Accession No. 07906-65

SID 65-1084

18

DEFINITION OF OPTICAL ATMOSPHERIC EFFECTS ON LASER PROPAGATION
VOLUME III

4 August 1965

NORTH AMERICAN AVIATION, INC.
SPACE and INFORMATION SYSTEMS DIVISION

FOREWORD

This document is the concluding report of Task I, Problem Definition, of the Laser Space Communications System (LACE) Study. It was prepared by the Space and Information Systems Division of North American Aviation, Inc. This report is submitted in accordance with requirements of Contract NASw-977, Supplemental Agreement, dated 15 February 1965.

TECHNICAL REPORT INDEX/ABSTRACT

ACCESSION NUMBER	07906-65
DOCUMENT SECURITY CLASSIFICATION	Unclassified
TITLE OF DOCUMENT	Laser Space Communications System (LACE) Study Task I - Problem Definition
LIBRARY USE ONLY	
AUTHOR(S)	H. E. Henry
CODE	
ORIGINATING AGENCY AND OTHER SOURCES	North American Aviation, Inc. Space and Information Systems Division
DOCUMENT NUMBER	SID 65-1084
PUBLICATION DATE	4 August 1965
CONTRACT NUMBER	NASw-977

DESCRIPTIVE TERMS

Laser Propagation
Laser Experiments
Atmospheric Optical Effects

ABSTRACT

This document comprises the conclusion of Task I, Problem Definition, of the Laser Space Communications System (LACE) Study. As such, it is a continuation and expansion of earlier published work under this task on the same contract.

The areas of study covered in this report include: (1) a brief review and updating of the summary of available measurements; (2) some additional cloud topics bearing on site selection and laser transmission; (3) an extension of the optical propagation theory to include temporal considerations; (4) a comparison of various theoretical propagation analyses; and (5) a method of acquiring and categorizing data related to the LACE study.

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INTRODUCTION

SCOPE

The overall objective of the Laser Space Communications System (LACE) Study is to provide a plan for the implementation of a comprehensive experimental program to determine atmospheric effects on laser propagation, with particular emphasis on effects related to optical space-ground communication. The present study effort is a continuation and expansion of earlier work under this contract and is divided into four tasks: Task I, Problem Definition; Task II, Experiment Specification; Task III, System Implementation Study; and Task IV, Program Specification. The results of all four tasks also will be summarized in a final report.

This report is concerned with the conclusion of earlier effort initiated under Task I. The earlier contractual effort on this task (References 1 and 2) concentrated on the determination of factors limiting optical earth-space communication, a survey of various atmospheric effects on optical transmission, and an analytical evaluation of the effect of signal transfer through the atmosphere on selected modulation techniques.

The areas of investigation included in this report were established with the following objectives: (1) an extension of the optical propagation theory to include temporal variations and their effects on system performance; (2) a comparison and evaluation of various different theoretical treatments; (3) a brief examination and updating of the measurement summary prepared in Reference 1; and (4) a study of the problem of the acquisition and dissemination of data.

Although this report is intended to satisfy the requirements of Task I, the Tasks II, III, and IV efforts, in order to be complete, also must cover subject matter related to the definition of the problem.

SUMMARY OF CONTENTS AND CONCLUSIONS

A rather complete review of the literature has pointed out the lack of measurements related to a theory that both describes the physical process causing fluctuation effects and is in a form suitable for laser communication system design.

A considerable amount of cloud-cover statistics is available in various forms that generally give average conditions for a given locale. However, computations of the conditional probability of sky coverage (e.g., What is the probability of sky coverage at an alternate site given a certain sky coverage at a prime site?) from one experimental site to another are not readily available. The explicit requirement for these data will be examined further in Task III.

A brief analysis of cloud transmittance indicated that for relatively thin clouds the major part of the transmitted signal is coherent, while for thick clouds the majority of the transmitted light is scattered.

The applicability of analytical work on wave propagation through a random media to the prediction of the time-power spectrum of the amplitude and phase fluctuations of a propagating wave is discussed, and recommendations for additional analysis are indicated.

The broad classifications of theoretical work of the following approaches are compared: (1) solutions based on Rytov's method; (2) propagation of mutual coherence; (3) ray optics: (4) two-level (tropopause, ground layer) generating mechanism; and (5) Born approximation. The general accuracy of the results by several workers, using Rytov's method, appear valid. Though some of the many details obtained by Tatarski are questioned (a completely revised and condensed text is recommended), it is believed that many of his results may be used as standards against which all other approaches may be judged. It was shown that Hufnagel and Stanley's result, using a solution to an equation for the mutual coherence function, is in perfect agreement with those obtained by Rytov's method, and may be considered a subcase of Tatarski's results. It is felt that ray optic methods probably can be trusted to provide a good approximation to phase problems, but not to amplitude problems. It is indicated that more experimental data are needed to verify the conclusions concerning the two-level generating mechanism, and that the Born approximation is not applicable.

A method of acquiring and categorizing data related to the LACE program is illustrated.

EXTENSION OF THE REVIEW OF TYPES OF AVAILABLE MEASUREMENTS

The review of types of measurements and limitations of atmospheric effects on optical propagation has been extended to include some additional work in measurements and theoretical analysis performed both in this country and abroad. This additional review of the literature serves further to reinforce previous conclusions on the lack of measurements related to a theory describing the physical process causing fluctuation effects.

The material covers reports on topics of index of refraction theory and measurement, atmospheric spectral conditions, atmospheric turbulence, turbidity measurements, and measurement of spatial coherence and noise spectra of the propagation of a laser beam through the atmosphere. In addition to the references reported here, attention is called to a recent Technical Note of the U.S. National Bureau of Standards (Reference 3), which presents a very comprehensive literature survey of observational, experimental, and theoretical aspects of optical scintillation. A later section of this task report also compares some of the theoretical approaches to the problem.

This review, together with that previously reported under Task I, provides a summary of available measurements bearing on the LACE problem definition.

Scattering by random variations in the index of refraction resulting from random variations in the temperature in the troposphere is considered in substantial detail in Reference 4. A number of typical optical paths were used to illustrate the problem, and it was pointed out that it is possible to show that multiple scattering contributions are of importance for all but very short paths in the troposphere.

A model of the atmosphere was used to account for turbulent motion and refractive-index fluctuations in the entire atmosphere in order to derive expressions for the correlation function and the spectrum of the intensity fluctuations of starlight (Reference 5). The shape of the refractive-index spectrum was assumed to be constant along a ray path, but the magnitude of the fluctuations was allowed to vary with altitude. There was good agreement between typical airborne measurement of refractive-index fluctuations and the magnitude of the refractive-index fluctuations required to account for the observed stellar scintillation data.

Atmospheric refractive index distributions were observed by aircraft over Japan for ten-day periods in the spring, summer and winter at elevations up to 3 kilometers by utilizing an airborne microwave refractometer (Reference 6). It was found that the seasonal fluctuation in $\overline{\Delta n^2}$ was so small that the ratio of $\overline{\Delta n^2}$ in July to $\overline{\Delta n^2}$ in February was only 8 decibels at most. Invisible refractivity clouds (characterized by several n-unit deviations) were detected during one-third to one-fifth of the flight paths during summer and winter.

The broadening of the spectral width of an electromagnetic wave passing through a medium with a randomly fluctuating refractive index is considered in Reference 7. The change in the frequency due to time-dependent fluctuations was studied by neglecting polarization and assuming that the scale of the fluctuations in space and time, respectively, are great compared to the wavelength and the time of oscillation of the fields. In addition, a monochromatic plane wave, traveling orthogonally through an infinite slab of a fluctuating medium, was assumed; by considering the fluctuations and the thickness of the slab to be small, a perturbation method was used. The ensemble average of the mean-square deviation of the frequency distribution of the outgoing wave was determined.

A technique for determining the vertical structure of atmospheric turbulence by utilizing balloon flights is described (Reference 8). A camera, pointing upward and mounted at the bottom of a string of 25 balloons, one above the others, photographed any distortions of the string due to atmospheric currents. The temperature also was measured. The associated change in refractive index is in good agreement with the theory of stellar scintillation, it is reported.

A means was developed for determining astronomical seeing by utilizing a reticle, consisting of a fine grid, placed in the focal plane of a telescope and recording the transmitted light with the aid of a photomultiplier and recorder (Reference 9). Any wandering of the image because of seeing conditions provides an a-c output from the photomultiplier. The grid was made with two crossed Ronchi gratings having a lattice with holes slightly smaller than the minimum size of a star image. The mean amplitude of the photomultiplier signal appeared to correlate with visual estimates of seeing conditions.

Spatial spectra of temperature fluctuations in the atmosphere were obtained by aircraft and tower measurements (Reference 10), and it is reported that the spectra in the 1000- to 3000-meter altitude range shows a clear power low dependence on wave number, with an exponent close to the 5/3 predicted by the Kolmogoroff-Obukhoff theory. At small wave numbers, the exponent deviated considerably from 5/3.

An expression is derived for a correlation function of the diffraction image formed by a paraxial focusing system subject to any fluctuations in the incident wave and to any ratio between the dimensions of the system and the correlation radius (Reference 11).

The limitation in the image quality of a large telescope due to atmospheric turbulence is pointed out in Reference 12. Although capable of yielding an essentially normal Airy disc, the atmospheric turbulence can spread out the image to the extent that the central maximum of the diffraction pattern can increase ten times.

A modulation index was developed (Reference 13) based on a series of experiments which studied the effect of turbulence on the transmission, near the ground, of visible and infrared radiation. This modulation index was defined as the normalized mean-square value of the fluctuation of the intensity of the detected radiation. It was found that dependence of the modulation index on the diameter of the entrance aperture of the radiometer-receiver is such that it varies inversely as the diameter.

The variation of turbidity with the time of the year, with air mass, and with latitude is presented in Reference 14, along with a summary of earlier determinations of a turbidity coefficient and a method of determining the wavelength dependence on the extinction by atmospheric aerosols.

Statistical data on the distribution of the turbidity factor at Karadog were analyzed in relation to vertical transparency for a measurement period of three years (Reference 15).

The correlation between t and σ (turbulence angle and mean square deviation of image motion) was considered (Reference 16).

The probability of receiving light from an object viewed through a turbulent atmophere was shown to follow a normal Gaussian distribution (Reference 17). The root-mean-square angular deflection of the points of any object were shown to be proportional to the square root of the object to observer distance.

An experimental apparatus is described which makes it possible to measure quantitatively the spatial coherence of various kinds of laser beams. The numerical values obtained for the spatial coherence of a helium-neon laser also are reported (Reference 18).

The noise spectrum of a horizontally polarized 6328Å laser beam over a 120-meter path was measured (Reference 19), and it was reported that the shape of the noise spectrum is unchanged when all of the detectable beam is received. Similarly, changes in diameter or geometrical divergence of the transmitted beam or changes in transmission distance did not affect the spectrum.

SOME ADDITIONAL CLOUD TOPICS RELATED TO LACE

CLOUD DISTRIBUTION CONSIDERATIONS

The first problem posed by clouds is how laser light propagates in them. It concerns the relationship of the intensity, coherence, polarization, and angular spread of the beam to the composition, structure, shape, turbulence, and thickness of the cloud. The answers to this problem should clarify the capability of laser communications systems through clouds and the resulting limitations placed on methods of signal transfer.

The second problem, which depends on the answers to the first, is how to avoid all or certain types of clouds. It concerns the statistics of cloud cover, heights, and thickness as a function of types. The answers should indicate the relative advantages of different station locations and the increased probability of getting through if alternate stations are available.

The main purpose for the analysis of these data was to gain insight into the gross aspects of cloud-cover implications of the location of laser experimental sites. The usefulness of these data beyond this point appears questionable for LACE. During the course of Task III, however, study will be devoted to determining exactly what cloud statistics are needed for implementation of the experimental program, and attempt (within the scope of contractual resources) to obtain as much as possible of the required statistics from the available data.

London (Reference 20), in a study of the atmospheric heat balance, summarizes the percentage of sky cover by all clouds and by certain types as function of latitude and season in the Northern Hemisphere. His results are quoted in Tables 1 through 5.

According to Table 1, the minimum cloud cover occurs around 20°N in the winter, moves north to about 30° in the summer, and then retreats as winter approaches again. A weaker minimum occurs near the pole during the winter. A maximum exists around 55°N in the winter, moves to about 70°N by late summer, intensifying all the while, and then retreats and weakens as winter approaches again.

Of course, detail has been lost in the averaging process. London indicates some of this detail by plotting total cloudiness on polar projections of the Northern Hemisphere. A study of the winter map reveals that the extremes range from 10 percent over the Sahara Desert to 85 percent over

Table 1. Percentage of Sky Cover by All Clouds

Latitude (°N)	Winter	April	Summer	October	Year
0-10	47	51	54	53	51.2
10-20	36	42	49	48	43.8
20-30	38	42	42	41	40.8
30-40	50	52	41	46	47.2
40-50	59	59	55	56	57.2
50-60	63	62	63	66	63.5
60-70	58	60	66	70	63.5
70-80	47	59	69	70	61.2
80-90	40	55	64	60	54.8
Northern Hemisphere Average	47.9	51.3	52.0	53.0	51.1

Table 2. Average Amount (%) and Height (km) of Cloud Types — Winter

Latitude (°N)	Ci		As		Ns		St		Cu		Cb	
	%	km	%	km	%	km	%	km	%	km	%	km
0-10	18.6	9.6	7.0	4.2	7.0	1.4	7.2	1.4	13.8	1.7	2.2	1.7
10-20	13.2	10.1	5.0	4.4	3.8	1.6	6.2	1.6	10.9	2.0	1.1	2.0
20-30	13.9	10.2	6.0	4.3	4.6	1.6	8.8	1.6	9.8	2.0	0.9	2.0
30-40	17.4	9.7	7.8	3.8	8.8	1.4	14.0	1.4	10.4	1.8	0.9	1.8
40-50	19.7	8.0	11.2	3.0	12.6	1.2	18.4	1.2	10.2	1.6	1.4	1.6
50-60	20.9	7.1	13.1	2.5	14.6	1.0	19.8	1.0	9.8	1.2	1.3	1.2
60-70	18.4	6.8	11.0	2.3	13.8	0.9	17.8	0.9	9.2	1.2	1.2	1.2
70-80	13.0	6.8	8.8	2.2	10.6	0.8	13.6	0.8	7.8	1.0	1.0	1.0
80-90	10.0	6.8	7.8	2.0	8.7	0.8	11.4	0.8	6.8	1.0	0.8	1.0

Table 3. Average Amount (%) and Height (km) of Cloud Types — Spring

Latitude (°N)	Ci %	Ci km	As %	As km	Ns %	Ns km	St %	St km	Cu %	Cu km	Cb %	Cb km
0-10	23.6	9.1	7.3	3.6	6.1	1.1	7.0	1.1	12.7	0.9	4.7	0.9
10-20	18.5	9.4	6.5	3.8	3.6	1.2	6.1	1.2	10.0	1.0	3.9	1.0
20-30	17.8	9.8	7.0	4.0	3.8	1.4	7.1	1.4	9.5	1.3	3.4	1.3
30-40	22.8	9.6	9.0	4.2	6.8	1.6	11.6	1.6	9.3	1.7	3.6	1.7
40-50	24.6	8.7	12.0	3.8	9.3	1.3	16.5	1.3	8.3	1.6	3.6	1.6
50-60	25.8	7.5	13.4	3.3	12.2	1.0	17.1	1.0	7.5	1.4	3.5	1.4
60-70	24.4	6.6	10.3	2.9	14.3	0.8	14.0	0.8	9.3	1.2	3.1	1.2
70-80	23.4	6.1	8.9	2.6	16.0	0.7	13.5	0.7	9.0	1.1	2.5	1.1
80-90	21.7	5.7	7.8	2.4	15.5	0.6	13.0	0.6	7.5	1.0	1.5	1.0

Table 4. Average Amount (%) and Height (km) of Cloud Types — Summer

Latitude (°N)	Ci %	Ci km	As %	As km	Ns %	Ns km	St %	St km	Cu %	Cu km	Cb %	Cb km
0-10	23.0	10.2	8.1	3.8	7.5	1.1	7.2	1.1	13.5	1.4	6.0	1.4
10-20	19.6	10.6	7.0	4.0	6.0	1.4	7.1	1.4	12.7	1.7	5.4	1.7
20-30	15.6	10.8	5.9	4.1	3.7	1.8	8.0	1.8	11.8	2.0	3.6	2.0
30-40	13.6	10.4	7.0	3.9	5.2	1.8	10.0	1.8	8.9	2.0	3.4	2.0
40-50	17.8	9.2	8.8	3.2	9.0	1.6	16.9	1.6	8.9	1.9	4.4	1.9
50-60	21.9	8.0	11.0	2.6	11.9	1.2	19.6	1.2	9.2	1.8	4.9	1.8
60-70	26.4	7.5	11.8	2.4	13.0	1.0	19.8	1.0	9.8	1.6	4.8	1.6
70-80	30.1	7.2	12.6	2.1	13.8	0.8	20.5	0.8	10.0	1.4	4.8	1.4
80-90	25.2	7.1	11.0	2.1	12.6	0.8	18.8	0.8	9.5	1.4	4.2	1.4

Table 5. Average Amount (%) and Height (km) of Cloud Types – Fall

Latitude (°N)	Ci		As		Ns		St		Cu		Cb	
	%	km	%	km	%	km	%	km	%	km	%	km
0-10	24.7	9.2	7.6	3.8	7.1	1.0	7.5	1.0	11.8	0.9	5.6	0.9
10-20	21.2	9.6	7.0	4.1	5.5	1.1	6.9	1.1	10.5	1.0	5.7	1.0
20-30	16.7	10.1	6.4	4.4	3.7	1.3	6.8	1.3	9.6	1.3	4.0	1.3
30-40	18.7	9.7	7.6	4.2	5.4	1.5	9.6	1.5	9.8	1.6	3.7	1.6
40-50	21.7	8.8	11.8	3.8	8.8	1.3	15.3	1.3	8.0	1.4	3.7	1.4
50-60	28.3	7.8	14.9	3.3	13.8	1.1	18.0	1.1	7.8	1.2	4.0	1.2
60-70	32.5	6.7	14.4	2.8	19.0	0.9	16.1	0.9	8.5	1.1	3.8	1.1
70-80	34.1	5.9	14.2	2.4	20.5	0.8	15.5	0.8	7.5	1.0	3.0	1.0
80-90	25.0	5.3	10.2	2.2	19.0	0.7	13.5	0.7	5.5	0.9	1.5	0.9

northern Scandanavia. Considerable longitudinal variation exists. At about 70° N, one can move westward from the maximum of 85 percent to a minimum of 30 percent in northern Canada, which is the northernmost minimum, being not right at the pole. Likewise, one can move eastward at 20°N from the minimum of 10 percent to a maximum of 60 percent at 125°W.

Tables 2 to 5 combine data and certain assumptions made partly for convenience and partly because of a lack of data. All high clouds are grouped together under Ci; all middle clouds are represented as As; and stratus and stratocummulus are both represented as St. The symbols Ns, Cu, and Cb represent nimbostratus, cumulus, and cumulonimbus, respectively. The bases of Cu and Cb have been assumed to be the same; the bases of St and Ns have been assumed to be the same; and the tops of Ns have been assumed to be the same as the bases of As. Lastly, middle and high clouds were assumed to be present in the proportion in the part of the sky shielded by low clouds as in the observable portion.

AN ANALYSIS OF LASER TRANSMISSION THROUGH A CLOUD

When light is described in its particle or photon form, the diffusion equation can be used to calculate the scattered (and direct) intensities. This has been done by Fritz (Reference 21). His calculations are for clouds that contain water particles which have a scattering function sharply peaked in the direction of forward scattering. Such a scattering function matches experimental data (Reference 22) quite well. For these calculations the incident flux was assumed to be plane wave and infinite with some arbitrary incident angle θ ($0° \leq \theta \leq 90°$). According to Fritz, these calculations are in fair agreement with some measured values. The results would be applicable to laser propagation through clouds when the cloud cover was extensive and the ratio of cloud thickness to the spatial spread of the laser beam was small. This would occur for down-link (space-to-earth) propagation or for earth-based systems having very wide beam angles.

Figure 1 is based on Fritz's work and gives the transmittance of clouds $T(\tau)$ versus the cloud's optical thickness for three zenith angles. The transmittance times the incident intensity on the cloud gives the intensity exiting from the cloud. In calculating the transmittance, the fact is used that in the frequency range of 0.4 - 0.7 microns, the particles are nonabsorbent, so that all the energy is scattered (Reference 23).

The intensity of light exiting from the cloud is composed of both direct and scattered light. If the original beam incident on the cloud was spatially coherent, the direct beam exiting from the cloud also will be

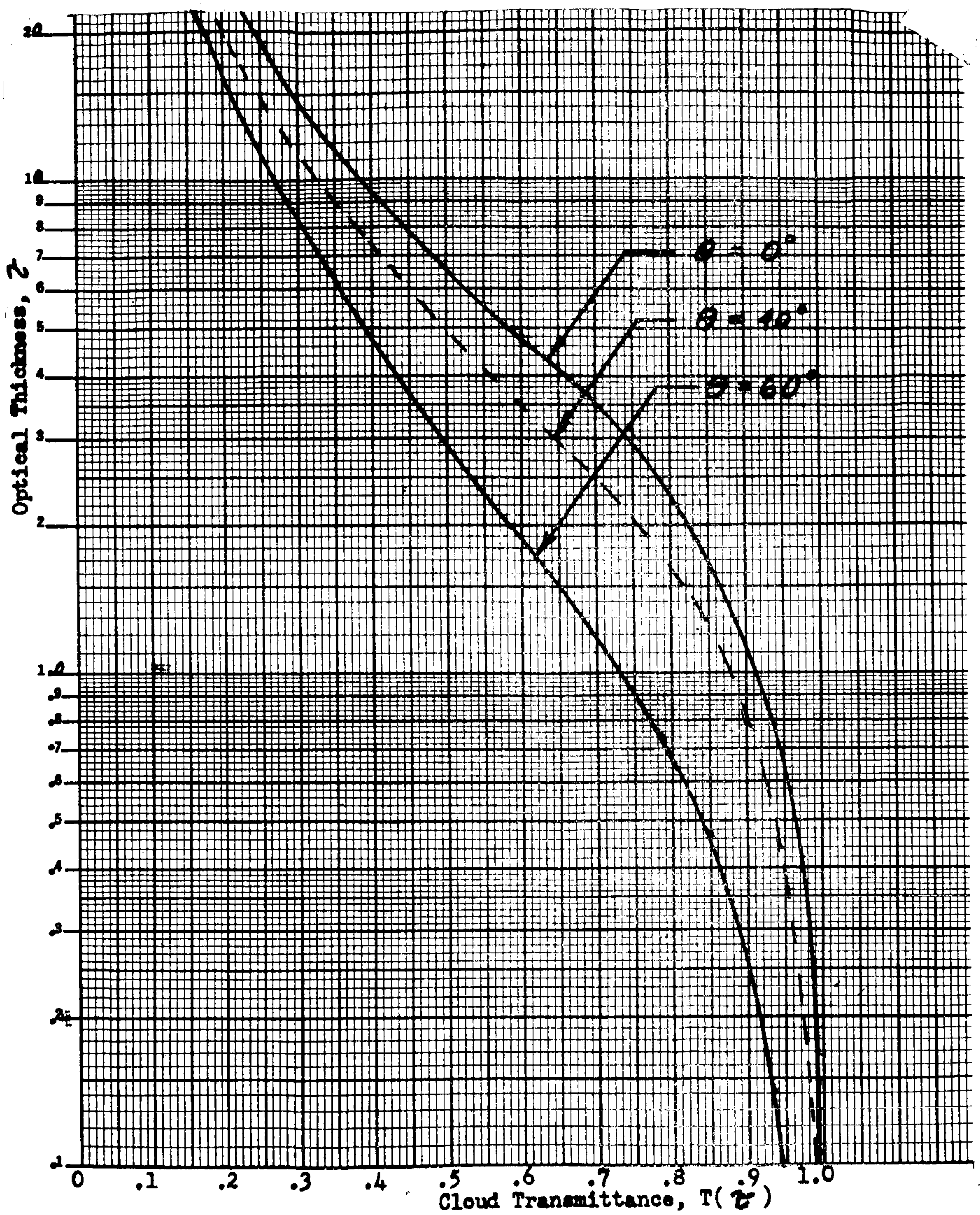

Figure 1. Cloud Transmittance vs Optical Thickness for Three Zenith Angles, θ

coherent. The scattered portion may have partial coherence but the degree is not known. The exiting transmitted light is

$$I_t = I_o e^{-\tau/\cos\theta} + I_s \tag{1}$$

where

I_t is the total light intensity

I_o is the incident light intensity

I_s is the scattered light

τ is the optical thickness of the cloud

θ is the angle of incidence measured to the vertical.

