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AN APPROXIMATE SOLUTION FOR LUMINOSITY DISTANCE IN ZERO-PRESSURE RELATIVISTIC MODEL UNIVERSES THAT HAVE THE U-PROPERTY

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Langley Station, Hampton, Va.

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AN APPROXIMATE SOLUTION FOR LUMINOSITY DISTANCE IN ZERO-PRESSURE RELATIVISTIC MODEL UNIVERSES THAT HAVE THE U-PROPERTY

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SUMMARY

Finite-density zero-pressure models of the universe based on general relativity are studied. These models have the U-property (time-dependent uniformity) and a metric subspace described by the Robertson-Walker metric. Highly accurate approximate closed-form expressions for the radial metric variable, the luminosity distance, the time of light travel, and the scale factor of the universe are obtained. The results are used to obtain new forms of the redshift-magnitude and count-magnitude relations that are useful for the analysis of observational data once the signs of the cosmical constant and curvature constant are known. The redshift-magnitude relation is of a form that permits the determination of the density parameter from apparent magnitude and redshift data.

INTRODUCTION

In order to find the model universe that best represents the observed universe, it is necessary to determine certain parameters from observational data. These parameters include the acceleration and density parameters. At present the most satisfactory expression to use for the determination of these parameters is the redshift-magnitude relation. Several forms of the redshift-magnitude relation have been derived and are presented in references 1 to 3. Two of these redshift-magnitude relations are based, either explicitly as in reference 1 or implicitly as in reference 2, on the assumption that a term called the cosmical constant is zero. Reference 3 presents a redshift-magnitude relation based on a model universe that has a nonzero cosmical constant. However, because of the assumption of zero material density used in deriving the model universes, some of the zero-density models presented in reference 3 are not suitable for the analysis of observational data. The unsuitable models occur, for the most part, when the acceleration parameter is less than or equal to zero. Because models with an acceleration parameter that is less than zero may be important in the analysis of observational data when evolutionary effects are considered, finite-density zero-pressure model universes that possess time-dependent uniformity (U-property) are studied to determine whether the catalog of redshift-magnitude relations for model universes with a nonzero cosmical constant could be improved.

The study resulted in a very good approximate expression for luminosity distance, and hence, for the redshift-magnitude relation, for model universes with a nonzero cosmical constant and finite density. The only model universes not included are those with a zero cosmical constant. For the model universe with a zero cosmical constant an exact solution exists. (See ref. 1.)

ANALYSIS

Distance in the observed universe is given by

$$m - K = 5 \log_{10} D_l + \Delta M_o + M_o - 5 \tag{1}$$

where m is the apparent magnitude of the source, K is the correction for redshift to apparent magnitude, M_o is the absolute magnitude of the equivalent local source, ΔM_o is the evolutionary correction to the apparent absolute magnitude, and D_l is the luminosity distance. (See appendix for a complete symbol list.) The luminosity distance is obtained from the model universe, which, in the present study, is based on general relativity. In relativistic model universes (ref. 2), D_l is given by

$$D_l = R_o(1 + \delta)S(\omega) \tag{2}$$

where R_o is the present value of the scale factor, δ is the redshift, ω is a function of the radial metric variable (ref. 4), and $S(\omega)$ is a function of ω that depends on the curvature of space. If the model also possesses zero pressure, time-dependent uniformity, and a metric subspace defined by the Robertson-Walker metric, $S(\omega)$ is given by

$$S(\omega) = \frac{1}{\sqrt{-k}} \sinh \sqrt{-k}\,\omega \tag{3}$$

where k is the curvature constant. In this case the variable ω is given by

$$\omega = \frac{c}{H_o R_o} \int_0^\delta \frac{d\delta}{\left[2\sigma_o(1 + \delta)^3 - \frac{kc^2}{R_o{}^2 H_o{}^2}(1 + \delta)^2 + \frac{\Lambda}{3H_o{}^2}\right]^{1/2}} \tag{4}$$

