NASA TECHNICAL NOTE

NASA TN D-2842

NASA TN D-2842

FACILITY FORM 602

N66-10323
(ACCESSION NUMBER)
(THRU)

45
(PAGES)
1
(CODE)

(NASA CR OR TMX OR AD NUMBER)
07
(CATEGORY)

GPO PRICE $ ____

CFSTI PRICE(S) $ 2.00

Hard copy (HC) ____

Microfiche (MF) .50

ff 653 July 65

# EQUIVALENT NOISE BANDWIDTH ANALYSIS FROM TRANSFER FUNCTIONS


*by Thomas J. Karras*

*Goddard Space Flight Center*

*Greenbelt, Md.*



NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • NOVEMBER 1965

NASA TN D-2842

# EQUIVALENT NOISE BANDWIDTH ANALYSIS FROM TRANSFER FUNCTIONS

By Thomas J. Karras

Goddard Space Flight Center
Greenbelt, Md.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151 – Price $2.00

N 66-10323

## ABSTRACT

This report clarifies the terms "equivalent noise bandwidth" (ENBW), "natural frequency" ($\omega_0$), and "3 db bandwidth" ($\omega_{3db}$), and demonstrates the relationships between these terms and the network transfer function. Two methods for calculating the ENBW from a filter transfer function are described. In illustration of these methods, three filters are analyzed. They are: (1) first order (Type 0) passive low pass filter, (2) second order (Type 1) phase lock loop tracking filter, and (3) third order (Type 2) phase lock loop tracking filter. Each of these filters possesses an attenuation characteristic of -6 db/octave at $\omega > 10\ \omega_0$. A comparison is made of the ENBW for a second order (Type 1) and third order (Type 2) phase lock loops, as a function of the damping factor ($\zeta$).

Author

CONTENTS

# EQUIVALENT NOISE BANDWIDTH ANALYSIS FROM TRANSFER FUNCTIONS

by

Thomas J. Karras

*Goddard Space Flight Center*

## INTRODUCTION

This report will clarify the concept of "equivalent noise bandwidth" (ENBW) and its methods of computation, given a system transfer function. Several transfer functions of filters will be analyzed, leading to the computation of their ENBW relative to the natural resonant frequency $(\omega_0)$ of the filter, and the "3 db down" bandwidth $(\omega_{3db})$.

If the ENBW of a filter is known, one may compute the signal-to-noise ratio improvement of the filter when used in a system. The use of the ENBW will be demonstrated by considering the processing of pulse frequency modulated data (Reference 1) by use of comb filters. Let a filter within the comb be a single-pole band-pass filter with a 3 db bandwidth of 100 cps and let each filter be separated by 100 cycles in center frequency from an adjacent filter. Figure 1 shows one such filter within the comb filter, used when the data frequencies range between 5 kc and 15 kc. In theory, one refers to a rectangular filter which has the same maximum gain and which passes the same average noise power from a white noise source as the single pole filter. The bandwidth of this ideal rectangular filter is called its equivalent noise bandwidth. Since the data frequency can be between 5 kc and 15 kc, the bandwidth of the noise passing into the comb filter is at least 10 kc.

The signal-to-noise improvement of the single pole filter in this comb filter is found by computing (S/N) improvement = $(S/N)_{out}/(S/N)_{in}$. Assuming that the input frequency is at the center of the filter $(S_{in} = S_0)$, then, the improvement becomes:

$$\frac{(S/N)_{out}}{(S/N)_{in}} = 10\log\left(\frac{\text{noise power input to the comb filter}}{\text{noise power passing through a single pole filter}}\right) \tag{1a}$$

$$= 10\log\frac{10{,}000}{\text{ENBW}} \tag{1b}$$

$$= 10\log\frac{10{,}000\text{ cps}}{\frac{\pi}{2}\,100\text{ cps}} = 18\text{ db improvement.} \tag{1c}$$

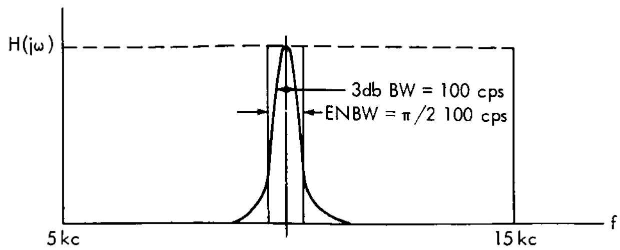


Figure 1–Single pole filter with its equivalent rectangular filter.

