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SPHERICAL HARMONIC ANALYSES FOR THE SPHEROIDAL EARTH, II

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The RAND Corporation
SANTA MONICA • CALIFORNIA

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PREFACE

This study is a continuation of RAND's efforts at a better representation of the earth's magnetic field by use of geometry that more accurately represents the shape of the earth. Besides being of interest to those studying geomagnetism, the method is of general interest in potential analyses of any data taken over the surface of the earth. The work was performed as part of a continuing study of particles and fields under Contract NASr-21(05) with the National Aeronautics and Space Administration.

ABSTRACT

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A spheroid approximates the earth more accurately than does a sphere. Before analyzing the data over the surface of the earth by spherical harmonic series, we must correct them to the appropriate values on a true sphere. The change in the earth's radius and latitude, and in the direction of the vector components must be considered. The corrections have been applied here to values derived from existing analyses of the earth's magnetic field. The changes are significant, the largest being about 120γ in the g_3^0 term.

author

I. INTRODUCTION

For some time geophysicists have realized that the earth is better approximated by a spheroid than by a sphere. As Schmidt (1889) pointed out, errors arise when analyzing geophysical data with spherical harmonic series over the nonspherical surface of the earth. While such series do correctly represent, on the surface of the earth, that component of the field from which they were derived, they are not true potential functions. Hence, the other components of the field cannot be determined from the analysis, nor can the field be extrapolated inward and outward through source-free regions. Schmidt analyzed the earth's magnetic field in ellipsoidal harmonics, but since the deviations from the spherical harmonics, until recently, were small compared to the errors in the data, these refinements could be ignored.

Now that the field is being determined somewhat more accurately, however, the effect of the spheroidal shape of the earth becomes more significant in analyses of the data. One or two current spherical harmonic analyses of the field do, in fact, correct for these deviations, referring data to a true sphere rather than to the earth's spheroidal surface. In order to have a consistent set of coefficients to measure secular changes in the field, it is desirable to update the older analyses to include these refinements where possible. The present paper forms the second of a series of two papers concerned with development of a methodology to this end. In the first paper (Kahle, et al., 1964, and references there cited), we indicated small corrections required in various published spherical harmonic analyses of the geomagnetic field in order that they be referred to a true sphere rather than the earth's surface. Only the difference in radius was considered, however. In this present paper, we continue these improvements by including the smaller but still significant corrections for latitude and direction.

Although we can of course extend our methods to an even more complex geoid, we will not attempt this here. Other methodologies are also available. For instance, the data could be fitted using ellipsoidal harmonics, but these are geometrically less suited to convenient physical interpretation. Cain, et al. (1964) have also shown the direct method of measuring the height to a satellite-borne magnetometer from a sphere instead of from a spheroid.

II. METHOD

The usual equations for analyzing the earth's magnetic field of internal origin in a spherical harmonic series are

$$V = a \sum_n \left(\frac{a}{r}\right)^{n+1} \sum_m (g_n^m \cos m\lambda + h_n^m \sin m\lambda)\, P_n^m (\cos\theta) \quad ,$$

$$X = \frac{1}{r}\frac{\partial V}{\partial \theta} = \sum_n \left(\frac{a}{r}\right)^{n+2} \sum_m (g_n^m \cos m\lambda + h_n^m \sin m\lambda)\, \frac{dP_n^m}{d\theta} (\cos\theta) \quad ,$$

$$Y = -\frac{1}{r \sin\theta}\frac{\partial V}{\partial \lambda}$$

$$= -\frac{1}{\sin\theta} \sum_n \left(\frac{a}{r}\right)^{n+2} \sum_m m\,(-g_n^m \sin m\lambda + h_n^m \cos m\lambda)\, P_n^m (\cos\theta) \quad ,$$

and

$$Z = \frac{\partial V}{\partial r} = -\sum_n (n+1) \left(\frac{a}{r}\right)^{n+2} \sum_m (g_n^m \cos m\lambda + h_n^m \sin m\lambda)\, P_n^m (\cos\theta) \quad .$$

It is usually assumed that on the surface of the earth $r = a =$ constant. Fields of external origin are ignored, since these are insufficiently defined by the survey data.

