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by Marten T. Landahl

Prepared under Contract No. NAS 1-3179 *by*
MCDONNELL AIRCRAFT CORPORATION
St. Louis, Mo.

for

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By Marten T. Landahl*
McDonnell Aircraft Corporation

SUMMARY

A theory is presented in which the pressure fluctuations in a turbulent boundary layer, or other almost parallel shear flow, are expressed in terms of integrals involving squares of the fluctuating velocity components. The analysis indicates that the resulting disturbance will be dominated by shear waves, i.e., the mean shear flow acts as a wave guide for the disturbances created by the turbulent breakdown. The wave propagation constants can be obtained by solving a modified Orr-Sommerfeld stability problem for the turbulent mean velocity profile. From these propagation constants one can then determine statistical properties of interest, e.g., how the cross-power spectral density varies with distance.

INTRODUCTION

Despite the considerable efforts devoted to the study of turbulent boundary layers and other shear flows in view of their great engineering and scientific interest, the inherent complexity of the phenomenon of non-isotropic turbulence has made theoretical progress very difficult.

The mixing-length hypotheses introduced by Prandtl (Reference 1) and extended by Taylor (Reference 2) and von Karman (Reference 3) have been successful in predicting, within empirical constants, the mean velocity distribution. Efforts to explain the observed behavior of the random velocity and pressure fluctuations have been less successful. The majority of papers published on the subject, to date, are variations on the approach suggested by Lighthill (Reference 4) for treating the noise from a turbulent jet. In this, the Navier-Stokes equation is written as a Poisson equation with the non-linear (Reynolds stress) terms and shear interaction terms treated as source terms (or rather, multipoles). Such an approach has been tried by, among others, Kraichnan (Reference 5) and by Lilley and Hodgson (Reference 6).

An interesting theoretical approach to a turbulent shear flow was presented by Malkus (Reference 7). He hypothesized that the velocity fluctuations must be selected within that class of marginally stable fluctuations which gives maximum viscous dissipation, and that the smallest allowable scale of motion is determined from the condition of

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stability. In this way he was able to predict a mean velocity profile for two-dimensional channel flow which was in good agreement with experimental observations.

The recognition that hydrodynamic stability plays a fundamental role for the structure of a turbulent boundary layer is, in the opinion of the present author, very important and deserves careful study. In the present report, an effort is made to relate the statistics of the pressure fluctuations in a turbulent shear flow to the stability characteristics of the flow.

The letter symbols used herein are defined in the appendix.

BASIC FORMULATION OF PROBLEM

We will consider a parallel, two-dimensional and incompressible flow field of the form

$$\vec{Q} = \vec{i}\,U(y) \tag{1}$$

upon which is superimposed an unsteady perturbation field

$$\vec{q}(x,y,z,t) = \vec{i}\,u + \vec{j}\,v + \vec{k}\,w \tag{2}$$

After substituting $\vec{Q} + \vec{q}$ into the Navier-Stokes equations and substracting the time average part, we obtain

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + v\frac{dU}{dy} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u + \frac{1}{\rho}\left(\frac{\partial\tau^{xx}}{\partial x} + \frac{\partial\tau^{xy}}{\partial y} + \frac{\partial\tau^{xz}}{\partial z}\right) \tag{3}$$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v + \frac{1}{\rho}\left(\frac{\partial\tau^{xy}}{\partial x} + \frac{\partial\tau^{yy}}{\partial y} + \frac{\partial\tau^{yz}}{\partial z}\right) \tag{4}$$

$$\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w + \frac{1}{\rho}\left(\frac{\partial\tau^{xz}}{\partial z} + \frac{\partial\tau^{yz}}{\partial y} + \frac{\partial\tau^{zz}}{\partial z}\right) \tag{5}$$

where

$$\begin{aligned}
\tau^{xx} &= \rho(\overline{u^2} - u^2) \\
\tau^{xy} &= \rho(\overline{uv} - uv) \\
\tau^{xz} &= \rho(\overline{uw} - uw) \\
\tau^{yy} &= \rho(\overline{v^2} - v^2) \\
\tau^{yz} &= \rho(\overline{vw} - vw) \\
\tau^{zz} &= \rho(\overline{w^2} - w^2)
\end{aligned} \tag{6}$$

and the bar denotes time average.

In obtaining (3) - (6), the continuity equation has been used which, for the fluctuating part, reads

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{7}$$

We will now derive from (3) - (5) and (7) a differential equation for p treating the quantities in (6) as known. First, we write (3) - (5) as follows:

$$L(u) + v\frac{dU}{dy} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + T^x \tag{3a}$$

$$L(v) = -\frac{1}{\rho}\frac{\partial p}{\partial y} + T^y \tag{4a}$$

$$L(w) = -\frac{1}{\rho}\frac{\partial p}{\partial z} + T^z \tag{5a}$$

where the differential operation L is defined by

$$L = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - \nu\nabla^2 \tag{8}$$

and the stresses T^x, T^y and T^z are given by the last bracketed terms in (3) - (5). Now we differentiate (3a) with respect to x, (4a) with respect to y, and (5a) with respect to z, and add them. Using (7), we then find

$$\nabla^2 p = -2\rho \frac{\partial v}{\partial x} \frac{dU}{dy} + \rho\left(\frac{\partial T^x}{\partial x} + \frac{\partial T^y}{\partial y} + \frac{\partial T^z}{\partial z}\right) \tag{9}$$

