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The Kinetic Energy of a Body of Revolution Moving in an Infinite Fluid

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Summary

A BODY of revolution is assumed to move rectilinearly in a perfect fluid, and the motion imparted to the fluid separated into its axial and transverse components. The velocity potentials for these two types of motion are well known and may be expressed in series of ellipsoidal harmonics. Analogous to a general theorem given by G. I. Taylor, it is then shown in a simple manner that the kinetic energy of the fluid external to the moving body depends only on the coefficients of the first terms of the aforementioned series and on the volume of the moving body.

Introduction

A general result obtained by G. I. Taylor[1] showed that if the velocity potential of the fluid external to a body moving without rotation be represented by a series of spherical harmonics, then the kinetic energy of the fluid depends only on the harmonic terms of the first degree in the expansion for the velocity potential and on the volume of the moving body. In the present note this interesting result is reexamined in connection with the rectilinear motion of bodies of revolution.

It is well known that, when a body of revolution moves without rotation, the motion imparted to the fluid may be regarded as made up of two component motions—an axial flow and a transverse flow. The respective velocity potentials are given by the following expressions:

$$\left.\begin{aligned}\Phi_a &= \sum_{n=1}^{\infty} A_n P_n(\mu) Q_n(\lambda)\\ \Phi_t &= \sum_{n=1}^{\infty} B_n P_n^1(\mu) Q_n^1(\lambda)\end{aligned}\right\} \tag{1}$$

where $P_n(\mu)$, $Q_n(\lambda)$ are respectively Legendre functions of degree n of the first and second kinds; and $P_n^1(\mu)$, $Q_n^1(\lambda)$ are respectively associated Legendre functions of degree n and order 1 of the first and second kinds.[2]

The kinetic energy T of the fluid may be obtained from the following surface integral:

$$\frac{2T}{\rho} = -\int_i\int \Phi \frac{\partial\Phi}{\partial n} d\sigma \tag{2}$$

where the suffix i indicates that the integration extends over the surface of the moving body and $\partial\Phi/\partial n$ denotes the component of the vector $\overline{\text{grad}}\ \Phi$ in the direction of the outward drawn normal to the surface of the moving body.

Furthermore if U, V are respectively the axial and transverse components of the velocity of the body then the kinetic energy of the fluid may be written as:

$$\frac{2T}{\rho} = AU^2 + BV^2 \tag{3}$$

the term involving UV vanishing because of the longi-

[1] G. I. Taylor, *The Energy of a Body Moving in an Infinite Fluid with an Application to Airships*, Proc. Roy. Soc., Series A. 120, August 1928, pp. 13-21.

[2] H. Lamb, *Hydrodynamics*, Fifth Edition, Cambridge University Press, 1924, pp. 132, 133.

tudinal symmetry of a body of revolution. In what follows it will be shown that A, B depend respectively only on the coefficients A_1, B_1 of the velocity potentials Φ_a, Φ_t and on the volume v of the body. The proof of this result involves the use of Green's second theorem, which states that

$$\int_i\int\left(\psi_1\frac{\partial\psi_2}{\partial n}-\psi_2\frac{\partial\psi_1}{\partial n}\right)d\sigma=\int_o\int\left(\psi_1\frac{\partial\psi_2}{\partial n}-\psi_2\frac{\partial\psi_1}{\partial n}\right)d\sigma \quad (4)$$

where the suffixes i, o denote respectively integration over the inner and outer bounding surfaces and where ψ_2, ψ_1 are two functions satisfying Laplace's equation in the region of integration.

Replacing Φ by $\Phi_a + \Phi_t$, Eq. (2) takes the form

$$\frac{2T}{\rho}=-\int_i\int\Phi_a\frac{\partial\Phi_a}{\partial n}d\sigma-\int_i\int\Phi_t\frac{\partial\Phi_t}{\partial n}d\sigma-2\int_i\int\Phi_t\frac{\partial\Phi_a}{\partial n}d\sigma \quad (5)$$

where the last term on the right-hand side is obtained by virtue of the reciprocal relation:

$$\int_i\int\Phi_a\frac{\partial\Phi_t}{\partial n}d\sigma=\int_i\int\Phi_t\frac{\partial\Phi_a}{\partial n}d\sigma$$

CALCULATION OF $\int_i\int\Phi_a\,\partial\Phi_a/\partial n\,d\sigma$

The boundary condition for axial flow is given by

$$-\frac{\partial\Phi_a}{\partial n}=l\,U$$

where l is the direction cosine with regard to the X axis of the normal drawn from the surface of the moving body into the fluid.