If the earth were a sphere, the coefficients (the g's and h's) could be determined from measurements of any one of the three field components X, Y, or Z. In practice, each of these is often analyzed separately; then the resulting coefficients may be averaged. Due to the geometry of the earth, however, the X, Y, and Z components as measured are not the derivatives of a potential in spherical coordinates as assumed. Hence, the three analyses should, and do, result in three <u>different</u> sets of coefficients. Much of the difference, it is true, comes from errors in the data, but part is intrinsic in the incorrect geometry. In order to correct these existing analyses, the coefficients from the X, Y, and Z analyses must be available separately.

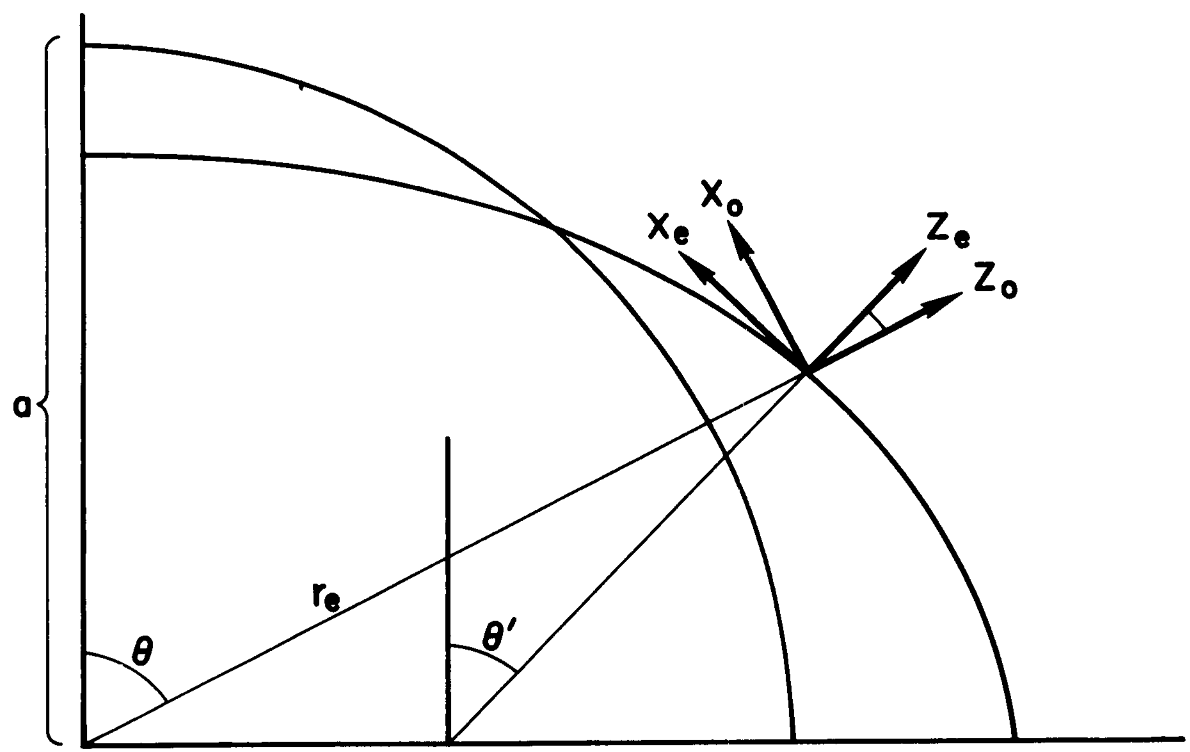
Figure 1 illustrates the differences between a spheroid representing the earth and a sphere. The most significant is the difference in radius. The radial distance to the earth's surface is