where c is the speed of light in a vacuum, σ_o is the density parameter, H_o is the Hubble parameter, and Λ is the cosmical constant. The constants $\frac{kc^2}{R_o{}^2}$ and Λ are given by

$$\frac{kc^2}{R_o{}^2} = H_o{}^2\left(3\sigma_o - q_o - 1\right) \tag{5}$$

$$\Lambda = 3H_o^2(\sigma_o - q_o) \tag{6}$$

where q_o is the acceleration parameter. Equations (5) and (6) are used to eliminate $\frac{kc^2}{R_o^2}$ and Λ from the integrand of equation (4), and the following equation results:

$$\omega = \frac{c}{H_o R_o}\int_0^{\delta} \frac{d\delta}{\left[2\sigma_o(1 + \delta)^3 - (3\sigma_o - q_o - 1)(1 + \delta)^2 + (\sigma_o - q_o)\right]^{1/2}} \tag{7}$$

This integral is elliptic and cannot be integrated in terms of simple functions. Mattig (ref. 1) simplified this integral by assuming that $\Lambda = 0$ and thus obtained simple closed-form solutions. In reference 3 the integral was simplified by assuming that the density parameter σ_o was zero and showed that under certain conditions the resulting zero-density models could be applied to the observable universe. The assumption that $\Lambda = 0$ deletes the constant term in equation (7), whereas the zero-density assumption affects terms of all orders, including the constant term. As shown in reference 3, there are certain zero-density models, primarily those with $0 \leqq q_o \leqq -1$, that cannot be used for the analysis of observational data. Because of this difficulty, equation (7) was studied to determine whether simple closed-form solutions that were good approximations of equation (7) could be obtained without setting the density equal to zero.

There are several approaches for obtaining approximate solutions in terms of simple functions. One of the approaches is to expand the integrand in a MacLaurin Series about $\delta = 0$, and another is to determine whether higher order terms involving density, such as $2\sigma_o\delta^3$, can be neglected. The second approach is used in the present analysis.

If the term $2\sigma_o(1 + \delta)^3$ in equation (7) could be neglected, $S(\omega)$, and consequently the luminosity distance, would be algebraic expressions. An analysis of the relative significance of the terms under the radical in equation (7), however, showed that the term $2\sigma_o(1 + \delta)^3$ was significant, and this approach was abandoned.

The denominator of equation (7) was expanded so that δ became the variable. The expansion gave

$$\omega = \frac{c}{H_o R_o}\int_0^{\delta} \frac{d\delta}{\left[2\sigma_o\delta^3 + (3\sigma_o + q_o + 1)\delta^2 + 2(q_o + 1)\delta + 1\right]^{1/2}} \tag{7a}$$

An analysis of the relative significance of the terms in the denominator of equation (7a) showed that for the presently accepted ranges of σ_o and q_o

$\left(0.016 \leqq \sigma_o \leqq 0.16\right.$ and $\left.0.5 \leqq q_o \leqq 2.5\right)$, the term $2\sigma_o\delta^3$ made a negligible contribution to the denominator when $\delta < 1$.

The general solution of equation (7a) with $2\sigma_o\delta^3 = 0$ is

$$\omega = \frac{1}{\sqrt{-k}}\sqrt{\frac{1 + q_o - 3\sigma_o}{1 + q_o + 3\sigma_o}}\left[\cosh^{-1}\frac{\left(1 + q_o + 3\sigma_o\right)\delta + \left(q_o + 1\right)}{\sqrt{q_o\left(q_o + 1\right) - 3\sigma_o}} - \cosh^{-1}\frac{q_o + 1}{\sqrt{q_o\left(q_o + 1\right) - 3\sigma_o}}\right] \tag{8}$$

