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AN EXTENSION OF THE GREEN'S THEOREM APPROACH TO HIGHER DIMENSIONS

by George W. Haynes

Perpared under Contract No. NAS 2-2351 *by*
MARTIN COMPANY
Denver, Colo.
for

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On the Optimality of a Totally Singular Vector Control: An Extension of the Green's Theorem Approach to Higher Dimensions*

Summary

An extension of the Green's theorem approach to higher dimensions has been derived for the determination of the optimality of totally singular vector controls governing n dimensional nonlinear systems with the (n-1) dimensional vector control appearing linearly. The essential condition that motivates the analysis is that the coefficients of the vector control, when viewed as tangent vectors, form a complete system of partial differential equations of order (n-1).

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1. Introduction

We are concerned with the optimality of totally singular vector controls governing dynamical systems of the form

$$\dot{x}_\alpha = A_\alpha(x) + B_{\alpha r}(x)u_r \quad (\alpha = 1 \ldots n, \; r = 1 \ldots (n-1))^* \qquad 1.1$$

and the extension of the Green's theorem approach[1,2] to higher dimensions to evaluate the optimality of such totally singular** vector controls. The problem of defining a control set for the dynamical system (1.1) is of paramount importance, because the singularity of a control is an inherent feature of the dynamical system and the function or functional to be extremized, and not the control set per se. It is not the intent here to rule out a singular control because of the limitations on the control imposed by a given control set. Therefore, the maximal control set which overcomes these limitations must necessarily include distributions. It should be noted that Kreindle[4] and Neustadt[5] have considered such control sets in their treatment of linear systems. However, for the nonlinear system considered, we shall effectively circumvent a difficult problem by replacing the dynamical system (1.1) by the equivalent pfaffian system

$$dx_\alpha = A_\alpha(x)dt + B_{\alpha r}(x)dy_r \qquad 1.2$$

where the control has the representation $u_r = \frac{dy_r}{dt}$ when it exists.

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* Greek letters will assume the values 1 to n, and Roman letters 1 to (n-1). The exceptions to this rule are noted where they occur.

** Following the usage in Reference 3, a totally singular vector control means that all the components of the control are singular.

The usual summation convention on repeated indices is used. The solutions to the pfaffian system (1.2) will be parametrized by $x(\sigma)$, $y(\sigma)$ and $t(\sigma)$ with $t(\sigma)$ monotone such that

$$dx_{\alpha}(\sigma) \overset{\sigma}{\equiv} A_{\alpha}(x(\sigma))\,dt(\sigma) + B_{\alpha r}(x(\sigma))\,dy_r(\sigma) \tag{1.3}$$

It is assumed that the vector $x(\sigma)$ has values confined to some simply connected region $D \subset R^n$, also $A_{\alpha}(x)$ and $B_{\alpha r}(x)$ are twice continuously differentiable in D. Furthermore, it is assumed that the system (1.1) is controllable, which implies that there does not exist a scalar function $W(t,x)$ such that the hypersurface $W(t,x)$ = constant contains all the solutions to the system (1.1) independent of the controls. From this can be inferred[6] that the system of n partial differential equations

$$\begin{aligned} \frac{\partial W}{\partial t}(t,x) + \frac{\partial W}{\partial x_{\alpha}}(t,x)\,A_{\alpha}(x) &= 0 \\ \frac{\partial W}{\partial x_{\alpha}}(t,x)B_{\alpha r}(x) &= 0 \end{aligned} \tag{1.4}$$

is not complete, so that another independent partial differential equation can be determined by the Poisson operator to yield the non-existence of a non-trivial $W(t,x)$, namely $\frac{\partial W}{\partial t}(t,x) = \frac{\partial W}{\partial x_{\alpha}}(t,x) = 0$.

The problem posed is to determine the control which steers the state from some initial point x^o to some final point x^f in minimum time. We shall now state a further condition which in essence is the sine qua non of the Green's theorem approach to higher dimensions.

Condition A

The columns of the $B_{\alpha r}(x)$ matrix are (n-1) linearly independent tangent vectors, furthermore the system of partial differential equations formed with the tangent vectors

$$\frac{\partial V}{\partial x_\alpha}(x)\, B_{\alpha r}(x) = 0 \qquad 1.5$$

is a complete system of order (n-1).

This condition has three important implications which will be developed in detail in later sections; however, for the purposes of motivation we shall briefly describe what these implications are:

1) Condition A guarantees the existence of a single unique pfaffian to system (1.1).

2) It provides a necessary condition for the existence of an optimal totally singular vector control. The sufficiency condition for the existence of an optimal totally singular vector control follows from the Green's theorem application.

3) On applying the n-dimensional Green's theorem to the single pfaffian, there results $\frac{n(n-1)}{2}$ hypersurfaces whose interpretation as singular hypersurfaces (assuming an analogy with the 2-dimensional Green's theorem approach) is doubtful since we need only (n-1) such hypersurfaces to specify the totally singular vector control. However, Condition A enables an integrability argument to be invoked and from this it can be shown that no more than (n-1) hypersurfaces are obtained which can then be interpreted as singular hypersurfaces.