If the input frequency is offset from the filter center frequency, the signal-to-noise ratio improvement becomes less than +18 db, since $S_0 < S_{in}$ along the skirts of the filter. For example, the improvement would be +15 db if the input frequency is at the crossover (3 db) point.

Another important use for the ENBW analysis is in computing the mean-square output noise voltage $\overline{\sigma^2}$, which is related to the ENBW by

$$\overline{\sigma^2} = (N_0/\pi)\int_0^\infty |H(j\omega)|^2 \, d\omega = 2N_0(BN1) \tag{2}$$

where $N_0$ is the single sided noise power density spectrum of the noise source in volts²/cycle and $H(j\omega)$ is the network transfer function, and BN1 is the one sided normalized ENBW.

The four basic relationships which can be utilized in solving for the ENBW for positive frequencies (one sided, BN1) given the transfer function of a filter are as follows:

$$BN1 = \int_0^\infty |H(j\omega)|^2 \, df \ (\text{cps}) \tag{3}$$

$$= \frac{1}{2\pi}\int_0^\infty |H(j\omega)|^2 \, d\omega \ (\text{cps}) \tag{4}$$

$$= \frac{1}{2\pi j}\int_0^{+j\infty} |H(j\omega)|^2 \, d(j\omega) \ (\text{cps}) \tag{5}$$

$$= \int_0^\infty |H(j\omega)|^2 \, d\omega \ (\text{rad/sec}) . \tag{6}$$

These four relationships are applicable when the low frequency or high frequency response of the filter possesses 0 db or unity gain. If the filter possesses a gain K at the low or high frequencies, then Equations 3 through 6 must be divided by $K^2$ or $|H(j\omega)|^2$ with $\omega = 0$ or $\omega = \infty$ for low or high pass filters respectively. For filters possessing a high Q (narrow band pass filters), then Equations 3 to 6 should be divided by the maximum value of $H(j\omega)$ squared $\left(|H(j\omega)|^2_{max}\right)$. This means there is a slight difference in the interpretation of the term ENBW for a high or low pass filter versus band pass filters, if the maximum gain of either is other than unity. This is to say that the ENBW $= BN1/|H(j\omega)|^2_{max}$ for a band pass filter; and the ENBW $= BN1/|H(j0)|^2$ for a low pass filter.

Each of the following filters will be analyzed for their ENBW: (1) first order (Type 0)* passive low pass filter; (2) second order (Type 1) phase lock loop tracking filter; and (3) third order (Type 2) phase lock loop tracking filter. The ENBW will be computed by two methods. Method 1 will be that of evaluating Equation 6 by calculating the residues of the poles in the upper half plane of the transfer function squared $(|H(j\omega)|^2)$ . Method 2 will be that of evaluating Equation 5 by the use of the table of integrals found in Reference 2. The low frequency gain for each filter in this report is unity or 0 db; hence, ENBW = BN1 in this report.

## EQUIVALENT NOISE BANDWIDTH ANALYSIS FOR A FIRST ORDER (TYPE 0) PASSIVE LOW PASS FILTER

The transfer function for the first order (1 pole), Type 0 (no zero's) passive low pass filter (shown in Figure 2) is

$$H(s) = E_0(s)/(E_{in}(s)) = \omega_0/(s + \omega_0), \tag{7}$$

where $\omega_0 = 1/RC$ (radians per second).

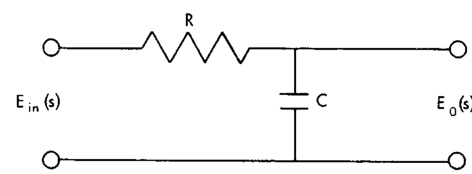

Figure 2–First order (Type 0) passive low pass filter.