This is the form employed by Kraichnan (Reference 5) and Lilley and Hodgson (Reference 6). These authors considered the right-hand side of (9) as known and then treated it as a Poisson equation. However, the first term on the right-hand side is linear in the disturbance quantities and should therefore be treated like the term $\nabla^2 p$. We therefore take the x-derivative of (4a) to obtain

$$L\left(\frac{\partial v}{\partial x}\right) = -\frac{1}{\rho} \frac{\partial^2 p}{\partial x \partial y} + \frac{\partial T^y}{\partial x} \tag{10}$$

Also, we divide (9) by $U' = dU/dy$ and apply the L-operator. This yields

$$L(\nabla^2 p/U') = -2\rho L\left(\frac{\partial v}{\partial x}\right) + \rho L\left[\left(\frac{\partial T^x}{\partial x} + \frac{\partial T^y}{\partial y} + \frac{\partial T^z}{\partial z}\right)\Big/ U'\right] \tag{11}$$

In this, we substitute $L\left(\frac{\partial v}{\partial x}\right)$ from (10). Hence

$$L(\nabla^2 p/U') - 2\frac{\partial^2 p}{\partial x \partial y} = -2\rho\frac{\partial T^y}{\partial x} + \rho L\left[\left(\frac{\partial T^x}{\partial x} + \frac{\partial T^y}{\partial y} + \frac{\partial T^z}{\partial z}\right)\Big/ U'\right] \tag{12}$$

This is now in a form that has all the terms which are linear in the perturbation pressure on the left hand side and all the quadratic terms on the right hand side. We may simplify it by neglecting terms like

$$L(U') = -\nu U''' \tag{13}$$

which are negligible for the extremely small values of viscosity of interest. Also, an allowable approximation should be

$$L\left(\frac{\partial T^x}{\partial x} + \frac{\partial T^y}{\partial y} + \frac{\partial T^z}{\partial z}\right) \simeq \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\frac{\partial T^x}{\partial x} + \frac{\partial T^y}{\partial y} + \frac{\partial T^z}{\partial z}\right) \tag{14}$$

since the neglected terms involve a product between the viscosity and quadratic terms. On the other hand, we may not be able to neglect terms like $\nu\nabla^4 p$, since such terms may be of importance in regions where p changes rapidly. Thus, we finally obtain

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 p - \nu\nabla^4 p - 2U'\frac{\partial^2 p}{\partial x \partial y} = q(x,y,z,t) \qquad (15)$$

where

$$q = \rho\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\frac{\partial T^x}{\partial x} + \frac{\partial T^y}{\partial y} + \frac{\partial T^z}{\partial z}\right) - 2\rho\ U'\frac{\partial T^y}{\partial x} \qquad (16)$$

The boundary conditions may be derived from those for the velocity fluctuations. Considering the case of a boundary layer, we have

$$u = v = w = 0 \quad \text{for} \quad y = 0 \quad \text{and} \quad y = \infty \qquad (17)$$

From (9) and (17), and the definitions of T^x, T^y and T^z, it then follows that

$$(\nabla^2 p)_{y=0} = 0 \qquad (18)$$

A second boundary condition on the surface may be obtained by differentiating (9) with respect to y. This gives

$$\left(\frac{\partial}{\partial y}\nabla^2 p\right)_{y=0} = -2\rho\,U'(0)\left(\frac{\partial^2 v}{\partial x \partial y}\right)_{y=0} \qquad (19)$$

Now, according to the equation of continuity (7) and (17)

$$\left(\frac{\partial v}{\partial y}\right)_{y=0} = -\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)_{y=0} = 0$$

Hence

$$\left(\frac{\partial}{\partial y}\nabla^2 p\right)_{y=0} = 0 \qquad (20)$$

In addition, perturbations must vanish far away from the surface so that

$$p = \frac{\partial p}{\partial y} = 0 \qquad \text{for} \quad y = \infty \qquad . \tag{21}$$

The boundary conditions (19) - (21) are sufficient to determine $p(x,y,z,t)$ uniquely for a given $q(x,y,z,t)$. In the case of a free shear flow the appropriate boundary conditions are (21) for $y = \pm\infty$.

The boundary value problem defined by (15) and (19) - (21) may be attacked by using Fourier transform techniques. For convergence purposes, assume that $p = 0$ and $q = 0$ for $|x|$, $|z|$ and $|t|$ greater than X, Z, and T, respectively, these being large numbers. Then define

$$\overline{\overline{P}}(y) = \iiint_{-\infty}^{\infty} e^{-i(\omega t + k_x x + k_z z)} \, p(x,y,z,t)\,dt\,dx\,dz \tag{22}$$

whereby the boundary value problem takes the following form:

$$i(k_x U + \omega)(\overline{\overline{P}}'' - k^2 \overline{\overline{P}}) - \nu\left(\frac{d^2}{dy^2} - k^2\right)^2 \overline{\overline{P}} - 2i\, k_x U' \overline{\overline{P}}' = \overline{\overline{Q}} \tag{23}$$

$$\overline{\overline{P}}''(0) - k^2 \overline{\overline{P}}(0) = 0 \tag{24a}$$

$$\overline{\overline{P}}'''(0) - k^2 \overline{\overline{P}}'(0) = 0 \tag{24b}$$

$$\overline{\overline{P}}'(\infty) = \overline{\overline{P}}(\infty) = 0 \tag{24c}$$

Here $k^2 = k_x^2 + k_z^2$, prime denotes differentiation with respect to y, and $\overline{\overline{Q}}(y)$ is the triple Fourier transform of q. The complete solution may be built up as a sum of a particular solution and the solution of the homogenous equation

$$i(k_x U + \omega)(\overline{\overline{P}}'' - k^2 \overline{\overline{P}}) - \nu\left(\frac{d^2}{dy^2} - k^2\right)^2 \overline{\overline{P}} - 2i\, k_x \overline{\overline{P}}' = 0 \tag{25}$$