where the coefficient of the integral has been written as $\dfrac{\sqrt{1 + q_o - 3\sigma_o}}{\sqrt{-k}}$ through the use of equation (5). In figure 1 is a comparison of $\dfrac{\omega H_o R_o}{c}$ given by equations (7a) and (8) for the model universe for which $\Lambda < 0$ and $k = -1$. For both models shown $\sigma_o = 0.08$; for one model $q_o = 0.5$, whereas for the other, $q_o = 2.5$. As shown in figure 1, differences between the exact and approximate forms cannot be detected for $q_o = 2.5$, and they are barely detectable for $q_o = 0.5$ at the higher values of δ. This fact is interesting because the term $2\sigma_o\delta^3$ was neglected on the basis that $\delta < 1$. The results shown in figure 1 indicate that, for present conditions, the term $2\sigma_o\delta^3$ is negligible out to $\delta = 1$, and the results of calculations (not shown) indicate that this condition extends appreciably beyond $\delta = 1$. This occurrence greatly extends the usefulness of the solution for ω.

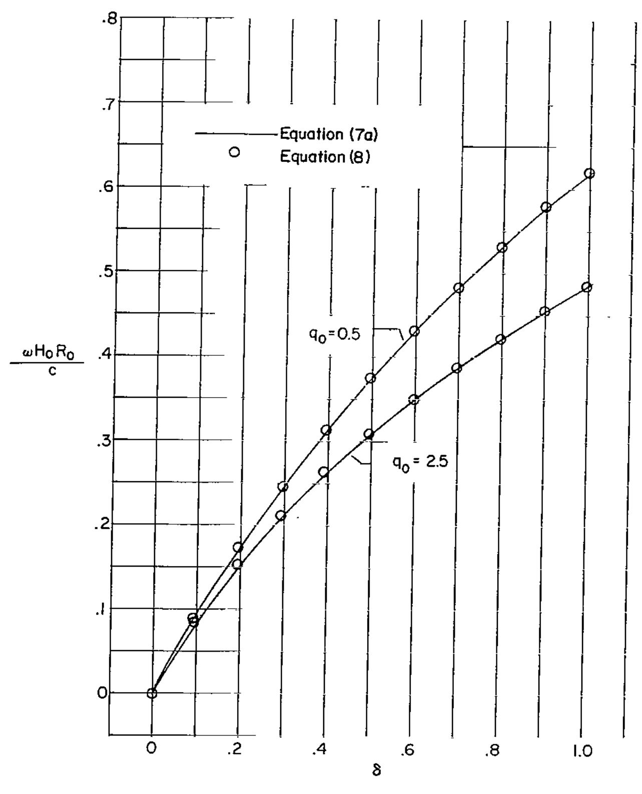

Figure 1.- Comparison of integration of equation 7(a) with integration of equation (8) for $\sigma_o = 0.08$.

The function $S(\omega)$ is obtained by substituting equation (8) into equation (3) and is

$$S(\omega) = \frac{1}{\sqrt{-k}} \sinh\left\{\sqrt{\frac{1 + q_o - 3\sigma_o}{1 + q_o + 3\sigma_o}}\left[\cosh^{-1} \frac{(1 + q_o + 3\sigma_o)\delta + (q_o + 1)}{\sqrt{q_o(q_o + 1) - 3\sigma_o}} - \cosh^{-1} \frac{q_o + 1}{\sqrt{q_o(q_o + 1) - 3\sigma_o}}\right]\right\} \tag{9}$$

Because, as shown in figure 1, the results for ω in the exact and approximate solutions are very nearly equal, there should be little difference in $S(\omega)$ for the exact and approximate expressions because the hyperbolic sine is operating on arguments that are almost equal for the exact and approximate solutions.

The luminosity distance is obtained by substituting equation (9) into equation (2) and is as follows:

$$D_l = \frac{c(1 + \delta)}{H_o\sqrt{1 + q_o - 3\sigma_o}}\left(\sinh\left\{\sqrt{\frac{1 + q_o - 3\sigma_o}{1 + q_o + 3\sigma_o}}\left[\cosh^{-1} \frac{(1 + q_o + 3\sigma_o)\delta + (q_o + 1)}{\sqrt{q_o(q_o + 1) - 3\sigma_o}} - \cosh^{-1} \frac{q_o + 1}{\sqrt{q_o(q_o + 1) - 3\sigma_o}}\right]\right\}\right) \tag{10}$$

In figure 2 is compared the approximate solution for D_l which is given by equation (10) with the exact solution for D_l which is obtained by using the value of ω given by equation (7a) in equations (2) and (3). As can be seen, the agreement is as good as that found for $\frac{\omega H_o R_o}{c}$. This approximation is good out to $\delta = 1$, as shown by figures 1 and 2, in spite of the fact that it was obtained under the restriction that $\delta < 1$. Unfortunately, this approximate solution holds only when $\Lambda \neq 0$; however, Mattig's work (ref. 1) provides an exact solution for $\Lambda = 0$.