It follows then that

$$-\int_i\int\Phi_a\frac{\partial\Phi_a}{\partial n}d\sigma=U\int_i\int l\Phi_a\,d\sigma$$

Suppose now the moving body to be enclosed by an outer surface, and apply Green's theorem by choosing $\psi_1 = Ux$ and $\psi_2 = \Phi_a$. Eq. (4) then becomes:

$$\int_i\int\left(-lU^2x-U\Phi_a\frac{\partial x}{\partial n}\right)d\sigma=\int_o\int\left(Ux\frac{\partial\Phi_a}{\partial n}-U\Phi_a\frac{\partial x}{\partial n}\right)d\sigma$$

Replacing $\int_i\int xl\,d\sigma$ by v (the volume of the body) and $\partial x/\partial n$ by l, the above equation becomes:

$$U\int_i\int l\Phi_a d\sigma=-U^2\cdot v-U\int_o\int\left(x\frac{\partial\Phi_a}{\partial n}-l\Phi_a\right)d\sigma \quad (6)$$

The choice of the outer surface is arbitrary but it suggests itself, in view of the fact that the velocity potentials are expressed in terms of spheroidal coordinates, to choose one of the members of the confocal system of closed coordinate surfaces (ellipsoids of revolution about the X axis) belonging to this system of coordinates, say the one defined by $\lambda = \lambda_0$. This being done the next step is to express the integrand of the surface integral on the right-hand side of Eq. (6) in terms of spheroidal coordinates.

The equations of transformation from rectangular Cartesian coordinates to spheroidal coordinates are as follows:

$$\left.\begin{aligned}x&=k\lambda\mu\\ y&=k(\lambda^2-1)^{1/2}(1-\mu^2)^{1/2}\cos\omega\\ z&=k(\lambda^2-1)^{1/2}(1-\mu^2)^{1/2}\sin\omega\end{aligned}\right\} \quad (7)$$

Then choosing as parametric curves of the surface $\lambda = \lambda_0$, the orthogonal coordinate lines ω = variable, μ = constant (circles with center on the X-axis) and ω = constant, μ = variable (ellipses formed by the intersection of the surface $\lambda = \lambda_0$ and the meridian planes ω = constant), it follows, since the element of surface is given by $ds_\omega\, ds_\mu$ and l by $\partial(y, z)/\partial(ds_\omega, ds_\mu)$ that:

$$l\,d\sigma=\frac{\partial(y,z)}{\partial(ds_\omega,ds_\mu)}\cdot ds_\omega, ds_\mu$$

and

$$\frac{\partial\Phi_a}{\partial n}d\sigma=\frac{\partial\Phi_a}{\partial s_\lambda}ds_\omega ds_\mu$$

From the equations of transformation (7) it may be shown that

$$ds_\mu=k\left(\frac{\lambda^2-\mu^2}{1-\mu^2}\right)^{1/2}d\mu$$
$$ds_\lambda=k\left(\frac{\lambda^2-\mu^2}{\lambda^2-1}\right)^{1/2}d\lambda$$
$$ds_\omega=k(1-\mu^2)^{1/2}(\lambda^2-1)^{1/2}d\omega$$

Hence

and

$$\left.\begin{aligned}l\,d\sigma&=k^2\mu(\lambda^2-1)\,d\omega d\mu\\ \frac{\partial\Phi_a}{\partial n}d\sigma&=k(\lambda^2-1)\frac{\partial\Phi_a}{\partial\lambda}d\omega d\mu\end{aligned}\right\} \quad (8)$$

Substituting for Φ_a from Eq. (1) and making use of Eq. (8), the integral over the outer surface appearing on the right-hand side of Eq. (6) becomes:

$$\int_o\int\left(x\frac{\partial\Phi_a}{\partial n}-l\Phi_a\right)d\sigma=k^2(\lambda_0^2-1)\sum_{n=1}^{\infty}A_n\int_0^{2\pi}\int_{-1}^{1}$$
$$\times\left[\lambda_0\left(\frac{dQ_n}{d\lambda}\right)_{\lambda=\lambda_0}-Q_n(\lambda_0)\right]\mu P_n(\mu)\,d\omega\,d\mu$$

According to the definitions of the zonal harmonics of the first and second kinds:

$$\mu=P_1(\mu)$$

and

$$\lambda_0\left(\frac{dQ_1}{d\lambda}\right)_{\lambda=\lambda_0}-Q_n(\lambda_0)=-\frac{1}{\lambda_0^2-1}$$

Furthermore

$$\int_{-1}^{1}P_1(\mu)P_n(\mu)=\begin{matrix}0\text{ if }n\neq1\\ \frac{2}{3}\text{ if }n=1\end{matrix}$$