The term $I_o e^{-\tau/\cos\theta}$ pertains to the part of the transmitted light that is still coherent. Defining $I_{coh} = I_o e^{-\tau/\cos\theta}$ Equation 1 becomes

$$\frac{I_s(\tau)}{I_{coh}(\tau)} = T(\tau)\, e^{\tau/\cos\theta} - 1 \tag{2}$$

This ratio is plotted versus τ and θ in Figure 2.

The optical thickness, τ, is related to the extinction coefficient β and the cloud thickness z by $\tau = \beta z$. This relation is plotted in Figures 3 and 4. The value of β has been calculated using measured values of droplet size and density distributions (Reference 22). For a cloud with an average density of 100 drops per cubic centimeter, a mode radius of 0.70 micron, the extinction coefficient is around 16 km^{-1}. Experimental data (Reference 24) gives for dense fog, a β of 13-16 km^{-1}. A good estimation of the range of β would give a β between 1 km^{-1} and 100 km^{-1} depending on droplet distribution. The graph shows values of $\beta < 1$ for convenience. The thickness of clouds varies, but averages of about 200 meters for alto-cumulus and 2000 meters for cumulo-nimbus are found (Reference 25). Using these facts, one finds that for thin, low density ($\beta \approx 1$ km^{-1}) clouds, τ will be on the order of 0.2, indicating that the major part of the transmitted light is still coherent. On the other hand, for fairly dense clouds ($\beta \approx 16$ km^{-1}) of average thickness, 200 m, $\tau \approx 3$ and the majority of transmitted light is scattered.

The lack of experimental data from which β and z could be found for various cloud types limits the usefulness of the curves in judging the transmittance of each cloud type. However, the curves do give values which seem reasonable.

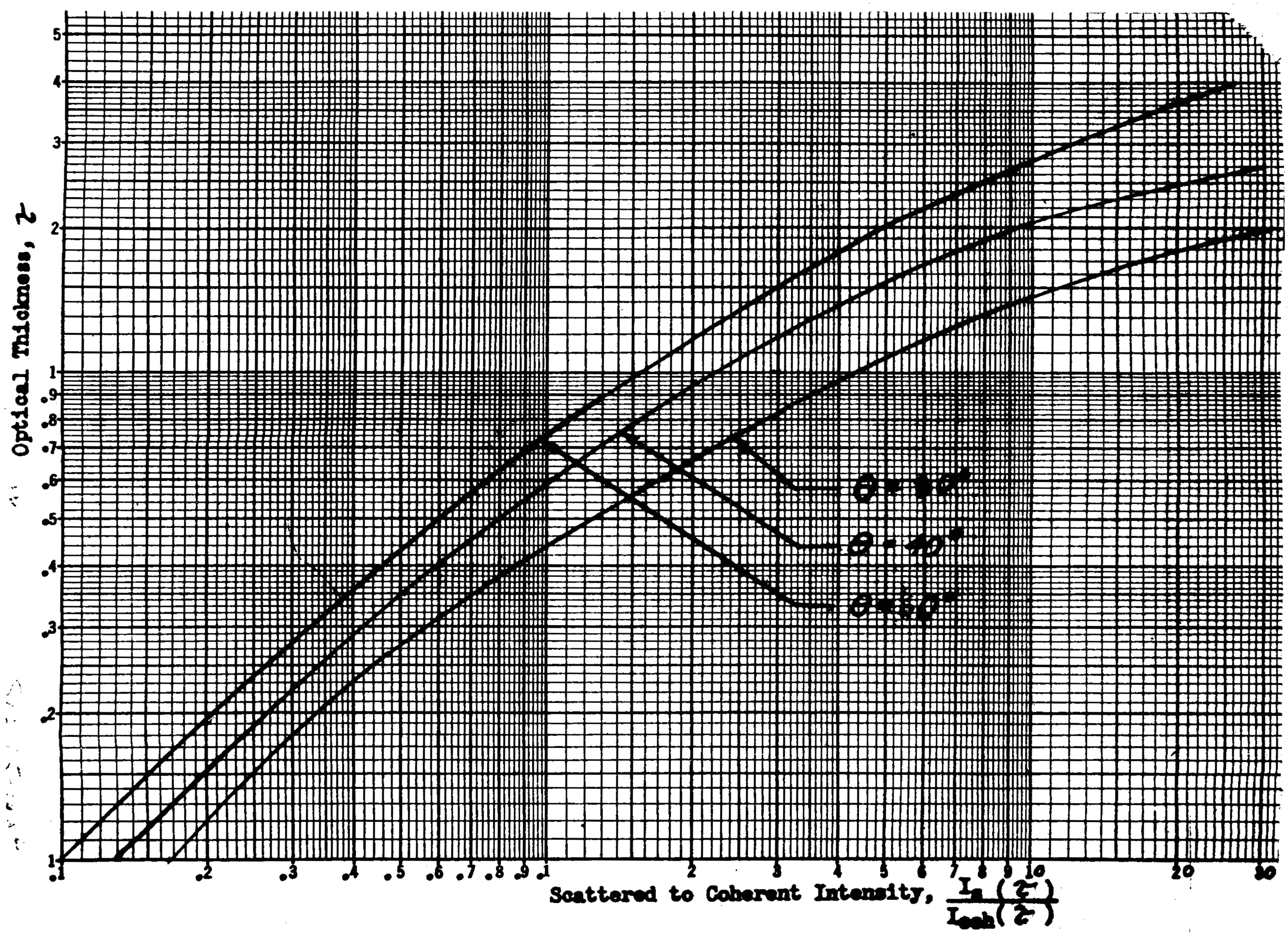

Figure 2. Ratio of Scattered to Coherent Intensity vs Optical Thickness τ, for Three Zenith Angles

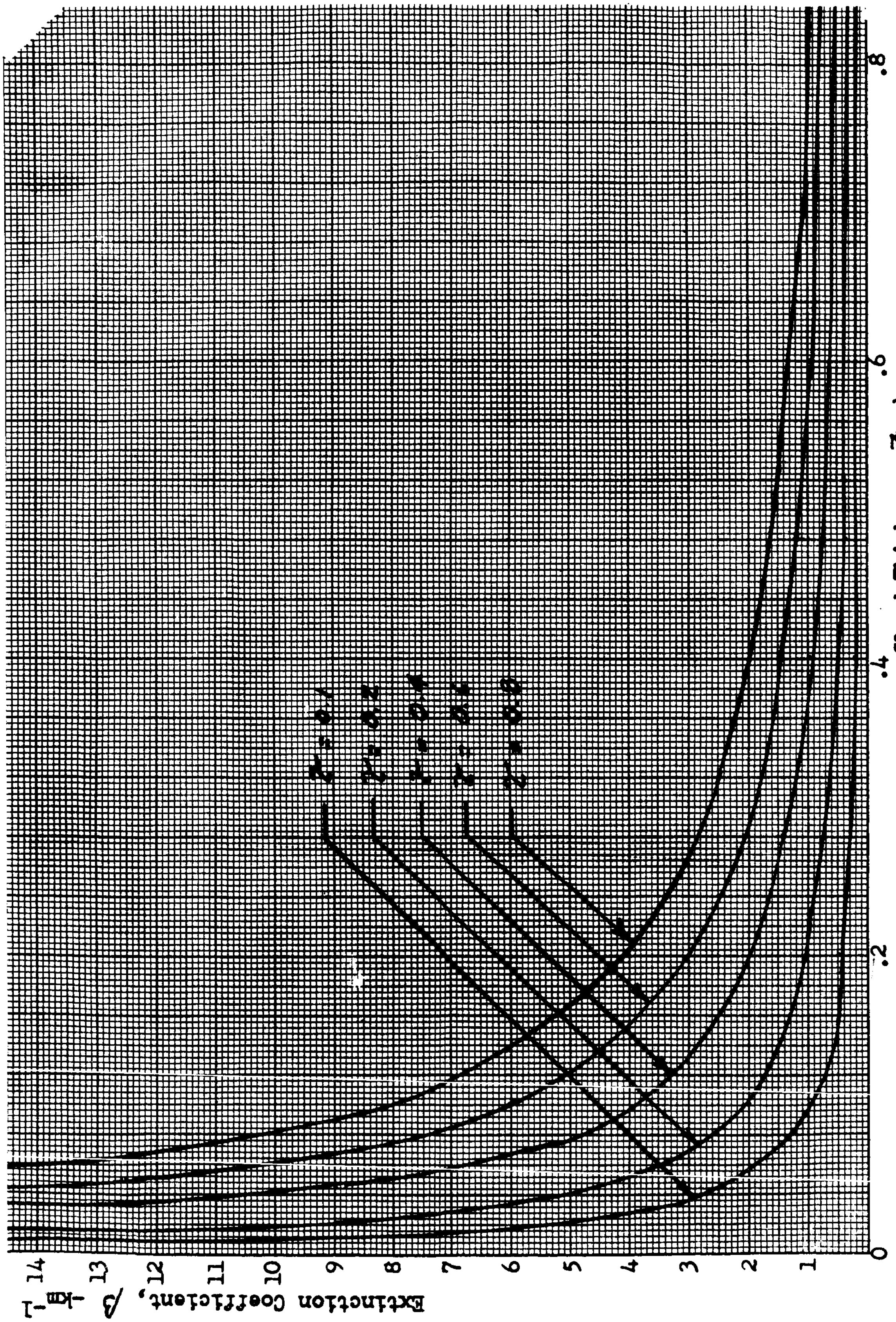

Figure 3. Extinction Coefficient, β, vs Cloud Thickness, z, With Optical Thickness τ (for small values of τ)

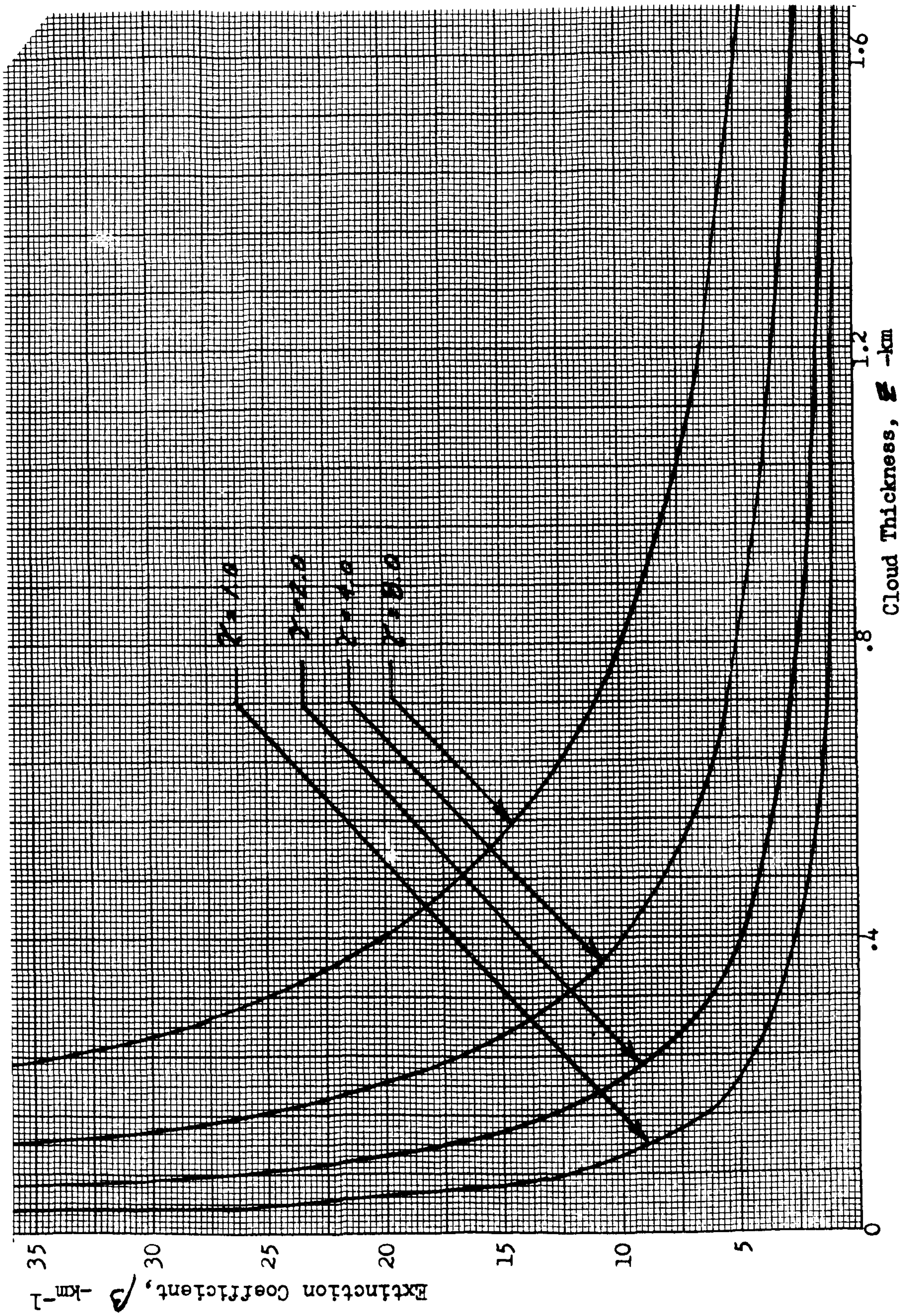

Figure 4. Extinction Coefficient, β, vs Cloud Thickness, z, With Optical Thickness τ (for large values of τ)

THEORETICAL CONSIDERATIONS OF TEMPORAL VARIATIONS OF A PROPAGATING WAVE

This section summarizes the work completed on the analysis of wave propagation through random media, and discusses the applicability of this theory to the prediction of the time-power spectrum of the amplitude and phase fluctuations of a propagating wave. A more complete discussion of the mathematical theory has been prepared by the Electro-Optical Laboratory in the series of Technical Memoranda 176, "Propagation of a Focused Laser Beam Through a Turbulent Atmosphere," 224, "Means, Variances, and Covariances for the Propagation of Laser Beams Through a Turbulent Atmosphere," and 118-A, "Theoretical Evaluation of the RMS Fluctuation of an Optical Heterodyne Signal with Random Wave Front Distortion." These technical memoranda are contained in Appendixes A, B, and C.

These studies are primarily concerned with the mathematics associated with the determination of the statistical behavior of an initially coherent beam of focused laser light propagating through a continuous random media which, while isotropic, is characterized by having a random point function for refractive index. Wave propagation through such a random medium is governed by a stochastic linear differential equation, the coefficients of which characterize the transmission medium. The solution to the stochastic wave equation is obtained by means of an iteration scheme. The statistical properties of the solution are then computed (viz., spatial correlation and structure functions, variance, and mean values of the logarithm of the relative amplitude and phase fluctuations of the wave function). The analyses have been kept sufficiently general by considering a focused laser beam parameterized with such quantities as initial spot diameter, beam spread, and an arbitrary radius of curvature of the focused wavefront as would be produced by the action of an ideal lens. The formulas, therefore, contain as limiting subcases the results of previous analyses for the plane wave and spherical wave (point source).

Wave propagation in a continuous random media may be described by the wave function $u(\vec{r})$, a solution of

$$\nabla^2 u(\vec{r}) + k^2 n^2(\vec{r})\, u(\vec{r}) = 0 \tag{3}$$

$$\psi_1(\vec{r}, t_2) = \frac{k^2}{2\pi} \iiint_{-\infty}^{+\infty} \mu(\vec{r}_o, t_2) \frac{u_o(\vec{r}_o)}{u_o(\vec{r})} \frac{\exp\left(ik|\vec{r}-\vec{r}_o|\right)}{|\vec{r}-\vec{r}_o|} d\vec{r}_o \tag{13}$$

If it is assumed that the fluctuation of the index of refraction at the point $\vec{r}_o$ at time t_1 coincides with itself at the point $\vec{r}_o - \vec{v}t$ at the time $t_2 = t_1 + t$, then

$$\left.\begin{aligned} \psi_1(\vec{r}, t_1) &= \frac{k^2}{2\pi} \iiint_{-\infty}^{+\infty} \mu(\vec{r}_o) \frac{u_o(\vec{r}_o)}{u_o(\vec{r})} \frac{\exp\left(ik|\vec{r}-\vec{r}_o|\right)}{|r - r_o|} d\vec{r}_o \\ \psi_1(\vec{r}, t_2) &= \frac{k^2}{2\pi} \iiint_{-\infty}^{+\infty} (\vec{r}_o - \vec{v}t) \frac{u_o(\vec{r}_o)}{u_o(\vec{r})} \frac{\exp\left(ik|\vec{r}-\vec{r}_o|\right)}{|\vec{r}-\vec{r}_o|} d\vec{r}_o \end{aligned}\right\} \tag{14}$$

Hence the time-dependent autocorrelation function is given again by equation (9) except that $\Phi(\sigma)$ must be modified so as to include the effects of wind velocity $\vec{v}(\vec{r}) = \vec{v}_n + \vec{v}_t$, where $\vec{v}_n$ and $\vec{v}_t$ are the normal and tangential components of $\vec{v}$, respectively:

$$C_{LL}(0, t) = \frac{-k^2}{8\pi^2} \text{Re} \int_0^z h(z_o) dz_o \iint_{-\infty}^{+\infty} \Phi(\vec{\sigma}, \vec{v}) \exp\left[1/4\sigma^2 \gamma(z, z_o)\right]$$

$$\left\{\exp\left[1/4\sigma^2 \gamma(z, z_o)\right] - \exp\left[1/4\sigma^2 \gamma^*(z, z_o)\right]\right\} d\vec{\sigma} + 0(\epsilon^3) \tag{15}$$

where it can be shown,

$$\Phi(\vec{\sigma}, \vec{v}) = \iiint_{-\infty}^{+\infty} \exp(i\vec{\sigma}\cdot\vec{\tau})\, C_n(\vec{\tau} - \vec{v}_n t)\, d\vec{\tau} \tag{16}$$

If it is assumed that $\vec{v}_n$ is independent of a displacement of $\vec{r}$ as $\vec{r}$ spans any plane normal to the direction of propagation (i.e., $\vec{v}_o = \vec{v}_n(z_o)$ a function of path length only), then we may replace $C_n(\vec{\tau} - \vec{v}_n t)$ by

$$C_n(\vec{\tau} - \vec{v}_n t) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{+\infty} \exp\left[-i\vec{\zeta}\cdot(\vec{\tau} - \vec{v}_n t)\right] \Phi(\vec{\zeta})\, d\vec{\zeta} \tag{17}$$

whence

$$\Phi(\vec{\sigma}, \vec{v}) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{+\infty} \exp(i\vec{\sigma}\cdot\vec{\tau})\, d\vec{\tau} \iiint_{-\infty}^{+\infty} \exp\left[-i\vec{\zeta}\cdot(\vec{\tau} - \vec{v}_n t)\right]\Phi(\vec{\zeta})\, d\vec{\zeta} \tag{18}$$

$$= \exp(i\vec{\sigma}\cdot\vec{v}_n t)\,\Phi(\sigma)$$

Substituting Equation 18 into Equation 15 gives

$$C_{LL}(0, t) = -\frac{k^2}{4\pi^2} \operatorname{Re} \int_0^z h(z_o)\, dz_o \iiint_{-\infty}^{+\infty} \exp(i\vec{\sigma}\cdot\vec{v}_n t)\,\Phi(\sigma)$$

$$\exp\left[1/4\,\sigma^2\,\gamma(z, z_0)\right] \left\{\exp\left[1/4\,\sigma^2\,\gamma(z, z_0)\right] - \exp\left[1/4\,\sigma^2\,\gamma^*(z, z_0)\right]\right\} d\vec{\sigma} + 0(\epsilon^3) \tag{19}$$

Performing the angular part of the $d\vec{\sigma}$ integration gives

$$C_{LL}(0, t) = -\frac{k^2}{2\pi} \operatorname{Re} \int_0^z h(z_o)\, dz_o \int_0^\infty \Phi(\sigma)\, J_o(\sigma v_n t)$$

$$\exp\left[1/4\,\sigma^2\,\gamma(z, z_0)\right] \left\{\exp\left[1/4\sigma^2\,\gamma(z, z_0)\right] - \exp\left[1/4\,\sigma^2\,\gamma^*(z, z_0)\right]\right\} d\sigma + 0(\epsilon^3) \tag{20}$$

We now compute the frequency (time) spectrum of the log-amplitude fluctuation of the wave. Denoting the spectral density of the fluctuation by $W_L(\omega)$, we have by definition

$$W_L(\omega) = 4 \int_0^\infty \cos(\omega t)\, C_{LL}(0, t)\, dt \tag{21}$$

Using the result

$$\int_0^\infty \cos(\omega t)\, J_o(\sigma v_n t)\, dt = \begin{cases} \left(\sigma^2 v_n^{\;2} - \omega^2\right)^{-1/2} & \omega < \sigma v \\ 0 & \omega > \sigma v \end{cases} \tag{22}$$

together with Equations 20 and 21, we obtain

$$W_L(\omega) = -\frac{2}{\pi} k^2 \,\mathrm{Re} \int_0^z h(z_o)\, dz_o \int_{\omega/v}^\infty \frac{\sigma \Phi(\sigma)}{(\sigma^2 v_n^{\;2} - \omega^2)^{1/2}}$$

$$\exp\left[1/4\, \sigma^2 \gamma(z, z_0)\right] \Big\{ \exp\left[1/4\, \sigma^2 \gamma(z, z_0)\right] - \exp\left[1/4\, \sigma^2 \gamma^*(z, z_0)\right] \Big\}\, d\sigma + O(\epsilon^3) \tag{23}$$

Further work in this area can be directed toward simplifying and evaluating the integrations appearing in Equation 23 for various physical situations of interest, toward considering the relationships between W_L and the spectrum of the intensity fluctuations, and, finally, toward considering the spectrum of the fluctuation of the signal collected by a large-diameter collector.

COMPARISON OF VARIOUS PROPAGATION ANALYSES

Numerous analyses of optical propagation have been performed by various workers with varying degrees of sophistication. This work has ranged from a discussion of the possibility that stellar scintillation is a purely physiological effect generated at the retina of the observer, to analysis of the propagation of the mutual coherence function under the assumption of the validity of the Kolmogoroff similarity theory of turbulence. It is the intent of this section to review briefly some of this work and show that there is a high degree of compatibility between the various results. Some of the compatibility is, in a sense, fortuitous. Where serious disagreement exists it has been described, and an attempt has been made to justify a choice between conflicting results.

Rather than attempt to review all the work which has been published in the analysis of optical propagation, the work of only a few persons has been treated in detail. It is hoped that a suitable selection has been made in order to provide broad coverage of all interesting approaches. The following broad classifications of work have been considered according to the approach: (1) solutions based on Rytov's method, (2) propagation of mutual coherence, (3) ray optics; (4) two-level (tropopause, ground layer) generating mechanism, and (5) Born approximation.

The order of these items is more or less the order of relative significance of the approach. As examples of the first approach, the work is considered of Tatarski (Reference 26), Chernov (Reference 27), and that of Fried, et al (References 28 and 29). The single most important example of the second approach is the work of Hufnagel and Stanley (Reference 30). As examples of the ray optics approach, the third group, the work of Bergmann (Reference 31) is considered for amplitude and phase effects, and of Duntley et al (Reference 32) for a slightly different view of phase effects. Of all the work that has gone into interpreting stellar scintillation as being due to a thin turbulent layer at about the altitude of the tropopause — which idea goes back to Rayleigh (Reference 33)—and loss of resolution as due to ground layer turbulence, the work of Barnhart (Reference 34) and co-workers is perhaps more interesting than most. While most of the aforementioned work is <u>ad hoc</u> model building, Barnhart attempts experimental verification. The fifth approach, the Born approximation is so poorly suited to the problem that, rather than discuss any worker who has made a serious attempt to use this approach, Tatarski's work will be considered, where he goes through the analysis just to point out its inapplicability.