2. Existence of a Totally Singular Vector Control

Let $\psi_\alpha(x)$ be a vector orthogonal to the columns of $B_{\alpha r}(x)$ that is

$$\psi_\alpha(x)\ B_{\alpha r}(x) \overset{x}{\equiv} 0 \qquad 2.1$$

Hence the pfaffian system (1.2) can be expressed as a single pfaffian, which is unique to within an arbitrary multiplicative factor by virtue of the linear independence of the columns of $B_{\alpha r}(x)$

$$\psi_\alpha(x) dx_\alpha = \psi_\alpha(x)\ A_\alpha(x) dt \qquad 2.2$$

Let $x = \phi^s(t)$, $t \in [t_o, t_f]$ represent parametrically a totally singular arc in state space, which transfers the state from $x^o = \phi^s(t_o)$ to $x^f = \phi^s(t_f)$. Hermes[3] has demonstrated that the totally singular arc automatically satisfies the pfaffian (2.2). The question now arises whether it is possible to obtain a parametric representation in state space of the same transfer (but not necessarily the same arc) by $x = x(\sigma)$, $\sigma \in [\sigma_o, \sigma_f]$ and t = constant, that is, satisfying the pfaffian

$$\psi_\alpha(x(\sigma))\ dx_\alpha(\sigma) \overset{\sigma}{\equiv} 0 \qquad 2.3$$

with $x^o = x(\sigma_o)$ and $x^f = x(\sigma_f)$.

If this is possible, then the totally singular vector control will not be optimum, since the transfer of the state vector from x^o to x^f can be synthesized by suitable impulses to achieve the transfer in zero time. Therefore the problem resolves down to the question of accessibility of points by trajectories satisfying the pfaffian

$$\psi_\alpha(x)dx_\alpha = 0 \qquad 2.4$$

The resolution of this question leads to the following lemma.

<u>Lemma 2.1</u>. <u>A necessary condition that an optimal totally singular vector control exists is that the pfaffian $\psi_\alpha(x)dx_\alpha = 0$ be integrable</u>.

<u>Proof</u>. The proof makes use of the following theorem[7] and its contrapositive which we shall state formally.

<u>Theorem (Caratheodory)</u>. If a pfaffian $\psi_\alpha(x)dx_\alpha = 0$ has the property that in every arbitrary close neighborhood of a given point $\bar{x}$ there exist points which are inaccessible from $\bar{x}$ by trajectories satisfying the pfaffian, then the pfaffian is integrable.

<u>Contrapositive</u>. If the pfaffian $\psi_\alpha(x)dx_\alpha = 0$ is not integrable, then there exists some neighborhood of a given point $\bar{x}$ in which all points are accessible by trajectories satisfying the pfaffian.

Assume the pfaffian is not integrable, then if the neighborhoods of accessibility from given points are closed, the conclusion of the lemma is immediate. The maximal neighborhood of accessibility from a given point will necessarily be open. Assume the neighborhoods to be open and consider a neighborhood of accessibility $\eta(\bar{x})$ of a given point $\bar{x}$ on the singular arc defined by $\bar{x} = \phi^s(\bar{t})$, $t_o < \bar{t} < t_f$. If the singular arc $x = \phi^s(t)$, $t \in [t_o, t_f]$ is contained within the neighborhood $\eta(\bar{x})$, then once again the conclusion of the lemma is immediate. If this is not the case, then there will exist a t_1 and a t_2 such that the points of interesection of the singular arc with the neighborhood $\eta(\bar{x})$ will be given by $x = \phi^s(t)$; $t_1 < t < t_2$. Consider the point $x' = \phi^s(t_1)$ and its neighborhood of accessibility $\eta(x')$. The intersection of $\eta(x')$ and $\eta(\bar{x})$ must be non-void, otherwise there will exist points of $\eta(x')$ which are inaccessible from x' and hence contradict the assumption that the pfaffian is not integrable. Therefore if the pfaffian is not integrable, then there will exist an $x(\sigma)$ satisfying $\psi_\alpha(x(\sigma))\,dx(\sigma) \overset{\sigma}{\equiv} 0$ that transfers the state vector from x^o to x^f in zero time and the singular arc will not be optimum, which completes the proof of the lemma.

If the pfaffian $\psi_\alpha(x)dx_\alpha = 0$ is integrable, then there exists a non-zero integrating factor $\mu(x)$ and a function $V(x)$ such that

$$\psi_\alpha(x) \overset{x}{\equiv} \mu(x) \frac{\partial V}{\partial x_\alpha}(x) \qquad 2.5$$

Therefore from (2.1) we have

$$\frac{\partial V}{\partial x_\alpha}(x)\; B_{\alpha r}(x) \overset{x}{\equiv} 0 \qquad 2.6$$

and from Condition A we are assured that such a $V(x)$ exists so that the pfaffian $\psi_\alpha(x)dx_\alpha = 0$ is integrable. It should be noted that the pfaffian

$$\psi_\alpha(x)dx_\alpha - \psi_\tau(x)A_\tau(x)dt = 0$$

is not integrable[3], otherwise this would contradict the assumption that the system (1.1) is controllable.

3. N-dimensional Green's Theorem

The problem of extremizing 2-dimensional line integrals of the form

$$I = \int_{x^o}^{x^f} \left[a_1(x_1,x_2)dx_1 + a_2(x_1,x_2)dx_2\right] \qquad 3.1$$

is a fairly simple one, since the relative optimality of two distinct trajectories may be compared directly under suitable smoothness conditions, by an application of Green's theorem. The unique feature of this approach when applied to two dimensional nonlinear systems, in which the control (single component) appears linearly[2], is that the projection of the singular arc in the state space is obtained immediately by $\omega(x_1,x_2) = 0$ where

$$\omega(x_1,x_2) = \frac{\partial}{\partial x_1} a_2(x_1,x_2) - \frac{\partial}{\partial x_2} a_1(x_1,x_2) \qquad 3.2$$

The utility of the method is immediately obvious since only simple algebraic manipulations are required to generate $\omega(x_1,x_2)$. Before applying Green's theorem to the control problem posed, we shall briefly review the analogue of Green's theorem in higher dimensions.