$$r_e = \frac{a_o}{(1 + \epsilon^2 \cos^2 \theta)^{\frac{1}{2}}} \quad ,$$

where according to Bomford (1962), Fischer (1960) finds a_o, the equatorial radius, is 6378.155 km, and ϵ^2 is an ellipticity parameter 0.00673863. The polar radius is 6356.773. Also important, however, is the difference between the measured (geodetic) colatitude θ' and the spherical coordinate θ (geocentric colatitude):

$$\cos \theta' = \frac{\cos \theta \, (1 + \epsilon^2)}{[1 + \epsilon^2 (z + \epsilon^2) \cos^2 \theta]^{\frac{1}{2}}} \quad .$$

VECTOR COMPONENTS OF THE EARTH'S FIELD



This amounts to about 0.2 degrees in middle latitudes. In this connection, we can see that the X_e (north) and Z_e (vertical) components of the field are not really measured in the direction of the spherical coordinates θ and r.

What is required is to know the components of the field (X_o and Z_o) in true spherical coordinates on a sphere of radius a, where they can then be correctly analyzed with spherical harmonics. For convenient comparison with previous results, we take the radius of this reference sphere to be the mean radius of the earth, a = 6371.2.

Knowing the measured X_e, Y_e, and Z_e on the surface of the earth, or the g's and h's derived from each component separately, the field components in the directions normal (Z_o) and tangent (X_o, Y_o) to the sphere (Fig. 1) are found from

$$X_o = X_e \cos\alpha - Z_e \sin\alpha \quad ,$$

$$Y_o = Y_e \quad ,$$

$$Z_o = X_e \sin\alpha + Z_e \cos\alpha \quad ,$$

and

$$\cos\alpha = \frac{(1 + \varepsilon^2 \cos^2\theta)}{[1 + \varepsilon^2(2 + \varepsilon^2)\cos^2\theta]^{\frac{1}{2}}} \quad .$$

These values still represent the field on the spheroid. They must be extrapolated inward or outward to the reference sphere. This is accomplished by multiplying the coefficients of the original (unrevised) analysis by the factors $(r_e/a)^{n+1}$ prior to the rotation. Since the original analyses do not represent true potential functions, this provides only an approximate extrapolation, but the error is of second order in ε^2 and hence negligible. In practice, both corrections can be accomplished with one computer operation. These new field values, in the proper directions and on the sphere, can then be analyzed correctly in terms of spherical harmonics.

III. APPLICATION

We have applied this procedure to three sets of geomagnetic data where coefficients for the X, Y, and Z components were available separately: the U.S. charts, epoch 1945 (Vestine et al., 1947); the U.S. charts, epoch 1955; and USSR charts, epoch 1955 (Vestine et al., 1963). The original and revised coefficients for the U.S. 1955 data, and the difference between the two sets of coefficients are given in Table 1. As is the usual practice, the coefficients obtained from the X and Y components of the field were averaged after the analysis. It should be noted that the unrevised coefficients differ slightly from the published coefficients cited above because of a second standard analysis having been made when better data and an improved computer program were available.

It can be seen that the revised g_1^0 term and g_3^0 term differ significantly from the original, by about 90γ and 120γ respectively, while the rest show smaller changes, 20γ or less. The geometry involved causes the error in the nth coefficient of the incorrect type analysis to show up as small values in the coefficients $n \pm 2$. That is, a pure centered-dipole field would have a large g_1^0 term and a small g_3^0 term in the incorrect analysis. Thus the earth's field, with its large dipole component, is most affected in these two terms. It is of interest to note that roughly 2/3 of the correction is due to the change in radius while 1/3 comes from the latitude and direction changes.