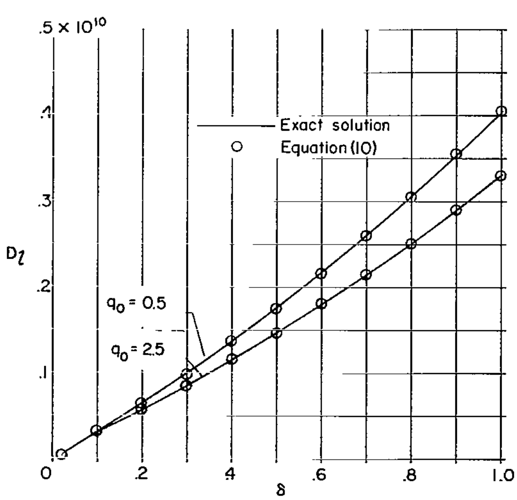

Figure 2.- Comparison of exact and approximate luminosity distances. $\sigma_0 = 0.08$; $H_0 = 100$ km/sec/Mpc.

Even though the solution for D_l given by equation (10) is a very good approximation to the exact solution for D_l, differences are introduced as δ increases above 1. Let ϵ be the difference between the exact and approximate forms of equation (9); then $D_{l,\epsilon}$, the difference in D_l due to the difference ϵ, is given by

$$D_{l,\epsilon} = \frac{c(1 + \delta)}{H_o\sqrt{1 + q_o - 3\sigma_o}} \epsilon \cosh \omega \tag{11}$$

where ω is given by equation (8). The difference ϵ could be read from a plot similar to figure 2 extended to higher values of δ; thus at the higher values of δ the deleted cubic term might have introduced a difference that does not show in figure 2. In reference 3 the difference between the values of $\frac{\omega H_o R_o}{c}$ for the finite- and zero-density models was 0.005 at $\delta = 1$. This difference, which corresponded to a difference in D_l of 10^8 parsecs, was considered acceptable. For equation (11) $D_{l,\epsilon}$ was assumed to be 10^8 parsecs, and ϵ was assumed to be 0.005. For these values and for $\sigma_o = 0.08$, equation (11) indicated that the errors would occur at $\delta = 2.25$ and $\delta = 4.95$ for $q_o = 0.5$ and $q_o = 2.5$, respectively. In these calculations H_o was taken as 100/km/sec/Mpc.

Equation (10) represents a good closed-form approximate solution for the luminosity distance in a zero-pressure finite-density model universe that possesses the U-property and has a nonzero cosmical constant. Unfortunately, there are many model universes covered by this equation. (See ref. 3.) Each of these universes involves a slightly different form of equation (10). In addition, singularities arise which must be handled as indeterminate forms. If the signs of k and Λ are known, equation (10) is useful in establishing the model universe, because only one specific form of equation (10) is necessary.

In addition to the luminosity distance, the time of light travel and the scale factor R are of interest. The fact that the term $2\sigma_o\delta^3$ can be neglected makes it possible to derive accurate closed-form expressions for the variables $t_o - t$ and R.