Let $x_\alpha = s_\alpha(z_1,z_2)$, $s_\alpha \in C^2$ define an orientable surface S in $D \subset R^n$; furthermore, let $z_1(\Theta)$ and $z_2(\Theta)$ define a Jordan curve Γ in S enclosing $\bar{S}$. We shall assume that the mapping s is one to one on S for every pair $(x_\alpha, x_\beta; x_\alpha \neq x_\beta)$ and the corresponding Jacobians $J_{\alpha\beta}$ do not change sign on S and can vanish only on a subset of S having zero Jordan content[8] (in R^2). Since S is orientable then there is a unique direction associated with the Jordan curve Γ and the direction of each projection of Γ on to the coordinate planes, denoted by $\Gamma_{\alpha\beta}$ will be determined by the sign of the corresponding Jacobian $J_{\alpha\beta}$.

Consider the line integral around Γ

$$I = \int_{\Gamma(x)} a_{\alpha}(x)dx_{\alpha} \tag{3.4}$$

where $a(x) \in C'$ in R^n.

Transforming the line integral to the surface coordinates (z_1, z_2) and using Green's theorem we obtain

$$I = \oint_{\Gamma(x(z))} \left\{ a_{\alpha}(s(z)) \frac{\partial s_{\alpha}}{\partial z_1}(z)dz_1 + a_{\alpha}(s(z)) \frac{\partial s_{\alpha}}{\partial z_2}(z)dx_2 \right\}$$

$$= \int_{\bar{S}} \frac{\partial a_{\alpha}}{\partial x_{\beta}} J_{\beta\alpha} \, dz_1 dz_2$$

Defining $\bar{S}_{\alpha\beta}$ to be the projection of $\bar{S}$ on to the coordinate planes, then taking the inverse transformation and renumbering α and β we obtain

$$I = \int_{\bar{S}_{\alpha\beta}} \omega_{\alpha\beta} \, dx_{\beta} \, dx_{\alpha} \qquad \begin{matrix} \alpha = 1 \ldots n \\ \beta = \alpha+1, \ldots, n \end{matrix} \tag{3.5}$$

where

$$\omega_{\alpha\beta} = \frac{\partial a_{\alpha}}{\partial x_{\beta}} - \frac{\partial a_{\beta}}{\partial x_{\alpha}} \tag{3.6}$$

In applying the higher dimensional Green's theorem to the control problem (1.1) we are immediately faced with a paradoxical situation.

In analogy with the two dimensional Green's theorem approach, if we interpret $\omega_{\alpha\beta} = 0$ as singular hypersurfaces, then there will exist $\frac{n(n-1)}{2}$ such hypersurfaces, whereas we need only (n-1) hypersurfaces to determine the (n-1) components of the totally singular control. When n=2 the number of hypersurfaces are the same, while for $n > 2$ we obtain too many hypersurfaces; however, since the pfaffian $\psi_\alpha(x)dx_\alpha = 0$ is integrable, it will be shown that no more than (n-1) of the $\frac{n(n-1)}{2}$ hypersurfaces $\omega_{\alpha\beta} = 0$ are independent.

4. On the Optimality of a Totally Singular Vector Control, by the n-dimensional Green's Theorem Approach

By virture of Condition A there exists a unique pfaffian to the control system (1.1) which can be expressed as

$$dt = \frac{\psi_\alpha(x)dx_\alpha}{\psi_\tau(x)A_\tau(x)} \qquad 4.1$$

Equivalently the pfaffian could be expressed by equation (2.5) as

$$dt = \frac{\frac{\partial V(x)}{\partial x_\alpha} dx_\alpha}{\frac{\partial V(x)}{\partial x_\tau} A_\tau(x)} \qquad 4.2$$

However, the determination of V(x) is inconsequential to the analysis; what is important is to generate the pfaffian (4.1) from the system of pfaffians (1.2) by the elimination of the differentials dy_r.

It has been assumed in equation (4.1) that $\psi_\tau(x)A_\tau(x) \neq 0$ in D. From the controllability requirements it is known that $\psi_\tau(x)A_\tau(x) \not\equiv 0$

in D otherwise the hypersurface V(x) = constant would contain all the solutions to equation (1.1) independent of the controls.

By equation (4.1) the time required to transfer the state from x^o to x^f through system (1.1) can be expressed as a line integral by

$$I = t_f - t_o = \int_{x^o}^{x^f} \frac{\psi_\alpha(x)dx_\alpha}{\psi_\tau(x)A_\tau(x)} \qquad 4.3$$

We now perform the usual ritual of comparing two trajectories joining x^o to x^f that project a Jordan curve Γ in state space.

It is assumed that the two trajectories can be transformed continuously into one another, so that a simply connected surface can be constructed containing the two trajectories. Denoting by I_1 and I_2 the respective costs to traverse the trajectories and accordingly, associating a sense of direction to Γ we have

$$I_1 - I_2 = \oint_\Gamma \frac{\psi_\alpha(x)dx}{\psi_\tau(x)A_\tau(x)}$$

On applying the n-dimensional Green's theorem we obtain

$$I_1 - I_2 = \int_{S_{\alpha\beta}} \omega_{\alpha\beta}\, dx_\alpha\, dx_\beta \qquad \begin{array}{l}\alpha = 1,\ldots,n\\ \beta = \alpha+1,\ldots,n\end{array} \qquad 4.4$$

where

$$\omega_{\alpha\beta}(x) = \frac{\partial}{\partial x_\beta}\left\{\frac{\psi_\alpha(x)}{\psi_\tau(x)A_\tau(x)}\right\} - \frac{\partial}{\partial x_\alpha}\left\{\frac{\psi_\beta(x)}{\psi_\tau(x)A_\tau(x)}\right\} \qquad 4.5$$

From the form of $\omega_{\alpha\beta}$ we have immediately

Lemma 4.1. *No more than (n-1) of the $\frac{n(n-1)}{2}$ hypersurfaces $\omega_{\alpha\beta}(x) = 0$ will be independent.*

Proof. The proof follows directly from the theorem[7] on the integrability of a pfaffian.