### Computing the 3-db Bandwidth

The frequency response for this filter is shown in Figure 3. The 3-db bandwidth $(\omega_{3db})$ is found by determining the value of $\omega$ where, $20 \log |H(j\omega)| = -3\text{db}$; hence,

$$20 \log \left| \frac{\omega_0}{s + \omega_0} \right| = -3\text{db}, \tag{8}$$

and

$$20 \log \frac{\omega_0}{\sqrt{\omega_{3db}^2 + \omega_0^2}} = -3\text{db}, \tag{9}$$

therefore,

$$\frac{\omega_0}{\sqrt{\omega_0^2 + \omega_{3db}^2}} = \frac{1}{\sqrt{2}} \tag{10}$$

and

$$\omega_{3db} = \omega_0 . \tag{11}$$

Thus, the 3-db bandwidth occurs at $\omega_0$ for the passive low pass filter.

*The terms "Type" and "Order" as used in this report are defined in Appendix I.

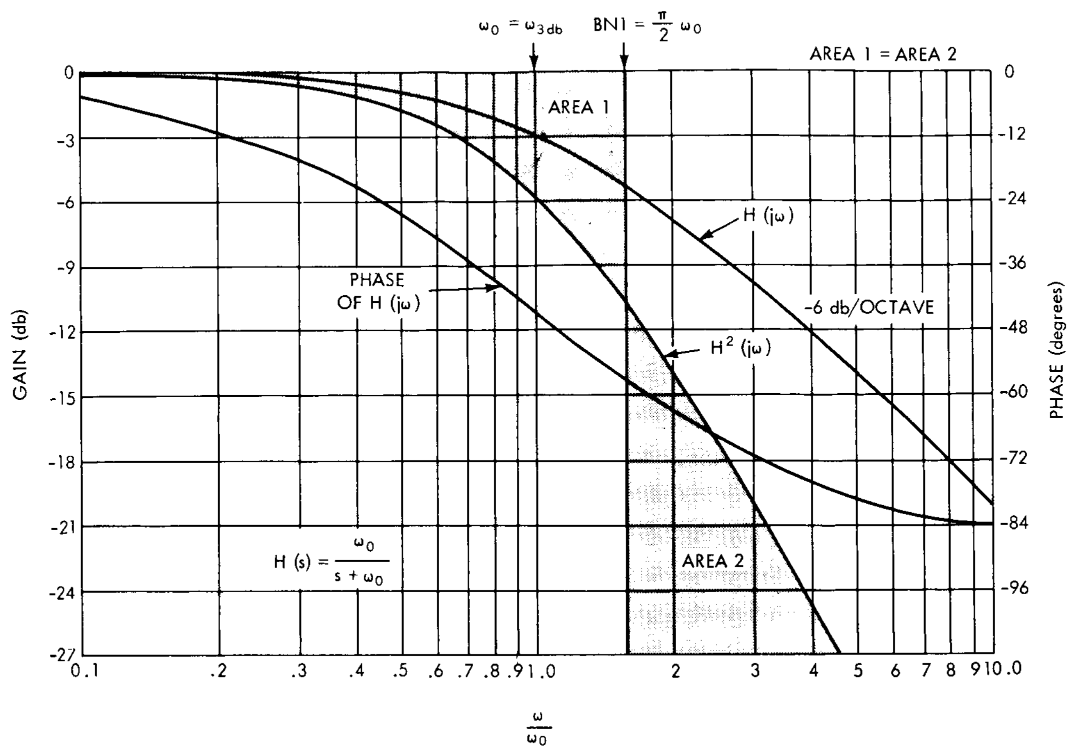


Figure 3–Frequency response for a first order (Type 0) passive low pass filter showing the ENBW, $\omega_{3db}$, and $\omega_0$ frequencies.

## Computing the ENBW

The equivalent noise bandwidth corresponds to a frequency (BN1) such that the area under the rectangle is equivalent to the area under the $H^2(j\omega)$ curve (Figure 3).