The integral for $t_o - t$ from reference 3 is

$$t_o - t = H_o^{-1} \int_0^\delta \frac{d\delta}{(1 + \delta)\left[2\sigma_o\delta^3 + (3\sigma_o + q_o + 1)\delta^2 + 2(q_o + 1)\delta + 1\right]^{1/2}} \tag{12}$$

where t_o is the present time and t is some other time in the past. When the term $2\sigma_o\delta^3$ is neglected, the expression integrates to

$$t_o - t = \frac{H_o^{-1}}{\sqrt{q_o - 3\sigma_o}}\left\{\sin^{-1} \frac{(3\sigma_o - q_o)R - 3\sigma_o R_o}{R_o\sqrt{q_o(q_o + 1) - 3\sigma_o}} - \sin^{-1}\left[\frac{-q_o}{\sqrt{q_o(q_o + 1) - 3\sigma_o}}\right]\right\} \tag{12a}$$

which is the general solution for $t_o - t$ in the model universe with a nonzero cosmical constant.

Figure 3 compares the numerical solution of the integral (eq. (12)) with the approximate solution given by equation (12a) for $k = -1$ and $\Lambda < 0$. In this figure $(t_o - t)H_o$ is plotted as a function of δ, and, as can be seen, the agreement between equations (12) and (12a) is excellent.

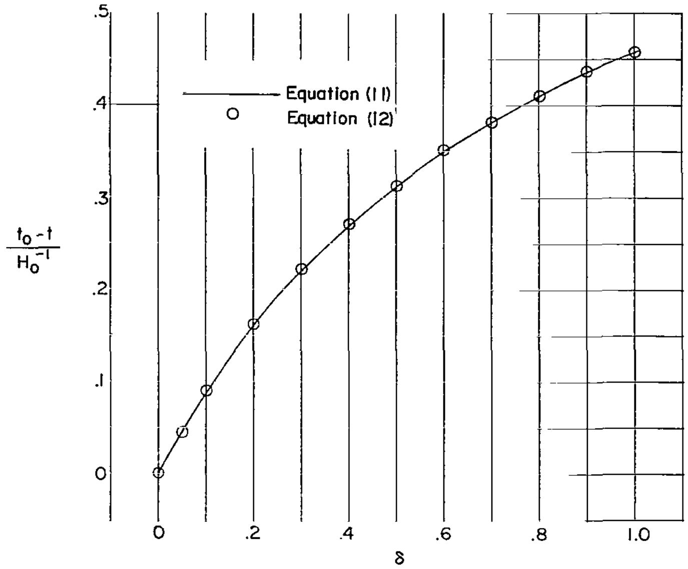

Figure 3.- Comparison of solutions for $t_0 - t/H_0^{-1}$ given by elliptic integral equation (11) and the approximate solution, equation (12).

The inversion of equation (12a) gives a solution for R, which is

$$R = \frac{R_o}{\sqrt{q_o - 3\sigma_o}}\left[\frac{q_o}{\sqrt{q_o - 3\sigma_o}} \cos \frac{\sqrt{q_o - 3\sigma_o}}{H_o^{-1}}(t_o - t) - \sin \frac{\sqrt{q_o - 3\sigma_o}}{H_o^{-1}}(t_o - t)\right] - \frac{3\sigma_o R_o}{q_o - 3\sigma_o} \tag{13}$$

RELATIONSHIPS CONNECTING OBSERVATION AND THEORY

The two most useful relationships connecting observation and theory are the redshift-magnitude relation and the count-magnitude relation. Of the two, the redshift-magnitude relation is more useful with present data and provides the stronger connection between observation and theory.

The redshift-magnitude relation is obtained by substituting equation (10) into equation (1):

$$m - K = 5 \log_{10}\left(\frac{c(1 + \delta)}{H_o\sqrt{1 + q_o - 3\sigma_o}} \sinh\left\{\sqrt{\frac{1 + q_o - 3\sigma_o}{1 + q_o + 3\sigma_o}}\left[\cosh^{-1} \frac{(3\sigma_o + q_o + 1)\delta + (q_o + 1)}{\sqrt{q_o(q_o + 1) - 3\sigma_o}} - \cosh^{-1} \frac{q_o + 1}{\sqrt{q_o(q_o + 1) - 3\sigma_o}}\right]\right\}\right) + M_o + \Delta M_o - 5 \tag{14}$$

To the first approximation ΔM_o is given by $-\frac{\dot{M}_o}{H_o}\delta$, $\dot{M}_o$ being the change in absolute magnitude per epoch due to evolutionary effects. (See ref. 2.) This form of the redshift-magnitude relation contains three undetermined constants, q_o, σ_o, and H_o, and can, in theory, be used with observational data in a least-squares computing process to find the best values of these parameters.

