck length. It is fairly evident that complete multiplexer analyses are not within the capability of the 1620 from a time standpoint.

Table 1. Two computer runs to obtain probability distribution of number of surviving points on multiplexer deck of length 10.

Points surviving (i)[a]	Probability of exactly i points surviving		Probability that the number of surviving points is less than i	
	Run 1[b]	Run 2[c]	Run 1	Run 2
10	0.2087	0.2087	0.999999999[d]	0.999999997[d]
9	0.1217	0.2441	0.7912	0.7912
8	0.0381	0.1424	0.6694	0.5470
7	0.0145	0.0691	0.6312	0.4045
6	0.0110	0.0455	0.6167	0.3355
5	0.0109	0.0424	0.6056	0.2898
4	0.0112	0.0440	0.5946	0.2474
3	0.0115	0.0465	0.5834	0.2033
2	0.0118	0.0493	0.5718	0.1567
1	0.0122	0.0521	0.5599	0.1074
0 (due to open failures)	0.0125	0.0552	0.5477	0.0552
0 (due to closed failures)	0.5351	0.0000		

[a] $(i)\ P\{E_s\} = 1.0$

[b] Run 1: $s = 0.90$; $m = 0.950$; $s_0 = 0.050$; $m_0 = 0.0250$; conditional probability of a switch open $= 0.50$; conditional probability of a memory element open $= 0.50$

[c] Run 2: $s = 0.90$; $m = 0.950$; $s_0 = 0.10$; $m_0 = 0.050$; conditional probability of a switch open $= 1.0$; conditional probability of a memory element open $= 1.0$

[d] Indicates results of a numerical check to verify the probability distribution and is an actual printout; other numerical results were truncated without rounding off.

Further study in the area of multiplexer reliability analysis can be divided into three parts:

1. The intrinsic analysis methods

2. The continual process of analysis verification by empirical methods

3. The mating of analysis methods with success criteria to improve the synthesis process

In the area of the analysis methods themselves, emphasis is required in improving the realism of the analysis on the basic deck model and should be expanded of course in scope to include models of other deck configurations. Another area which needs great emphasis is the organization of computer subroutines for frequently used operations (such as convolution) which can be speedily combined to form a program for a specific multiplexer configuration. Reliability analysis in the multiplexer area has been extremely slow in the past, to the point of having negligible value in the design process. More speed for the analysis process, combined with some degree of realism should be a specific goal in organizing computer programs.

Although empirical results take time to be generated, their value in verification and improvement of basic analysis techniques cannot be denied. Consequently this second area for further study should receive due emphasis. Statistical testing of multiplexer configurations, the basic elements, and the devices themselves to obtain failure-effects data on all levels of the multiplexer are a necessary part of this area.

Analyses results, per se, do not yield significant inputs to a design process unless they can be mated with success criteria. Obviously, if the only success criteria for a multiplexer specified no degradation in the transfer function of the multiplexer, the analysis in this Report would not be required; it would only be necessary to count the parts and compute failure rates. However, it is certainly logical to assume other kinds of success criteria will be required. The outlining of these success criteria in mathematical terms and which approximate intuitive notions on what success should be are not readily available at present. The output of the analyses here is amenable to such criteria as the expected value of the number of surviving points, the expected value of some function of the number of surviving points, or the probability that the number of successful points will not drop below some specified number. Though one of the deck models has provision for retaining the identity of individual points, most of the output results do mask the identity of individual points and consequently their individual importance. If success criteria based upon some weighting function of the input data are desirable, then the type of analysis considered here would have to be extended. In this type of analysis there is a possibility for assigning individual data points weights greater than one and deriving output figures of merit for specific deck or multiplexer configurations. The question of usable success criteria and analysis methods which are compatible with them will certainly not be answered by such general discussions and remains a necessary area for further studies.