This possesses four linearly independent solutions. For large y, for which $U(y) = U_\infty = \text{const}$, they behave like

$$\bar{\bar{P}}_{1,2} = e^{\mp k y} \qquad (26a,b)$$

$$\bar{\bar{P}}_{3,4} = e^{\mp y \sqrt{\frac{i(k_x U_\infty + \omega)}{\nu} + k^2}} \qquad (27c,d)$$

Of these, $\bar{\bar{P}}_1$ and $\bar{\bar{P}}_3$ decay for large distances whereas $\bar{\bar{P}}_2$ and $\bar{\bar{P}}_4$ grow. $\bar{P}_1$ and $\bar{P}_2$ vary relatively slowly through the shear layer ("inviscid" solutions) whereas $\bar{P}_3$ and $\bar{P}_4$ vary rapidly ("viscous" solutions). The slowly varying solutions $\bar{P}_1$ and $\bar{P}_2$ may be obtained to a good approximation from the "inviscid" equation

$$(k_x U + \omega)(\bar{\bar{P}}'' - k^2 \bar{\bar{P}}) - 2k_x U' \bar{\bar{P}}' = 0 \qquad (28)$$

However, special care in applying (28) must be exercised when k_x and ω have opposite signs, because then there may be a point $y = y_c$, for which $k_x U + \omega = 0$, and the equation thus becomes singular. It is known from hydrodynamic stability theory that the inviscid solutions are not given uniformly by the asymptotic form (28) for $\nu \longrightarrow 0$ in the whole complex y-plane but only in the sector (see Figure 1) (assuming $k_x > 0$, $\omega < 0$).

$$-7\pi/6 < \arg (y - y_c) < \pi/6$$

outside a region of radius

$$|y - y_c| = \epsilon \qquad (29)$$

where $\epsilon = 0(\nu^{1/3})$, (Reference 8, page 127). Viscosity will therefore always be of importance even in the limit of $\nu \longrightarrow 0$, in a sector and region around the point $y = y_c$. In addition, one of the viscous solutions (in this case, P_3) will be needed to satisfy the no-slip condition at the wall but will be important only near $y \simeq 0$. Outside these regions, however, the inviscid equation (28) should describe the flow accurately.

To obtain an approximate particular solution of (23), it should therefore be sufficient to consider the inviscid form

$$(k_x U + \omega)(\bar{\bar{P}}''_p - k^2 \bar{\bar{P}}) - 2k_x U' \bar{\bar{P}}'_p = -i \bar{\bar{Q}} \qquad (30)$$

A solution is given formally by

$$\bar{\bar{P}}_p = i \int_y^{\infty} \frac{\bar{\bar{Q}}(y_1)\,[\bar{\bar{P}}_1(y_1)\,\bar{\bar{P}}_2(y) - \bar{\bar{P}}_2(y_1)\,\bar{\bar{P}}_1(y)]}{(k_x U + \omega)\,\Delta(y_1)}\, dy_1 \tag{31}$$

where $\Delta(y)$ is the Wronskian:

$$\Delta = \bar{\bar{P}}_2\,\bar{\bar{P}}'_1 - \bar{\bar{P}}_1\,\bar{\bar{P}}'_2 \tag{32}$$

To obtain Δ we differentiate (32) with respect to y and use the differential equation (28) to express the second derivatives in terms of the function and its first derivative. This yields

$$\frac{d\Delta}{dy} = \frac{2k_x U'}{k_x U + \omega}\,\Delta$$

from which it follows that

$$\Delta = C\,(k_x U + \omega)^2 \tag{33}$$

where C is a constant. For positive k_x and negative ω such that $|\omega/k_x| < 1$, the integral in (30) will thus have a pole of order three. In carrying out the integration, it is necessary to consider the Stokes phenomenon (29), so that the path of integration must be extended analytically into the complex y-plane in such a way that it passes only through regions in which the asymptotic solution for $\bar{\bar{P}}_1$ and $\bar{\bar{P}}_2$ are valid. Thus, the path must encircle the critical point y_c below it

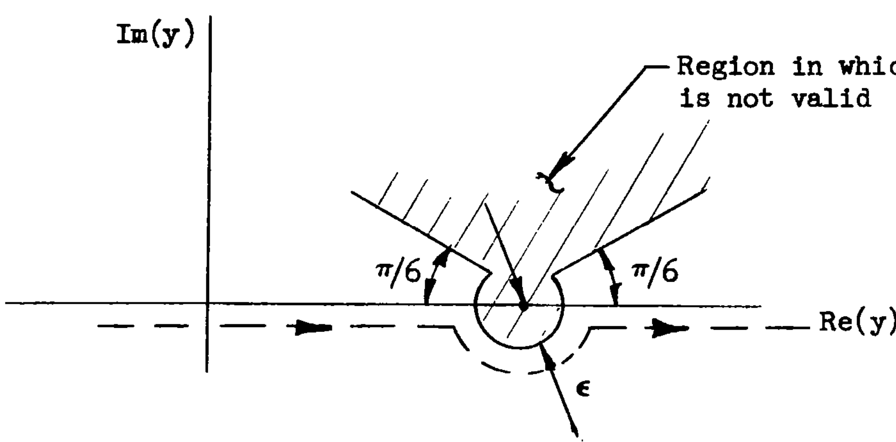

Figure 1. Region of Validity for (28) and Path of Integration for (31) and (34) (for $k_x > 0$, $\omega < 0$)

(see Figure 1). Assuming that $\bar{\bar{Q}}$ and its first two derivatives are finite everywhere the integral will be convergent. For negative k_x and positive ω the conjugate path must be taken.