Theorem. A necessary and sufficient condition that the pfaffian $a_\alpha(x)dx_\alpha = 0$ be integrable is

$$a_\alpha(x)\left(\frac{\partial a_\beta(x)}{\partial x_\gamma} - \frac{\partial a_\gamma(x)}{\partial x_\beta}\right) + a_\beta\left(\frac{\partial a_\gamma(x)}{\partial x_\alpha} - \frac{\partial a_\alpha(x)}{\partial x_\gamma}\right)$$

$$+ a_\gamma\left(\frac{\partial a_\alpha(x)}{\partial x_\beta} - \frac{\partial a_\beta(x)}{\partial x_\gamma}\right) \overset{x}{\equiv} 0$$

Since the pfaffian $\Psi_\alpha(x)dx_\alpha = 0$ is integrable, then the pfaffian $\frac{\Psi_\alpha(x)dx_\alpha}{\Psi_\tau(x)A_\tau(x)} = 0$ also is integrable. Applying the integrability test yields

$$\Psi_\alpha(x)\,\omega_{\beta\gamma}(x) + \Psi_\beta(x)\,\omega_{\gamma\alpha}(x) + \Psi_\gamma(x)\,\omega_{\alpha\beta}(x) \overset{x}{\equiv} 0 \qquad 4.6$$

from which it follows that only (n-1) of the $\frac{n(n-1)}{2}$ hypersurfaces $\omega_{\alpha\beta} = 0$ are independent.

The hypersurfaces $\omega_{\alpha\beta} = 0$ can now be interpreted as singular hypersurfaces, since their common intersection (assuming it exists) yields the totally singular arc. This equivalence is demonstrated in the next section. The importance of the n-dimensional Green's theorem approach is in the simple algorithms it provides for the determination of the singular hypersurfaces and the totally singular arc. Once this has been accomplished, then it is a relatively simple matter to construct a two parameter group as indicated in Section 3, which contains the totally singular arc for some values of the parameters and then use the 2-dimensional Green's theorem to evaluate globally the optimality of the totally singular arc. Since we are primarily interested in evaluating the optimality of the totally singular arc, it is tacitly assumed that the singular hypersurfaces have a common intersection that can be represented in terms of a single parameter $x = x(\sigma)$ so that $\omega_{\alpha\beta}(x(\sigma)) \overset{\sigma}{\equiv} 0$. If this is not the case, then the control will not be totally singular; that is, all components of the control are singular. We shall illustrate the method with an obvious example. Consider the system

$$\dot{x}_1 = u_1; \qquad \dot{x}_2 = u_2; \qquad \dot{x}_3 = \frac{1}{x_1^2 + x_2^2 + x_3^2} \qquad 4.7$$

and the problem is to transfer the state from $[0,0,1]$ to $[0,0,2]$ in minimum time. It is obvious that the system of partial differential equations (1.5) is a complete system of order 2 thus satisfying Condition A. The line integral (4.3) is

$$I = \int_{[0,0,1]}^{[0,0,2]} (x_1^2 + x_2^2 + x_3^2)dx_3 \qquad 4.8$$

so that $\omega_{12} = 0$; $\omega_{13} = 2x_1$; $\omega_{23} = 2x_2$. The singular hypersurfaces are given by the planes $x_1 = 0$; $x_2 = 0$ and the singular arc can be parametrized by $x_1 = 0$; $x_2 = 0$; $x_3 = \sigma$. Let the representation of the surface S (see Figure 1), containing the totally singular arc, in terms of the two parameters z_1 and z_2 be

$$\begin{aligned} x_1 &= z_1 \cos \phi \\ x_2 &= z_1 \sin \phi \qquad z_1 \geq 0 \\ x_3 &= z_2 \end{aligned} \tag{4.9}$$

so that $z_1 = 0$, $z_2 = \sigma$ are the values of the parameters yielding the singular arc.