RYTOV'S METHOD

Rytov's method is based on starting with Maxwell's equation in scalar form with the inhomogeneity and recasting it as an equation in the complex phase, which represents the phase and log-amplitude corrections to the unperturbed wave function. The key to Rytov's method is the dropping of terms quadratic in the refractive index correction and quadratic in the first derivative of the phasor. Chernov uses the solution to this equation directly and obtains an integral expression for the statistics of the complex phasor. This is the same method used in the original work on LACE. The difference in NAA's work is centered on the evaluation of multiple dimensional integrals. Chernov has to restrict attention to horizontal paths to suppress a dependence and simplify an integration; NAA used "steepest descent" techniques, permitting nonhorizontal paths to be considered but raising problems of the validity of asymptotic series results. Tatarski uses the same solution as Chernov but immediately returns to the differential equation (having used the solution only to justify an approximation involving dropping one of the terms in the differential equation for the phasor) and then studies the differential equation by means of a Fourier-Stieltjes integral. After sufficient manipulation, Tatarski arrives at useful expressions for the statistics of the phasor. Of the three, only Tatarski provides really useful expressions for log-amplitude statistics for all propagation path lengths, while only Tatarski and NAA-S&ID provide useful results for the phase fluctuation statistics. The differences among the three are, however, merely those of the degree of completion of integral evaluation. There is no real contradiction between the results of any of the workers using Rytov's method.

At an early point in all of the above works the assumption is made that what happens at an angle far off the straight-line propagation path can contribute and that the "small angle approximation" can be made. In a recent work (Reference 35) a spectral domain technique was used to solve the differential equation for the complex phasor (after making Rytov's approximation but avoiding the explicit spatial solution) and the small scattering angle approximation was able to be eliminated. At a fairly late point in the computation, the mathematical approximation was made instead that the wave number of the radiation is large compared to the highest significant wave number of the turbulence. This, however, is just a matter of clearing up details. The results are identical to Tatarski's.

It is our belief that results obtained by a rigorous application of Rytov's method are correct. For statistics of propagation of an infinite plane wave (explicitly the phase and log-amplitude correlation functions), the results as presented by Tatarski can be considered a standard against which other results may be judged. The use of the Kolmogoroff hypotheses for data on the turbulence spectrum is sufficiently well established that any use of less sophisticated turbulence models (such as a Gaussian correlation function for turbulence) is wasted effort. For this reason, special emphasis is placed on some of Tatarski's results as well as on equivalent results of NAA. It is not felt that any of the above works represents a significant contribution to the problem of propagation of anything other than an infinite plane wave, as for example, a collimated finite diameter beam. In this regard, the somewhat incomplete work of Schmeltzer (Reference 36) is recommended.

To recapitulate comments on Rytov's methods, faith is maintained in the general accuracy of the results and the general agreement of the results of the several workers using this method. Though some of the many details obtained by Tatarski are questioned (and a completely revised and condensed text is recommended), many of his results may be used as standards against which to judge all other approaches.

PROPAGATION OF MUTUAL COHERENCE

Hufnagel and Stanley have taken the approach that rather than try to solve the inhomogeneous Maxwell's equation and compute the propagation statistics from the solution, they will modify the equation so that it becomes a differential equation in the statistics. They obtain an equation for the mutual coherence function which they solve.

Utilizing the Kolmogoroff similarity hypotheses, and applying the central limit theorem to justify treating the random variable as a Gaussian variable, they find that the mutual coherence function $M(\rho)$ is given as

$$M(\rho) \doteq \exp\left\{-1/2\left[2.91\rho^{5/3}k^2 \int_{\text{path}} d\Lambda \; C_N^2(\Lambda)\right]\right\} \tag{24}$$

where C_N^2 is the turbulence structure constant, k is the wave number, and the integration is over the path of propagation. This will be shown to be equivalent to the result

$$D_A(\rho) + D_s(\rho) = 2.91\, k^2 \rho^{5/3} \int_{\text{path}} d\Lambda \, C_N^2 \tag{25}$$

taken from Equation 7.98 of Tatarski's work. Here $D_A(\rho)$ is the log-amplitude structure function, and $D_s(\rho)$ is the phase structure function. It will be assumed that Gaussian distributions are applicable to phase and log-amplitude fluctuations.

Let V_1 and V_2 denote the wave functions at points 1 and 2, and let ℓ_1 and ℓ_2 and ϕ_1 and ϕ_2 denote the log-amplitude and phase at these points. Let ρ be the distance between these points.

Then

$$\left.\begin{aligned} V_1 &= \exp(ikz)\exp(\ell_1 + i\phi_1) \\ V_2 &= \exp(ikz)\exp(\ell_2 + i\phi_2) \\ M(\rho) &= \langle V_1^* V_2 \rangle \\ &= \exp\left[\ell_1 + \ell_2 + i(\phi_1 - \phi_2)\right] \end{aligned}\right\} \tag{26}$$

Let ℓ denote the average value of ℓ_1 and of ℓ_2. Then

$$M(\rho) = \langle \exp\left[\ell_1 + \ell_2 - 2\bar{\ell} + i(\phi_1 - \phi_2)\right]\rangle \exp(2\bar{\ell}) \tag{27}$$

Now the exponent is a Gaussian random variable with zero mean. It has been shown (Reference 37) that under this condition

$$M(\rho) = \exp 2\bar{\ell} \exp\left\{1/2 \langle [\ell_1 + \ell_2 - 2\bar{\ell} + i(\phi_1 - \phi_2)]^2 \rangle\right\} \tag{28}$$

Simple manipulation, noting such facts as that $\langle \phi_1 \ell_2 \rangle$ is equal to $\langle \phi_2 \ell_1 \rangle$ yields the result,

$$\begin{aligned} \langle \ell_1 + \ell_2 - 2\bar{\ell} + i(\phi_1 - \phi_2)^2 \rangle &= \langle \ell_1^2 \rangle + \langle \ell_2^2 \rangle + 2\langle \ell_1 \ell_2 \rangle \\ &\quad - 4\bar{\ell}^2 - \langle (\phi_1 - \phi_2)^2 \rangle \\ &= 2\,C_A(0) + 2\,C_A(\rho) - 4\bar{\ell}^2 - D_s(\rho) \\ &= -D_s(\rho) - D_A(\rho) - 4\bar{\ell}^2 + 4\,C_A(0) \end{aligned} \tag{29}$$

where C_A (ρ) is the correlation function for log-amplitude. Thus,

$$M(\rho) = \exp\left[2\bar{\ell} - 2\bar{\ell}^2 + 2C_A(0)\right] \exp\left\{-1/2\left(D_A(\rho) + D_S(\rho)\right)\right\} \tag{30}$$

It has been proven elsewhere (Reference 37) that

$$C_A(0) + \bar{\ell} - \bar{\ell}^2 \equiv 0 \tag{31}$$

if the log-amplitude has a Gaussian distribution. Consequently,

$$M(\rho) = \exp\left\{-1/2\left(D_A(\rho) + D_S(\rho)\right)\right\} \tag{32}$$

It is now apparent, comparing this with equations 24 and 25, that Hufnagel and Stanley's result is in perfect agreement with those obtained by Rytov's method and that Hufnagel and Stanley's result may be considered as a subcase of Tatarski's results.

In this regard, it is interesting to recall that Hufnagel and Stanley have questioned the accuracy of Rytov's method. NAA has previously argued against this (Reference 38). Now, it is interesting to note that their results also argue against this. In fact, considering the question in point, some rather interesting conclusions can be drawn. The question was whether one is justified in dropping the term quadratic in the first derivative of the complex phasor. We showed that this could be justified since we dropped, along with that term, the term quadratic in the refractive index fluctuation, and the two terms cancelled to one higher order. Hufnagel and Stanley have dropped out the latter term alone. Since both terms have the same order of magnitude, the same argument should apply against dropping either separately (but not against dropping both). Yet, dropping only one term, Hufnagel and Stanley arrive at the same result as is obtained by dropping both. Apparently, the problem is very insensitive to this type of approximation.

RAY OPTICS

The paper by Bergmann starts with the eikonal equation, the standard starting point for the geometric or ray optic approach. The procedure appears entirely rigorous, once the applicability of the eikonal equation is granted. Bergmann (Reference 31, Equation 44) obtains the results that the phase variance is

$$B_s\ (0) \simeq 2\ L \int_0^{\infty} B_N\ (\rho)\ d\rho \tag{33}$$

and that the log-amplitude variance, Equation (50), is given by

$$B_A\ (0)\ \simeq 1/15\ L^3 \int_0^{\infty} \nabla^2 \nabla^2\ B_N\ (\rho)\ d\rho \tag{34}$$

If we examine Tatarski's results, and restrict attention to short distance propagation so that the ray optic approximation can be considered valid, i.e., L small, we can obtain good agreement with Bergmann's results.

Starting with Tatarski's equation 7.63

$$B_s\ (\rho) = 2\pi^2\ k^2\ L \int_0^{\infty} J_o\ (\sigma\rho) \left(1 + \frac{k}{\sigma^2 L} \sin \frac{\sigma^2 L}{k}\right) \Phi_N(\sigma)\, \sigma\, d\sigma \tag{35}$$

and letting L be small

$$B_s\ (0) \simeq 2\pi^2 k^2 L \int_0^{\infty} 2\ \Phi_N\ (\sigma)\ \sigma d\sigma \qquad \text{(L small)} \tag{36}$$

Since

$$\Phi_N\ (\sigma) = \frac{1}{2\pi^2 \sigma} \int_0^{\infty} B_N\ (\rho)\ \sin\ (\sigma\rho)\ \rho\, d\rho \tag{37}$$

and

$$\int_0^{\infty} \sin\ (\sigma\rho)\ d\sigma = \frac{1}{\rho} \tag{38}$$

it can be seen that Tatarski's work can be reduced to

$$B_s\ (0) = 2k^2 L \int_0^{\infty} B_N\ (\rho)\ d\rho \qquad \text{(L small)} \tag{39}$$

in perfect agreement with Bergmann, for short propagation paths. For long propagation paths, Tatarski (Reference 26, Equation 7.65) obtains the result that

$$B_s\ (0) = k^2\ L \int_0^\infty B_N\ (\rho)\ d\rho \qquad \text{(L large)} \tag{40}$$

which differs from Bergmann's result only by a factor of two. This factor-of-two error will be found in all ray optic treatments. With the exception of this factor, ray optic treatment of phase fluctuation can generally be trusted. It should be noted, however, that the ray optic method is not self-justifying. It is justified by its agreement with the more accurate Rytov's method solution.

In the case of log-amplitude fluctuations, it will be seen that the ray optic treatment gives a result, which can be derived in the limit of small L, from the more exact result of Rytov's method but is, none the less, not a very good approximation to what is needed in most cases. Starting with Tatarski's Equation 7.62

$$B_A\ (\rho) = 2\pi\ ^2k^2L \int_0^\infty J_o\ (\sigma\ \rho) \left(1 - \frac{k}{\sigma\ ^2L}\ \sin \frac{\sigma\ ^2L}{k}\right) \Phi_N\ (\sigma)\ \sigma\ d\sigma \tag{41}$$

and letting L be small, a power series expansion can be used for sin $(\frac{\sigma^2 L}{k})$, giving

$$B_A\ (0) = 2\pi\ ^2k^2L \int_0^\infty \left[\left(\frac{\sigma^2 L}{k}\right)^2 \Big/ 3\,!\right] \Phi_N\,(\sigma)\sigma d\sigma \tag{42}$$

Again using the definition of $\Phi_N\ (\sigma)$ in terms of $B_N\ (\rho)$ as a Fourier transform and noting that $\int \sigma^4\ \Phi_N\,(\sigma)\ d\sigma$ corresponds to $\nabla^2\nabla\ ^2\ B_N\ (\rho)$, just as $\int \Phi_N\ (\sigma)\ d\sigma$ corresponds to $B_N\ (\rho)$, it can be seen that Tatarski's result implies that

$$B_A\ (0) \simeq k^2\ L^3 \int_0^\infty \nabla^2\nabla\ ^2\ B_N\ (\rho)\ d\rho \tag{43}$$

which is in agreement with the form of Bergmann's result but disagrees slightly in the coefficient. Even more obvious agreement with Bergmann's results can be seen by referring to Reference 28, Equation 19 of Section V, which was also derived from Rytov's solution. But this agreement is only valid for very short propagation paths. The range of validity is for

$$\frac{\sigma^2 L}{k} \ll 1 \tag{44}$$

Since

$$k = 2\pi/\lambda \tag{45}$$

and we must consider values of σ as large as

$$\sigma = 2\pi/\ell_o \tag{46}$$

where ℓ_o is the inner scale of turbulence, characteristically less than one cm, then

$$L \ll \; 30 \text{ meters } (\lambda = 0.5 \text{ microns}) \tag{47}$$

Outside this range, as we have shown (Reference 28), the results are not even a good approximation to the correct results. It is shown there that the equivalent of the ray optic result must be considered as the first term of an asymptotic series. That is, the total series diverges, but the first few terms may be a good approximation to the result sought. However, it has been shown that for large L even the first term is part of the divergent, rather than the good approximation part of the series. The nonsignificance of the result may further be seen by considering the fact that

$$\int \nabla^2 \nabla^2 \; B_N (\rho) \, d\rho \tag{48}$$

is very strongly a function of the inner scale of turbulence, ℓ_o, while the accurate results presented by Tatarski indicate that $B_A(0)$ is independent of ℓ_o. Also, the correct value of $B_A(0)$ is proportional to L to the eleven-sixths power, compared to the third (eighteen-sixths) power of Bergmann's result.

We conclude therefore, that ray optic methods can probably be trusted to provide a good approximation to phase problems but not to amplitude problems. Since optical resolution is principally a function of wavefront distortion (a phase effect) we would expect a ray optic treatment of optical resolution through the atmosphere to give good results.

Duntley, et al (Reference 32) treated the resolution problem by ray optics and concluded that the image of a point source will show an angular spread with a Gaussian distribution and a standard deviation which increases as the square-root of the propagation path length. Taking the transform of the spread function, the atmospheric modulation function is obtained, which from Duntley's result we conclude should also be Gaussian with a standard deviation that increases as the square-root of the propagation path length. Taking the transform of the spread function, the atmospheric modulation function is obtained, which from Duntley's result it is concluded should be Gaussian, with a standard deviation that varies inversely as the square-root of the path length. (The treatment given in Duntley's paper is crude enough so that there is no point in trying to relate the proportionality constant of the standard deviation to atmospheric turbulence parameters.) From Hufnagel's results on image statistics, it is known that the atmospheric modulation transfer function is equal to the mutual coherence of the wave evaluated at a separation ρ equal to the wave length, λ , times the image spatial frequency, f. As shown previously, the mutual coherence is equal to

$$\exp \left\{- 1/2 \left[D_A (\lambda f) + D_s (\lambda f)\right]\right\} \tag{49}$$

where the structure functions have a sum equal to

$$2.91 \, k^2 (\lambda f)^{5/3} \int_{\text{path}} C_N{}^2 \, d\Lambda \tag{50}$$

The atmospheric modulation transfer function can then be written as

$$\exp \left\{- 3.44 \left(\frac{\lambda f}{r_o}\right)^{5/3}\right\} \tag{51}$$

where r_o is defined as

$$r_o = \left\{ \frac{6.88}{2.91} k^2 \int_{\text{path}} C_N{}^2 \, d\Lambda \right\}^{3/5} \tag{52}$$

It is obvious that r_o is inversely proportional to the path length to the minus three-fifths power. To compare this result with Duntley's, the approximation is made that the five-thirds power can be replaced by six-thirds, so that an image transfer function which was Gaussian in frequency would be expected, as Duntley predicts. But the standard deviation would now be expected to vary inversely with the three-fifths power of path length, rather than the one-half power that Duntley predicts. These differences in power are small, and Duntley's result can be accepted as within the expected reliability of a geometric optic approach to the problem (i. e., it is not exact, but also not seriously in error).

To seek the source of the difference in power, it is necessary only to note Duntley's analysis, working in the geometric optic approximation, is essentially equivalent to a short propagation path so that $D_S(\rho) \gg D_A(\rho)$. Furthermore, the method of ray tracing Duntley uses is equivalent to assuming that the inner scale of turbulence ℓ_o is fairly large. This is necessary for the ray Duntley traces to maintain its integrity (i. e., it wanders but it doesn't spread). As shown in Reference 39 for values of $\rho < \ell_o$, the phase structure function, $D_S(\rho)$, is quadratic in ρ. Tatarski (Equation 6.64 on page 151) indicates the same result. It can be seen that Duntley's result is compatible with, but only an approximation to, the more exact results obtained by Rytov's method.

TWO-LEVEL GENERATING MECHANISM

It has been reported that optical resolution of a tracking system can be markedly improved by raising the system several meters above the ground (Reference 40). It is generally agreed that atmospheric turbulence close to the ground can produce severe effects on optical propagation. There is no reason to question this conclusion; such ground effects are unusually strong, compared to turbulence in the rest of the atmosphere, but any well designed optical system will be raised above these effects. It has also been suggested (Reference 33) that a turbulent layer at the tropopause is responsible for stellar scintillation. However, a large body of experimental data suggest that the scintillation-generating mechanism is in fact distributed throughout the atmosphere in accordance with the data on the altitude dependence of C_N^2 given by Hufnagel and Stanley (Reference 30). Nonetheless, the experimental work of Barnhart et al (Reference 34) warrants serious consideration, because it tends to confirm the idea that stellar scintillation is generated at an altitude close to the tropopause. They measured the speed and direction of drift of stellar scintillation shadow bands and found that this could be correlated with the wind velocity in the vicinity of the tropopause. S&ID recently carried out a calculation of the contribution to the log-amplitude variance of starlight due to turbulence at various levels of the atmosphere based on Hufnagel's data for C_N^2. It was found that the contribution curve is broadly peaked, with the peak near 10,000 feet and with a significant portion of the curve extending out to 30,000 feet. This would tend to accommodate Barnhart's data into a distributed turbulence model. However, there can be no question that the fit is not really comfortable and more experimental data are needed.

BORN APPROXIMATION

Tatarski discusses the application of the Born approximation to the optical propagation problem. As Tatarski remarks, however, the validity of the Born approximation is based on the assumption that most of the radiation is unscattered. Scintillation effects are strong, however, and most of the beam is scattered. It may therefore be concluded that the Born approximation is not applicable. It may be possible that a multiple scattering Born approximation can give satisfactory results, but most experience in strong scattering problems tends to argue against this. If the scattering is strong the Born approximation cannot be used.

DATA ACQUISITION AND DISSEMINATION

The problem of data acquisition and dissemination was examined briefly to provide background information for the tasks under this study, and to seek means for providing a maximum exchange of scientific information.

The main result of this effort was the preparation and recommended use of a LACE questionnaire (Appendix D). It could provide (1) a means of categorizing the various backgrounds and interests relating to this program, (2) a check-off list for acquiring data during personal interviews with individuals or groups participating in this field, and (3) the basis of a mail survey into current work and future plans of individuals or organizations with regard to theoretical and applied analysis, laboratory and field experimentations, experimental systems definition, and availability of resources.

NAA is presently utilizing the questionnaire as a part of the second application.

REFERENCES

1. Definition of Optical Atmospheric Effects on Laser Propagation: Vol. 1, NAA/S&ID, SID 64-1894-1 (9 Oct 1964).

2. Experimental Laser Space Communications Program: Task I, Problem Definition, Vol. 2 NAA/S&ID, SID 64-1894-2 (9 Oct 1964).

3. Meyer-Arendt, J.R., and C.B. Emmanuel, Optical Scintillation: A Survey of the Literature, U.S. Department of Commerce, National Bureau of Standards, Technical Note 225 (1965).

4. Bugnolo, D. S., On the Propagation of Electromagnetic Waves at Optical Frequencies in the Troposphere, Electromagnetic Theory and Antennas Symposium, Copenhagen (1962).

5. Reiger, S. H., "Starlight Scintillation and Atmospheric Turbulence," Astronomical Journal, Vol. 68, No. 6 (Aug 1963).

6. Fukushima, M. H. Iriye, K. Akita, and S. Miura, "Preliminary Study of the Spatial Distribution of Atmospheric Refractive Index from Aircraft Observations," Journal of Radio Research Laboratories (Japan), Vol. II (March 1964).

7. Andersen, E., "On-the-Line Broadening of Light in a Random Medium," Arkir for Fysik (Sweden), Vol. 26, Paper 1 (1964).

8. Dollfus, A., "A New Technique for Stratosphere Balloons and a Direct Study of the Origin of Stellar Scintillations," Comptes Rendus (France), Vo. 255, No. 1 (2 July 1962).

9. Babcock, H. W., "Instrumental Recording of Astronomical Seeing," Publications of the Astronomical Society of the Pacific, Vol. 75 (Feb 1963).

10. Tsvang, L. R., "Some Characteristics of the Spectra of Temperature Pulsations in the Boundary Layer of the Atmosphere," Bulletin of the Academy of Sciences of the U.S.S.R., Geophysics Series, No. 10, (Jan 1964).

11. Shifrin, Y. S., "Correlation Characteristics of the Diffraction Image Formed by a Focusing System, "Soviet Physics-Acoustics, Vol. 8, No. 4, (April-June 1963).

12. Texerau, J., "Limitations to the Image Quality of a Large Telescope," Applied Optics, Vol. 2, No. 1, (Jan 1963).

13. Edward, B.N. and R.R. Steer, "Effects of Atmospheric Turbulence on the Transmission of Visible and Near Infrared Radiation," Applied Optics, Vol. 4, No. 3 (March 1965).

14. Angstrom, A., "Techniques of Determining the Turbidity of the Atmosphere," Tellus, Vol. 13, No. 2 (May 1961).

15. Shifrin, Y.S., and G.L. Shubora, "The Statistical Characteristics of the Vertical Transparency of the Atmosphere," Bulletin of the Academy of Sciences, U.S.S.R., Geophysics Series, No. 2 (June 1964).

16. Darchiya, S.P., "Some Problems of the Method of Astroclimate Investigation," Astronomicheskii Zhurnal (U.S.S.R.), Vol. 41, No. 1 (1964).

17. Duntley, S.Q., W.H. Culver, F. Richey, and R.W. Preiscndorfer, "Reduction of Contrast by Atmospheric Boil," Journal of the Optical Society of America, Vol. 53, No. 3 (March 1963).

18. Bertolochi, M, B. Daino, and S. Setts, "On the Measurement of the Spatial Coherence of a Laser Beam," Nuovo Cimento (Italy), Vol. 33, No. 6 (Sept 1964).

19. Subramanian, M. and J.A. Collinson, "Modulation of Laser Beams by Atmospheric Turbulence," Bell System Technical Journal, 44 (March 1965).

20. London, I., A study of the Atmospheric Heat Balance, Final Report. New York University, AD117227 (1957).

21. Fritz, S., "Scattering of Solar Energy by Clouds of Large Drops," Journal of Meteorology, Vol. 11 (1954), pp. 291-300.

22. Pritchard, B.S. and W.G. Elliot, Journal of the Optical Society of America, 50, 191 (1960).

23. Deirmedjian, D., "Scattering and Polarization Properties of Water Clouds and Haze in the Visible and Infrared," Applied Optics (Feb. 1944), pp 187-196.

24. Eldridge, R. G. and J. C. Johnson, "Distribution of Irradiance in Haze and Fog," Journal of the Optical Society of America, 52, 787 (1962).

25. Humphreys, W. J., Physics of the Air. New York: Dover Books (1964).

26. Tatarski, V. I., Wave Propagation in a Turbulent Medium, New York: McGraw-Hill Book Co., Inc. (1961)

27. Chernov, L. A. Wave Propagation in a Randon Medium. New York: McGraw-Hill Book Co., Inc., I. D. Section 16 (1960).

28. Fried, D. L., and J. D. Cloud, Atmospheric Turbulence and its Effect on Laser Communication Systems. NAA/S&ID, Electro-Optical Laboratory Technical Memo. No. 91 (June 1964).

29. Fried, D. L. and M. F. Sternberg, Extension of Results for Phase Log-Amplitude Correlation: Higher-Order Terms in the Near-Field Series and Coverage of the Far-Field. NAA/S&ID, Electro-Optical Laboratory Technical Memo No. 125 (July 1964).