Figure 4 shows the effect of varying q_o when H_o = 100 km/sec/Mpc, $\sigma_o = 0.08$, and $M_o = 20.56$. The separation of these curves varies from 0.127^m between the curves for $q_o = 0.5$ and $q_o = 1.0$ to 0.078^m between the curves for $q_o = 2.5$ and $q_o = 3.0$ at $\delta = 1$. This separation is slightly smaller than the separation for the zero-density model universe (ref. 3), and indicates that either observation must be extended beyond $\delta = 1$ or the precision must be improved before a model of the universe can be defined. The separations are still less than the best root mean square of the residuals found in reference 3. The calculations in reference 3 are considered to be applicable to this case because there is little or no effect of σ_o on the separation curve out to redshift values of 0.46.

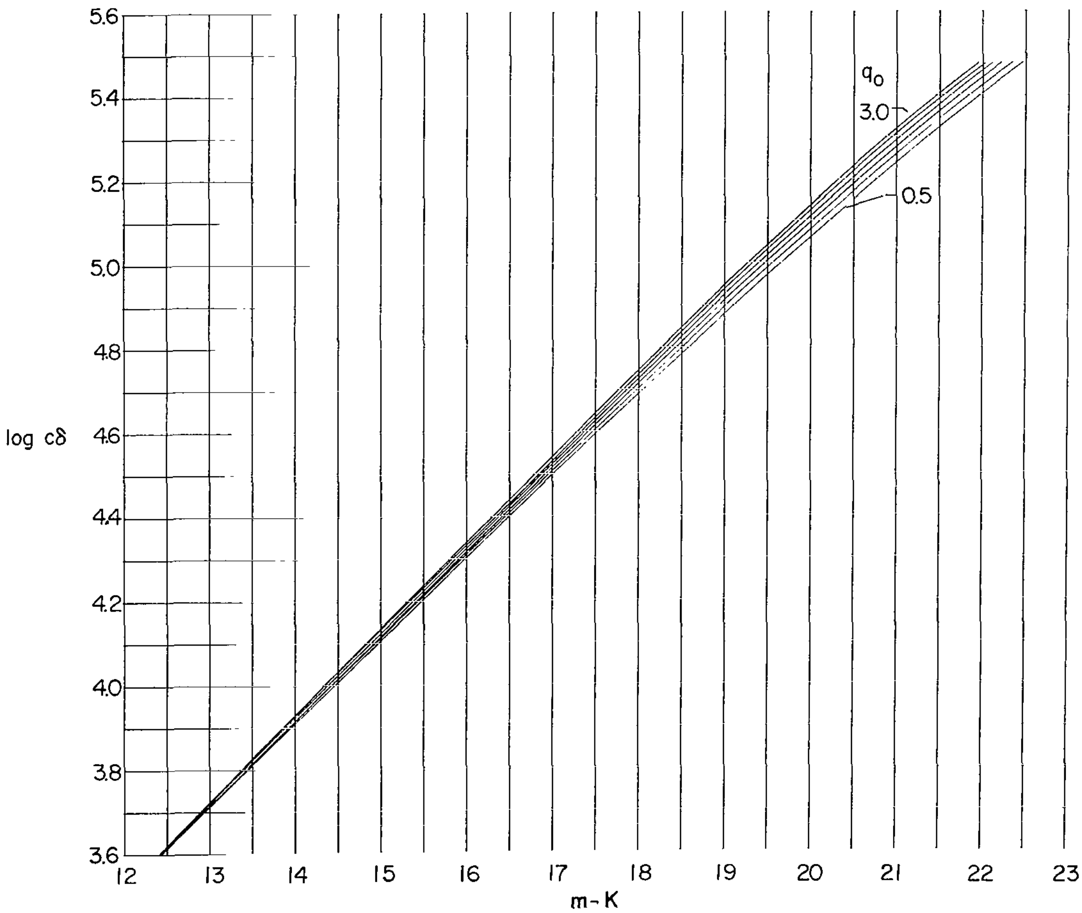

Figure 4.- Effect of varying q_o on redshift-magnitude relation (eq. (14)). H_o = 100 km/sec/Mpc; $\sigma_o = 0.08$; $M_o = 20.56$; $\Delta M = 0$.