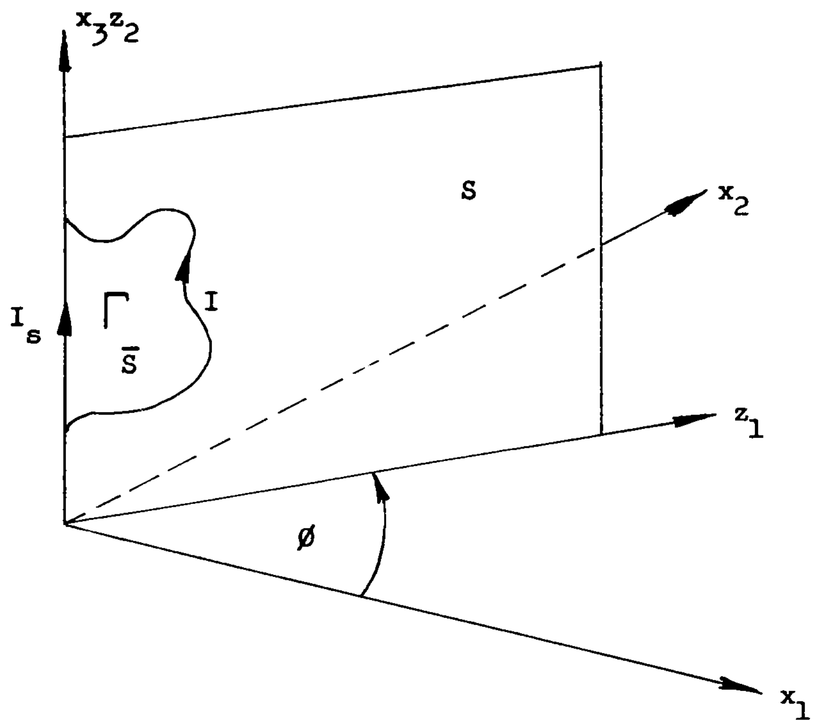

Figure 1

Denoting by I_s the cost along the totally singular arc, and by I the cost along any other arc contained in S, then

$$I - I_s = \int_{\Gamma} (x_1^2 + x_2^2 + x_3^2) dx_3 = \int_{\Gamma} (z_1^2 + z_2^2) dz_2 \qquad \text{by (4.9).}$$

Hence applying the 2-dimensional Green's theorem yields

$$I - I_s = \int_{S} 2z_1 \, dS \geq 0$$

so that the totally singular arc is optimum relative to the comparison trajectories contained in the family of surfaces given by (4.9).

A more general family of surfaces, similar to (4.9), can be described in terms of a vector valued parameter $\emptyset$ by

$$x_1 = z_1 f_1(z_2;\emptyset)$$

$$x_2 = z_1 f_2(z_2;\emptyset) \qquad 4.10$$

$$x_3 = z_2$$

where f_1 and f_2 are scalar functions of the vector $\emptyset$. Hence, for $z_1 \geq 0$ we have

$$I - I_s = \oint_{\Gamma} \left\{ z_1^2 \left| f_1^2(z_2;\emptyset) + f_2^2(z_2;\emptyset) \right| + z_2^2 \right\} dz_2$$

$$= \int_{S} 2z_1 \left| f_1^2(z_2;\emptyset) + f_2^2(z_2;\emptyset) \right| dS \geq 0$$

and for $z_1 \leq 0$

$$I - I_s = -\oint_{\Gamma} \left\{ z_1^2 \left| f_1^2(z_2;\emptyset) + f_2^2(z_2;\emptyset) \right| + z_2^2 \right\} dz_2$$

$$= -\int_{S} 2z_1 \left| f_1^2(z_2;\emptyset) + f_2^2(z_2;\emptyset) \right| dS \geq 0$$

so that once again the totally singular arc is optimum relative to the comparison trajectories contained in the family of surfaces given by (4.10). The existence and construction of a family of surfaces containing all possible comparison trajectories will be left as an open question.

5. Equivalence Between the Hypersurfaces $\omega_{\alpha\beta} = 0$ and the Totally Singular Problem

Treating the time optimal problem (1.1) by the conventional methods of optimization[9], the Hamiltonian is

$$H(x,p) = 1 + p_\alpha \left[A_\alpha(x) + B_{\alpha r}(x) u_r \right] \qquad 5.1$$

where p is the costate and is determined by the Euler-Lagrange equations

$$\dot{p}_\alpha = - \frac{\partial H}{\partial x_\alpha}(x,p) = - p_\gamma \left[\frac{\partial A_\gamma}{\partial x_\alpha}(x) + \frac{\partial B_{\alpha r}}{\partial x_\alpha}(x) u_r \right] \qquad 5.2$$

The singular problem giving rise to the totally singular control $u_r^s(t)$ occurs when

$$p_\alpha(t)\, B_{\alpha r}\left(\varphi^s(t)\right) \overset{t}{\equiv} 0 \qquad 5.3$$

is satisfied together with

$$\frac{d}{dt} \varphi_\alpha^s(t) \overset{t}{\equiv} A_\alpha\left(\varphi^s(t)\right) + B_{\alpha r}\left(\varphi^s(t)\right) u_r^s(t) \qquad 5.4$$

$$\frac{d}{dt} p_\alpha(t) \overset{t}{\equiv} - p_\gamma(t) \left[\frac{\partial A_\gamma}{\partial x_\alpha}\left(\varphi^s(t)\right) + \frac{\partial B_{\gamma r}}{\partial x_\alpha}\left(\varphi^s(t)\right) u_r^s(t) \right] \qquad 5.5$$

Furthermore, the Hamiltonian is a constant along the extremals and this constant is zero by virtue of the transversality condition, to yield

$$1 + p_{\alpha}(t)\, A_{\alpha}\left(\varphi^{s}(t)\right) \overset{t}{\equiv} 0 \qquad 5.6$$

Differentiating equation (5.3) with respect to time and using equations (5.4) and (5.5) to simplify we obtain the following set of (n-1) equations which also has to be satisfied.

$$p_{\alpha}(t)\left[\frac{\partial A_{\alpha}}{\partial x_{\gamma}}\left(\varphi^{s}(t)\right) B_{\gamma r}\left(\varphi^{s}(t)\right) - \frac{\partial B_{\alpha r}}{\partial x_{\gamma}}\left(\varphi^{s}(t)\right) A_{\gamma}\left(\varphi^{s}(t)\right)\right] \overset{t}{\equiv} 0 \qquad 5.7$$