Therefore

$$\int_o\int\left(x\frac{\partial\Phi_a}{\partial n}-l\Phi_a\right)d\sigma=-\frac{4}{3}\pi k^2A_1$$

or from Eq. (6)

$$-\int_i\int\Phi_a\frac{\partial\Phi_a}{\partial n}d\sigma=-U^2\cdot v+\frac{4}{3}\pi\,Uk^2A_1 \quad (9)$$

CALCULATION OF $\int_t\int \Phi_t \frac{\partial \Phi_t}{\partial n} d\sigma$

The boundary condition for transverse flow is given by

$$-\frac{\partial \Phi_t}{\partial n} = mV$$

where m is the direction cosine with regard to the Y-axis of the outward drawn normal to the surface of the moving body. Hence

$$-\int_t\int \Phi_t \frac{\partial \Phi_t}{\partial n} d\sigma = V\int_t\int m\Phi_t d\sigma$$

Then choosing $\psi_1 = Vy$ and $\psi_2 = \Phi_t$, Green's theorem yields the following:

$$\int_t\int \left(Vy\frac{\partial \Phi_t}{\partial n} - V\Phi_t \frac{\partial y}{\partial n}\right) d\sigma = \int_0\int \left(Vy\frac{\partial \Phi_t}{\partial n} - V\Phi_t \frac{\partial y}{\partial n}\right) d\sigma$$

and noting that

$$\frac{\partial y}{\partial n} = m, \quad \int_t\int myd\sigma = v$$

it follows that

$$V\int_t\int m\Phi_t d\sigma = -V^2 . v - V\int_0\int \left(y\frac{\partial \Phi_t}{\partial n} - m\Phi_t\right)d\sigma \quad (10)$$

Again, suppose the outer surface to be a spheroid defined by $\lambda = \lambda_0$. Then

$$md\sigma = \frac{\partial(z, x)}{\partial(ds_\omega, ds_\mu)} ds_\omega\, ds_\mu = k^2\lambda\,(\lambda^2-1)^{\frac{1}{2}}\,(1-\mu^2)^{\frac{1}{2}} cos\omega d\omega d\mu$$

and

$$\frac{\partial \Phi_t}{\partial n} d\sigma = k(\lambda^2-1)\frac{\partial \Phi_t}{\partial \lambda} d\omega\, d\mu \quad (11)$$

Substituting for Φ_t from Eq. (1) and making use of Eq. (11), the integral over the outer surface appearing on the right-hand side of Eq. (10) becomes:

$$\int_0\int \left(y\frac{\partial \Phi_t}{\partial n} - m\Phi_t\right) d\sigma = k^2\,(\lambda_0^2-1)^{\frac{1}{2}} \sum_{n=1}^{\infty} B_n \left[(\lambda_0^2-1)\times \left(\frac{dQ_n^1}{d\lambda}\right)_{\lambda=\lambda_0} - \lambda_0 Q_n^1(\lambda_0)\right] \int_0^{2\pi}\int_{-1}^{1} (1-\mu^2)^{\frac{1}{2}} P_n^1(\mu)\, cos^2\omega d\omega\, d\mu$$

According to the definitions of the associated Legendre functions of the first and second kinds:

$$(1-\mu^2)^{\frac{1}{2}} = P_1^1(\mu)$$

and

$$(\lambda_0^2-1)\left(\frac{dQ_1^1}{d\lambda}\right)_{\lambda=\lambda_0} - \lambda_0 Q_1^1(\lambda_0) = \frac{2}{(\lambda_0^2-1)^{\frac{1}{2}}}$$

Also

$$\int_{-1}^{1} P_1^1(\mu)\, P_n^1(\mu)\, d\mu = \begin{cases} 0 \text{ if } n\neq 1 \\ \frac{4}{3} \text{ if } n=1 \end{cases}$$

Therefore

$$\int_0\int \left(y\frac{\partial \Phi_t}{\partial n} - m\Phi_t\right) d\sigma = \frac{8}{3}\pi k^2 B_1$$

or from Eq. (10)

$$-\int_t\int \Phi_t \frac{\partial \Phi_t}{\partial n} d\sigma = -V^2 v - \frac{8}{3}\pi V k^2 B_1 \quad (12)$$

CALCULATION OF $\int_t\int \Phi_t \frac{\partial \Phi_a}{\partial n} d\sigma$

The boundary condition for axial motion being $-\partial\Phi_a/\partial n = lU$, it follows that

$$-\int_t\int \Phi_t \frac{\partial \Phi_a}{\partial n} d\sigma = U\int_t\int l\Phi_t d\sigma$$