30. Hufnagel, R. E., and N. R. Stanley, Journal of the Optical Society of America, 54, 52 (1964).

31. Bergmann, P. G., Phys. Rev., 70, 486 (1946).

32. Duntley, S. Q., et al, Journal of the Optical Society of America, 53, 351 (1963).

33. Lord Rayleigh (J. W. Strutt), "On the Theory of Stellar Scintillation," Phil. Mag. 36, 129 (1893).

34. Barnhart, et al, Investigation of Upper Air Turbulence by the Method of Analyzing Stellar Scintillation Shadow Patterns . . . AFCRC TR 39-291, AD 231-378 (1959).

35. Fried, D. L., and R. A. Schmeltzer, Statistics of Plane Wave Propagation in the Turbulent Atmosphere, to be presented at the 1965 International Symposium on Antennas and Propagation, Washington, D. C.

36. Schmeltzer, R. A., Statistics of Plane Wave Propagation in the Turbulent Atmosphere -- A New Solution and Consensus Amongst Several Different Solutions. NAA/S&ID, Electro-Optics Laboratory Technical Memo No. 225 (June 1965).

37. Fried, D. L., The Relationship Between Random Optical Wave Front Distortion and Optical System Performance. NAA/S&ID, Electro-Optical Laboratory Technical Memo No. 89 (April 1964).

38. Fried, D. L., On the Validity of Rytov's Method, NAA/S&ID, Electro-Optical Laboratory Technical Memo No. 142 (Sept 1964).

39. Fried, D. L., The Angular Scintillation Correlation Function (Appendix) NAA/S&ID, Electro-Optical Laboratory, Technical Memo No. 131 (Aug 1964).

40. Kiepenheuer, K. O., Experience and Experiments With Solar Seeing, Proceedings of Symposium on Solar Seeing, Rome (1962), pp 49-54.

APPENDIX A

PROPAGATION OF A FOCUSED LASER BEAM THROUGH A TURBULENT ATMOSPHERE

This appendix is Technical Memorandum No. 176, prepared by R.A. Schmeltzer of Electro-Optical Laboratory of NAA/S&ID, and dated January 1965. This work was performed partially under NASw-977 and partially on NAA/S&ID research and development funds.

PROPAGATION OF A FOCUSED LASER BEAM THROUGH A TURBULENT ATMOSPHERE

A fundamental limitation to communications by coherent light transmission through the atmosphere is the unavoidable effects created by normal atmospheric turbulence. The presence of irregular inhomogeneities in the atmosphere causes a random fluctuation of the refractive index of the transmission medium. This fluctuation distorts and bends the light, impressing undesirable modulation on the transmitted beam until eventually all coherence is destroyed. Propagation through such a transmission medium can only be described statistically.

Thus the problem is determining how light propagates through a random medium, i. e., ascertaining a relationship between certain physically important statistical quantities, which describe the coherence or random distortion of the optical wave front, and the statistical fluctuations of the refractive index of the turbulent atmosphere.

More specifically, the purpose of this report is to derive a formula for the correlation (or structure) function for the amplitude and phase fluctuation of a wave produced by a focused laser beam as the beam propagates through a turbulent atmosphere having a prescribed correlation (or structure) function. For a description of a valid mathematical model characterizing a turbulent atmosphere, the results of the confirmed theory developed by Kolmogoroff were used. The use of the correlation (or structure) function for describing the random perturbations of an optical wavefront has been demonstrated to be highly significant for predicting the performance of optical systems, as well as being easily measured. Such parameters as angular resolution, angular and amplitude scintillations, and efficiency of coherent heterodyne detectors all depend directly upon the correlation function of the perturbed optical wavefront.

The results achieved are significant in that previously many questions which could only be answered qualitatively or intuitively may now be answered quantitatively also. The relationships presented can be used to realistically evaluate modes of propagation where the statistics of the medium are slowly varying along the path of propagation, e. g., space to earth, earth to space, and earth to earth communications through the atmosphere.

The analysis has been kept sufficiently general by considering a focused laser beam with parameters of such quantities as initial beam spot diameter, beam spread (a function of path length), and an arbitrary radius of curvature of the focused wavefront as would be produced by an ideal lens. The formulae therefore contain as limiting special cases, the results of previous analyses for the plane wave and spherical wave (point source).

In spite of the many tempting ramifications one can now consider at this point, none have been presented in order to expedite the publication of the present results.

ANALYSIS

In the following analysis, some of the statistical properties of a focused beam of light propagating through a randomly turbulent atmosphere are considered. The initial light disturbance, such as would be produced at the exit pupil plane of a transmitting laser, shall be described by a scalar function of radial position vector $\vec{\rho}$

$$u(\vec{\rho}) = \exp\left(-\rho^2/2a_o^2 + ik\,\rho^2/2R\right) \tag{1}$$

situated in the z = o plane of a cylindrical coordinate system, where a_o is a measure of the initial beam diameter, and R is the radius of curvature of the wavefront produced by the action of an ideal lens. The harmonic time dependence exp $(i\omega t)$ is suppressed. The random wave propagation of the light beam produced by this disturbance is described by a scalar wave function $u(\vec{r})$ with $\vec{r}$ a three-dimensional position vector. The function $u(\vec{r})$ satisfies the reduced wave equation,

$$\nabla^2 u(\vec{r}) + k^2 n^2(\vec{r})\, u(\vec{r}) = 0, \quad z \geqslant 0 \tag{2}$$

Here, the refractive index $n(\vec{r})$ characterizes the transmission medium and is of the form

$$n(\vec{r}) = 1 + \mu(\vec{r}), \quad \mu \ll 1 \tag{3}$$

The turbulence of the atmosphere causes random fluctuations of the refractive index. Thus in the ensuing analysis, $\mu(\vec{r})$ is regarded as a random function.

If the atmosphere were perfectly homogeneous and nonrandom, then $\mu(\vec{r}) = 0$ and the light beam would propagate unperturbed through the atmosphere. The index of refraction would be constant and not random. In this case, the propagating wave at a distance z from the laser's exit pupil plane is found by applying Green's theorem to the region $z \geqslant 0$

$$u_o(\vec{r}) = \frac{1}{4\pi} \iint_{-\infty}^{+\infty} u_o(\vec{\rho}_o) \frac{\partial G}{\partial z_o}(\vec{r}; \vec{\rho}_o)\, d\vec{\rho}_o, \tag{4}$$

where the Green function $G(\vec{r};\vec{r}_o)$ vanishes on $z_o = 0$ and satisfies

$$\nabla^2 G + k^2 G = -4\pi\delta(r - r_o), \quad z \geqslant 0 \tag{5}$$

Hence

$$G(\vec{r};\vec{r}_o) = \frac{\exp(ik|\vec{r} - \vec{r}_o|)}{|\vec{r} - \vec{r}_o|} - \frac{\exp(ik|\vec{r} - \vec{r}'_o|)}{|\vec{r} - \vec{r}'_o|}$$

$$|\vec{r} - \vec{r}_o| = \sqrt{|\vec{\rho} - \vec{\rho}_o|^2 + |z - z_o|^2}$$

$$|\vec{r} - \vec{r}'_o| = \sqrt{|\vec{\rho} - \vec{\rho}_o|^2 + |z + z_o|^2}$$

so that

$$\frac{\partial G(\vec{r};\vec{\rho}_o)}{\partial z_o} = -2ikz \frac{\exp(ik|\vec{r} - \vec{\rho}_o|)}{|\vec{r} - \vec{\rho}_o|^2} \left[1 - \frac{1}{ik|\vec{r} - \vec{\rho}_o|} \right] \tag{6}$$

It is perfectly reasonable to make the approximations for visible optics that $k = 2\pi/\lambda \gg 1/z$. Under these conditions 4 becomes

$$u_o(\vec{r}) = \frac{kz}{2\pi i} \iint_{-\infty}^{+\infty} \frac{\exp\left[-\rho_o^2/2a_o^2 + ik\rho_o^2/2R + ik|\vec{r} - \vec{\rho}_o|\right] d\vec{\rho}_o}{|\vec{r} - \vec{\rho}_o|^2} \tag{7}$$

Since we are interested in the light disturbances in the region centered about the principle ray, the following can be expanded to

$$|\vec{r} - \vec{\rho}_o| \sim z + |\vec{\rho} - \vec{\rho}_o|^2/2z \; - - -$$

neglecting higher order terms. Then

$$u_o(\vec{r}) \sim \frac{k\, e^{ikz}}{2\pi z} \iint_{-\infty}^{+\infty} \exp\left[-\rho_o^{\,2}/2a_o^{\,2} + ik\rho^2/2R + \frac{ik}{2z}\,|\vec{\rho} - \vec{\rho}_o|^2\right] d\vec{\rho}_o$$

$$u_o(\vec{r}) \sim \frac{-ika^2 \exp(-k^2 a^2)}{Z} \exp\left[ik(Z + \rho^2/2Z)\right], \quad z \gg a_o \gg k \tag{8}$$

where

$$Z = z - ika^2, \quad 1/a^2 = 1/a_o^{\,2} - ik/R. \tag{9}$$

Next to be considered is a beam propagating through an inhomogeneous medium having a randomly fluctuating index of refraction. In this case, the beam of light is bent and the wavefront is distorted as the beam randomly propagates through the atmosphere until eventually, at a sufficiently long distance from the transmitter, the coherence of the beam is completely destroyed.

Before proceeding further, the meaning of random wave propagation should be reviewed. A random medium is characterized by a family of media having a family of index refraction functions, $n(\vec{r};\omega)$, depending on a parameter ω, together with a probability distribution specifying the probabilities of various members of the family. Wave propagation in a random medium refers then to propagation in each member medium, together with the probability of each member medium. This probability, when associated with the wave motion in each medium, characterizes a random wave motion.

A turbulent atmosphere can be characterized by a family of index of refraction functions $n(\vec{r};\omega)$, depending on a parameter ω. Here ω is a point in probability space which identifies the individual family member for which a probability measure $dP(\omega)$ is defined. If the problem is well posed, then $n(\vec{r};\omega)$ is measurable with respect to $dP(\omega)$ and a solution, $u(\vec{r};\omega)$ exists for each ω. The wave function $u(\vec{r};\omega)$ satisfies a stochastic differential equation of the form 2 for each value of ω. This family of solutions, with the probabilities for each ω, $dP(\omega)$, then characterizes the random wave propagation.

The mean value of a random function $f(\vec{r};\omega)$ denoted $\langle f(\vec{r})\rangle$ is defined as the ensemble average of all possible values the function can assume at the point $\vec{r}$, i.e.,

$$\langle f(\vec{r}) \rangle = \int_{-\infty}^{+\infty} f(\vec{r};\omega)\, dP(\omega) \tag{10}$$

The mean value of the refractive index shall be assumed equal to unity, hence $\langle \mu\ (\vec{r}) \rangle = 0$. The correlation function of a function $f(\vec{r};\omega)$ is defined to be

$$B_f\ (\vec{r}, \vec{r}') = \int_{-\infty}^{+\infty} \left[f(\vec{r};\omega) - \langle f(\vec{r}) \rangle \right] \cdot \left[f(\vec{r}';\omega) - \langle f(\vec{r}') \rangle \right] dP(\omega)$$

$$= \left\langle \left[f(\vec{r}) - \langle f(\vec{r}) \rangle \right] \cdot \left[f(\vec{r}') - \langle f(\vec{r}') \rangle \right] \right\rangle \tag{11}$$

In what follows, it is assumed that $\langle \mu\ (\vec{r}) \rangle = 0$. Hence

$$B_\mu\ (\vec{r}, \vec{r}') = \langle \mu\ (\vec{r})\ \mu\ (\vec{r}') \rangle \tag{12}$$

A random function is said to be stationary when its correlation function depends only on the difference between the two points, i. e.,

$$B_f(\vec{r};\vec{r}') = B_f\ (\ |\vec{r} - \vec{r}'|\ ) \tag{13}$$

For each value of ω, the index of refraction $n(\vec{r};\omega)$ is an ordinary nonrandom function of position. Therefore it is permissible to assume the existence of a Fourier spectral expansion for each ω

$$N(\vec{t}, z_o;\omega) = \iint_{-\infty}^{+\infty} \exp\ (-i\vec{t} \cdot \vec{\rho}_o)\ \mu\ (\vec{r}_o;\omega)\ d\vec{\rho}_o \tag{14}$$

The random function $\mu\ (\vec{r}_o;\omega)$ can then be represented by a Fourier integral

$$\mu\ (\vec{r}_o, z_o;\omega) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{+\infty} \exp\ (i\vec{t} \cdot \vec{\rho}_o)\ N(\vec{t}, z_o;\omega)\ d\vec{t} \tag{15}$$

Multiplying Equation 14 by its complex conjugate and averaging the result, we find that

$$\langle N(t, z_o)N^*(t', z_o)\rangle = \iiiint_{-\infty}^{+\infty} \exp(-i\vec{t}'\cdot\vec{\rho}_o + i\vec{t}'\cdot\vec{\rho}_o')\langle\mu(\vec{r}_o)\,\mu(\vec{r}_o')\rangle\, d\vec{\rho}_o\, d\vec{\rho}_o' \tag{16}$$

We now assume that the correlation function of μ is stationary with respect to $\vec{\rho}$, i.e.,

$$\langle\mu(\vec{r}_o)\,\mu(\vec{r}_o')\rangle = B_\mu(\vec{r}_o, \vec{r}_o') = B_\mu(|\vec{\rho}_o - \vec{\rho}_o'|, z_o, z_o') \tag{17}$$

Making the change of variables

$$\vec{\zeta} = \vec{\rho}_o - \vec{\rho}_o', \quad \vec{\eta} = \frac{1}{2}(\vec{\rho}_o + \vec{\rho}_o')$$

and then integrating with respect to $\vec{\eta}$, we obtain

$$\langle N(\vec{t}, z_o)N^*(\vec{t}', z_o')\rangle = (2\pi)^2\delta(\vec{t} - \vec{t}') \iint_{-\infty}^{+\infty} \exp(-i\vec{t}\cdot\vec{\zeta})\, B(\zeta, z_o, z_o')\, d\vec{\zeta} \tag{18}$$

The last integral is a Fourier transform of the correlation function of μ. Introducing the function

$$F_\mu(t, z_o, z_o') = \iint_{-\infty}^{+\infty} \exp(-i\vec{t}\cdot\vec{\zeta})\, B_\mu(\zeta, z_o, z_o')\, d\vec{\zeta} \tag{19}$$

we have

$$N(\vec{t}, z_o)N^*(\vec{t}', z_o') = (2\pi)^2\delta(\vec{t} - \vec{t}')\, F_\mu(t, z_o, z_o') \tag{20}$$

A similar result is found for an arbitrary random function ψ. Thus if we define a Fourier spectral expansion for ψ

$$\Psi(\vec{s}, z; \omega) = \iint_{-\infty}^{+\infty} \exp(-i\vec{s}\cdot\vec{\rho})\, \psi(\vec{r}; \omega)\, d\vec{\rho} \tag{21}$$

and assume lateral stationarity of the correlation function of ψ, then

$$\langle \Psi(\vec{s}, z)\Psi^*(s', z')\rangle = (2\pi)^2 \delta(\vec{s} - \vec{s}') F_\psi(s, z, z') \tag{22}$$

where

$$F\psi(s, z, z') = \iint_{-\infty}^{+\infty} \exp(i\vec{s} \cdot \vec{\zeta}) <\psi\psi^*> d\vec{\zeta} \tag{23}$$

Let the real and imaginary parts of ψ be denoted by X and S respectively.

$$\langle \psi\psi^* \rangle = \langle X^2 + S^2 \rangle$$

Hence

$$F\psi(s, z, z') = F_X(s, z, z') + F_S(s, z, z')$$

so that

$$\langle \Psi(\vec{s}, z)\Psi^*(\vec{s}', z')\rangle = (2\pi)^2 \delta(\vec{s} - \vec{s}') \left\{F_X(s, z, z') - F_S(s, z, z')\right\} \tag{24}$$

A similar expression is obtained for another correlation function,

$$\mathrm{Re}\langle \Psi(\vec{s}, z, z')\Psi(-\vec{s}', z, z')\rangle = (2\pi)^2 \delta(\vec{s} - \vec{s}') \left\{F_X(s, z, z') - F_S(s, z, z')\right\}. \tag{25}$$

Combining Equations 24 and 25 and integrating the result with respect to $\vec{s}'$ gives the important relation

$$F_{X,S}(s, z, z') = \frac{1}{2(2\pi)^2} \mathrm{Re} \iint_{-\infty}^{+\infty} \left\{\langle \Psi(\vec{s}, z)\Psi^*(\vec{s}', z')\rangle \right.$$

$$\left. \pm \langle \Psi(\vec{s}, z)\Psi(-\vec{s}', z')\rangle\right\} d\vec{s}' \tag{26}$$

where the upper sign belongs with X, the lower sign with S. This relation provides a means for computing the spectral expansion $F_{X,S}(s, z, z')$ of the correlation $B_{X,S}(\zeta, z, z')$ of X and S, the real and imaginary parts of a laterally stationary random function ψ, from a knowledge of its lateral two-dimensional spectral expansion $\Psi(s, z, z')$.

Let us now consider the solution $u(\vec{r};\omega)$ of the stochastic differential Equation 2. Following Rytov's method, we look for a solution of the form

$$u(\vec{r};\omega) = u_o(\vec{r}) \exp\psi(\vec{r};\omega) \tag{27}$$

where $u_o(\vec{r})$ is often called the coherent wave, a solution for $\mu = 0$, e.g., that given by 8. Thus ψ is a measure of the random departure of the propagating wave from its unperturbed value $u_o(\vec{r})$. If we separate ψ into its real and imaginary parts

$$\psi(\vec{r};\omega) = X(\vec{r};\omega) + iS(\vec{r};\omega) \tag{28}$$

we have

$$X(\vec{r};\omega) = \log \left|\frac{u(\vec{r};\omega)}{u_o(\vec{r})}\right|$$

$$S(\vec{r};\omega) = \operatorname{Im} \log \left|\frac{u(\vec{r};\omega)}{u_o(\vec{r})}\right|$$

so that X and S represent the fluctuation in the logarithm of the relative amplitude and phase of the propagating wave. Substituting Equation 27 into Equation 2 we find, after some manipulation,

$$\nabla^2(\psi u_o) + k^2(\psi u_o) + \left[(\nabla\psi)^2 + 2k^2\mu + k^2\mu^2\right] u_o = 0 \tag{29}$$

If it is assumed that $1/k\,|\nabla\psi| \sim |\mu|$, or $1/k\,|\nabla\psi| \ll 1$,

then Equation 29 becomes

$$\nabla^2(\psi u_o) + k^2 u_o = -2k^2\mu\, u_o \tag{30}$$

subject to the requirement that the perturbation changes of the logarithm of the relative amplitude and phase, over distances of the order of a wavelength λ, be small. This restriction does not impose any limitation on the total change of these quantities. Equation 30 is an inhomogeneous Helmholtz equation whose solution is

$$\psi(\vec{r};\omega) = k^2/2\pi \iiint_{-\infty}^{+\infty} \mu\,(\vec{r}_o;\omega)\,\frac{u_o(\vec{r}_o)}{u_o(\vec{r})}\,\frac{\exp(ik|\vec{r} - \vec{r}_o|)}{|\vec{r} - \vec{r}_o|}\,d\vec{r}_o \tag{31}$$

It seems reasonable to assume that only those inhomogeneities which are located within a cone, with its vertex at the receiving point $\vec{r}$ with a small aperture, should contribute significantly to the evaluation of Equation 31. Inside this cone region

$$|\vec{r} - \vec{r}_o| \sim |z - z_o| + \frac{1}{2}\frac{|\vec{\rho} - \vec{\rho}_o|^2}{|z - z_o|} \tag{32}$$

Reflections should also play a minor role, and therefore the integration to the region between the transmitter and receiver is limited, i. e., $0 < z_o < z$. Therefore setting

$$\frac{\exp\left(ik|\vec{r} - \vec{r}_o|\right)}{|\vec{r} - \vec{r}_o|} \sim \left(\frac{\exp\left[ik(z - z_o)\right]}{(z - z_o)}\right)\left(\exp\left[\frac{ik|\vec{\rho} - \vec{\rho}_o|^2}{2(z - z_o)}\right]\right) \tag{33}$$

upon substituting Equations 8 and 33 into 31, the following is obtained:

$$\psi(\vec{r};\omega) = \frac{k^2}{2\pi}\int_0^z \frac{Z}{Z_o}\frac{dz_o}{(z - z_o)}\iint_{-\infty}^{+\infty} \mu\,(\vec{r}_o;\omega)\exp\left[\frac{ikZZ_o}{2(z - z_o)}\left(\frac{\vec{\rho}_o}{Z_o} - \frac{\vec{\rho}}{Z}\right)^2\right] d\vec{\rho}_o \tag{34}$$

If Equation 15 is substituted into 34, the order of $d\vec{t}$ and $d\vec{\rho}_o$ integrations can be interchanged allowing the $d\vec{\rho}_o$ integration to be carried out explicitly. Thus

$$\psi(\vec{r};\omega) = \frac{k^2}{(2\pi)^3}\int_0^z \frac{Z}{Z_o}\frac{dz_o}{(z - z_o)}\iint_{-\infty}^{+\infty} N(\vec{t}, z_o;\omega)\, d\vec{t}$$

$$\times \iint_{-\infty}^{+\infty} \exp\left\{\frac{ikZZ_o}{2(z - z_o)}\left[\frac{\vec{\rho}_o}{Z_o} - \frac{\vec{\rho}}{Z}\right]^2 + i\vec{t}\cdot\vec{\rho}_o\right\} d\vec{\rho}_o \tag{35}$$

The last double integral in Equation 35 is easily evaluated. Performing the angular integration first,

$$\iint_{-\infty}^{+\infty} \exp\left[\frac{ikZZ_o}{2(z-z_o)}\left(\frac{\vec{\rho}_o}{Z_o} - \frac{\vec{\rho}}{Z}\right)^2 + i\vec{t}\cdot\vec{\rho}_o\right] d\vec{\rho}_o$$

$$= 2\pi \int_o^{\infty} \exp\left[\frac{ik}{2(z-z_o)}\left(\frac{Z}{Z_o}\rho_o^2 + \frac{Z_o}{Z}\rho^2\right)\right] J_o\left(\rho_o\left|\frac{k\vec{\rho}}{z-z_o} + \vec{t}\right|\right)\rho_o \, d\rho_o$$

$$= \frac{2\pi(z-z_o)Z_o}{-ikZ} \times \exp\left\{\frac{iZ_o}{Z}\left[\vec{t}\cdot\vec{\rho} - \frac{t^2}{2k}(z-z_o)\right]\right\} \tag{36}$$

Hence 35 becomes

$$\psi(\vec{r};\omega) = \frac{ik}{(2\pi)^2} \int_o^z dz_o \iint_{-\infty}^{+\infty} N(\vec{t}, z_o;\omega) \exp\left\{\frac{iZ_o}{Z}\left[\vec{t}\cdot\vec{\rho} - \frac{t^2}{2k}(z-z_o)\right]\right\} d\vec{t} \tag{37}$$

The spectral expansion for $\psi(\vec{r};\omega)$ is given by Equation 21. If Equation 37 is multiplied by $\exp(-i\vec{s}\cdot\vec{\rho})$ and integrated over the entire $\vec{\rho}$ plane, it is found that

$$\Psi(\vec{s}, z;\omega) = ik \int_o^z dz_o \iint_{-\infty}^{+\infty} \exp(-i\vec{s}\cdot\vec{\rho})\, d\vec{\rho}$$

$$\times \frac{1}{(2\pi)^2} \iint_{-\infty}^{+\infty} \exp\left(i\vec{t}\cdot\vec{\rho}\frac{Z_o}{Z}\right)\left[N(\vec{t}, z_o, \omega)\exp\left(\frac{-t^2}{2k}\frac{Z_o}{Z}(z-z_o)\right)\right] d\vec{t} \tag{38}$$

The inner four-fold integral in Equation 38 is Fourier's repeated integral operating on the square bracketed function. Therefore it is found that Ψ is simply

$$\Psi(\vec{s}, z;\omega) = ik \int_o^z \left(\frac{Z}{Z_o}\right)^2 N\left(\vec{s}\frac{Z}{Z_o}, z_o;\omega\right) \exp\left[-\frac{is^2}{2k}\frac{Z}{Z_o}(z-z_o)\right] dz_o \tag{39}$$

Now some of the statistical properties of $\Psi(\vec{s}, z; \omega)$ are computed. Multiplying Equation 39 by its complex conjugate,

$$\Psi^*(\vec{t}, z; \omega) = -ik \int_0^z \left(\frac{Z^*}{Y_o^*}\right)^2 N\left(\frac{-\vec{t}Z^*}{Y_o^*}, y_o; \omega\right) \exp\left[\frac{+it^2}{2k} (z-y_o) \frac{Z^*}{Y_o^*}\right] dy_o \tag{40}$$

Interchanging the order of taking the mean value with integration, which is assumed permissible, the following is obtained

$$\langle \Psi(\vec{s}, z; \omega) \Psi^*(\vec{t}, z; \omega) \rangle =$$

$$k^2 \int_0^z \int_0^z \left(\frac{Z}{Z_o} \frac{Z^*}{Y_o^*}\right)^2 \exp\left\{\frac{-i}{2k}\left[s^2 (z-z_o)\frac{Z}{Z_o} - t^2 (z-y_o) \frac{Z^*}{Y_o^*}\right]\right\}$$

$$\times \left\langle N\left(\frac{\vec{s}Z}{Z_o}, z_o\right) N\left(\frac{-\vec{t}Z^*}{Y_o^*}, y_o\right)\right\rangle dz_o\, dy_o \tag{41}$$

where $Z_o = z_o - ika^2$, $Y_o = y_o - ika^2$ and a is complex Equation 9.