APPENDIX A

Verification of $P_N D\{i\}$ as a Probability Distribution

In the body of this Report, $P_N D\{i\}$ had the form

$$P_N D\{0\} = 1 - [(s+s_o)(m+m_o)]^N P\{E_s\} + P\{E_s\} \times \left[(ms_o)^N + \sum_{r=1}^{N} m^{N-r}(m_o)^r \sum_{k=r}^{N} \binom{k-1}{r-1} (s+s_o)^k (s_o)^{N-k}\right]$$

and

$$P_N D\{i\} = P\{E_s\}\left[m^N \binom{N}{N-i} s^i (s_o)^{N-i} + s^i \sum_{r=1}^{N-i} m^{N-r}(m_o)^r \sum_{k=r}^{N-i}\binom{k-1}{r-1} \times \binom{N-k}{N-i-k}(s+s_o)^k (s_o)^{N-i-k}\right]$$

for $i = 1, 2, \cdots, N$. It is to be verified that

$$\sum_{i=0}^{N} P_N D\{i\} = 1$$

This can be demonstrated by proving that

$$\sum_{i=0}^{N}\left[m^N \binom{N}{N-i} s^i (s_o)^{N-i} + s^i \sum_{r=1}^{N-i} m^{N-r}(m_o)^r \sum_{k=r}^{N-i}\binom{k-1}{r-1} \times \binom{N-k}{N-i-k}(s+s_o)^k (s_o)^{N-i-k}\right] \quad \text{(A-1)}$$

is equal to

$$[(s+s_o)(m+m_o)]^N$$

Using a change of variable $i = N - j$ to put the left side statement (A-1) in more convenient form, with some rearrangement, gives

$$\sum_{j=0}^{N} m^N \binom{N}{j} s^{N-j}(s_o)^j + \sum_{j=0}^{N}\sum_{r=1}^{j}\sum_{k=r}^{j} m^{N-r}(m_o)^r \times \binom{k-1}{r-1}\binom{N-k}{j-k}(s+s_o)^k s^{N-j}(s_o)^{j-k}$$

Examining the binomial coefficients and limits in the last expression, it will be seen that the following is true:

$$r \leq k \leq j \leq N$$

Therefore, first sum over the j indices assuming fixed values of k and r, then sum over k assuming a fixed value of r, and finally sum over r from 1 to N. The first step yields

$$m^N (s+s_o)^N + \sum_{r=1}^{N} m^{N-r}(m_o)^r \sum_{k=r}^{N}\binom{k-1}{r-1} \times \sum_{j=k}^{N}\binom{N-k}{j-k} s^{N-j}(s_o)^{j-k}(s+s_o)^k$$

The last inner summation of the second term is

$$\sum_{j=k}^{N}\binom{N-k}{j-k} s^{N-j}(s_o)^{j-k}(s+s_o)^k$$

By a translation of variable, i.e., $x = j - k$,

$$\sum_{x=0}^{N-k}\binom{N-k}{x} s^{N-k-x}(s_o)^x (s+s_o)^k = (s+s_o)^{N-k}(s+s_o)^k = (s+s_o)^N$$

Therefore, statement (A-1) becomes

$$m^N (s+s_o)^N + (s+s_o)^N \sum_{r=1}^{N} m^{N-r}(m_o)^r \sum_{k=r}^{N}\binom{k-1}{r-1}$$

Now summing over k involves the last summation in the second term or

$$\sum_{k=r}^{N}\binom{k-1}{r-1}$$

Letting $k - r = y$, for this term

$$\sum_{y=0}^{N-r}\binom{y+r-1}{r-1} = \binom{N}{r}$$

from the relation

$$\sum_{\gamma=0}^{M}\binom{\gamma+r-1}{r-1} = \binom{M+r}{r}$$

with $\gamma = y$, $M = N - r$ (Ref. 1). Therefore, statement (A-1) reduces to

$$m^N (s+s_o)^N + (s+s_o)^N \sum_{r=1}^{N}\binom{N}{r} m^{N-r}(m_o)^r = m^N(s+s_o)^N + (s+s_o)^N[(m+m_o)^N - m]^N = [(s+s_o)(m+m_o)]^N$$

which is what was to be proved.

APPENDIX B
Verification of $P_N D\{i, x_i\}$ as a Probability Distribution

In the body of this Report, $P_N D\{i,x_i\}$ had the form

$$P_N D\{i,x_i\} = P\{E_s\}\binom{x_i-1}{i-1} s^i m^{x_i} (s_o)^{x_i-i}\Big[(ms_o)^{N-x_i} + \sum_{r=1}^{N-x_i} m^{N-x_i-r}(m_o)^r \sum_{k=r}^{N-x}\binom{k-1}{r-1}(s+s_o)^k (s_o)^{N-x_i-k}\Big]$$

$$i = 1, 2, \cdots, N;\; x_i = i, i+1, \cdots, N$$

and

$$P_N D\{0,x_0\} \overset{\Delta}{=} P_N D\{0\} = 1 - [(s+s_o)(m+m_o)]^N P\{E_s\}$$