It follows that the coefficients

$$\left[\frac{\partial A_{\alpha}\,\varphi^{s}(t)}{\partial x_{\gamma}} B_{\gamma r_1}\left(\varphi^{s}(t)\right) - \frac{\partial B_{\alpha r_1}}{\partial x_{\gamma}}\left(\varphi^{s}(t)\right) A_{\gamma}\left(\varphi^{s}(t)\right)\right]$$

are either zero or some linear combination of $B_{\alpha r_2}\left(\varphi^{s}(t)\right)$ otherwise equations (5.3) and (5.7) would imply a trivial result for p (t). To demonstrate the equivalence between the hypersurfaces $\omega_{\alpha\beta} = 0$ and the singular problem, we shall for convenience take the pfaffian in the equivalent form (4.2)

$$dt = \frac{\dfrac{\partial V(x)}{\partial x_{\alpha}}\, dx_{\alpha}}{\dfrac{\partial V(x)}{\partial x_{\tau}}\, A_{\tau}(x)} \qquad 5.8$$

and recall that V(x) satisfies the complete system of partial differential equations

$$\frac{\partial V(x)}{\partial x_\alpha} B_{\alpha r}(x) \overset{x}{\equiv} 0 \tag{5.9}$$

of order (n-1). From the definition of the hypersurface $\omega_{\alpha\beta} = 0$ we have for the equivalent pfaffian form (5.8)

$$\omega_{\alpha\beta}\left|\varphi^s(t)\right| = \frac{1}{\left\{\frac{\partial V}{\partial x_\tau}\left|\varphi^s(t)\right| A_\tau\left|\varphi^s(t)\right|\right\}^2}$$

$$\cdot\left\{\frac{\partial V}{\partial x_\beta}\left|\varphi^s(t)\right|\left[\frac{\partial^2 V}{\partial x_\tau \partial x_\alpha}\left|\varphi^s(t)\right| A_\tau\left|\varphi^s(t)\right| + \frac{\partial V}{\partial x_\tau}\left|\varphi^s(t)\right| \frac{\partial A_\tau}{\partial x_\alpha}\left|\varphi^s(t)\right|\right]\right.$$

$$\left. - \frac{\partial V}{\partial x_\alpha}\left|\varphi^s(t)\right| \left[\frac{\partial^2 V}{\partial x_\tau \partial x_\beta}\left|\varphi^s(t)\right| A_\tau\left|\varphi^s(t)\right| + \frac{\partial V}{\partial x_\tau}\left|\varphi^s(t)\right| \frac{\partial A_\tau}{\partial x_\beta}\left|\varphi^s(t)\right|\right]\right\} \tag{5.10}$$

The factor $1\Big/\left\{\frac{\partial V}{\partial x_\tau}\left|\varphi^s(t)\right| A_\tau\left|\varphi^s(t)\right|\right\}^2$ may be neglected since by assumption

$$\frac{\partial V}{\partial x_\tau}(x)\ A_\tau(x) \neq 0$$

From (5.9) we have

$$\frac{\partial V}{\partial x_\alpha}\left|\varphi^s(t)\right| B_{\alpha r}\left|\varphi^s(t)\right| \overset{t}{\equiv} 0 \tag{5.11}$$

which can be identified with equations (5.3) by defining

$$p_\alpha(t) \overset{t}{\equiv} \lambda(t) \frac{\partial V}{\partial x_\alpha}\left|\varphi^s(t)\right| \tag{5.12}$$

where $\lambda(t)$ is a non-zero multiplier that has to be determined. Substituting this form (5.12) into the Euler-Lagrange equations (5.5) yields

$$\frac{d}{dt}\lambda(t)\cdot\frac{\partial V}{\partial x_\alpha}\left|\varphi^s(t)\right| + \lambda(t)\left[\frac{\partial^2 V}{\partial x_\alpha \partial x_\gamma}\left|\varphi^s(t)\right|\cdot A_\gamma\left|\varphi^s(t)\right| + \right.$$

$$\left. + \frac{\partial V}{\partial x_\gamma}\left|\varphi^s(t)\right|\cdot\frac{\partial A_\gamma}{\partial x_\alpha}\left|\varphi^s(t)\right|\right] \overset{t}{\equiv} 0 \qquad 5.13$$

Using this result we find from equations (5.10) that

$$\omega_{\alpha\beta}\left|\varphi^s(t)\right| = \frac{1}{\left\{\frac{\partial V}{\partial x_\tau}\left|\varphi^s(t)\right| A_\tau\ \varphi^s(t)\right\}^2}$$

$$\cdot\left[-\frac{\partial V}{\partial x_\alpha}\left|\varphi^s(t)\right|\frac{\partial V}{\partial x_\beta}\left|\varphi^s(t)\right|\frac{d\lambda(t)}{dt}\frac{1}{\lambda(t)}\right.$$

$$\left. + \frac{\partial V}{\partial x_\alpha}\left|\varphi^s(t)\right|\frac{\partial V}{\partial x_\beta}\left|\varphi^s(t)\right|\frac{d\lambda(t)}{dt}\frac{1}{\lambda(t)}\right] \overset{t}{\equiv} 0$$

thus showing the equivalence between $\omega_{\alpha\beta}\left|\varphi^s(t)\right| = 0$ and the singular problem. Some further consequences of this equivalence are as follows. Mutliplying equation (5.13) by $B_{\alpha r}$ and summing and invoking (5.11) yields

$$\lambda(t)\left[\frac{\partial^2 V\ \varphi^s(t)}{\partial x_\alpha\ \partial x_\gamma} A_\gamma\left|\varphi^s(t)\right| B_{\alpha r}\left|\varphi^s(t)\right|\right.$$

$$\left. + \frac{\partial V}{\partial x_\gamma}\left|\varphi^s(t)\right|\frac{\partial A_\gamma}{\partial x_\alpha}\left|\varphi^s(t)\right| B_{\alpha r}\left|\varphi^s(t)\right|\right] \overset{t}{\equiv} 0 \qquad 5.14$$