These new coefficients do represent a potential function, so the field in the source-free space near the earth's surface can be deter-

Table 1

CORRECTIONS APPLIED TO VARIOUS

ANALYSES, IN GAMMAS

		Vestine, *et al.*, U.S. 1955					
		Original		Revised		Difference	
n	m	g_n^m	h_n^m	g_n^m	h_n^m	$\|\Delta g_n^m\|$	$\|\Delta h_n^m\|$
1	0	-30521		-30429		92	
	1	- 2198	5827	- 2210	5832	12	5
2	0	- 1471		- 1463		8	
	1	3053	-1855	3059	-1855	6	0
	2	1368	478	1374	477	6	1
3	0	1224		1343		119	
	1	- 1726	- 607	- 1716	- 627	10	20
	2	1300	287	1300	287	0	0
	3	916	- 29	916	- 29	0	0
4	0	862		869		7	
	1	640	174	622	185	18	11
	2	437	- 243	431	- 244	6	1
	3	- 443	- 80	- 444	- 80	1	0
	4	354	- 160	354	- 160	0	0
5	0	- 152		- 162		10	
	1	383	82	397	87	14	5
	2	223	99	214	97	9	2
	3	- 16	- 7	- 21	- 7	5	0
	4	- 176	- 171	- 176	- 171	0	0
	5	- 55	154	- 55	153	0	1
6	0	6		- 4		10	
	1	121	- 60	114	- 63	7	3
	2	- 43	150	- 47	152	4	2
	3	- 271	- 13	- 268	- 12	3	1
	4	- 32	- 19	- 34	- 18	2	1
	5	23	- 46	24	- 46	1	0
	6	- 151	- 46	- 151	- 46	0	0

Table 1 (Continued)

		Vestine, et al., 1945				Vestine, et al., U.S.S.R. 1955			
		Original		Revised		Original		Revised	
N	m	g_n^m	h_n^m	g_n^m	h_n^m	g_n^m	h_n^m	g_n^m	h_n^m
1	0	-30567		-30475		-30497		-30406	
	1	- 2136	5836	- 2148	5841	- 2128	5904	- 2141	5909
2	0	- 1265		- 1257		- 1417		- 1408	
	1	2971	-1669	2977	-1670	2981	-1883	2987	-1884
	2	1561	529	1566	527	1587	376	1591	374
3	0	1154		1274		1163		1282	
	1	- 1738	- 511	- 1729	- 531	- 1784	- 539	- 1776	- 558
	2	1226	188	1226	189	1247	244	1248	245
	3	903	89	904	88	816	4	817	3
4	0	924		931		960		968	
	1	786	134	768	144	782	98	765	109
	2	545	- 283	538	- 285	527	- 314	520	- 315
	3	- 392	- 80	- 393	- 80	- 366	- 50	- 368	- 50
	4	357	- 122	357	- 122	370	- 101	370	- 101
5	0	- 223		- 232		- 263		- 272	
	1	301	25	316	30	303	113	317	118
	2	195	89	186	87	171	112	162	110
	3	- 48	9	- 52	9	- 87	1	- 91	1
	4	- 162	- 131	- 162	- 131	- 164	- 142	- 164	- 142
	5	- 88	110	- 88	111	- 78	109	- 78	109
6	0	53		42		56		44	
	1	109	- 61	100	- 64	90	- 91	81	- 92
	2	- 26	133	- 32	136	3	136	- 3	139
	3	- 269	- 18	- 266	- 17	- 274	- 7	- 271	- 6
	4	0	8	- 2	8	- 6	18	- 8	19
	5	30	- 12	30	- 12	15	- 13	15	- 13
	6	- 114	30	- 114	30	- 109	- 26	- 109	- 26

mined. Calculations of the field at a height of 500 km, using these coefficients differ from the unrevised field by over 200γ, this difference is of the same order as that estimated previously by Cain, et al. (1964). Conjugate points can differ by more than 30 km. Thus, the differences are seen to be significant.

These corrections are now being estimated for some of the earlier analyses of the earth's magnetic field, in the interests of uniformity and accuracy of representations of the field as a function of time.

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