An alternative form may be obtained through series expansion about $y = y_c$. This yields

$$\bar{\bar{P}}_p = \frac{i}{k^2 k_x} \left[\frac{d}{dy} \left(\frac{\bar{\bar{Q}}}{U'} \right) \right]_c + \frac{i \bar{\bar{Q}}_c}{2k_x U'} (y - y_c)$$

$$+ i \bar{\bar{P}}_2(y) \int_y^{\infty} \frac{(k_x U + \omega) \bar{\bar{Q}}_1(y_1) \bar{\bar{P}}_1(y_1)}{\Delta (y_1)} dy_1$$

$$+ i \bar{\bar{P}}_1(y) \int_0^y \frac{(k_x U + \omega) \bar{\bar{Q}}_1(y_1) \bar{\bar{P}}_2(y_1)}{\Delta (y_1)} dy_1 \qquad (34)$$

where index c refers to conditions at $y = y_c$ and where

$$\bar{\bar{Q}}_1 = i \left\{ \bar{\bar{Q}} - \bar{\bar{Q}}_c \left[\frac{U'}{U'_c} - \frac{U''_c}{k_x U'^2_c} (k_x U + \omega) + \frac{k^2}{2k_x U'_c} (y-y_c)(k_x U+\omega) \right] \right.$$

$$\left. - \frac{\bar{\bar{Q}}'_c}{k_x U'_c} (k_x U + \omega) \right\} \Big/ (k_x U + \omega)^2 \qquad (35)$$

The range of integration has now been modified in such a way that $\bar{\bar{P}}_p$ will vanish for $y \to \infty$. Provided $\bar{\bar{Q}}'(y_c)$ and $\bar{\bar{Q}}''(y_c)$ are finite, $\bar{\bar{Q}}_1$ is continuous at y_c, and the singularity in the integral thus is of the simple-pole type. Here, again, the path of integration must be extended analytically as shown in Figure 1. One can show that the integrals give a contribution of order $(y - y_c)^3 \ln(y - y_c)$ near y_c. When y_c is small, the first two terms of (34) alone would probably constitute an acceptable approximation. Equation (34) emphasizes the importance of the disturbances near the critical region.

Since $\bar{\bar{P}}_p$ as given above vanishes for $y \to \infty$, a complete solution for the pressure Fourier transform that vanishes at infinity is given by

$$\bar{\bar{P}} = C_1 \bar{\bar{P}}_1 + C_3 \bar{\bar{P}}_3 + \bar{\bar{P}}_p \qquad (36)$$

Applying the boundary conditions (24a) and (24b), we obtain

$$C_1 = \left(\frac{\bar{\bar{\nabla}}^2 \bar{\bar{P}}_3 \bar{\bar{\nabla}}^2 \bar{\bar{P}}'_p - \bar{\bar{\nabla}}^2 \bar{\bar{P}}_p \bar{\bar{\nabla}}^2 \bar{\bar{P}}'_3}{\bar{\bar{D}}}\right)_{y=0} \tag{37}$$

$$C_3 = \left(\frac{\bar{\bar{\nabla}}^2 \bar{\bar{P}}_p \bar{\bar{\nabla}}^2 \bar{\bar{P}}'_1 - \bar{\bar{\nabla}}^2 \bar{\bar{P}}_1 \bar{\bar{\nabla}}^2 \bar{\bar{P}}'_p}{\bar{\bar{D}}}\right)_{y=0} \tag{38}$$

where

$$\bar{\bar{D}}(y) = \bar{\bar{\nabla}}^2 \bar{\bar{P}}_1 \bar{\bar{\nabla}}^2 \bar{\bar{P}}'_3 - \bar{\bar{\nabla}}^2 \bar{\bar{P}}_3 \bar{\bar{\nabla}}^2 \bar{\bar{P}}'_1 \tag{39}$$

and

$$\bar{\bar{\nabla}}^2 = \frac{d^2}{dy^2} - k^2 \tag{40}$$

This can be simplified considerably by remembering that, since $\bar{\bar{P}}_3$ is a rapidly varying function, the highest-order derivative of $\bar{\bar{P}}_3$ will dominate in any of the above expressions in the limit of $\nu \longrightarrow 0$. In fact, one may conclude from hydrodynamic stability theory (Reference 8) that

$$\bar{\bar{P}}'_3/\bar{\bar{P}}_3 = O(k/\nu)^{1/3} = O(\bar{\bar{P}}'''_3/\bar{\bar{P}}''_3)$$

Also, from (28) and (34) and the fact that $\bar{\bar{Q}}(0)$ and $\bar{\bar{Q}}'(0)$ are zero, we can show that for both $\bar{\bar{P}}_1$ and $\bar{\bar{P}}_p$ the following formulas hold:

$$(\bar{\bar{\nabla}}^2 \bar{\bar{P}})_{y=0} = 2k_x U'(0) \bar{\bar{P}}'(0)/\omega \tag{41}$$

$$(\bar{\bar{\nabla}}^2 \bar{\bar{P}}')_{y=0} = \left(\frac{2k_x U'(0)}{\omega}\right)\left\{\left(\frac{k_x U'(0)}{\omega} + \frac{U''(0)}{U'(0)}\right) \bar{\bar{P}}'(0) + k^2 \bar{\bar{P}}(0)\right\} \tag{42}$$