Differentiating equation (5.9) which is an identity in x with respect to x_γ and multiplying by $A_\gamma(x)$ and summing, gives

$$\frac{\partial^2 V(x)}{\partial x_\alpha \partial x_\gamma} B_{\alpha r}(x) A_\gamma(x) + \frac{\partial V}{\partial x_\alpha}(x) \frac{\partial B_{\alpha r}(x)}{\partial x_\gamma} A_\gamma(x) \overset{x}{\equiv} 0 \qquad 5.15$$

By virtue of this result equation (5.14) becomes

$$\lambda(t) \frac{\partial V}{\partial x_\gamma}\left(\varphi^s(t)\right)\left[\frac{\partial A_\gamma}{\partial x_\alpha}\left(\varphi^s(t)\right) B_{\alpha r}\left(\varphi^s(t)\right)\right.$$

$$\left. - \frac{\partial B_{\gamma r}}{\partial x_\alpha}\left(\varphi^s(t)\right) A_\alpha\left(\varphi^s(t)\right)\right] \overset{t}{\equiv} 0 \qquad 5.16$$

which is easily recognized as equation (5.7).

Finally, to complete the equivalence, if we multiply equation (5.13) by $A_\alpha\left(\varphi^s(t)\right)$ and sum we obtain

$$\frac{d}{dt} \lambda(t) \frac{\partial V}{\partial x_\alpha}\left(\varphi^s(t)\right) A_\alpha\left(\varphi^s(t)\right)$$

$$+ \lambda(t) A_\alpha\left(\varphi^s(t)\right) \frac{\partial}{\partial x_\alpha}\left[A_\gamma\left(\varphi^s(t)\right) \frac{\partial V}{\partial x_\gamma}\left(\varphi^s(t)\right)\right] \overset{t}{\equiv} 0$$

Using equation (5.4) the above equation becomes

$$\frac{d}{dt}\left\{\lambda(t) \frac{\partial V}{\partial x_\alpha}\left(\varphi^s(t)\right) A_\alpha\left(\varphi^s(t)\right)\right\}$$

$$- \lambda(t) B_{\alpha r}\left(\varphi^s(t)\right) u_r(t) \frac{\partial}{\partial x_\alpha}\left[A_\gamma\left(\varphi^s(t)\right) \frac{\partial V}{\partial x_\gamma}\left(\varphi^s(t)\right)\right] \overset{t}{\equiv} 0$$

However the second term is zero by equation (5.14) so that the above equation can be integrated directly to yield.

$$\lambda(t) \frac{\partial V}{\partial x_\alpha} \left| \varphi^s(t) \right| A_\alpha \left| \varphi^s(t) \right| = \text{constant.}$$

This result is equivalent to equation (5.6), the constancy of the Hamiltonian, and determines the multiplier $\lambda(t)$.

6. Minimization of a Functional

The n-dimensional Green's theorem approach described in the previous sections can be applied to minimizing functionals of the form

$$I = \int_{t_o}^{t_f} L\left| x(t) \right| dt \qquad 6.1$$

The problem is to determine a control $u_r(t)$ which by (1.1) transfers the state from x_o to x_f with no restrictions on t_f (t_f free), such that I is minimized.

Since there is no precise statement about the reachable set for the system (1.1) given, some restrictions must be placed on L(x). This is necessary because it could transpire that if for some $u_r(t)$ the solution to equations (1.1) formed a closed curve in a region of state space where L(x) is negative, than I can assume any value whatsoever simply by traversing the closed curve an arbitrary number of times. The existence theorem of Markus and Lee[10] circumvents this problem by placing a restriction on t_f. However, we cannot include such a restriction without destroying the equivalence between the hypersurfaces $\omega_{\alpha\beta}(x) = 0$ and the singular problem. We shall assume that $L(x) \geq 0$ in D so that the problem becomes equivalent to one of minimum time.

By use of the pfaffian (4.1) equation (6.1) can be expressed as

$$I = \int_{x_o}^{x_f} \frac{L(x)\,\psi_\alpha(x)dx}{\psi_\alpha(x)A_\alpha(x)} \qquad 6.2$$

For this form of the line integral the singular hypersurfaces are given by

$$\omega_{\alpha\beta}(x) = \frac{\partial}{\partial x_\beta}\left\{\frac{L(x)\;\psi_\alpha(x)}{\psi_\tau(x)A_\tau(x)}\right\} - \frac{\partial}{\partial x_\alpha}\left\{\frac{L(x)\,\psi_\beta(x)}{\psi_\tau(x)A_\tau(x)}\right\} = 0$$

and the arguments given in Section 4 regarding the number of hypersurfaces still apply, since the pfaffian

$$\frac{L(x)\psi_\alpha(x)}{\psi_\tau(x)A_\tau(x)}\,dx_\alpha = 0$$

is integrable.