As in Appendix A, it must be proved that

$$\sum_{i=1}^{N}\sum_{x_i=i}^{N} P_N D\{i,x_i\} + P_N D\{0,x_0\} = 1$$

which can be demonstrated by proving that

$$\frac{1}{P\{E_s\}}\sum_{i=1}^{N}\sum_{x_i=i}^{N} P_N D\{i,x_i\} + (ms_o)^N + \sum_{r=1}^{N} m^{N-r}(m_o)^r \sum_{k=r}^{N}\binom{k-1}{r-1}(s+s_o)^k (s_o)^{N-k} \qquad \text{(B-1)}$$

is equal to

$$[(s+s_o)(m+m_o)]^N$$

Reversing the order of summation and noting that there are only three terms involved in the summation over i, namely,

$$\sum_{x_i=1}^{N}\sum_{i=1}^{x_i}\binom{x_i-1}{i-1} s^i (s_o)^{x_i-i} \cdots \qquad \text{(B-2)}$$

statement (B-1) can be reduced immediately to one summation over x_i by letting $y = i - 1$. Thus statement (B-1) becomes

$$\sum_{x_i=1}^{N}\sum_{y=0}^{x_i-1}\binom{x_i-i}{y} s^{y+1}(s_o)^{x_i-1-y} \cdots$$

$$= \sum_{x_i=1}^{N} s(s+s_o)^{x_i-1} \cdots$$

after successively using the relation (Ref. 2)

$$a + ar + ar^2, \cdots, n \text{ terms} = a\frac{(r^n-1)}{r-1}$$

for the inner summations, statement (B-1) becomes

$$\sum_{x_i=1}^{N} s(s+s_o)^{x_i-1} m^{x_i}\left[(ms_o)^{N-x_i}(1 - y + y(z)^{N-x_i}) + (ms_o)^N (1 - y + y(z)^{N-x_i})\right]$$

where

$$y = \left(\frac{m_o}{m}\right)\left(\frac{s+s_o}{s_o}\right)\left(\frac{1}{z-1}\right)$$

and

$$z = \left(\frac{s+s_o}{s_o}\right)\left(\frac{m+m_o}{m}\right)$$

After an interminable amount of algebraic reduction, the last set of equations finally reduces to

$$[(s+s_o)(m+m_o)]^N$$

which is what was to be proved.

APPENDIX C
Generating Functions — A Quick Refresher

Consider M independent probability distributions

$$p_1\{X_1\}, p_2\{X_2\}, \cdots, p_j\{X_j\}, \cdots, p_M\{X_M\}$$

where X_j is a random variable which can assume integer values over some defined range. For the cases considered in this Report, X_j will only assume non-negative values $0, 1, 2, \cdots, N_j$, where N_j is the maximum value which X_j may assume.

Suppose the probability distribution for the sum of the random variables was desired, say

$$Z = X_1 + X_2 +, \cdots, + X_j +, \cdots, + X_M$$

then

$$p\{Z\} = p_1\{X_1\} * p_2\{X_2\} *, \cdots, * p_j\{X_j\} *, \cdots, * p_M\{X_M\}$$

where $*$ denotes the convolution operation. The variable Z will assume discrete values from 0 to

$$\sum_{\text{All } j} N_j$$

Define

$$A_j(S) \triangleq \sum_{X_j=0}^{N_j} p_j\{X_j\} S^{X_j}$$

which is a polynomial in S with $N_j + 1$ terms. This is called the generating function for the probability distribution $p_j\{X_j\}$. Now the generating function for $p\{Z\}$, say $B(S)$, is

$$B(S) = \prod_{j=1}^{M} A_j(S) = \sum_{k=0}^{k=\sum_{\text{All } j} N_j} p(Z = k) S^k \qquad \text{(C-1)}$$

The coefficients of S^k terms in this polynomial are, as can be seen from Eq. (C-1), the probabilities that the variable Z will assume the value k. Of course

$$\sum_{\text{All } k} p(Z = k) = 1$$

The generating function furnishes a useful tool in deriving probability distributions for the sum of random variables and essentially converts the task of outcome enumeration into multiplication of polynomials with the added convenience of a built-in bookkeeping system. For application in this Report, the random variables can be considered to be the number of data points surviving in a multiplexer deck. Consequently, operations with generating functions are extremely useful in compiling probability distribution for the number of surviving points in more than one deck.

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