By substitution into (37) and (38) it then appears that the viscous solution gives a contribution to (36) which is of order $(\nu/k)^{1/3}$ compared to the contribution from $\bar{\bar{P}}_1$ and hence may be neglected. The result for the pressure on the surface may then be written as follows:

$$\bar{\bar{P}}(0) = \bar{\bar{P}}_p(0)\left\{1 - \left(\frac{\bar{\bar{P}}'_p(0)}{\bar{\bar{P}}_p(0)}\right)\left[\frac{1 - \left(\bar{\bar{P}}''_3(0)/\bar{\bar{P}}'''_3(0)\right)\left(\frac{k_xU'(0)}{\omega} + \frac{U''(0)}{U'(0)} + k^2\frac{\bar{\bar{P}}_p(0)}{\bar{\bar{P}}'_p(0)}\right)}{\bar{\bar{\mathcal{D}}}}\right]\right\} \tag{43}$$

where

$$\bar{\bar{\mathcal{D}}} = \frac{\bar{\bar{P}}'_1(0)}{\bar{\bar{P}}_1(0)}\left\{1 - \frac{\bar{\bar{P}}''_3(0)}{\bar{\bar{P}}'''_3(0)}\left[\frac{k_xU'(0)}{\omega} + \frac{U''(0)}{U'(0)} + k^2\frac{\bar{\bar{P}}_1(0)}{\bar{\bar{P}}'_1(0)}\right]\right\} \tag{44}$$

Here some terms proportional to $\bar{\bar{P}}''_3/\bar{\bar{P}}'''_3$ have been retained although this ratio is of order $(\nu/k)^{1/3}$ and therefore small. The reason for this is that is multiplies the factor $k_xU'(0)/\omega$ which may be very large. This can happen when the critical point is close to the surface. Using as an approximation the first two terms of (34), we obtain that

$$k^2\,\bar{\bar{P}}_p(0)/\bar{\bar{P}}'_p(0) \cong 2\left[\frac{d}{dy}\left(\ell n\,\frac{\bar{\bar{Q}}}{U'}\right)\right]_c \tag{45}$$

which is of order unity. When $|k_xU'(0)/\omega|$ is large, this term may therefore be neglected. On the other hand, when $|k_xU'(0)/\omega|$ is not large, the whole term proportional to $\bar{\bar{P}}''_3(0)/\bar{\bar{P}}'''_3(0)$ may be neglected altogether. Thus (43) may be simplified further to

$$\bar{\bar{P}}(0) = \bar{\bar{P}}_p(0) - \bar{\bar{P}}'_p(0)\;\frac{1 - (\bar{\bar{P}}''_3(0)/\bar{\bar{P}}'''_3(0)\left(\frac{k_xU'(0)}{\omega} + \frac{U''(0)}{U'(0)}\right)}{\bar{\bar{\mathcal{D}}}} \tag{46}$$

From (31) one may obtain $\bar{\bar{P}}_p(0)$ and $\bar{\bar{P}}'_p(0)$. Thus

$$\bar{\bar{P}}'_p(0) = i\int_y^\infty \frac{\bar{\bar{Q}}(y_1)\,[\bar{\bar{P}}_1(y_1)\,\bar{\bar{P}}'_2(0) - \bar{\bar{P}}_2(y_1)\,\bar{\bar{P}}'_1(0)]}{(k_x\,U + \omega)\Delta\;\;(y_1)}\,dy_1 \tag{47}$$

For $\omega/k_xU'(0)$ small in magnitude and negative we may use (34) to approximate $\bar{\bar{P}}_p(0)$ and $\bar{\bar{P}}'_p(0)$. Thus

$$\bar{\bar{P}}_p(0) = \frac{i}{k^2 k_x} \left[\frac{d}{dy}\left(\frac{\bar{\bar{Q}}}{U'}\right)\right]_c - \frac{i\,\bar{\bar{Q}}_c\, y_c}{2k_x\, U'_c} + O(y_c^3 \ln y_c) \tag{48}$$

$$\bar{\bar{P}}'_p(0) = \frac{i\,\bar{\bar{Q}}_c}{2k_x\, U'_c} + O(y_c^2 \ln y_c) \tag{49}$$

From the solution for $\bar{\bar{P}}(0)$, (46), we may, finally, calculate statistical properties of interest. Thus, the cross-spectral density $S^p(\xi, \zeta, \omega)$ for the surface pressure in two points, separated by the distance ξ in the x-direction and by ζ in the z-direction, is obtained from

$$S^p = \lim_{X,Z,T \to \infty} \left(\frac{1}{8XZT}\right) \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\bar{\bar{P}}(0)|^2 e^{i(k_x \xi + k_z \zeta)}\, dk_x\, dk_z \tag{50}$$

It is possible to write this in form of a quadruple integral involving the power spectral density S^q of the "source term", q, but little will be gained by this at present.

THE IMPORTANCE OF EIGENVALUES - WAVE-GUIDE MODEL

We have in (50) an integral relation that would in principle allow us to calculate the cross-power spectrum for the pressure, p, if the appropriate statistical quantities of the velocity fluctuations were known. The formidable complexity of (50) would, of course, in practice preclude its use in any such calculation. Nevertheless, its mathematical structure can give us important mathematical and physical insight into the problem.