Figure 5 shows the effect of σ_o on the redshift-magnitude relation when $q_o = 1.5$. The curves for $\sigma_o = 0.016$ and $\sigma_o = 0.16$ approximately correspond to the limits of Oort's density range (see ref. 5) and the separation is about 0.07^m at $\delta = 1$. Increasing σ_o to 0.80 gave a separation of 0.286^m at $\delta = 1$. These results indicate that, up to $\sigma_o = 0.016$, density has little effect on the redshift-magnitude relation. This conclusion confirms the results of reference 3 which indicate that up to $\sigma_o = 0.16$, the finite- and zero-density redshift-magnitude relations are the same. The curve for $\sigma_o = 0.16$ is separated very little from the curves for $\sigma_o = 0.04$ and $\sigma_o = 0.016$ in figure 5. This separation is not evident in reference 3 because of the scale used in plotting the figure, and for practical purposes it is negligible. It thus appears that the value of σ_o must exceed 0.16 before the effect of density becomes appreciable.

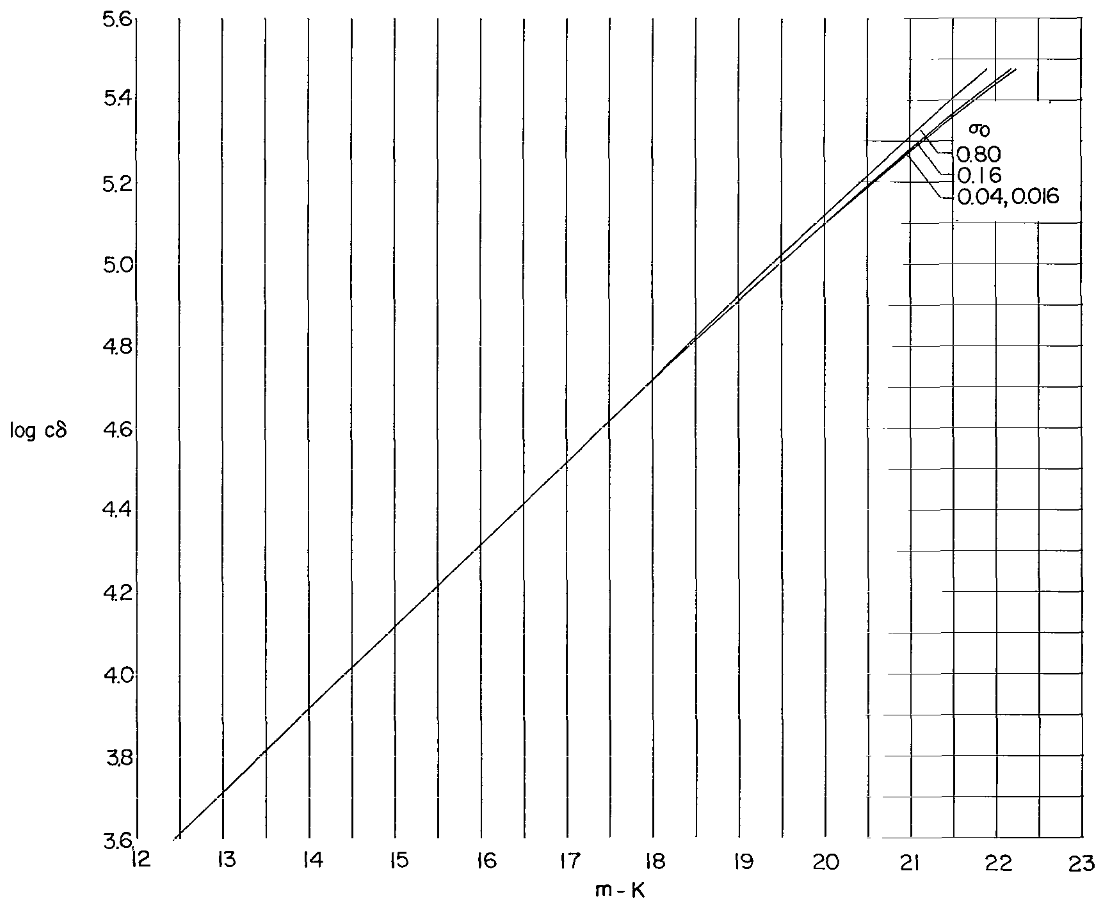

Figure 5.- Effect of varying σ_0 on redshift-magnitude relation (eq. (14)). H_0 = 100 km/sec/Mpc; $q_0 = 1.5$; $M_0 = -20.56$; $\Delta M = 0$.

Another relationship that is useful in the analysis of observational data and that comes from the model universe is the count-magnitude relation. In the past, the count-magnitude relation has not been very satisfactory for the analysis of observational data because the distance surveyed by the 200-inch telescope is not large enough to give an adequate data sample. (See ref. 6.) However, with the advent of the space telescope, this situation may change.

The count-magnitude relation (ref. 3) is

$$N(m) = \frac{2\pi n R_o^3}{(-k)Q_o}\left(\sinh \sqrt{-k}\,\omega \cosh \sqrt{-k}\,\omega - \sqrt{-k}\,\omega\right) \tag{15}$$

where ω is given by equation (8). In equation (15), $N(m)$ is the number of galaxies brighter than a given apparent magnitude m, n is the number of galaxies per unit volume, and Q_o is the number of square degrees in the celestial sphere.

This count-magnitude relation for the finite-density model, unlike the one for the zero-density model (ref. 3), is very close to the exact count-magnitude relation because of the agreement of the exact and approximate values of ω and R_o.

CONCLUDING REMARKS