From Equation 20

$$\left\langle N\left(\frac{\vec{s}Z}{Z_o}, z_o\right) N\left(\frac{-\vec{t}Z^*}{Y_o^*}, y_o\right)\right\rangle = \left\langle N\left(\frac{\vec{s}Z}{Z_o}, z_o\right) N^*\left(\frac{\vec{t}Z}{Z_o}, z_o\right)\right\rangle$$

$$= \left(2\pi \frac{Y_o^*}{Z^*}\right)^2 \delta\left(\vec{t} + \frac{\vec{s}Z\, Y_o^*}{Z_o Z^*}\right) F_n\left(\vec{s}Z/Z_o, z_o, y_o\right) \tag{42}$$

Substituting Equation 42 into 41 and integrating with respect to $\vec{t}$ gives

$$\iint_{-\infty}^{+\infty} \langle \Psi (\vec{s}, z) \Psi * (\vec{t}, z) \rangle \, d\vec{t} =$$

$$(2\pi k)^2 \int_0^z \int_0^z \left(\frac{Z}{Z_o}\right)^2 \exp \left\{ \frac{-is^2}{2k} \left[\frac{Z}{Z_o} (z-z_o) \right.\right.$$

$$\left.\left. - \left(\frac{Z}{Z_o}\right)^2 \frac{Y_o^*}{Z*} (z-y_o) \right] \right\} F_n \left(\frac{sZ}{Z_o}, z_o, y_o \right) dz_o \, dy_o \qquad (43)$$

Likewise,

$$\iint_{-\infty}^{+\infty} \langle \Psi (\vec{s}, z) \Psi (-\vec{t}, z) \rangle \, d\vec{t} =$$

$$- (2\pi k)^2 \int_0^z \int_0^z \left(\frac{Z}{Z_o}\right)^2 \exp \left\{ \frac{-is^2}{2k} \left[\frac{Z}{Z_o} (z-z_o) \right.\right.$$

$$\left.\left. + \left(\frac{Z}{Z_o}\right)^2 \frac{Y_o}{Z} (z-y_o) \right] \right\} F_n \left(\frac{sZ}{Z_o}, z_o, y_o \right) dz_o \, dy_o \qquad (44)$$

From Equation 26,

$$F_{X,S}(\vec{s}, z) = \frac{1}{2}k^2 \, \mathrm{Re} \int_0^z \int_0^z \exp\left[\frac{-is^2}{2k}(z-z_o)\frac{Z}{Z_o}\right]\left\{\exp\left[\frac{is^2}{2k}\left(\frac{Z}{Z_o}\right)^2 \frac{Y_o^*}{Z*}\right.\right.$$

$$\left.(z-y_o)\right] \mp \exp\left[\frac{-is^2}{2k}\left(\frac{Z}{Z_o}\right)^2\left(\frac{Y_o}{Z}\right)(z-y_o)\right]\Bigg\}\left(\frac{Z}{Z_o}\right)^2 F_n\left(s\frac{Z}{Z_o}, z_o, y_o\right)$$

$$dz_o \, dy_o \tag{45}$$

But

$$B_{X,S}(\rho, z) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{+\infty} \exp(i\vec{s}\cdot\vec{\rho}) \, F_{X,S}(\vec{s}, z) \, d\vec{s} \tag{46}$$

Hence, if we substitute

$$\left(\frac{Z}{Z_o}\right)^2 F_n(sZ/Z_o, z_o, y_o) = \iint_{-\infty}^{+\infty} F_n(t, z_o, y_o) \quad \delta(s - \vec{t}Z_o/Z) \quad d\vec{t} \tag{47}$$

into Equation 45, multiply by exp $(i\vec{s}\cdot\vec{\rho})$ and integrate with respect to $d\vec{s}$, the following is obtained

$$B_{X,S}(\rho, z) = \frac{\mp k^2}{8\pi^2} \iint_{-\infty}^{+\infty} \iint_{0\,0}^{z\,z} F_n(t, z_o, y_o) \exp(i\vec{\rho}\cdot\vec{t}Z_o/Z)$$

$$\times \exp\left[\frac{-it^2}{2k}\frac{Z_o}{Z}(z-z_o)\right]\left\{\exp\left[\frac{-it^2}{2k}\frac{Y_o}{Z}(z-y_o)\right] \mp \text{conj.}\right\}$$

$$dz_o \, dy_o \, d\vec{t} \tag{48}$$

Performing the angular integration

$$B_{X,S}(\rho, z) = \mp \frac{k^2}{4\pi} \operatorname{Re} \int_0^\infty t\,dt \int_0^z\int_0^z J_o(\rho t Z_o/Z)\, F_n(t, z_o, y_o)$$

$$\times \exp\left[\frac{-it^2}{2k}\frac{Z_o}{Z}(z-z_o)\right]\left\{\exp\left[\frac{-it^2}{2k}\frac{Y_o}{Z}(z-y_o)\right] \mp \text{conj.}\right\} dz_o\, dy_o \tag{49}$$

Equation 49 can be simplified further if it is assumed that the two-dimensional spectral expansion of the refractive index fluctuations is a slowly, smoothly varying function of position along the path. Thus, in any localized region along the path of propagation, the function $F_n(t, z_o, y_o)$ behaves as if the medium were homogeneous. Writing $F_n(t, z_o, y_o)$ in the form

$$F_n(t, z_o, y_o) = F_n(t, z_o; |z_o - y_o|) \tag{50}$$

assuming

$$\frac{F_n(t, z_o; |z_o - y_o|)}{F_n(t, z_o; 0)} << 1 \quad \text{for} \quad |z_o - y_o| \geq \zeta_c \tag{51}$$

where ζ_c is the correlation distance, and making the change of variables

$$z_o - y_o = \zeta \quad \text{or} \quad Y_o = Z_o - \zeta \;; \quad dy_o = -d\zeta$$

Equation 49 becomes

$$B_{X,S}(\rho, z) = \mp \frac{k^2}{4\pi} \text{ Re} \int_0^\infty t\, dt \int_0^z dz_o \int_{-|z-z_o|}^{z_o} d\zeta \; J_o\left(\rho t Z_o/Z\right)$$

$$F_n(t, z_o; \zeta) \exp\left[\frac{-it^2}{2k} \frac{Z_o}{Z} (z-z_o)\right] \left\{ \exp\left[\frac{-it^2}{2kZ} (Z_o - \zeta)\right.\right.$$

$$\left.\left.(Z - Z_o + \zeta)\right] \mp \text{conj.}\right\} \tag{52}$$

Since the function $F_n(t, z_o; \zeta)$ falls off rapidly for ζ beyond the correlation distance ζ_c, $\zeta \ll z_o$ is in the important region of integration. Therefore, the following can be written

$$(Z_o - \zeta)\,(Z - Z_o + \zeta) \sim Z_o(Z - Z_o)$$

and the integration with respect to ζ can be extended between the limits $-\infty$ and $+\infty$. Hence Equation 52 becomes

$$B_{X,S}(\rho, z) = \mp \frac{k^2}{4\pi} \text{ Re} \int_0^\infty t\, dt \int_0^z dz_o \, J_o(\rho t Z_o/Z) \int_{-\infty}^{+\infty}$$

$$F_n(t, z_o; \zeta)\, d\zeta \times \exp\left[\frac{-it^2}{2k} \frac{Z_o}{Z} (z-z_o)\right] \left\{ \exp\left[\frac{it^2}{2k} \frac{Z_o}{Z} (z-z_o)\right]\right.$$

$$\left. \mp \text{conj.}\right\} \tag{53}$$

Now

$$\int_{-\infty}^{\infty} F_n(t, z_o, \zeta)\, d\zeta = \iiint_{-\infty}^{+\infty} B_n(r; z_o) \exp(-i\vec{t} \cdot \vec{\rho})\, d\vec{r} \tag{54}$$

is the three-dimensional spectral expansion of the correlation function for the refractive index fluctuation. Introducing the notation

$$\Phi(t, z_o) \equiv \Phi\left(\sqrt{t_1^2+t_2^2+t_3^2}\, ; z_o\right) \equiv \iiint_{-\infty}^{+\infty} B_n(r; z_o) \quad \exp(-i\vec{t} \cdot \vec{r})\, d\vec{r} \tag{55}$$

then

$$\int_{-\infty}^{+\infty} F_n(t, z_o, \zeta)\, d\zeta = \Phi\left(\sqrt{t_1^2+t_2^2},\ z_o\right) = \Phi(t, z_o) \tag{56}$$

and Equation 53 finally becomes

$$B_{X,S}(\rho, z) = \mp \frac{k^2}{4\pi} \int_0^{\infty} t\, dt \int_0^{z} \Phi(t, z_o)\, J_o(\rho t Z_o/Z) \times \exp\left[-\frac{it^2}{2k}\frac{Z_o}{Z}(z-z_o)\right] \left\{\exp\left[\frac{it^2}{2k}\frac{Z}{Z_o}(z-z_o)\right] \mp \text{conj.}\right\} dz_o \tag{57}$$

Equation 57 relates the correlation function of the amplitutde and phase fluctuations of the wave in the z plane to the three-dimensional spectral density of the correlation function of the refractive index fluctuation along the path of propagation.

The value of the correlation function $B_{X,S}(\rho, z)$ at $\rho = 0$ gives the mean square fluctuation of the logarithm of the wave amplitude and mean square phase fluctuation. The parameter Z_o/Z can be separated into its real and imaginary parts:

$$\frac{Z_o}{Z} = \left[\frac{Z_o}{Z}\right]_R + i\left[\frac{Z_o}{Z}\right]_I = \left[\frac{z_o - ika^2}{z - ika^2}\right] \tag{58}$$

where

$$\left[\frac{Z_o}{Z}\right]_R = \frac{z_o}{z}\left[\frac{1 + k^2 a_o^4 (1/z + 1/R)(1/z_o + 1/R)}{1 + k^2 a_o^4 (1/z + 1/R)^2}\right] \tag{59}$$

$$\left[\frac{Z_o}{Z}\right]_I = \frac{ka_o^2}{z}\left[\frac{(1 - z_o/z)}{1 + k^2 a_o^4 (1/z + 1/R)^2}\right] \tag{60}$$

Then

$$B_{X,S}(0, z) = \frac{k^2}{4\pi}\int_0^\infty t\,dt \int_0^z \Phi(t, z_o) \exp\left\{-t^2 a_o^2 \left[\frac{(1 - z_o/z)^2}{1 + k^2 a_o^4 (1/z + 1/R)^2}\right]\right\}$$

$$\times\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}^2 \left[\frac{t^2}{2k} z_o(1 - z_o/z)\right]\left\{\frac{1 + k^2 a_o^4 (1/z + 1/R)(1/z_o + 1/R)}{1 + k^2 a_o^4 (1/z + 1/R)^2}\right\} dz_o \tag{61}$$

The three-dimensional spectral expansion $\Phi(t, z_o)$ of the correlation function $B_n(r;z_o)$ of the refractive index fluctuation is obtained from Equation 55. Since the medium is isotropic, the function $B_n(r;z_o)$ depends only on r. Thus, in the integral Equation 55, we can introduce spherical coordinates and carry out the angular integrations.

MEANS, VARIANCES, AND COVARIANCES FOR THE PROPAGATION OF LASER BEAMS THROUGH A TURBULENT ATMOSPHERE

A fundamental limitation to communications by coherent light transmission through the atmosphere is the unavoidable effects created by normal atmospheric turbulence. The presence of random inhomogeneities in the atmosphere causes a random fluctuation of the refractive index of the transmission medium, which, in turn, distorts and bends the light, impressing undesirable modulation on the propagating light until eventually all coherence is destroyed. The present paper is devoted primarily with the mathematics associated with the determination of the statistical behavior of an initially coherent beam of focused light propagating through a random medium, which, while isotropic, is characterized by having a random point function for refractive index. Wave propagation in such a random continuous medium is referred to as turbulent scattering. We shall not be concerned with the theoretics of the statistical mechanics of the random medium, but for this purpose we turn rather to the results of the confirmed theory of Kolmogoroff for a physical description of the statistics characterizing a turbulent atmosphere. The theoretical approach used here is from the stochastic aspects of the problem. Wave propagation in a random continuous medium is formulated as being governed by a stochastic linear differential equation, the coefficients of which characterize the transmission medium. The solution to the stochastic differential equation is then obtained by means of an iteration scheme and is expressed in the form of a power series in a perturbation parameter. The statistical properties of the stochastic solution are then computed, viz., correlation and structure functions, variance, mean value, etc.

The use of the structure function for describing the random perturbations of an optical wavefront has been aptly demonstrated to be a highly significant quantity for predicting the performance of optical systems, as well as being easily measured. Such parameters as angular resolution, angular and amplitude scintillations, and efficiency of coherent heterodyne detectors all depend directly upon the structure function of the distorted optical wavefront. It is believed that the results achieved here are significant in that many questions which were previously answerable only qualitatively or intuitively should now be amenable to quantitative study. The relationships presented can be used to evaluate realistically paths of propagation where the statistics of the medium are slowly varying, e. g., space to earth, earth to space, and earth to earth communications through the atmosphere.

The analysis has been kept sufficiently general by considering a focused laser beam limited with such quantities as initial beam spot diameter, beam spread (a function of path length), and an arbitrary radius of curvature of the focused wavefront as would be produced by the action of an ideal lens. The formulae, therefore, contain as limiting special cases, the results of previous analyses for the plane wave and spherical wave (point source).

In spite of the many tempting ramifications one can now consider at this point, none are presented here in order to expedite the publication of the results obtained so far.

NON-RANDOM WAVE PROPAGATION

Before illustrating the techniques used to analyze wave propagation in random media, we shall consider the nature of a focused beam of light propagating in a non-random medium, and thereby obtain an equation for the wave function of an unperturbed beam of light. This formulation shall subsequently serve to demonstrate the application of our results with a specific example of the effects of randomness on a prescribed wave function. The application chosen is of considerable importance in determining the fate of communication by coherent laser light transmission through the atmosphere.

For this purpose, let $u_o(\vec{r})$ represent the unperturbed wave function describing the propagation of a collimated beam of light such as would be emitted by a focused transmitting laser, where $\vec{r} = \vec{z} + \vec{\rho}$ is a position vector in a cylindrical coordinate system, $\vec{z}$ the axial coordinate vector, and $\vec{\rho}$ the radial coordinate vector. The wave function $u_o(\vec{r})$ is chosen to represent the propagation of a light disturbance as produced by a laser, $u_o(\vec{\rho})$ which is assumed to have a Gaussian amplitude distribution and a focused wave front

$$u_o(\vec{r})\Big]_{z=o} = u_o(\vec{\rho}) = \exp\left\{-\frac{\rho^2}{2\alpha_o^2} + \frac{ik}{2R}\rho^2\right\} \tag{1}$$

and is located at the exit pupil plane $z = 0$. Here α_o is a parameter measuring the effective beam radius, and R, the radius of curvature of the wavefront of the beam produced, say, by the action of an ideal lens. This initial disturbance propagates into the space $z > 0$, and this propagating wave shall be described by the wave function $u_o(\vec{r})$. Throughout this work, we shall take for granted the monochromaticity of our solution and suppress the harmonic time factor $\exp(i\omega t)$.

Accordingly, the wave function $u_o(\vec{r})$ satisfies the reduced wave equation,

$$\nabla^2 u_o(\vec{r}) + k^2 u_o(\vec{r}) = 0\,, \qquad z > 0 \tag{2}$$

Here, the refractive index is taken as unity and $k = 2\pi/\lambda$. To solve Equation 2 subject to the boundary condition Equation 1, we apply Green's theorem to the interior region bounded by a large hemisphere and the plane $z = 0$. The integral over the hemisphere can be made arbitrarily small, and we obtain

$$u_o(\vec{r}) = \frac{1}{4\pi} \int d\vec{\rho}_o \quad u_o(\vec{\rho}_o) \frac{\partial}{\partial z_o} G(\vec{r}\,;\vec{\rho}_o) \tag{3}$$

where the integral extends over the entire $z = 0$ plane, and the Green's function $G(\vec{r}\,;\vec{r}_o)$ vanishes on $z = 0$ satisfying

$$\nabla^2 G + k^2 G = -4\pi\,\delta(\vec{r} - \vec{r}_o), \quad z \geq 0 \tag{4}$$

Hence,

$$G(\vec{r}, \vec{r}_o) = \frac{\exp\left(ik\left|\vec{r} - \vec{r}_o\right|\right)}{\left|\vec{r} - \vec{r}_o\right|} - \frac{\exp\left(ik\left|\vec{r} - \vec{r}\,'_o\right|\right)}{\left|\vec{r} - \vec{r}\,'_o\right|} \tag{5}$$

where

$$\left|\vec{r} - \vec{r}_o\right| = \sqrt{\left|\vec{\rho} - \vec{\rho}_o\right|^2 + \left|z - z_o\right|^2}$$

$$\left|\vec{r} - \vec{r}\,'_o\right| = \sqrt{\left|\vec{\rho} - \vec{\rho}_o\right|^2 + \left|z + z_o\right|^2}$$

so that

$$\frac{\partial}{\partial z_o} G(\vec{r}, \vec{\rho}_o) = \lim_{z_o \to 0} \frac{\partial}{\partial z_o} \left\{ \frac{\exp\left(ik\left|\vec{r} - \vec{r}_o\right|\right)}{\left|\vec{r} - \vec{r}_o\right|} - \frac{\exp\left(ik\left|\vec{r} - \vec{r}\,'_o\right|\right)}{\left|\vec{r} - \vec{r}\,'_o\right|} \right\}$$

$$= 2 \lim_{z_o \to 0} \frac{\partial}{\partial z_o} \left\{ \frac{\exp\left(ik\left|\vec{r} - \vec{r}_o\right|\right)}{\left|\vec{r} - \vec{r}_o\right|} \right\}$$

$$= -2 \lim_{z_o \to 0} \frac{\partial}{\partial z} \left\{ \frac{\exp\left(ik\left|\vec{r} - \vec{r}_o\right|\right)}{\left|\vec{r} - \vec{r}_o\right|} \right\}$$

$$= -2 \frac{\partial}{\partial z} \frac{\exp\left(ik\left|\vec{r} - \vec{\rho}_o\right|\right)}{\left|\vec{r} - \vec{\rho}_o\right|} \tag{6}$$

Hence

$$u_o(\vec{r}) = -\frac{1}{2\pi}\frac{\partial}{\partial z}\int d\vec{\rho}_o \quad u_o(\vec{\rho}_o)\,\frac{\exp\ ik\left|\vec{r}-\vec{\rho}_o\right|}{\left|\vec{r}-\vec{\rho}_o\right|} \tag{7}$$

where

$$u_o(\vec{\rho}_o) = \exp\left\{-\frac{\rho_o^2}{2\alpha^2}\right\} \tag{8}$$

and

$$\frac{1}{\alpha^2} = \frac{1}{\alpha_o^2} - \frac{ik}{R} \tag{9}$$

Since we are interested in the wave function in the region centered about the principal ray, one can expand

$$\left|\vec{r}-\vec{\rho}_o\right| \sim z + \frac{\left|\vec{\rho}-\vec{\rho}_o\right|^2}{2z} + \ldots \tag{10}$$

and neglect higher order terms. It is perfectly reasonable to make the approximation for visible optics that $k = 2\pi/\lambda \gg 1/z$. Under these conditions, Equation 7 becomes

$$u_o(\vec{r}) \sim \frac{k\exp(ikz)}{2\pi zi}\int d\vec{\rho}_o \exp\left\{-\frac{\rho_o^2}{2\alpha^2} + \frac{ik}{2z}\left|\vec{\rho}-\vec{\rho}_o\right|^2\right\} \tag{11}$$

which evaluated becomes

$$u_o(\vec{r}) \sim -\frac{ik\alpha^2\exp(-k^2\alpha^2)}{Z}\ \exp\left\{ik\left(Z+\frac{\rho^2}{2Z}\right)\right\},\ z \gg \alpha_o \gg \frac{1}{k} \tag{12}$$

where

$$Z \equiv z - ik\alpha^2 \tag{13}$$

Equation 12 represents the wave function for the Gaussian beam of focused light as it propagates through a non-random medium with an index of refraction of unity.

WAVE PROPAGATION IN RANDOM CONTINUOUS MEDIA

Let us consider next what happens when the beam propagates through an inhomogeneous medium having a randomly fluctuating index of refraction. In this case, the beam of light is bent and the wavefront is distorted as the beam propagates until eventually, at sufficiently large distances, the coherence of the beam is completely destroyed.

Before proceeding, let us recall what is meant by wave propagation in a random continuous medium. A random medium is characterized by a family of media having a family of index of refraction functions, $n(\vec{r}, \omega)$, depending on a position vector $\vec{r}$, and a parameter ω identifying a particular member medium of the family, together with a probability distribution specifying the probability of various members of the family. Wave propagation in a random medium then refers to propagation in each member medium together with the probability of each member medium. This probability, when associated with the wave motion in each medium, characterizes a random wave motion.

Wave propagation in continuous random media can be studied by considering the solution for the wave function $u(\vec{r}\,;\omega)$ for each member medium with a refractive index function $n(\vec{r}\,;\omega)$. Each member solution is governed by a stochastic linear differential equation of the form

$$\nabla^2 \ddot{u}(\vec{r}\,;\omega) + k^2 n^2(\vec{r}\,;\omega)\; u(\vec{r}\,;\omega) = 0 \tag{14}$$

Here, the coefficient $n(\vec{r}\,;\omega)$ characterizes the random propagation medium, as a family of index of refraction functions depending on a parameter ω, a point in probability space which identifies the individual family member, for which a probability measure $dP(\omega)$ is defined. If the problem is well posed, then $n(\vec{r}\,;\omega)$ is measurable with respect to $dP(\omega)$, and there exists a solution, $u(\vec{r}\,;\omega)$ for each ω. The wave functions $u(\vec{r}\,;\omega)$ satisfy Equation 14 for each value of ω. This family of solutions, with the probability for each value of ω, $dP(\omega)$, then characterizes the wave propagation in a continuous random medium.

In studying the statistics of wave propagation in random media, we shall need to define certain statistical properties of a random function.