Then choosing $\psi_1 = Ux$ and $\psi_2 = \Phi_t$, Green's theorem yields:

$$\int_t\int \left(Ux\frac{\partial \Phi_t}{\partial n} - U\Phi_t \frac{\partial x}{\partial n}\right) d\sigma = \int_0\int \left(Ux\frac{\partial \Phi_t}{\partial n} - U\Phi_t \frac{\partial x}{\partial n}\right) d\sigma$$

and since

$$-\frac{\partial \Phi_t}{\partial n} = mV \text{(on the surface of the moving body) and } \frac{\partial x}{\partial n} = l$$

it follows that

$$U\int_t\int l\Phi_t d\sigma = -UV\int_t\int mxd\sigma - U\int_0\int \left(x\frac{\partial \Phi_t}{\partial n} - l\Phi_t\right) d\sigma \quad (13)$$

According to Gauss' theorem

$$\int_t\int mxd\sigma = \int\int_\tau\int \frac{\partial x}{\partial y} d\tau = 0$$

Furthermore, when the integrand of the outer surface integral appearing on the right-hand side of Eq. (13) is expressed in terms of spheroidal coordinates it will be seen that cos ω appears to the first power only.

Hence, since $\int_0^{2\pi} \cos \omega d\omega = 0$ it follows that:

$$-\int_t\int \Phi_t \frac{\partial \Phi_a}{\partial n} d\sigma = U\int_t\int l\Phi_t d\sigma = 0,$$

or that the product term vanishes in the expression for the kinetic energy of the fluid.

This result could have been obtained by noting that the integral $\int\int \Phi_t \frac{\partial \Phi_a}{\partial n} d\sigma$ is proportional to the product UV. Then, because of the longitudinal symmetry of a body of revolution if V is replaced by $-V$ the kinetic energy of the fluid is unaltered and therefore the coefficient of UV must vanish.

EXPRESSION FOR THE KINETIC ENERGY T

Substituting from Eqs. (9), (12) into Eq. (5), the kinetic energy of the fluid is given by:

$$\frac{2T}{\rho} = \left(\frac{\frac{4}{3}\pi k^3}{v}\frac{A_1}{kU} - 1\right) v U^2 + \left(-\frac{\frac{8}{3}\pi k^3}{v}\frac{B_1}{kV} - 1\right) v V^2 \quad (14)$$

or comparing term for term with Eq. (3):

$$\left.\begin{aligned} A &= \left(\frac{\frac{4}{3}\pi k^3}{v}\frac{A_1}{kU}-1\right)v = k_a\cdot v \\ B &= \left(-\frac{\frac{8}{3}\pi k^3}{v}\frac{B_1}{kV}-1\right)v = k_t\cdot v \end{aligned}\right\} \quad (15)$$

Therefore

$$T = k_a\cdot\frac{1}{2}MU^2 + k_t\cdot\frac{1}{2}MV^2$$

where M is the mass of fluid displaced by the body and

$$\left.\begin{aligned} k_a &= \frac{\frac{4}{3}\pi k^3}{v}\frac{A_1}{kU}-1, \text{ the inertia coefficient for axial flow} \\ k_t &= -\frac{\frac{8}{3}\pi k^3}{v}\frac{B_1}{kV}-1, \text{ the inertia coefficient for transverse flow.} \end{aligned}\right\} \quad (16)$$

These remarkably simple expressions for the coefficients of inertia k_a, k_t, have already been obtained by C. Ferrari[3] by a method based on a consideration of the sink-source and doublet distributions along the axis of symmetry of the moving body in conjunction with Munk's theorem on sinks and sources.[4]

Conclusion

The results of this note, contained in Eq. (16), have been obtained without the necessity of introducing extraneous features to the flow, such as sinks and sources inside the moving body. It is to be further noted that rotational motion was not included. The reason for excluding this type of motion is suggested by the choices necessary for the arbitrary function ψ_1 appearing in Eq. (4). Thus, for axial flow ψ_1 was taken equal to Ux and for transverse flow equal to Vy. It is known that Ux and Vy are the velocity potentials of the rectilinear flows which must be superposed on the fluid and moving body in order to render the latter stationary. However, the rotational flow which must be superposed on the fluid and a rotating body, in order to fix the body in the fluid, does not possess a velocity potential and it therefore appears impossible to treat this type of motion in a manner analogous to that for rectilinear motion.

[3] Carlo Ferrari, *Sul campo aerodinamico attorno ad un solido siluriforme,* Memorie della R. Accademia delle Scienze di Torino, Serie II, vol. LXVII, N. 4, 1932.

[4] Carl Kaplan, *Potential Flow about Elongated Bodies of Revolution,* T. R. No. 516, N.A.C.A., 1935.