Experiments show that the pressure fluctuations in a turbulent boundary layer is only a small fraction, ϵ, of the free-stream dynamic pressure with ϵ being of the order of a few percent. It is evident from the equation of motion (3) - (5) that the velocity fluctuations must be of the order ϵ times the free stream velocity. The non-linear driving stress term q will thus be of order ϵ^2 and $|\bar{\bar{P}}(0)|^2$ consequently of order ϵ^4. Thus, (50) expresses S^p, a quantity of order ϵ^2, in form of an integral of a quantity of order ϵ^4. The only way such an expression can be meaningful is that the integrand is highly singular so that the integration introduces a factor of order ϵ^2. In the absence of singularities of $\bar{\bar{P}}_p(0)$ and $\bar{\bar{P}}_p(0)$, such would be the case if $\bar{\bar{\mathcal{D}}}$ has zeros on or near the real k_x-axis, because then the integrand will have a double pole (see (46)). Let these zeros be located at

$$k_x = \alpha_n \tag{51}$$

where each zero (eigenvalue)

$$\alpha_n = \alpha_{nR} + i\,\alpha_{nI}$$

is a function of k_z as well as of ω, ν, and the mean velocity distribution, and whose imaginary part for reasons of convergence must be assumed to be negative. Near an eigenvalue, the denominator $\bar{\bar{\mathcal{D}}}$ varies with k_x approximately as

$$\bar{\bar{\mathcal{D}}} \cong K_n(k_x - \alpha_n) \tag{52}$$

where K_n is a (complex) constant. Substituting (52) into (46) and then into (50) and carrying out the integration over k_x past the double pole under the assumption that the magnitude of α_{nI} is small, we obtain

$$SP = \frac{1}{4\pi} \sum_n \int_{-\infty}^{\infty} \frac{A_n(k_z)}{|\alpha_{nI}(k_z)|} e^{i\,\alpha_{nR}\,\xi + \alpha_{nI}|\xi| + i\,k_z\zeta}\, dk_z \tag{53}$$

where

$$A_n = \lim_{X,Z,T\to\infty} \left\{ \frac{\left[\left| \bar{\bar{P}}'_p(0) \right|^2 \left| 1 - \frac{\bar{\bar{P}}''_3(0)}{\bar{\bar{P}}'''_3(0)} \left(\frac{k_x U'(0)}{\omega} + \frac{U''(0)}{U'(0)} \right) \right|^2 \right]_{k_x=\alpha_{nR}(k_z)}}{(8XZT)|K_n|^2} \right\} \tag{54}$$

It is anticipated from the results of hydrodynamic stability theory that for the eigenvalues, y_c will be small (and positive), so that (49) will constitute a reasonably accurate approximation for $\bar{\bar{P}}'_p(0)$. Then (54) simplifies to

$$A_n \cong \frac{\bar{\bar{S}}_c^{\,q}(\alpha_{nR}, k_z)}{4\alpha_{nR}^2 (U'_c)^2 |K_n|^2} \left[1 - \frac{\bar{\bar{P}}''_3(0)}{\bar{\bar{P}}'''_3(0)} \left(\frac{k_x U'(0)}{\omega} + \frac{U''(0)}{U'(0)} \right) \right]_{k_x=\alpha_{nR}(k_z)} \tag{55}$$

where

$$\bar{\bar{S}}_c^{\,q}(\alpha_{nR}, k_z) = \lim_{X,Z,T\to\infty} \left(\frac{1}{8XZT} \right) \left| \bar{\bar{Q}}_c \right|^2 \tag{56}$$

is the double Fourier transform of the power spectral density of the driving source term for pairs of points at the distance y_c from the surface. Here y_c is defined as the value of y for which

$$-\omega/\alpha_{nR} = U \tag{57}$$

If $|\alpha_{nI}|$ now is small, of the order of the square of the disturbance velocities, S^p may become of the order of the disturbance velocities square, despite the fact that A_n is of the order (disturbance velocity)^4. The following physical interpretation thus emerges: Equation (53) represents a random pressure field obtained by superposition of waves of random phases with propagation constants given by $\alpha_n(k_z;\omega)$ and wave fronts oriented at an angle

$$\theta = \tan^{-1} \frac{k_z}{\alpha_{nR}} \tag{58}$$

to the free stream. Hence the boundary layer acts like a wave guide to the random pressure pulses that are created through the non-linear breakdown of turbulent eddies. Thus, the main contribution will come from those wave modes that are poorly attenuated and consequently may accumulate over large distances.

It is suggestive that the driving term for each wave number involves predominately the disturbances at the distance y_c from the surface for which the mean velocity in the boundary layer is equal to the wave velocity.

In order for the turbulence to sustain itself, the present analysis requires that $|\alpha_{nI}|$ must be small. Hence, a turbulent boundary layer should have a small stability margin for a range of frequencies and wave numbers. This conclusion differs slightly from Malkus' (Reference 7) hypothesis, since he assumed that there should be neutral stability over a range of wave numbers.

Although no conclusion can be drawn for (53) regarding the variation of S^p with lateral separation distance, ζ, without knowledge of how the residue term A_n varies with k_z (which is not available from the present theory) one may still use it for determining to a reasonable approximation how S^p varies with streamwise separation distance, ξ. Consider the averaged values

$$\tilde{S}^p(\xi) = \int_{-\infty}^{\infty} S^p(\xi,\zeta)\,d\zeta \tag{59}$$

Then, making use of the theory of Fourier transforms, we find that

$$\tilde{S}^p(\xi) = \frac{1}{2} \sum \frac{A_n(0)}{|\alpha_{nI}(0)|} e^{i \xi \alpha_{nR}(0) + |\xi| \alpha_{nI}(0)} \tag{60}$$

In other words, only $k_z = 0$ needs be investigated. For a boundary layer without an inflexion point there is likely to be only one lightly damped eigenvalue $\alpha_n = \alpha_R + i\alpha_I$ for each value of ω and k_z. Hence

$$\frac{\tilde{S}^p(\xi)}{\tilde{S}_p(0)} \cong e^{i\xi\alpha_R(0) + |\xi| \alpha_{nI}(0)} \tag{61}$$

Now, it is likely that the ζ-dependence does not vary greatly with ξ so that, approximately,

$$\frac{S^p(\xi,0)}{S^p(0,0)} = \frac{\tilde{S}^p(\xi)}{\tilde{S}^p(0)} = e^{i\xi\alpha_R(0) + |\xi| \alpha_I(0)} \tag{62}$$

Hence the variation of the cross spectrum with streamwise distance for zero lateral separation distance is determined approximately from the propagation constant for the least attenuated mode.