Similarly, the equivalence between the hypersurfaces $\omega_{\alpha\beta}(x) = 0$ and the singular problem follows from Section 5 with minor modifications. The totally singular arc $\varphi^s(t)$ with the totally singular control $u_r^s(t)$ satisfy

$$\frac{d}{dt}\varphi_\alpha{}^s(t) \overset{t}{\equiv} A_\alpha\, \varphi^s(t) + B_{\alpha r}\left(\varphi^s(t)\right) u_r{}^s(t)$$

$$\frac{d}{dt}p_\alpha(t) \overset{t}{\equiv} -\frac{\partial L}{\partial x_\alpha}\left(\varphi^s(t)\right) - p_\gamma(t)\left[\frac{\partial A_\gamma}{\partial x_\alpha}\left(\varphi^s(t)\right) + \frac{\partial B_{\gamma r}}{\partial x_\alpha}\left(\varphi^s(t)\right) u_r{}^s(t)\right]$$

and

$$p_\alpha(t)\, B_{\alpha r}\left(\varphi^s(t)\right) \overset{t}{\equiv} 0$$

The Hamiltonian is a constant along the extremals and the constant is zero by virtue of the transversality condition and the final time t_f being unspecified, so that

$$L\left(\varphi^s(t)\right) + p_\alpha(t) A_\alpha\left(\varphi^s(t)\right) \overset{t}{\equiv} 0$$

From these equations it can be shown that

$$\omega_{\alpha\beta}\left(\varphi^s(t)\right) \overset{t}{\equiv} 0$$

and hence the methods described can be used to evaluate the optimality of the totally singular arc.

7. Some Examples

In applying the Green's theorem technique to a specific example, it is not necessary to determine beforehand if the system (1.1) is controllable, because if the system (1.1) is not controllable, then the pfaffian (4.1) is integrable, and the integrability conditions are given by

$$\omega_{\alpha\beta}(x) \equiv 0 \qquad 7.1$$

Consider the following system

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2u_1 \\ \dot{x}_2 &= x_2 - x_1u_1 + x_3u_2 \\ \dot{x}_3 &= x_3 - x_2u_2 \end{aligned} \qquad 7.2$$

The system of partial differential equations (1.5) associated with the system (7.2), namely

$$x_2 \frac{\partial V}{\partial x_1} - x_1 \frac{\partial V}{\partial x_2} = 0$$

$$x_3 \frac{\partial V}{\partial x_2} - x_2 \frac{\partial V}{\partial x_3} = 0$$

is a complete system of order 2, thus satisfying Condition A. The corresponding pfaffian (4.1) is

$$dt = \frac{x_1dx_1 + x_2dx_2 + x_3dx_3}{x_1^2 + x_2^2 + x_3^2} \qquad 7.3$$

and it is immediately obvious that $\omega_{12}(x) \equiv \omega_{13}(x) \equiv \omega_{23}(x) \equiv 0$ so that the pfaffian (7.3) is integrable. Therefore the system (7.2) is not controllable, since the hypersurface $W(t,x) \equiv (x_1^2 + x_2^2 + x_3^2)e^{-2t}$ = constant contains all the solutions independent of the controls.

On the other hand, it is most important to check whether Condition A is satisfied before applying the Green's theorem technique. It does not follow that, if the required number of hypersurfaces are obtained, then Condition A is automatically satisfied, as demonstrated by the following counter-example. The system equations are

$$\dot{x}_1 = x_2 + u_1 + x_3 u_2$$

$$\dot{x}_2 = x_3^2 + x_2 u_2$$

$$\dot{x}_3 = -x_2 x_1 + x_1 u_2$$

and the pfaffian (4.1) is

$$dt = \frac{dx_2}{(x_2^2 + x_3^2)} - \frac{x_2 dx_3}{x_1(x_2^2 + x_3^2)}$$

so that

$$\omega_{12} = 0$$

$$\omega_{13} = \frac{x_2}{x_1^2(x_2^2 + x_3^2)}$$

$$\omega_{23} = \frac{x_2^2 - x_3^2 + 2x_3 x_1}{x_1(x_2^2 + x_3^2)}$$

Therefore, it would appear that the singular hypersurfaces are given by

$$x_2 = 0$$

$$x_3 - 2x_1 = 0 \qquad (x_1 \neq 0,\ x_3 \neq 0)$$

thus yielding the correct number of hypersurfaces despite the fact that Condition A is not satisfied. However, the fallacy of this result is readily apparent, since the controls $u_1(t)$ and $u_2(t)$ yielding the arc $x_1 = \sigma$; $x_2 = 0$, $x_3 = 2\sigma$ do not exist without violating $x_1 \neq 0$; $x_3 \neq 0$.

References

1. A. Miele, Application of Green's Theorem to the Extremization of Linear Integrals, Symposium on Vehicle Systems Optimization, Garden City, L.I., New York, 1961, pp. 26-35.

2. H. Hermes and G. Haynes, On the Nonlinear Control Problem with Control Appearing Linearly, J.S.I.A.M. Control, Ser. A, Vol. 1, No. 2, 1963.

3. H. Hermes, Controllability and the Singular Problem, J.S.I.A.M. Control, Ser. A, Vol. 2, No. 2,

4. E. Kreindler, Contributions to the Theory of Time Optimal Control, J. Franklin Inst., 275 (1963), pp. 314-344.

5. L.W. Neustadt, Optimization, A Moment Problem, and Nonlinear Programming, J.S.I.A.M. Control, Ser. A, Vol. 2, No. 1, (1964).

6. R. Hermann, On the Accessibility Problem in Control Theory International Symposium on Nonlinear Differential Equations and Nonlinear Mechanics, Ed. by J.P. LaSalle and S. Lefschetz, Academic Press, 1963.

7. I.N. Sneddon, Elements of Partial Differential Equations, McGraw-Hill, 1957.

8. T.M. Apostol, Mathematical Analysis, Addison-Wesley Publishing Company, Inc., 1957.

9. R.E. Kalman, The Theory of Optimal Control and the Calculus of Variations, RIAS Technical Report 6-13, 1961.

10. L. Marks and E.B. Lee, Optimal Control for Nonlinear Processes, Arch. Rational. Mech. Anal., 8 (1961), pp. 36-58.