*Method 1: ENBW Analysis of the First Order (Type 0) Passive Low Pass Filter by Evaluating the Residues.*

From Equation 6, given below

$$BN1 = \int_0^\infty |H(j\omega)|^2 \, d\omega \ (\text{rad/sec}) \tag{6}$$

where $H(j\omega) = \omega_0/(\omega_0 + j\omega)$, we see that BN1 becomes, in succession,

$$BN1 = \int_0^\infty \frac{\omega_0^2}{\omega^2 + \omega_0^2} \, d\omega \ , \tag{12}$$

$$\mathrm{BN1} = \int_{-\infty}^{\infty} \frac{\omega_0^2}{\omega^2 + \omega_0^2}\, d\omega , \tag{13}$$

$$= \frac{\omega_0^2}{2} \left[ 2\pi j \sum \text{Residues of } \omega \text{ in the upper half plane} \right] . \tag{14}$$

The poles of $|H(j\omega)|^2$ occur when $\omega^2 + \omega_0^2 = 0$. Therefore, the poles are $\omega_1 = j\omega_0$ and $\omega_2 = -j\omega_0$. We find that by appropriate substitution BN1 becomes

$$\mathrm{BN1} = \frac{\omega_0^2}{2} \left[ 2\pi j \; (\mathrm{R1}) \right] . \tag{15}$$

The residue R1 is the value of $|H(j\omega)|^2$ evaluated at the single pole located in the upper half plane (Reference 3), where

$$\mathrm{R1} = \left. \frac{(\omega - \omega_1)}{(\omega - \omega_1)(\omega - \omega_2)} \right|_{\omega = \omega_1} , \tag{16}$$

$$= \frac{1}{\omega_1 - \omega_2} = \frac{1}{2j\omega_0} . \tag{17}$$

Therefore, the relation for BN1 is now

$$\mathrm{BN1} = \frac{\omega_0^2}{2} \left[ 2\pi j \left( \frac{1}{2j\omega_0} \right) \right] , \tag{18a}$$

$$= \frac{\pi}{2}\, \omega_0 \ (\text{rad/sec}) = \frac{\omega_0}{4} \ (\text{cps}) = \frac{\pi}{2}\, f_0 \ (\text{cps}). \tag{18b}$$

*Method 2: ENBW Analysis of the First Order (Type 0) Passive Low Pass Filter by Use of the Table of Integrals (Reference 2).*

The integrals to be evaluated in obtaining the ENBW by integration are in the form

$$I_N = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} |H(s)|^2 \, ds , \tag{19}$$

$$= \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} \frac{c(s)\,\overline{c(s)}}{d(s)\,\overline{d(s)}} \, ds , \tag{20}$$

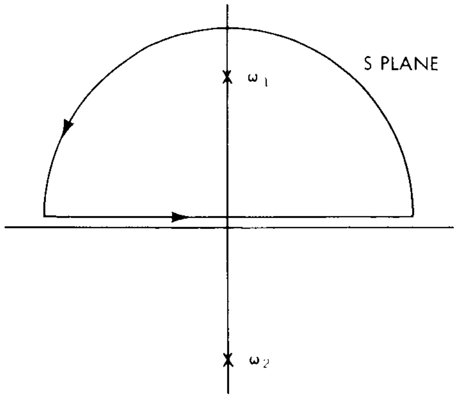


Figure 4–Poles of $1/(\omega^2 + \omega_0^2)$.

where

$$H(s) = \frac{c(s)}{d(s)}, \quad \overline{c(s)} = c(-s), \text{ and } \overline{d(s)} = d(-s) \tag{21a}$$

and

$$c(s) = c_{N-1}\, s^{N-1} + \cdots\cdots + c_0 , \tag{21b}$$

$$d(s) = d_N\, s^N + \cdots\cdots + d_0 . \tag{21c}$$

Note that $d(s)$ has zeros in the left half plane only, and the highest power of $c(s)$ is at least one degree less than the highest power of $d(s)$.