The mean value of a random function $f(\vec{r}\,;\omega)$ denoted $\langle f(\vec{r})\rangle$ is defined as the ensemble average of all possible values the function can assume at the point $\vec{r}$, weighted by a probability measure $dP(\omega)$, i. e.,

$$\langle f(\vec{r})\rangle = \int_{-\infty}^{\infty} f(\vec{r};\omega)\, dP(\omega) \tag{15}$$

The covariance of a random function is defined by

$$C_f(\vec{\tau}) = \int_{-\infty}^{\infty} \left[f(\vec{r}+\vec{\tau};\omega) - \langle f(\vec{r}+\vec{\tau})\rangle \right] \left[f(\vec{r};\omega) - \langle f(\vec{r})\rangle \right] dP(\omega)$$

$$= \langle \{ f(\vec{r}+\vec{\tau}) - \langle f(\vec{r}+\vec{\tau})\rangle \} \{ f^*(\vec{r}) - \langle f^*(\vec{r})\rangle \} \rangle \tag{16}$$

Thus, if $\langle f(\vec{r})\rangle = 0$, then

$$C_f(\vec{\tau}) = \langle f(\vec{r}+\vec{\tau})\, f(\vec{r})\rangle \tag{17}$$

The mean square continuity of random functions means

$$\lim_{\tau \to 0} \langle | f(\vec{r}+\vec{\tau}) - f(\vec{r}) |^2 \rangle = 0 \tag{18}$$

The function

$$D_f(\vec{\tau}) = \langle | f(\vec{r}+\vec{\tau}) - f(\vec{r}) |^2 \rangle \tag{19}$$

is called the structure function and is the basic characteristic of a random process with stationary increments. In a sense, the value of $D_f(\vec{\tau})$ characterizes the intensity of those fluctuations of $f(\vec{r})$ with periods which are smaller than or comparable with τ. The connection between the structure function D_f and the covariance function C_f for a random function $f(\vec{r};\omega)$ whose mean value is zero is clear from

$$D_f(\vec{\tau}) = \langle | f(\vec{r}+\vec{\tau}) - f(\vec{r}) |^2 \rangle$$

$$= \langle \{ f(\vec{r}+\vec{\tau}) - f(\vec{r}) \} \{ f(\vec{r}+\vec{\tau}) - f(\vec{r}) \} \rangle = 2 \left[C_f(0) - C_f(\vec{\tau}) \right] \tag{20}$$

It is interesting to observe that from this result it follows that the continuity of the covariance function at $\vec{\tau} = 0$ in the ordinary sense implies the continuity of $f(\vec{r})$ in the mean square sense. Thus $f(\vec{r})$ must have finite variance if mean square properties are to hold. In the case where $C_f(\infty) = 0$, which is usually the case, we have

$$C_f(\vec{\tau}) = \frac{1}{2} \left\{ D_f(\infty) - D_f(\vec{\tau}) \right\} \tag{21}$$

A random function is said to be globally stationary when its covariance function depends only on the distance between the two points being correlated, i. e.,

$$C_f(\vec{r}, \vec{r}') = C_f(|\vec{r} - \vec{r}'|) \tag{22}$$

In the ensuing analysis, it is assumed that the mean value of the index of refraction function $\langle n(\vec{r}) \rangle$ is unity. The correlation function for the refractive index is accordingly

$$C_n(|\vec{r} - \vec{r}'|) = \langle (n(\vec{r}) - 1)(n(\vec{r}') - 1) \rangle \tag{23}$$

Here we have expressed C_n as a function of the distance $|\vec{r} - \vec{r}'|$ between the two points being correlated, implying global stationarity. This is an appropriate notation when the transmission medium is both homogeneous and isotropic in space.

We shall also need to know the expectation value of the gradient of the index of refraction, denoted by

$$\langle \nabla n(\vec{r}) \cdot \nabla n(\vec{r}') \rangle \tag{24}$$

This quantity can be expressed in terms of the covariance function C_n. To show this, we represent $\langle \nabla n(\vec{r}) \cdot \nabla n(r') \rangle$ by the following limit process

$$\begin{aligned}
&= \lim_{t,\, t' \to 0} \langle \nabla_t n(\vec{r} + \vec{t}) \cdot \nabla_{t'} n(\vec{r}' + \vec{t}') \rangle \\
&= \lim_{t,\, t' \to 0} \langle \nabla_t \cdot \nabla_{t'} \; (n(\vec{r} + \vec{t}) - 1) \; (n(\vec{r}' + \vec{t}') - 1) \rangle \\
&= \lim_{t,\, t' \to 0} \nabla_t \cdot \nabla_{t'} \; C_n(|\vec{r} + \vec{t} - \vec{r}' - \vec{t}'|) \\
&= -\nabla^2 C_n(|\vec{r} - \vec{r}'|)
\end{aligned} \tag{25}$$

We next define what is meant by a locally stationary process as contrasted to one which is globally stationary (over all space). The concept of local stationarity has considerably more physical significance here, since the atmosphere is actually homogeneous locally. Thus the statistics vary smoothly and slowly as a function of position. A process shall be said to be locally stationary if the covariance functions can be put into the form

$$C_f(\vec{r}, \vec{r}') = h\left[\frac{1}{2}(\vec{r}+\vec{r}')\right] C_f(|\vec{r} - \vec{r}'|) \tag{26}$$

$$C_{\nabla f}(\vec{r}, \vec{r}') = -h\left[\frac{1}{2}(\vec{r}+\vec{r}')\right] \nabla^2 C_f(|\vec{r} - \vec{r}'|) \tag{27}$$

where h is a non-negative function, and C_n is a globally stationary function. Thus h $(\vec{r})$ may vary over the whole real line, while $C_f(\vec{r})$, which is invariant under translation, can be thought of as describing its local behavior. The index of refraction function $n(\vec{r}; \omega)$ shall be assumed to be locally stationary, and therefore Equations 26 and 27 are appropriate representations for its covariance functions.

PERTURBATION SOLUTION OF THE EQUIVALENT SPATIAL RICCATI EQUATION

Wave propagation in a continuous random medium shall be considered here as being governed by the stochastic linear differential equation

$$\nabla^2 u(\vec{r}, \omega) + k^2 n^2(\vec{r}; \omega) u(\vec{r}, \omega) = 0 \tag{28}$$

with a random function $n(\vec{r}; \omega)$ characterizing the transmission medium. Equation 28 represents a family of equations depending upon a probability parameter ω, each member of which possesses a solution $u(\vec{r}; \omega)$. In what follows, the probability parameter ω will be omitted throughout.

Applying the Rytov transformation takes Equation 28 into the equivalent nonlinear spatial Riccati equation

$$\nabla^2 \Psi + (\nabla\Psi)^2 = -k^2 n^2(\vec{r}) \tag{29}$$

where

$$\Psi(\vec{r}) = \ln A(\vec{r}) + i\phi(\vec{r}), \tag{30}$$

$A(\vec{r})$ and $\phi(\vec{r})$ being the "amplitude" and "phase" associated with $u(\vec{r})$ as

$$u(\vec{r}) = A(\vec{r}) \exp i\phi(\vec{r}) \tag{31}$$

Now $n(\vec{r})$ may be written as

$$n(\vec{r}) = 1 + \epsilon\mu(\vec{r}) \tag{32}$$

where the parameter ϵ has been inserted as a measure of the deviation of the refractive index from its mean value unity.

It seems reasonable to suppose that Ψ can be represented in a power series in ϵ

$$\Psi(\vec{r}\,;\epsilon) = \psi_o(\vec{r}) + \epsilon\,\psi_1(\vec{r}) + \epsilon^2\psi_2(\vec{r}) + \ldots \tag{33}$$

Substituting Equation 33 into 29 then gives

$$\sum_m \epsilon^m \left\{\nabla^2\psi_m + 2\nabla\psi_o\cdot\nabla\psi_m + \sum_{p=1}^{m-1}\nabla\psi_p\cdot\nabla\psi_{m-p}\right\}$$

$$= -k^2 - \epsilon\,2k^2\mu - \epsilon^2k^2\mu^2, \quad m = 0, 1, 2, 3 \ldots \tag{34}$$

Equating to zero the coefficient of each power of ϵ, we obtain

$$\nabla^2\psi_o + \nabla\psi_o\cdot\nabla\psi_o = -k^2 \tag{35}$$

$$\nabla^2\psi_1 + 2\nabla\psi_o\cdot\nabla\psi_1 = -2k^2\mu \tag{36}$$

$$\nabla^2\psi_2 + 2\nabla\psi_o\cdot\nabla\psi_2 = -k^2\mu^2 - \nabla\psi_1\cdot\nabla\psi_1 \tag{37}$$

$$\nabla^2\psi_m + 2\nabla\psi_o\cdot\nabla\psi_m = -\sum_{p=1}^{m-1}\nabla\psi_p\cdot\nabla\psi_{m-p}, \quad m=3, 4, 5 \ldots \tag{38}$$

Equation 35 gives the solution of Equation 1 for the case $\mu \equiv 0$. Thus, if $\mu_o = \exp\Psi_o$, then Equation 35 is equivalent to

$$\nabla^2 u_o + k^2 u_o = 0 \tag{39}$$

Equations 36 through 38 are a system that can be solved successively for the Ψ_m's. To accomplish this, we introduce the functions

$$W_m = \psi_m \exp\psi_o, \quad m=1, 2, 3, \ldots \tag{40}$$

Using Equation 35, we find

$$\nabla^2 W_m = \nabla^2 (\psi_m \exp \psi_o)$$

$$= \exp \psi_o \left\{ \nabla^2 \psi_m + 2\nabla\psi_o \cdot \nabla\psi_m + \psi_m \left[\nabla^2 \psi_o + \nabla\psi_o \cdot \nabla\psi_o \right] \right\}$$

$$= \exp \psi_o \left\{ \nabla^2 \psi_m + 2\nabla\psi_o \cdot \nabla\psi_m \right\} - k^2 W_m$$

so that Equation 36 through 38 can be written as

$$\nabla^2 W_1 + k^2 W_1 = - 2k^2 \mu u_o \tag{41}$$

$$\nabla^2 W_2 + k^2 W_2 = - \left\{ k^2 \mu^2 + \nabla(W_1 \exp - \psi_o) \cdot \nabla (W_1 \exp -\psi_o) \right\} u_o \tag{42}$$

$$\nabla^2 W_m + k^2 W_m = - \sum_{p=1}^{m-1} \nabla(W_p \exp -\psi_o) \cdot \nabla (W_{m-p} \exp -\psi_o) u_o \tag{43}$$

$$m = 3, 4, 5 \ldots$$

The solutions of sets 41 through 43 are given iteratively by

$$W_1(\vec{r}) = \frac{k^2}{2\pi} \int \mu (\vec{r}_o) u_o (\vec{r}_o) \frac{\exp\left(i k \left|\vec{r} - \vec{r}_o\right|\right)}{\left|\vec{r} - \vec{r}_o\right|} d\vec{r}_o \tag{44}$$

$$W_2 (\vec{r}) = \frac{1}{4\pi} \int \left\{ k^2 \mu^2 + \nabla(W_1 \exp -\psi_o) \cdot \nabla (W_1 \exp -\psi_o) \right\} u_o(\vec{r}_o) \frac{\exp\left(ik \left|\vec{r}-\vec{r}_o\right|\right)}{\left|\vec{r}-\vec{r}_o\right|} d\vec{r}_o \tag{45}$$

$$W_m(\vec{r}) = \frac{1}{4\pi} \sum_{p=1}^{m-1} \int \nabla(W_p \exp -\psi_o) \cdot \nabla (W_{m-p} \exp -\psi_o) u_o(\vec{r}_o) \frac{\exp\left(ik\left|\vec{r}-\vec{r}_o\right|\right)}{\left|\vec{r}-\vec{r}_o\right|} d\vec{r}_o \tag{46}$$

$$m = 3, 4, 5 \ldots$$

where the integration is taken over that part of space from which scattered waves arrive at the point $\vec{r}$. From Equation 40

$$\psi_1(\vec{r}) = \frac{k^2}{2\pi} \int \mu (\vec{r}_o) \frac{u_o(\vec{r}_o)}{u_o(\vec{r})} \frac{\exp\left(ik \left|\vec{r}-\vec{r}_o\right|\right)}{\left|\vec{r}-\vec{r}_o\right|} d\vec{r}_o \tag{47}$$

$$\psi_2(\vec{r}) = \frac{1}{4\pi}\int\left\{k^2\mu^2 + \nabla\psi_1 \cdot \nabla\psi_1\right\} \frac{u_o(\vec{r}_o)}{u_o(\vec{r})} \frac{\exp\left(ik\left|\vec{r}-\vec{r}_o\right|\right) d\vec{r}_o}{\left|\vec{r}-\vec{r}_o\right|} \tag{48}$$

$$\psi_m(\vec{r}) = \frac{1}{4\pi}\sum_{p=1}^{m-1}\int \nabla\psi_p \cdot \nabla\psi_{m-p} \frac{u_o(\vec{r}_o)}{u_o(\vec{r})} \frac{\exp\left(ik\left|\vec{r}-\vec{r}_o\right|\right) d\vec{r}_o}{\left|\vec{r}-\vec{r}_o\right|} \tag{49}$$

$$m=3, 4, 5 \ldots$$

Equations 47 through 49 together with the solution of Equation 39 represent the general solution for wave propagation in a random continuous medium.

The solution of Equation 28 can be written in the form

$$u(\vec{r}) = u_o(\vec{r}) \exp \psi(\vec{r}) \tag{50}$$

where $u_o(r)$ may be thought of as an unperturbed wave function propagating as if the transmission medium were non-random, and where $\psi(\vec{r})$ is a measure of the fluctuation

$$\psi = \Psi - \psi_o = \ln\left|\frac{A(\vec{r})}{A_o(\vec{r})}\right| + i\left\{\phi(\vec{r}) - \phi_o(\vec{r})\right\} \tag{51}$$

of the amplitude and phase from their unperturbed values, $A_o(\vec{r})$ and $\phi_o(\vec{r})$ respectively. The fluctuation function ψ is computed from

$$\psi(\vec{r}) = \epsilon\,\psi_1(\vec{r}) + \epsilon^2\psi_2(\vec{r}) + \epsilon^3\psi_3(\vec{r}) + \ldots \tag{52}$$

where the ψ_m's are given by Equations 47 through 49.

MEAN VALUE FOR PHASE AND LOG-AMPLITUDE

The mean value of Ψ, denoted by $\langle\Psi\rangle$ is found by taking the ensemble average of each term in Equation 5.

$$\langle\Psi(\vec{r})\rangle = \langle\psi_o(\vec{r})\rangle + \epsilon\langle\psi_1(\vec{r})\rangle + \epsilon^2\langle\psi_2(\vec{r})\rangle + \ldots \tag{53}$$

Since $\psi_o(\vec{r})$ is non-random, $\langle\psi_o(\vec{r})\rangle = \psi_o(\vec{r})$. The ensemble average of $\psi_1(\vec{r})$ may be computed from Equation 47 by interchanging the order of integrating and averaging, which is assumed permissible. Then

$$\langle\psi_1(\vec{r})\rangle = \frac{k^2}{2\pi}\int\langle\mu(\vec{r}_o)\rangle \frac{u_o(\vec{r}_o)}{u_o(\vec{r})} \frac{\exp\left(ik\left|\vec{r}-\vec{r}_o\right|\right) d\vec{r}_o}{\left|\vec{r}-\vec{r}_o\right|}$$

Since $\langle \mu(\vec{r}_o) \rangle = 0$, we have $\langle \psi_1(\vec{r}) \rangle = 0$. From Equation 48, we find

$$\langle \psi_2(\vec{r}) \rangle = \frac{k^2}{4\pi} \int \left\{ \langle \mu^2(\vec{r}_o) \rangle + \frac{\langle \nabla\psi_1(\vec{r}_o) \nabla\psi_1(\vec{r}_o) \rangle}{k^2} \right\} \frac{u_o(\vec{r}_o)}{u_o(\vec{r})} \frac{\exp\left(ik\left|\vec{r}-\vec{r}_o\right|\right)}{\left|\vec{r}-\vec{r}_o\right|} d\vec{r}_o \tag{54}$$

Therefore, the mean value of Ψ is given by

$$\langle \Psi(\vec{r}) \rangle = \psi_o(\vec{r}) + \frac{\epsilon^2 k^2}{4\pi} \int \langle \mu^2(\vec{r}_o) \rangle \frac{u_o(\vec{r}_o)}{u_o(\vec{r})} \frac{\exp\left(ik\left|\vec{r}-\vec{r}_o\right|\right)}{\left|\vec{r}-\vec{r}_o\right|} d\vec{r}_o + M(\vec{r}) + O(\epsilon^3) \tag{55}$$

where

$$M(\vec{r}) = \frac{1}{4\pi} \int \langle \nabla\psi_1(\vec{r}_o) \cdot \nabla\psi_1(\vec{r}_o) \rangle \frac{u_o(\vec{r}_o)}{u_o(\vec{r})} \frac{\exp\left(ik\left|\vec{r}-\vec{r}_o\right|\right)}{\left|\vec{r}-\vec{r}_o\right|} d\vec{r}_o \tag{56}$$

and ψ_1 is given by Equation 47.

Multiplying Equation 47 by $u_o(\vec{r})$ and taking the gradient of the product gives

$$\nabla\left[\psi_1(\vec{r})\, u_o(\vec{r})\right] = \frac{k^2}{2\pi} \int \mu(\vec{r}_o)\, u_o(\vec{r}_o)\, \nabla_r \left\{ \frac{\exp\left(ik\left|\vec{r}-\vec{r}_o\right|\right)}{\left|\vec{r}-\vec{r}_o\right|} \right\} d\vec{r}_o$$

$$= -\frac{k^2}{2\pi} \int \mu(\vec{r}_o)\, u_o(\vec{r}_o)\, \nabla_{\vec{r}_o} \left\{ \frac{\exp\left(ik\left|\vec{r}-\vec{r}_o\right|\right)}{\left|\vec{r}-\vec{r}_o\right|} \right\} d\vec{r}_o \tag{57}$$

Integrating by parts

$$\nabla\left[\psi_1(\vec{r})\, u_o(\vec{r})\right] = \frac{k^2}{2\pi} \int \nabla \left\{ \mu(\vec{r}_o)\, u_o(\vec{r}_o) \right\} \frac{\exp\left(ik\left|\vec{r}-\vec{r}_o\right|\right)}{\left|\vec{r}-\vec{r}_o\right|} d\vec{r}_o \tag{58}$$

Next, we use the identity

$$\nabla\psi_1(\vec{r}) = \frac{\nabla\left[\psi_1(\vec{r})\, u_o(\vec{r})\right]}{u_o(\vec{r})} - \psi_1(\vec{r})\, \nabla\psi_o(\vec{r})$$

and Equation 47 to obtain

$$\nabla \psi_1(\vec{r}) = \frac{k^2}{2\pi} \int \left\{ \frac{\nabla[\mu(\vec{r}_o)\, u_o(\vec{r}_o)]}{u_o(\vec{r}_o)} - \mu(\vec{r}_o)\, \nabla \psi_o(\vec{r}) \right\} \frac{u_o(\vec{r}_o)}{u_o(\vec{r})} \frac{\exp\left(ik\,|\vec{r}-\vec{r}_o|\right) d\vec{r}_o}{|\vec{r}-\vec{r}_o|}$$

$$= \frac{k^2}{2\pi} \int \left\{ \nabla\mu(\vec{r}_o) + \mu(\vec{r}_o)[\nabla \psi_o(\vec{r}) - \nabla \psi_o(\vec{r}_o)] \right\} \tag{59}$$

$$\times \frac{u_o(\vec{r}_o)}{u_o(\vec{r})} \frac{\exp\left(ik\,|\vec{r}-\vec{r}_o|\right)}{|\vec{r}-\vec{r}_o|}\, d\vec{r}_o$$

It is perfectly permissible to make the assumption for visible optics that $k = 2\pi/\lambda >> 1/\vec{r}_o$ and

$$\left| \frac{\nabla \mu(\vec{r}_o)}{\mu(\vec{r}_o)} \right| >> \left| \nabla\psi_o(\vec{r}) - \nabla \psi_o(\vec{r}_o) \right| \tag{60}$$

Hence Equation 59 becomes

$$\nabla\psi_1(\vec{r}) = \frac{k^2}{2\pi} \int \nabla \mu(\vec{r}_o)\, \frac{u_o(\vec{r}_o)}{u_o(\vec{r})} \frac{\exp\left(ik\,|\vec{r}-\vec{r}_o|\right) d\vec{r}_o}{|\vec{r}-\vec{r}_o|} \tag{61}$$

The wave function u_o is chosen to represent the unperturbed wave function of the focused laser beam given by Equation 12. Thus,

$$\frac{u_o(\vec{r}_o)}{u_o(\vec{r})} = \frac{Z}{Z_o} \exp ik \left\{ z_o - z + \frac{1}{2}\left(\frac{\rho_o^2}{Z_o} - \frac{\rho^2}{Z} \right) \right\} \tag{62}$$

It seems reasonable that only those inhomogeneities that are located within a cone with vertex at $\vec{r}$ with a small aperture should contribute significantly to the evaluation of Equation 61. Inside this cone,

$$\frac{\exp\left(ik\,|\vec{r}-\vec{r}_o|\right)}{|\vec{r}-\vec{r}_o|} \sim \frac{\exp\left(ik\,|z-z_o|\right)}{|z-z_o|} \exp\left\{ \frac{ik}{2} \frac{|\vec{\rho}-\vec{\rho}_o|^2}{|z-z_o|} \right\} \tag{63}$$

Reflections should also play a minor role, and we shall consequently limit the integration to the region $o<z_o<z$. Therefore, substituting Equation 62 and 63 into 61, we find that $\nabla\psi_1$ can be represented by a simpler expression than Equation 61:

$$\nabla\psi_1(\vec{r}) = -\frac{ik}{\pi}\int_o^z \frac{dz_o}{\gamma(z,z_o)}\int d\vec{\rho}_o \quad \nabla\mu(\vec{r}_o)\ \exp\left\{\frac{(\vec{\rho}_o - \vec{\rho}Z_o/Z)^2}{\gamma(z,z_o)}\right\} \tag{64}$$

where the $d\vec{\rho}_o$ integration is taken over the entire two-dimensional vector space of $\vec{\rho}_o$ and

$$\gamma(z,z_o) = \frac{2(z-z_o)Z_o}{ikZ} \tag{65}$$

Consequently,

$$\langle\nabla\psi_1(\vec{r}_o)\cdot\nabla\psi_1(\vec{r}_o)\rangle = -\frac{k^2}{\pi^2}\int_o^{z_o}\int_o^{z_o}\frac{dz_1\ dz_2}{\gamma(z_o,z_1)\,\gamma(z_o,z_2)}\int d\vec{\rho}_1 \tag{66}$$

$$\int d\vec{\rho}_2\langle\nabla\mu(\vec{r}_1)\cdot\nabla\mu(\vec{r}_2)\rangle\cdot\exp\left\{\frac{(\vec{\rho}_1-\vec{\rho}_o Z_1/Z_o)^2}{\gamma(z_o,z_1)}+\frac{(\vec{\rho}_2-\vec{\rho}_o Z_2/Z_o)^2}{\gamma(z_o,z_2)}\right\}$$

Now from Equation 27

$$\langle\nabla\mu(\vec{r}_1)\cdot\nabla\mu(\vec{r}_2)\rangle = -h\left[\frac{1}{2}(z_1+z_2)\right]\nabla^2 C_n(|\vec{r}_1-\vec{r}_2|) \tag{67}$$

which is appropriate here, since the refractive index of the medium can be considered locally stationary. Substituting Equation 67 into 66 and introducing the change of variables

$$\vec{\zeta} = \vec{\rho}_1 - \vec{\rho}_2\ , \quad \vec{\eta} = \frac{1}{2}(\vec{\rho}_1+\vec{\rho}_2) \tag{68}$$

yields

$$\langle \nabla\psi_1(\vec{r}_o) \cdot \nabla\psi_1(\vec{r}_o)\rangle = +\frac{k^2}{\pi^2}\int_0^{z_o}\int_0^{z_o} \frac{dz_1 \; dz_2 \; h\left[\frac{1}{2}(z_1+z_2)\right]}{\gamma(z_o, z_1)\;\gamma(z_o, z_2)}\int d\vec{\zeta}\,\nabla^2$$

$$C_n\left[\left\{(z_1-z_2)^2+\zeta^2\right\}^{1/2}\right] \cdot \int d\vec{\eta} \quad \exp\left\{\frac{(\vec{\eta}+\vec{a})^2}{\gamma(z_o, z_1)} + \frac{(\vec{\eta}+\vec{b})^2}{\gamma(z_o, z_2)}\right\} \tag{69}$$

where

$$\begin{aligned} \vec{a} &= \frac{1}{2}\vec{\zeta} - \vec{\rho}_o \; Z_1/Z_o \\ \vec{b} &= -\frac{1}{2}\vec{\zeta} - \vec{\rho}_o \; Z_2/Z_o \end{aligned} \tag{70}$$