Relativistic finite-density zero-pressure models of the universe that possess the U-property and have a metric subspace that is described by the Robertson-Walker metric have been studied. Approximate expressions for the radial metric variable and the luminosity distance have been derived. Up to redshifts of 1, the approximate solutions do not differ from the exact solution obtained through the use of a high-speed digital computer. An expression for the difference in the luminosity distances that is valid for any redshift was obtained. In addition, similarly precise closed-form solutions have been obtained for the travel time of light from galaxies and the scale factor of the universe.

These results are used to obtain a redshift-magnitude relation that holds for all finite-density zero-pressure model universes that have the U-property except for those with a zero cosmical constant. This expression is difficult to use because of the many models the solution contains and because of the singularities that arise. However, if the signs of the cosmical constant and the curvature constant are known, the expression becomes very useful for the analysis of observational data. This form of the redshift-magnitude relation permits the calculation of the density parameter from apparent magnitude and redshift data. Lastly, a potentially useful form of the count-magnitude relation is also derived.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., June 23, 1965.

APPENDIX

SYMBOLS

c	speed of light in vacuum
D_l	luminosity distance
H_o	Hubble parameter
K	redshift correction to apparent magnitude
k	curvature constant
M_o	absolute magnitude
ΔM_o	evolutionary correction to absolute magnitude
$\dot{M}_o$	rate of change of absolute magnitude for present epoch, $\left(\frac{dM_o}{dt}\right)_{t=t_o}$
m	apparent magnitude
$N(m)$	number of galaxies brighter than apparent magnitude m
n	number of galaxies per unit volume
Q_o	number of square degrees in celestial sphere
q_o	acceleration parameter
R	scale factor
R_o	present value of scale factor
$S(\omega)$	function of ω that depends on curvature of space
t	time before present
t_o	present time
δ	redshift
Λ	cosmical constant
σ_o	density parameter
ω	function of radial metric variable

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