RELATION TO THE HYDRODYNAMIC STABILITY PROBLEM

The reader familiar with the theory of hydrodynamic stability will have noticed that the eigenvalue problem defined by (24) and (25) is really nothing but the classical hydrodynamic stability problem in disguise. To show this, introduce in the usual fashion non-dimensional distances referred to the boundary layer thickness, velocities made dimensionless through division by the free-stream velocity (hence $U(y) = 1$ for $y > 1$), and

$$\begin{aligned} k_x &= \alpha \\ k_z &= \beta \\ \omega &= -\alpha c \end{aligned} \tag{63}$$

In addition, set

$$\bar{\nabla}^2 \bar{P} = 2\alpha^2 U'(y)\,\Phi(y) \tag{64}$$

Then, by applying the operator $\bar{\nabla}^2 = d^2/dy^2 - (\alpha^2 + \beta^2)$ to (23), it reduces (after omission of some unimportant terms proportional to $\nu\bar{\nabla}^2 U'(y)$) to

$$(U - c)\left[\Phi'' - (\alpha^2 + \beta^2)\Phi\right] - U''\Phi =$$

$$= \frac{1}{i\alpha R}\left[\frac{d^2}{dy^2} - (\alpha^2 + \beta^2)\right]^2\Phi \tag{65}$$

with the boundary conditions

$$\Phi(0) = \Phi'(0) = 0 \tag{66}$$

$$\Phi(\infty) = \Phi'(\infty) = 0 \tag{67}$$

Here R is the Reynolds number based on boundary layer thickness and free-stream velocity. Equations (65) - (67) define the classical Orr-Sommerfeld problem for the stream function $\Phi \exp\left[i\alpha(x - ct) + i\beta z\right]$ for the stability of an infinite wave of wave numbers α and β, and phase velocity c. A considerable amount of literature exists on this problem (Reference 8). In the preceding analysis, this information has already made specific use of, in particular for the general behavior of the four linearly independent solutions as $R \longrightarrow \infty$.

The present problem differs from the classical one in that we here seek a complex eigenvalue α_n for real frequencies $\omega = -\alpha c$, whereas in the Orr-Sommerfeld problem a complex wave speed, c, for real wave numbers is sought. When the eigenvalues are nearly real, there exists a simple relationship between the two cases (Reference 9). Only the case $\beta = 0$ need be specifically treated since the results for $\beta \neq 0$ may be obtained from this case by Squire's (Reference 10) transformation.

Although the methods developed for the laminar stability problem should, in principle, be applicable to the present problem also, there are important differences which in fact require substantial developments before numerical values of the eigenvalues $\alpha_n(\omega)$ may be obtained. Apart from the necessity for considering complex wave numbers, these will in the present problem be typically of the order of ten times larger in magnitude than in the laminar stability problem. The Reynolds number will also be extremely large, typically of the order 10^5. It is not clear whether this would justify neglecting viscous terms altogether because of the large velocity gradient near the wall. Finally, the mean velocity profile is highly curved near the wall, a complication that does not normally arise in the laminar stability problem. It is therefore likely that a purely numerical approach would have the best chances of success. Even this will be beset by difficulties concerning numerical stability and accuracy due to the extremely large parameters appearing.*

DISCUSSION AND CONCLUSIONS

The basic idea introduced in the present report is that the turbulent fluctuations in a boundary layer may be represented by a superposition of shear waves of random phase and orientation, each triggered by the non-linear interaction of the fluctuating velocities themselves. Since a parallel shear flow may admit wave propagation modes that are very lightly damped, a triggering effect that is of the order of the square of the velocity fluctuations is sufficient to maintain a fluctuating field of an order of magnitude greater than the "triggering stresses" themselves, thus making the turbulent field self-maintained. The basic physical mechanism that makes this possible is, of course, the hydrodynamic coupling to an infinite source of energy, namely the free stream.

The propagation constants for the shear waves are obtainable from the solution of a modified Orr-Sommerfeld stability problem. Knowing the wave length and attenuation coefficient for each frequency, one can then determine statistical properties of interest like cross-power spectra for the fluctuating pressures. These expressions contain cross-power spectra of quadratic combinations of the fluctuating velocities themselves, which are, of course, not known beforehand. However, it is demonstrated that results of practical interest can nevertheless be obtained like how the cross-power spectra vary with separation distance. The absolute magnitude of the pressure fluctuations cannot be directly obtained from the present theory since this depends on the non-linear turbulent breakdown process which is not specifically considered.