The $d\vec{\eta}$ integration can be performed explicitly by substituting the result

$$\int d\vec{\eta} \quad \exp\left\{\frac{(\vec{\eta}+\vec{a})^2}{\gamma_a} + \frac{(\vec{\eta}+\vec{b})^2}{\gamma_b}\right\}$$

$$= -\pi\,\frac{\gamma_a\gamma_b}{\gamma_a+\gamma_b} \quad \exp\left\{\frac{(\vec{a}-\vec{b})^2}{\gamma_a+\gamma_b}\right\} \tag{71}$$

Thus Equation 69 becomes

$$\langle\nabla\psi_1(\vec{r}_o) \cdot \nabla\psi_1(\vec{r}_o)\rangle = -\frac{k^2}{\pi}\int_0^{z_o}\int_0^{z_o}\frac{dz_1\,dz_2 \quad h\left[\frac{1}{2}(z_1+z_2)\right]}{\gamma(z_o, z_1)+\gamma(z_o, z_2)}\int d\vec{\zeta}\nabla^2$$

$$C_n\left[\left\{(z_1-z_2)^2+\zeta^2\right\}^{1/2}\right] \cdot \exp\left\{\frac{(\vec{\zeta} - \vec{\rho}_o\,(z_1-z_2)/Z_o)^2}{\gamma(z_o, z_1)+\gamma(z_o, z_2)}\right\} \tag{72}$$

Applying the approximations for visible optics of Equations 62 and 63 to the integral representing $M(\vec{r})$ of Equation 56 gives

$$M(\vec{r}) = \frac{1}{2\pi ik}\int_0^{z}\frac{dz_o}{\gamma(z, z_o)}\int d\vec{\rho}_o\langle\nabla\psi_1(\vec{r}_o) \cdot \nabla\psi_1(\vec{r}_o)\rangle\exp\left\{\frac{(\rho_o-\rho Z_o/Z)^2}{\gamma(z, z_o)}\right\}$$

(73)

Substituting Equation 72 into 73

$$M(\vec{r}) = \frac{ik}{2\pi^2} \int_0^z \frac{dz_o}{\gamma(z, z_o)} \int_0^{z_o} \int_0^{z_o} \frac{dz_1\, dz_2\, h\left[\frac{1}{2}(z_1 + z_2)\right]}{\gamma(z_o, z_1) + \gamma(z_o, z_2)} \int d\vec{\zeta}\, \nabla^2 C_n$$

$$\left[\left\{(z_1 - z_2)^2 + \zeta^2\right\}^{1/2}\right] \cdot \int d\vec{\rho}_o \exp\left\{\frac{(\vec{\rho}_o - \vec{\rho}\, Z_o/Z)^2}{\gamma(z, z_o)} + \frac{(\vec{\rho}_o - \vec{\zeta}\, Z_o/(z_1 - z_2))^2}{\left(\frac{Z_o}{z_1 - z_2}\right)^2 \left[\gamma(z_o, z_1) + \gamma(z_o, z_2)\right]}\right\} \tag{74}$$

The $d\vec{\rho}_o$ integration is performed using Equation 71

$$M(\vec{r}) = \frac{-ik}{2\pi} \int_0^z dz_o \int_0^{z_o} \int_0^{z_o} \frac{\left[dz_1\, dz_2\, h\left[\frac{1}{2}(z_1 + z_2)\right]\right]}{\Lambda} \tag{75}$$

$$\cdot \int d\vec{\zeta}\, \nabla^2 C_n \left[\left\{(z_1 - z_2)^2 + \zeta^2\right\}^{1/2}\right] \exp\left\{\frac{(\vec{\xi} - \vec{\rho}(z_1 - z_2)/Z)^2}{\Lambda}\right\}$$

where

$$\Lambda = \gamma(z_o, z_1) + \gamma(z_o, z_2) + \left(\frac{z_1 - z_2}{Z_o}\right)^2 \gamma(z, z_o) \tag{76}$$

We next introduce the change of variables

$$S = \frac{1}{2}(z_1 + z_2), \quad t = z_1 - z_2 \tag{77}$$

into Equation 75

$$M(\vec{r}) = -\frac{ik}{2\pi} \int_0^z dz_o \int_0^{z_o} \frac{h(s)\, ds}{\Lambda} \int_{-z_o + 2\left|S - 1/2 z_o\right|}^{z_o - 2\left|S - 1/2 z_o\right|} dt \int d\vec{\zeta}\, \nabla^2 C_n \tag{78}$$

$$(t^2 + \zeta^2)^{1/2} \exp \frac{(\vec{\zeta} - \vec{\rho}\, t/Z)^2}{\Lambda}$$

and into Equation 76

$$\Lambda = 2\gamma(z_o, S) + \frac{t^2}{ikZ_o} \left(1 - 2\frac{Z_o}{Z}\right) \tag{79}$$

The covariance function $C_n(\zeta)$ falls off rapidly to zero for points separated by a distance ζ larger than a_c which we call a correlation distance. Now since a_c is a relatively small distance compared to path length, there are several approximations that can be made that will simplify the evaluation of Equation 78 considerably.

Since

$$\left| z_o - 2\left| S - 1/2\, z_o \right| \right| \gg a_o \gtrsim t \tag{80}$$

for values of S larger than a few times a_o from the end points S=0 and S = z_o, the limits in the dt integral in Equation 78 can be extended between $-\infty$ and $+\infty$ with negligible error.

Likewise, the coefficient Λ of Equation 79 has a very weak dependence on t. We shall therefore set t = 0 in Equation 79. Consequently Equation 78 can be expressed more simply as

$$M(\vec{r}) = -\frac{ik}{4\pi} \int_o^z dz_o \int_o^{z_o} \frac{h(s)ds}{\gamma(z_o, s)} \int_{-\infty}^{\infty} dt \int d\vec{\zeta}\, \nabla^2 C_n \left[(t^2 + \zeta^2)^{1/2} \right] \cdot \exp\left\{ \frac{(\vec{\zeta} - \vec{\rho}t/Z)^2}{2\gamma(z_o, s)} \right\} \tag{81}$$

Examination of the relative magnitudes of the terms in exponential in Equation 81 shows that for $\vec{\rho}/Z \ll 1$, the dependence on $\vec{\rho}$ is quite weak. For this reason, we shall set $\vec{\rho}$=o and neglect this dependence entirely. This also serves to eliminate the dependence of the exponential on t. Consequently, the dt integration can be carried out, assuming C_n is prescribed. Thus

$$M(z) = -ik \int_o^z dz_o \int_o^{z_o} \frac{h(s)\, ds}{\gamma(z_o, s)} \int_o^{\infty} \zeta f_o(\zeta) \exp\left\{ \frac{\zeta^2}{2\gamma(z_o, s)} \right\} d\zeta \tag{82}$$

where

$$f_o(\zeta) = \int_0^\infty \nabla^2 C_n\left[(t^2+\zeta^2)^{1/2}\right] dt \tag{83}$$

Proceeding with the evaluation of Equation 55, we note that the variance of the index of refraction is given by

$$\epsilon^2 \langle \mu^2(\vec{r}_o)\rangle = h(z_o)\; C_n(0) \tag{84}$$

To evaluate the volume integral in Equation 55, we use this result together with Equations 62 and 63,

$$\frac{\epsilon^2 k^2}{4\pi}\int \langle\mu^2\rangle \frac{u_o(\vec{r}_o)}{u_o(\vec{r})} \frac{\exp\left(ik\left|\vec{r}-\vec{r}_o\right|\right) d\vec{r}_o}{\left|\vec{r}-\vec{r}_o\right|}$$

$$= -\frac{ik\, C_n(o)}{2\pi}\int_0^z \frac{h(z_o)\, dz_o}{\gamma(z,z_o)} \int d\bar{\rho}_o \exp\left\{\frac{(\vec{\rho}_o - \vec{\rho} Z_o/Z)^2}{(z,z_o)}\right\}$$

$$= -\frac{ik\, C_n(o)}{2}\int_0^z \frac{h(z_o)\, dz_o}{\gamma(z,z_o)} \int_0^\infty \exp\left(t/\gamma(z,z_o)\right) dt$$

$$= +\frac{ik}{2}\, C_n(o) \int_0^z h(z_o)\; dz_o \tag{85}$$

Therefore, the mean value of the fluctuation function ψ is given by

$$\langle\psi(z)\rangle = \frac{ik}{2}\, C_n(o) \int_0^z h(z_o)\, dz_o$$

$$- ik \int_0^z dz_o \int_0^{z_o} \frac{h(s)\, ds}{\gamma(z_o, s)} \int_0 \zeta f_o(\zeta) \exp\left(\frac{\zeta^2}{2\gamma(z_o,s)}\right) d\zeta + 0(\epsilon^3) \tag{86}$$

COVARIANCE FUNCTIONS

Let us now compute the lateral convariance functions for the logarithm of the relative amplitude

$$C_{LL}(\vec{\tau}) \equiv \left\langle \log \left| \frac{A(\vec{r})}{A_o(\vec{r})} \right| \cdot \log \left| \frac{A(\vec{r}+\vec{\tau})}{A_o(\vec{r}+\vec{\tau})} \right| \right\rangle \tag{87}$$

and of the phase fluctuation

$$C_{\phi\phi}(\vec{\tau}) \equiv \left\langle \left(\phi(\vec{r}) - \phi_o(\vec{r})\right) \left(\phi(\vec{r}+\vec{\tau}) - \phi_o(\vec{r}+\vec{\tau})\right) \right\rangle \tag{88}$$

where $\vec{\tau}$ is a radial position vector in the plane normal to the z-axis. These can be expressed as

$$C_{LL}(\vec{\tau}) = \frac{1}{2} \mathrm{Re} \left\{ C_{\psi\psi^*}(\vec{\tau}) + C_{\psi\psi}(\vec{\tau}) \right\} \tag{89}$$

$$C_{\phi\phi}(\vec{\tau}) = \frac{1}{2} \mathrm{Re} \left\{ C_{\psi\psi^*}(\vec{\tau}) - C_{\psi\psi}(\vec{\tau}) \right\} \tag{90}$$

where

$$C_{\psi\psi^*}(\vec{\tau}) = \left\langle \left(\psi(\vec{r}) - \langle \psi(\vec{r}) \rangle\right) \left(\psi^*(\vec{r}+\vec{\tau}) - \langle \psi^*(\vec{r}+\vec{\tau}) \rangle\right) \right\rangle \tag{91}$$

$$C_{\psi\psi}(\vec{\tau}) = \left\langle \left(\psi(\vec{r}) - \langle \psi(\vec{r}) \rangle\right) \left(\psi(\vec{r}+\vec{\tau}) - \langle \psi(\vec{r}+\vec{\tau}) \rangle\right) \right\rangle \tag{92}$$

which can easily be computed to the second order in ϵ from our explicit solution Equation 52. Now from Equation 47, since ψ_1 is linear in μ and since we assume that $\langle \mu \rangle = 0$, it follows that $\langle \psi_1 \rangle = 0$. Hence,

$$C_{\psi\psi^*}(\vec{\tau}) = \epsilon^2 \langle \psi_1(\vec{r}) \psi_1^*(\vec{r}+\vec{\tau}) \rangle + O(\epsilon^3) \tag{93}$$

$$C_{\psi\psi}(\vec{\tau}) + \epsilon^2 \langle \psi_1(\vec{r}) \psi_1(\vec{r}+\vec{\tau}) \rangle + O(\epsilon^3) \tag{94}$$

The covariance of $\psi_1(\vec{r})$ shall be computed from Equation 47 and the approximations implicit in Equations 62 and 63. Thus, we represent $\psi_1(\vec{r})$ as in Equation 64:

$$\psi_1(\vec{r}) = \frac{-ik}{\pi} \int_0^z \frac{dz_0}{\gamma(z, z_0)} \int d\vec{\rho}_o\, (\vec{r}_o) \exp \left\{ \frac{(\vec{\rho}_o - \vec{\rho} Z_o/Z)^2}{\gamma(z, z_0)} \right\} \tag{95}$$

Hence

$$R_e\, C_{\psi\psi^*}(\vec{\tau}) = \frac{e^2k^2}{\pi^2}\, \mathrm{Re}\int_0^z\int_0^z \frac{dz_1\, dz_2}{\gamma(z,z_1)\,\gamma^*(z,z_2)} \int d\vec{\rho}_1 \int d\vec{\rho}_2 \,\langle \mu(\vec{r}_1)\,\mu(\vec{r}_2)\rangle$$

$$\exp \frac{(\vec{\rho}_1 - \vec{\rho}\, Z_1/Z)^2}{\gamma(z.z_1)} + \frac{(\vec{\rho}_2 - (\vec{\rho}+\vec{\tau})\, Z_2{}^*/Z)^2}{\gamma^*(z,z_2)} + O(\epsilon^3) \qquad (96)$$

$$\mathrm{Re}\, C_{\psi\psi}(\vec{\tau}) = -\frac{e^2k^2\mathrm{Re}}{\pi^2}\int_0^z\int_0^z \frac{dz_1\, dz_2}{\gamma(z,z_1)\,\gamma^*(z,z_2)} \int d\vec{\rho}_1 \int d\vec{\rho}_2 \,\langle \mu(\vec{r}_1)\,\mu(\vec{r}_2)\rangle$$

$$\exp \left\{ \frac{(\vec{\rho}_1 - \vec{\rho}\, Z_1/Z)^2}{\gamma(z,z_1)} + \frac{(\vec{\rho}_2 - (\vec{\rho}+\vec{\tau})\, Z_2/Z)^2}{\gamma(z,z_2)} \right\} + O(\epsilon^3) \qquad (97)$$

Upon inserting Equation 26 into Equations 96 and 97 and introducing the change of variable of Equation 68, we find

$$\mathrm{Re}\, C_{\psi\psi^*}(\vec{\tau}) = \frac{k^2}{\pi^2}\, \mathrm{Re}\int_0^z\int_0^z \frac{dz_1\, dz_2\, h\left[1/2\,(z_1+z_2)\right]}{\gamma(z,z_1)\,\gamma^*(z,z_2)} \int d\vec{\zeta}\; C\eta \left[\left\{(z_1-z_2)^2 + \zeta^2\right\}^{1/2}\right]$$

$$\int d\vec{\eta}\, \exp \frac{(\vec{\eta} + 1/2\,\vec{\zeta} - \vec{\rho}\, Z_1/Z)^2}{\gamma(z,z_1)} + \frac{(\vec{\eta} - 1/2\,\vec{\zeta} - (\vec{\rho}+\vec{\tau})\, Z_2{}^*/Z)^2}{\gamma^*(z,z_2)} + O(\epsilon^3) \qquad (98)$$

$$\mathrm{Re}\, C_{\psi\psi}(\vec{\tau}) = -\frac{k^2}{\pi^2}\int_0^z\int_0^z \frac{dz_1\, dz_2\, h\left[1/2\,(z_1+z_2)\right]}{\gamma(z,z_1)\,\gamma(z,z_2)} \int d\vec{\zeta}\; Cn \left[\left\{(z_1-z_2)^2 + \zeta^2\right\}^{1/2}\right]$$

$$\int d\vec{\eta}\, \exp \left\{ \frac{(\vec{\eta} + 1/2\,\vec{\zeta} - \vec{\rho}\, Z_1/Z)^2}{\gamma(z,z_1)} + \frac{(\vec{\eta} - 1/2\,\vec{\zeta} - (\vec{\rho}+\vec{\tau})\, z_2/z)^2}{\gamma(z,z_2)} \right\} + O(\epsilon^3) \qquad (99)$$

Using the result given by Equation 71 permits the $d\vec{\eta}$ integration to be performed explicitly:

$$\operatorname{Re} C_{\psi\psi}{}^{*}(\vec{\tau}) = -\frac{k^2}{\pi} \operatorname{Re} \int_0^{z_1}\int_0^{z_2} \frac{dz_1\, dz_2\, h\left[1/2\,(z_1+z_2)\right]}{\gamma(z, z_1) + \gamma^{*}(z, z_2)} \int d\vec{\zeta}\, C_n\left[\left\{(z_1 - z_2)^2 + \zeta^2\right\}^{1/2}\right]$$

$$X \exp\left\{\frac{(\vec{\zeta} - \vec{\rho}\, Z_1/Z + (\vec{\rho}+\vec{\tau})\, Z_2{}^{*}/Z^{*})^2}{\gamma(z, z_1) + \gamma^{*}(z, z_2)}\right\} + O(\epsilon^3) \tag{100}$$

$$\operatorname{Re} C_{\psi\psi}(\vec{\tau}) = +\frac{k^2}{\pi} \operatorname{Re} \int_0^{z}\int_0^{z} \frac{dz_1\, dz_2\, h\left[1/2\,(z_1+z_2)\right]}{\gamma(z, z_1) + \gamma(z, z_2)} \int d\vec{\zeta}\, C_n\left[\left\{z_1 - z_2{}^2 + \zeta^2\right\}^{1/2}\right]$$

$$X \exp\left\{\frac{(\vec{\zeta} - \vec{\rho}\, Z_1/Z + (\vec{\rho}+\vec{\tau})\, Z_2/Z)^2}{\gamma(z, z_1) + \gamma(z, z_2)}\right\} + O(\epsilon^3) \tag{101}$$

Assuming that the observation point $\vec{r} = z$ is in the far field, i.e., $z \gg |k\alpha^2|$, then examination of the terms in the exponentials of Equations 100 and 101 shows that the dependence on $\vec{\rho}$ is quite weak when $\vec{\rho} \gg z$. We shall therefore set $\vec{\rho} = o$. Introducing the change of variables of Equation 77 and extending the limits of the resulting dt integration between $-\infty$ and $+\infty$ yields

$$\operatorname{Re} C_{\psi\psi}{}^{*}(\vec{\tau}) = -\frac{k^2}{\pi} \operatorname{Re} \int_0^{z} h(s)\, ds \int_{-\infty}^{\infty} dt \int d\vec{\zeta}\, \frac{C_r\left[(t^2 + \zeta^2)^{1/2}\right]}{\gamma(z, s + 1/2\, t) + \gamma^{*}(z, s - 1/2\, t)}$$

$$\exp\left\{\frac{(\vec{\zeta} + \vec{\tau}\,(s - 1/2\, t + ik\alpha^{*})/Z^{*})^2}{\gamma(z, s + 1/2 t) + \gamma^{*}(z, s - 1/2 t)} + \right\} + O(\epsilon^3) \tag{102}$$

$$\text{Re } C_{\psi\psi}(\tau) = \frac{k^2}{\pi} \text{ Re} \int_0^z h(s)\, ds \int_{-\infty}^{\infty} dt \int \overrightarrow{d\zeta} \frac{C_n\left[(t^2+\zeta^2)^{1/2}\right]}{\gamma(\zeta_1\ s + 1/2\ t) + \gamma(\zeta_1\ s - 1/2\ t)}$$

$$\exp\left\{\frac{(\vec{\zeta} + \vec{\tau}\,(s - 1/2t - ik\alpha^2)/Z)^2}{\gamma(z,\ s+1/2\ t) + \gamma(z,\ s - 1/2\ t)}\right\} + O(\epsilon^3) \qquad (103)$$

Now

$$\gamma(z,\ s+1/2\ t) + \gamma^*(z,\ s-1/2\ t) =$$

$$\frac{2\left[(z-s)^2 + 1/4\ t^2\right] \text{ Re}\,(k\alpha^2) + it\left[z^2 - k^2|\alpha|^4 - 2\,s\,(z + \text{Im}\,(k\alpha^2))\right]}{-\,1/2\ k\ zz^*} \qquad (104)$$

and

$$\gamma(z,\ s+1/2\ t) + \gamma(z,\ s-1/2\ t) = 2\ \gamma(z,\ s) - \frac{t^2}{ikZ} \qquad (105)$$

Since $C_n\left[(t^2+\zeta^2)^{1/2}\right]$ falls to zero rapidly for values of t beyond a correlation distance a_c, the dt integrations in Equations 102 and 103 are significant in the rather limited range $0 < t < a_c$. Hence, the coefficients of Equations 104 and 105 have a very weak dependence upon t. Hence, there is negligible error introduced by setting t = o in Equations 104, 105 and in the exponential terms of Equations 102 and 103. If we do this, and introduce the functions

$$f_c(\zeta) = \int_0^{\infty} C_n\left[(t^2+\zeta^2)^{1/2}\right] dt \qquad (106)$$

then

$$\text{Re } C_{\psi\psi^*}(\tau) = -\frac{k^2}{\pi} \text{ Re} \int_0^{\infty} \frac{h(s)\, ds}{\text{Re } \gamma(z,\ s)} \int \overrightarrow{d\zeta}\ f_c(\zeta)$$

$$\exp\left\{\frac{(\vec{\zeta} + \vec{\tau}\,(s - ik\alpha^2)^*/Z^*)^2}{2 \text{ Re } \gamma(z,\ s)}\right\} + O(\epsilon^3) \qquad (107)$$

$$\mathrm{Re}\, C_{\psi\psi}(\vec{\tau}) = \frac{k^2}{\pi}\, \mathrm{Re} \int_0^z \frac{h(s)\, ds}{\gamma(z, s)} \int d\vec{\zeta}\, f_c(\zeta) \exp\left\{\frac{(\vec{\zeta} + \vec{\tau}(s - ikx^2)/Z)^2}{2\,\gamma(z, s)}\right\} + O(e^3) \tag{108}$$

Therefore, from Equations 89 and 90

$$C_{LL}(\vec{\tau}) = \frac{k^2}{2\pi}\, \mathrm{Re} \int_0^z h(z_o)\, dz_o \int d\vec{\zeta}\, f_c(\zeta)$$

$$\left[\frac{1}{\gamma(z, z_o)} \exp\left\{\frac{(\vec{\zeta} + \vec{\tau}\, Z_o/Z)^2}{2\,\gamma(z, z_o)}\right\} - \frac{1}{\mathrm{Re}\,\gamma(z, z_o)} \exp\left\{\frac{(\vec{\zeta} + \vec{\tau}\, Z_o^*/Z^*)^2}{2\,\mathrm{Re}\,\gamma(z, z_o)}\right\} + O(\epsilon^3)\right] \tag{109}$$

$$C\phi\phi(\vec{\tau}) = -\frac{k^2}{2\pi}\, \mathrm{Re} \int_0^z h(z_o)\, dz_o \int d\vec{\zeta}\, f_c(\zeta)$$

$$\left[\frac{1}{\gamma(z, z_o)} \exp\left\{\frac{(\vec{\zeta} + \vec{\tau}\, Z_o/Z)^2}{2\,\gamma(z, z_o)}\right\} + \frac{1}{\mathrm{Re}\,\gamma(z, z_o)} \exp\left\{\frac{(\vec{\zeta} + \vec{\tau}\, Z_o^*/Z)^2}{2\,\mathrm{Re}\,\gamma(z, z_o)}\right\} + O(\epsilon^3)\right] \tag{110}$$

To obtain the corresponding structure functions, we introduce the function

$$f_D(\tau) = \int_0^\infty \left\{Dn\left[(t^2 + \zeta^2)^{1/2}\right] - D_n(t)\right\} dt = 2\left\{f_c(0) - f_c(\zeta)\right\} \tag{111}$$

whence from Equations 20, 109, and 110

$$D_{LL}(\vec{\tau}) = \frac{k^2}{\pi} \operatorname{Re} \int_0^z h(z_o)\, dz_o \int d\vec{\zeta}\, f_d(\zeta)$$

$$\left\{ \frac{1}{\gamma(z, z_o)} \left[\exp\left\{ \frac{\zeta^2}{2\gamma(z, z_o)} \right\} - \exp\left\{ \frac{(\vec{\zeta} + \vec{\tau} Z_o/Z)^2}{2\gamma(z, z_o)} \right\} \right] \right.$$

$$\left. - \frac{1}{\operatorname{Re} \gamma(z, z_o)} \left[\exp\left\{ \frac{\zeta^2}{2 \operatorname{Re} \gamma(z, z_o)} \right\} - \exp\left\{ \frac{(\vec{\zeta} + \vec{\tau} Z_o^*/Z^2)^2}{2 \operatorname{Re} \gamma(z, z_o)} \right\} \right] \right\} + O(\epsilon^3) \tag{112}$$

$$D_{\phi\phi}(\vec{\tau}) = -\frac{k^2}{\pi} \operatorname{Re} \int_0^z h(z_o)\, d z_o \int d\vec{\zeta}\, f_D(\vec{\zeta})$$

$$\left\{ \frac{1}{\gamma(z, z_o)} \left[\exp\left\{ \frac{\zeta^2}{2\gamma(z, z_o)} \right\} - \exp\left\{ \frac{(\vec{\zeta} + \vec{\tau} Z_o/Z)^2}{2\gamma(z, z_o)} \right\} \right] \right.$$

$$\left. + \frac{1}{\operatorname{Re} \gamma(z, z_o)} \left[\exp\left\{ \frac{\zeta^2}{2 \operatorname{Re} \gamma(z, z_o)} \right\} - \exp\left\{ \frac{(\vec{\zeta} + \vec{\tau} Z_o^*/Z^*)^2}{2 \operatorname{Re} \gamma(z, z_o)} \right\} \right] \right\} + O(\epsilon^3) \tag{113}$$