There are several of the specific assumptions made in the analysis that need to be carefully investigated. The assumption of a parallel shear flow is probably fairly good as regards the determination of the wave propagation constants. However, if an individual wave persists over a distance over which the shear layer thickness may vary substantially, this variation must be taken into account in determining the total disturbance in one point, and the simple statistical analysis employed which assumed homogenity in the x-direction, will not be valid. One case for which this is apparently so is for a shear flow profile with an inflexion point (as would be obtained for a boundary layer in a strong adverse pressure gradient). Then there would be frequency ranges in which the wave propagation constant has a negative imaginary part, i.e., the wave will be unstable. According to the simplified analysis employed here, the total fluctuation will become infinite since there will be contributions from waves originating at upstream infinity with consequently infinite amplification. This obviously meaningless result is due to the neglect of variation in shear layer thickness. When

* An initial unsuccessful attempt to apply a computational program that has worked well for the laminar stability problem confirms this.

this variation is taken into account one finds that an unstable wave of a given frequency will, as the shear layer grows, eventually travel into a stable region, and any given point will therefore in reality receive amplified waves only from a limited region. A statistical analysis of such a case would, of course, be considerably more complicated than the present one.

A particularly difficult question to answer is whether the Orr-Sommerfeld equation (or the counterpart for pressure presently used) is really appropriate to use for determining the wave modes in the present turbulent case. The fluctuations will "smear out" the critical layer, and, in view of the importance of this in stability theory, there remains a question whether a linear equation is adequate for describing the intricate behavior of this region. Similarly, the assumption of a smooth variation of the quadratic "source term" through the critical region for each wave number made in the analysis needs investigation considering the large derivatives of the eigenfunction for the u-perturbation velocity predicted by the stability theory.

Finally it is conceivable that turbulence is maintained by a much more intricate process than proposed in the present simple model. For example, the main mechanism could be a weak non-linear interaction between disturbances with a cumulative effect over large distances, but here again it is likely that the travelling wave modes will play a dominant role.

The present investigation points up the desirability for a careful investigation of the stability characteristics of a turbulent boundary-layer velocity profile. Numerical results for the wave propagation constants for various frequencies could be compared to experimental results to see whether the actual measured cross-power spectral densities give evidence of waves corresponding to the theoretical predictions. For reasons explained in the preceding section, such calculations are considerably more difficult than for stability problem. Nevertheless, such an undertaking would be worthwhile despite the difficulties anticipated, becuase even if the validity of the presently proposed conceptual model should not be confirmed by the results, the stability characteristics of the turbulent velocity profile would by itself be extremely interesting and quite relevant to the understanding of the turbulence problem.

APPENDIX

SYMBOLS

A_n	residue term (equation (54))
c	wave velocity
k	wave number $\sqrt{k_x^2 + k_z^2}$
k_x, k_z	wave numbers in x,z directions respectively
$L(u)$	differential operator (equation (8))
p	pressure
$\overline{\overline{P}}$	triple Fourier transform of pressure
q	quadratic source term (equation (16))
$\vec{q}$	perturbation velocity vector
$\overline{\overline{Q}}(y)$	Fourier transform of q
R	Reynolds number
$S^p(\xi,\zeta,\omega)$	cross-spectral density of surface pressure (equation (50))
S^q	power spectral density of q
T^x, T^y, T^z	turbulent stresses
u, v, w	perturbation velocity components
U_∞	free stream velocity
$U(y)$	mean velocity of parallel shear flow
x, y, z	Cartesian coordinates
y_c	critical point for wave
α,β	wave numbers
α_n	complex eigenvalues $= \alpha_R + i\alpha_I$
Δ	Wronskian $= \overline{\overline{P}}_2 \overline{\overline{P}}_1' - \overline{\overline{P}} \overline{\overline{P}}_2'$

APPENDIX

θ	orientation angle of wave front
ν	kinematic viscosity
ξ, ζ	separation distances in x and z directions, respectively
ρ	density of fluid
τ^{xx}, τ^{xy}, etc.	Reynolds stress terms (equation (6))
$\Phi(y)$	stream function
ω	frequency
∇^2	Laplacian operator $= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
$\bar{\bar{\nabla}}^2$	Laplacian operator after Fourier transform $= \frac{d^2}{dy^2} - (\alpha^2+\beta^2)$

REFERENCES

1. Prandtl, L. (1925) Bericht über Untersuchunger zur Ansgebildeten Turbuleng, Z. Angrew, Math. Mech. 5.

2. Taylor, G. I. (1932) The Transport of Vorticity and Heat Through Fluids in Turbulent Motion, Proc. Roy, Soc., London, A-135.

3. von Karman, T.N. (1930) Mechanische Ähnlichkeit und Turbulenz, Noch. Ges. Wiss. Gottingen, Math, Phys. Klasse, 58.

4. Lighthill, M. T. (1952) On Sound Generated Aerodynamically - Part I, Proc. Roy. Soc. A, 211, 564.

5. Kraichnan, R. H. (1956) Pressure Field Within Homogenous Anistropic Turbulence, J. Acoust. Soc. Amer. 28, 64.

6. Lilley, G. M. & Hodgson, T. H. (1960) On Surface Pressure Fluctuations in Turbulent Boundary Layers, AGARD Rep. 276.

7. Malkus, W. V. R. (1956) Outline of a Theory of Turbulent Shear Flow, J. Fluid Mech., 1, 521.

8. Lin, C. C. (1955) The Theory of Hydrodynamic Stability, Cambridge University Press.

9. Gaster, M. E., A Note on the RelationBetween Temporally-Increasing and Spatially-Increasing Disturbances in Hydrodynamic Stability, J. Fluid Mech., 14, 222

10. Squire, H. B., On the Stability of Three-Dimensional Disturbances of Viscous Flow Between Parallel Walls, Proc. Roy. Soc. A 142, 621