SPECTRAL REPRESENTATIONS

The means and covariances computed above may be cast into an alternative form of representation, which depends upon the spectral function Φ defined by a three dimensional Fourier integral of C_n:

$$\Phi(\sigma) \equiv \int d\vec{r}\, \exp(i \vec{\sigma} \cdot \vec{r})\, C_n(r) \tag{114}$$

rather than on the "f" functions of Equations 83, 106, and 111. By Fourier's integral identity, C_n may be expressed as

$$C_n(r) = \frac{1}{(2\pi)^3} \int d\vec{\sigma}\, \exp(-i \vec{\sigma} \cdot \vec{r})\, \Phi(\sigma) \tag{115}$$

Since

$$\nabla^2 \ C_n(r) = \frac{-1}{(2\pi)^3} \int d\vec{\sigma} \ \exp(-i\vec{\sigma}\cdot\vec{r}) \ \sigma^2 \ \Phi(\sigma) \tag{116}$$

then

$$f_o(\zeta) = 1/2 \int_{-\infty}^{\infty} \frac{dz}{(2\pi)^3} \int d\vec{\sigma} \ \exp(-i\vec{\sigma}\cdot\vec{r}) \ \sigma^2 \ \Phi(\sigma)$$

$$= \frac{1}{8\pi^2} \int d\vec{\sigma} \ \exp(-i\vec{\sigma}\cdot\vec{\zeta'}) \ \sigma^2 \ \Phi(\sigma) \tag{117}$$

Likewise

$$f_c(\zeta) = \frac{1}{8\pi^2} \int d\vec{\sigma} \ \exp(-2\vec{\sigma}\cdot\vec{\zeta}) \ \Phi(\sigma) \tag{118}$$

Substituting Equation 117 into Equation 86, interchanging the order of the integrations and performing the resulting $d\zeta$ integration gives

$$\langle\psi(z)\rangle = \frac{ik}{2} \ C_n(O) \int_0^z h(z_o)\,dz_o + \frac{ik}{8\pi} \int_0^z dz_o \int_0^{z_o} h(s)$$

$$F\left[-1/2\,\gamma\,(z_o,\ s)\right] \ ds \ + O(\epsilon^3) \tag{119}$$

where we have introduced F, the Laplace transform of $\sigma\Phi(\sigma^{1/2})$:

$$F(\rho) = \mathcal{L}\left\{\sigma\Phi(\sigma^{1/2})\right\} \equiv \int_0^{\infty} \sigma\Phi(\sigma^{1/2}) \exp(-\rho\sigma)d\sigma \tag{120}$$

Likewise, we substitute Equation 118 into Equations 109 and 110 and perform the $d\zeta$ integrations and upon rearranging, find

$$C_{LL}\ (\vec{\tau}) = -\frac{k^2}{8\pi} \mathrm{Re} \int_0^z h(z_o)dz_o \int_0^\infty \Phi(\sigma^{1/2})\ J_o\ (\sigma^{1/2} \tau Z_o/Z) \exp\left[1/4\sigma\Upsilon(z, z_o)\right]$$

$$X \left\{\exp \left[1/4\ \sigma\ \Upsilon\ (z, z_o)\right] - \exp \left[1/4\ \sigma\Upsilon^* (z, z_o)\right]\right\} d\sigma + O\ (\epsilon^3) \quad (121)$$

$$C_{\Phi\Phi}\ (\vec{\tau}) = + \frac{k^2}{8\pi} \mathrm{Re} \int_0^z h(z_o)\ dz_o \int_0^\infty \Phi\ (\sigma^{1/2})\ J_o\ (\sigma^{1/2} \tau Z_o/Z)$$

$$\exp\left[1/4\ \sigma\Upsilon (z,z_o)\right]\left\{\exp\left[1/4\sigma\Upsilon (z, z_o)\right] + \exp\left[1/4\ \sigma\Upsilon^*(z, z_o)\right]\right\} d\sigma + O(\epsilon^3)$$

(122)

APPENDIX C

THEORETICAL EVALUATION OF THE RMS FLUCTUATION OF AN OPTICAL HETERODYNE SIGNAL WITH RANDOM WAVEFRONT DISTORTION

This appendix is Technical Memorandum 118-A, prepared by M. F. Sternberg and D. L. Fried of the NAA/S&ID Electro-Optical Laboratory in March 1965. This work is a continuation of Technical Memorandum 118, and was performed partially under NASw-977 and partially under NAA/S&ID research and development funds.

RMS FLUCTUATION OF AN OPTICAL HETERODYNE SIGNAL WITH RANDOM WAVEFRONT DISTORTION

This document is a continuation of Technical Memorandum (TM) 118, Optical Heterodyne Detection of an Atmospherically Distorted Signal Wave Front. The notation used are the same as the notations in TM 118. Equations referred to by number may refer to equations in TM 118.

This document will be concerned with the mean square heterodyne signal strength, $\langle S'^2 \rangle$.

The quantity computed can be used to evaluate the RMS modulation noise in a heterodyne receiver. Unlike the case treated in TM 118, an exact treatment is not possible and an approximation is necessary to reduce the problem to reasonable size. While the approximation appears to be fairly accurate, only heuristic arguments will be provided to justify it.

MEAN SQUARE SIGNAL

Starting with Equation 5.5 for the instantaneous heterodyne signal S',

$$S' = 2\,(\eta A_o A_s G)^2 \iint d\vec{x}\; d\vec{x}'\; W(x, D)\, W(x', D) \times \cos\left(\phi_s(\vec{x}) - \phi_s(\vec{x}')\right) \tag{5.5}$$

where we have set the load resistance R equal to unity for convenience, and writing the squared signal S'^2 which contains the product of two double integrals as a four-fold integral, we get

$$S'^2 = 4\,(\eta A_o A_s G)^4 \iiiint d\vec{x}\; d\vec{x}'\; d\vec{y}\; d\vec{y}'\; W(x, D)\, W(x', D) \times W(y, D)\, W(y', D) \cos\left[\phi(\vec{x}) - \phi(\vec{x}')\right] \cos\left[\phi(\vec{y}) - \phi(\vec{y}')\right] \tag{9.1}$$

In Equation 9.1 and the following, we have dropped the subscript s in $\phi_s \longrightarrow \phi$, and are using y and y' for the magnitudes of $\vec{y}$ and $\vec{y}'$. W(x, D) has been used in place of W(x) to make the circle diameter, D, more explicit and permit us to use other circle diameters. Since

$$\cos a \cos b = 1/2 \left\{ \cos (a + b) + \cos (a - b) \right\} \tag{9.2}$$

we note that

$$S'^2 = 4 (\eta A_o A_s G)^4 \iiiint d\vec{x}\, d\vec{x}'\, d\vec{y}\, d\vec{y}'\, W(x,D)\, W(x',D)\, W(y,D)\, W(y',D)$$
$$\times 1/2 \left\{ \cos \left[\phi(\vec{x}) - \phi(\vec{x}') + \phi(\vec{y}) - \phi(\vec{y}') \right] + \cos \left[\phi(\vec{x}) - \phi(\vec{x}') + \phi(\vec{y}') - \phi(\vec{y}) \right] \right\} \tag{9.3}$$

We see that if we rewrite the integral as the sum of two integrals, each containing one of the cosines, then interchange the symbols $\vec{y}$ and $\vec{y}'$ in one of the intregals, and recombine the two integrals, we get

$$S'^2 = 4 (\eta A_o A_s G)^4 \iiiint d\vec{x}\, d\vec{x}'\, d\vec{y}\, d\vec{y}'\, W(x,D)\, W(x',D)\, W(y,D)\, W(y',D)$$
$$\times \cos \left[\phi(\vec{x}) - \phi(\vec{x}') + \phi(\vec{y}) - \phi(\vec{y}') \right] \tag{9.4}$$

Taking the ensemble average of both sides, commuting integration and averaging so that on the right side the brackets enclose only the cosines, and then using the relationship discussed in Footnote 11 of TM 118, we get

$$\langle S'^2 \rangle = 4 (\eta A_o A_s G)^4 \iiiint d\vec{x}\, d\vec{x}'\, d\vec{y}\, d\vec{y}'\, W(x,D)\, W(x',D)\, W(y,D)$$
$$W(y',D) \times \exp \left\{ -1/2 \left\langle \left[\phi(\vec{x}) - \phi(\vec{x}') + \phi(\vec{y}) - \phi(\vec{y}') \right]^2 \right\rangle \right\} \tag{9.5}$$

Using the Equation

$$(a-b+c-d)^2 = (a-b)^2 + (a-d)^2 - (a-c)^2 + (b-c)^2 - (b-d)^2 + (c-d)^2 \tag{9.6}$$

to expand the square in Equation 9.5, expressing the average of the sum as the sum of the averages, and recognizing the phase structure function in such expressions as $\langle(\phi(\vec{x}) - \phi(\vec{y}'))^2\rangle$, we get

$$\langle S'^2\rangle = 4\ (\eta A_o A_s G)^4 \iiiint d\vec{x}\, d\vec{x}'\, d\vec{y}\, d\vec{y}'\ W(x, D)\ W(x', D)\ W(y, D)$$

$$W(y', D) \times \exp\left\{1/2\left[D(|\vec{x} - \vec{x}'|) + D(|\vec{x} - \vec{y}'|) - D(|\vec{x} - \vec{y}|) + D(|\vec{x}' - \vec{y}|) - D(|\vec{x}' - \vec{y}'|) + D(|\vec{y} - \vec{y}'|)\right]\right\} \quad (9.7)$$

We will now offer heuristic reasons for making the replacement

$$D(|\vec{x} - \vec{x}'|) + D(|\vec{x} - \vec{y}'|) - D(|\vec{x} - \vec{y}|) + D(|\vec{x}' - \vec{y}|) - D(|\vec{x}' - \vec{y}'|) + D(|\vec{y} - \vec{y}'|) \simeq D(|\vec{x} - \vec{x}' + \vec{y} - \vec{y}'|) \quad (9.8)$$

We base our argument on Equation 4.4 which says that

$$D(r) = 6.88\ (r/r_o)^{5/3}$$

i.e., that D(r) varies as the five-thirds power of r, and on the fact that there is only a 16-2/3 percent difference between five-thirds and six-thirds, so that we may approximate the relationship in Equation 9.6 with

$$(a-b+c-d)^{5/3} \simeq (a-b)^{5/3} + (a-d)^{5/3} - (a-c)^{5/3} + (b-c)^{5/3} - (b-d)^{5/3} + (c-d)^{5/3} \quad (9.9)$$

We further argue the approximate validity of Equation 9.9 by noting that for n = 1,

$$(a-b+c-d)^n = (a-b)^n + (a-d)^n - (a-c)^n + (b-c)^n - (b-d)^n + (c-d)^n \quad (9.10)$$

as well as for n = 2. Thus, if we study the error in Equation 9.10 for various values of n, we see that the error vanishes for n = 1 and n = 2, which leads one to believe that the relative error never gets very large for $1 < n < 2$. This, together with the fact that $(2-5/3)/2 = 0.166$, makes Equation 9.9 quite plausible. This plausibility, together with Equation 4.4, constitutes our heuristic argument for Equation 9.8.

Substituting Equation 9.8 into Equation 9.7, we get

$$\langle S'^2 \rangle = 4\,(\eta A_o A_s G)^4 \iiiint d\vec{x}\; d\vec{x}'\; d\vec{y}\; d\vec{y}'\; W(x, D)\, W(x', D)\, W(y, D)\, W(y', D) \times \exp\left\{ -1/2\, D(|\vec{x} - \vec{x}' + \vec{y} - \vec{y}'|)\right\} \tag{9.11}$$

We now make the substitutions

$$\vec{\rho} = \vec{x} - \vec{x}' \tag{9.12a}$$

$$\vec{\rho}' = 1/2\,(\vec{x} + \vec{x}') \tag{9.12b}$$

$$\vec{\sigma} = \vec{y}' - \vec{y} \tag{9.12c}$$

$$\vec{\sigma}' = 1/2\,(\vec{y}' + \vec{y}) \tag{9.12d}$$

in Equation 9.11 giving

$$\langle S'^2 \rangle = 4\,(\eta A_o A_s G)^4 \iiiint d\vec{\rho}\; d\vec{\rho}'\; d\vec{\sigma}\; d\vec{\sigma}'\; W(|\vec{\rho}' + 1/2\,\vec{\rho}|, D)\, W(|\vec{\rho}' - 1/2\,\vec{\rho}|, D) \times W(|\vec{\sigma}' + 1/2\,\vec{\sigma}|, D)\, W(|\vec{\sigma}' - 1/2\,\vec{\sigma}|, D)\, \exp\left\{-1/2\, D\,(|\vec{\rho} - \vec{\sigma}|)\right\} \tag{9.13}$$

The $\vec{\rho}'$ and $\vec{\sigma}'$ integrations are independent of the structure function and can be performed explicitly. The result for each integration is given in Equation 5.11. We may write

$$\langle S'^2 \rangle = 4(\eta A_o A_s G)^4 \iint d\vec{\rho}\, d\vec{\sigma}\, W(\rho, 2D)\, W(\sigma, 2D)\, K_o(\rho, D) \times K_o(\sigma, D)$$

$$\exp\left\{-1/2\, D(|\vec{\rho} - \vec{\sigma}|)\right\} \tag{9.14}$$

where $K_o(r, D)$ is defined in Equation 5.10.

$$K_o(r,D) = 1/2 \left\{ D^2 \cos^{-1}\left(\frac{r}{D}\right) - r\sqrt{D^2 - r^2} \right\} \tag{5.10}$$

Again we make a change of variables, this time

$$\vec{u} = \vec{\rho} - \vec{\sigma} \tag{9.15a}$$

$$\vec{v} = 1/2\,(\vec{\rho} + \vec{\sigma}) \tag{9.15b}$$

and denote the components of $\vec{v}$ parallel and perpendicular to $\vec{u}$ by p and q respectively. Substituting $\vec{u}$ and p, q into Equation 9.14, we note that the angular integration in $\vec{u}$ can be performed giving a factor of 2π. We get

$$\langle S'^2 \rangle = 8\pi(\eta A_o A_s G)^4 \int_o^{2D} u\, du\, K_1\left(\frac{u}{2}, D\right) \exp\left\{-1/2\, D(u)\right\} \tag{9.16}$$

where

$$K_1\left(\frac{u}{2} D\right) = \int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} dq\, W\left(\sqrt{(p + 1/2\,u)^2 + q^2},\, 2D\right) W\left(\sqrt{(p - 1/2u)^2 + q^2},\, 2D\right)$$

$$\times K_o\left(\sqrt{(p + 1/2\,u)^2 + q^2},\, D\right) K_o\left(\sqrt{(p - 1/2\,u)^2 + q^2},\, D\right) \tag{9.17}$$

The upper limit of 2D on the integration in u in Equation 9.16 is redundant in the sense that $K_1(u, D)$ obviously vanishes when $u > 2D$, as can be seen from the W functions in Equation 9.17. The region of integration in Equation 9.17, as determined by the two W - functions is the area of overlap of two circles of diameter 2D, each with its center on the p-axis, with the separation between centers equal to u, i.e., the centers are at (p, q) = (±1/2 u, 0). The value of the integral is four times the value of the integration in the first quadrant. The limits on the first quadrant are

$$0 \leq p \leq D - 1/2\, u \text{ and } 0 \leq q \leq \sqrt{D^2 - (p + 1/2\, u)^2}$$

Thus

$$K_1\,(u, D) = 4 \int_0^{D - \frac{u}{2}} dp \int_0^{\sqrt{D^2 - (p + \frac{u}{2})^2}} dq\; K_o \left(\sqrt{(p + \frac{u}{2})^2 + q^2}, D\right) K_o\left(\sqrt{(p - \frac{u}{2})^2 + q^2}, D\right) \tag{9.18}$$

Making the substitutions

$$\ell = u/2D \tag{9.19a}$$

$$m = p/D \tag{9.19b}$$

$$n = q/D \tag{9.19c}$$

and noting that

$$K_o\,(r, D) = D^2 K_o\,(\frac{r}{D}, 1) \tag{9.20}$$

so that

$$K_o\left(\sqrt{(p \pm \frac{u}{2})^2 + q^2}, D\right) = D^2 K_o\left(\sqrt{(m \pm \ell)^2 + n^2}, 1\right) \tag{9.21}$$

and that consequently

$$K_1\,(\frac{u}{2}, D) = 4\, D^6 \int_0^{1-\ell} dm \int_0^{\sqrt{1 - (m + \ell)^2}} dn\; K_o \sqrt{(m + \ell)^2 + n^2}, 1)$$

$$\times\, K_o\left(\sqrt{(m - \ell)^2 + n^2}, 1\right) = 4\, D^6\; K_1\;(\ell, 1) \tag{9.22}$$

We are led to the equation

$$\langle S'^2 \rangle = \frac{\pi}{2} \left(4\eta \; A_o A_s G \, D^2\right)^4 \int_o^1 \ell \, d\ell \, K_1 \left(\ell, 1\right) \exp \left\{ -1/2 \, D \left(2 \, D\ell\right) \right\} \tag{9.23}$$

Now introducing the phase structure function as defined by Equation 4.4, i.e.

$$D(r) = 6.88 \; (r/r_o)^{5/3}$$

we get

$$\langle S'^2 \rangle = \frac{\pi}{2} \left(4\eta A_o A_s G \; r_o^2\right)^4 \left(\frac{D}{r_o}\right)^8 \int_o^1 \ell \, d\ell \, K_1 \left(\ell, 1\right) \exp \left\{ -10.92 \left(\frac{D}{r_o}\right)^{5/3} \ell^{5/3} \right\} \tag{9.24}$$

Casting Equation 5.12 in similar form for comparison, we have

$$\langle S' \rangle = \pi \left(2\eta A_o A_s G \; r_o^2\right)^2 \left(\frac{D}{r_o}\right)^4 \int_o^1 \ell \, d\ell \; K_o \left(\ell, 1\right) \exp \left\{ -3.44 \left(\frac{D}{r_o}\right)^{5/3} \ell^{5/3} \right\} \tag{9.25}$$

The functions $K_o(\ell, 1)$ and especially $K_1(\ell, 1)$ will be tabulated later. For the present, we note simply that the mean square signal amplitude modulation noise due to wavefront distortion is δ^2.

$$\delta 2 = \frac{\langle S'^2 \rangle - \langle S' \rangle^2}{\langle S' \rangle^2} = \frac{8}{\pi} \frac{\displaystyle\int_o^1 \ell \, d\ell \, K_1 \left(\ell, 1\right) \exp \left\{ -10.92 \left(\frac{D}{r_o}\right)^{5/3} \ell^{5/3} \right\}}{\left[\displaystyle\int_o^1 \ell \, d\ell \; K_o \left(\ell, 1\right) \exp \left\{ -3.44 \left(\frac{D}{r_o}\right)^{5/3} \ell^{5/3} \right\}\right]^2} \tag{9.26}$$

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APPENDIX D

DATA ACQUISITION QUESTIONNAIRE

This questionnaire was prepared to be used as a tool during data acquisition and assimilation in connection with the LACE study contract.

Name of organization	Address	Phone No.

1.0 Are you engaged in any technical activity associated with laser communications and propagation? Yes ______ No _____

2.0 What percentage of this technical activity is theoretical? ________ experimental? ______ development? ______ other? ____________

*Explain ____

3.0 Which categories would best describe your main area of laser application?

components _______ acquisition and tracking ______ Other ____

subsystems _______ ranging _______

systems ________ communication _______

Explain ____

4.0 Describe your present laser activity in general terms.

5.0 List your principle investigators in laser technology (name, title, group, address, telephone no.).

6.0 Describe your past and present laser study and development programs.

6.1 List Government contracts and independent research.

6.2 What types of theoretical analyses have been performed?

6.3 What types of experimental programs (laboratory, field)?

6.4 Do you have an analytical or experimental program concerned with optical propagation in a turbulent atmosphere? Summarize approach, equipment used, significant results, problems etc.

6.5 List laser technology papers presented, publications, reports (give source and identification), and include copies or abstracts, if possible.

7.0 What do you consider are the most significant problems yet to be solved to advance the laser state of the art and to derive major exploitation of the use of lasers in space communication, acquisition and tracking, mapping, etc.

8.0	ANALYTICAL APPROACH, DETAILED (Use additional paper when necessary.)
8.1	Which optical characteristics (as affected by the atmosphere) do you study: wavelength, monochromaticity, spatial coherence, polarization, mutual coherence? Discuss.
8.2	Which beam properties do you consider: spread, intensity, irradiance, direction, phase? Discuss.
8.3	Is attention restricted to infinite plane waves, or have other wave-forms (especially finite collimated beams) been considered? Discuss.
8.4	What mathematical approach do you use in analyzing laser propagation in a turbulent atmosphere, and how do you justify any approximations made physically?
8.4.1	Have analyses been made in spatial or in spectral domains? Discuss.
8.4.2	What types of approximations were used (ray tracing, small- and smooth-perturbation theory, renormalization, etc.)? Discuss.

8.5	What standard reference works (if any) were used for a departure point?	
8.6	Which atmospheric effects do you look at: absorption, scattering, refraction, turbulence, depolarization (polarization fluctuations)? In what manner?	
8.7	In turbulence studies, which statistical properties of the atmosphere do you consider most important? For what reason?	
8.8	Which meteorological parameters are used to specify the optical state of the atmosphere?	
8.9	How are these meteorological parameters measured so as to specify experimental conditions?	
8.10	Do you have a theory relating atmospheric parameters and propagation effects? If so, explain.	
8.11	Is the turbulence model you use homogeneous or inhomogeneous? Discuss.	

8.12	What data do you use to predict attenuation in the atmosphere?
8.13	Do you study sky background: spectral radiance (as a function of wavelength), polarization, zenith angle, meteorological conditions, and the positions of the sun, moon, and prominent stars? If so, explain and discuss.
8.14	Are the light paths studied horizontal, vertical, or slant; and are the end points both terrestrial, one earth and one aircraft, both aircraft, or one earth and one space vehicle? Discuss.
8.15	Have you considered aberration and doppler shift in light beams transmitted between ground and satellite? If so, how?
8.16	What degree of consistency exists between your work and other theoretical analyses? Which results are beyond question, and which require further investigation due to disagreements among various theories? Discuss.
8.17	Has your experience with the analytical work caused you to re-assess the relative value or difficulty of your original laser program objectives? If so, how?

8.18	Describe your future plans for theoretical analysis in the area of atmospheric effects on optical propagation.
9.0	EXPERIMENTAL APPROACH
9.1	What propagation experiments have you performed? Give the objectives, significance, method of measurement, parameters measure, equipment used (functional block diagram, if possible), location, and results.
9.2	Has basic propagation analysis and/or predicted system performance been verified (in any degree) with available experimental data? If so, how?
9.3	In outdoor experiments, how have you allowed for different conditions of weather, terrain, and topology?
9.4	Have you considered satellite experiments? If so, how? What experiments and measurement techniques would you use?
9.5	What other platforms have you used or considered for experiments? Discuss.

10.0 SYSTEMS ANALYSIS

Discuss the components and systems that have been analyzed and used in your analytical and measurement studies. Specify the significant parameters of the transmitters (power, wavelength, beam width, temporal and spatial coherence, etc.), receivers (detector characteristics, collection systems, coherent versus incoherent, photographic systems, etc.), beam steering, modulation, data handling. Were system design and performance studies carried out independently or in conjunction with basic propagation studies? How were the components used and predicted performance ascertained?

11.0 RESOURCES

Discuss in detail the equipment and facilities you have or plan to have in the future that could be used in a laser propagation experimental program. Specify the types and performance of such equipment as tracking and receiving electro-optics, mounts, data links, flight platforms, etc. Indicate the location and availability of laboratory and field experimental ranges.