The integrals which will be used in this report are for $N = 1$, 2, and 3; these integrals are solved with the following algebraic relationships. The solution for higher values of N are given in Reference 2.

$$N = 1,\ I_1 = \frac{c_0^2}{2\, d_0\, d_1} \tag{22}$$

$$N = 2,\ I_2 = \frac{c_1^2\, d_0 + c_0^2\, d_2}{2\, d_0\, d_1\, d_2} \tag{23}$$

$$N = 3,\ I_3 = \frac{c_2^2\, d_0\, d_1 + \left(c_1^2 - 2\, c_0\, c_2\right) d_0 d_3 + c_0^2\, d_2\, d_3}{2\, d_0\, d_3\, \left(-d_0\, d_3 + d_1\, d_2\right)} \tag{24}$$

The equivalent noise bandwidth for the passive low pass filter can be found by using Equations 5 and 22. Since $H(s) = \omega_0/(s + \omega_0)$ and $N = 1$, the ENBW is as follows:

$$BN1 = \frac{1}{2\pi j} \int_0^{+j\infty} |H(j\omega)|^2\, d(j\omega) \text{ cps} . \tag{5}$$

If $|H(j\omega)|^2$ is an even function of $\omega$ ($\omega$ can be replaced by $-\omega$ and the integrand remains the same), then BN1 becomes, successively,

$$BN1 = \frac{1}{2}\left[\frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} |H(j\omega)|^2\, d(j\omega)\right] \text{ cps}, \tag{25}$$

$$BN1 = \frac{1}{2}\, I_1 , \tag{26a}$$

$$BN1 = \frac{1}{2}\left(\frac{c_0^2}{2\, d_0\, d_1}\right), \tag{26b}$$

where $N = 1$ in $H(s) = \omega_0/s + \omega_0 - c(s)/d(s)$, and $c_0 = \omega_0$, $d_1 = 1$, $d_0 = \omega_0$. From this, we may see that $BN1$ reduces to the same form as expressed in Equation 18b.

## EQUIVALENT NOISE BANDWIDTH ANALYSIS FOR A SECOND ORDER (TYPE 1) PHASE LOCK LOOP TRACKING FILTER

The transfer function for a second order (2 poles), Type 1 (1 zero) phase lock loop tracking filter shown in Figure 5 is

$$H(s) = \frac{2\zeta\omega_0\, s + \omega_0^2}{s^2 + 2\zeta\omega_0\, s + \omega_0^2}, \tag{27}$$

where $\omega_0$ is the natural resonant frequency and $\zeta$ is the damping factor. Appendix A contains a development of the transfer function $H(s)$.

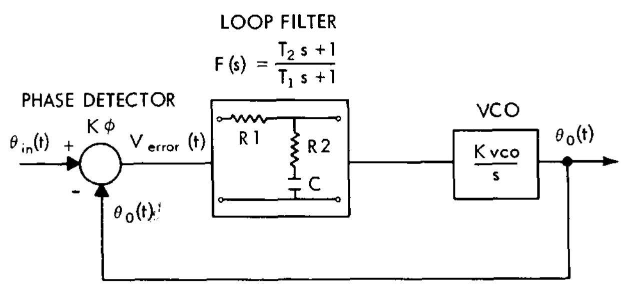


Figure 5–Second order (Type 1) phase lock loop tracking filter.

The frequency response for this phase lock loop is shown in Figure 6 for $\zeta = .707$.

### Computing the 3-db Bandwidth

The 3-db bandwidth $(\omega_{3db})$ can be found by finding the value of $\omega$ for which

$$20 \log |H(j\omega)| = -3 \text{ db},$$

so that

$$20 \log \left| \frac{\omega_0^2 + j\,2\zeta\omega_0\, \omega_{3db}}{\left(\omega_0^2 - \omega_{3db}^2\right) + j\,2\zeta\omega_0\, \omega_{3db}} \right| = -3 \text{ db}. \tag{28}$$

Therefore, we obtain the expression

$$\frac{\omega_0^4 + 4\zeta^2\omega_0^2\,\omega_{3db}^2}{\left(\omega_0^2 - \omega_{3db}^2\right)^2 + 4\zeta^2\omega_0^2\,\omega_{3db}^2} = \frac{1}{2}. \tag{29}$$

A typical phase lock loop has a damping factor of $\zeta = .707$, for which case Equation 29 reduces to

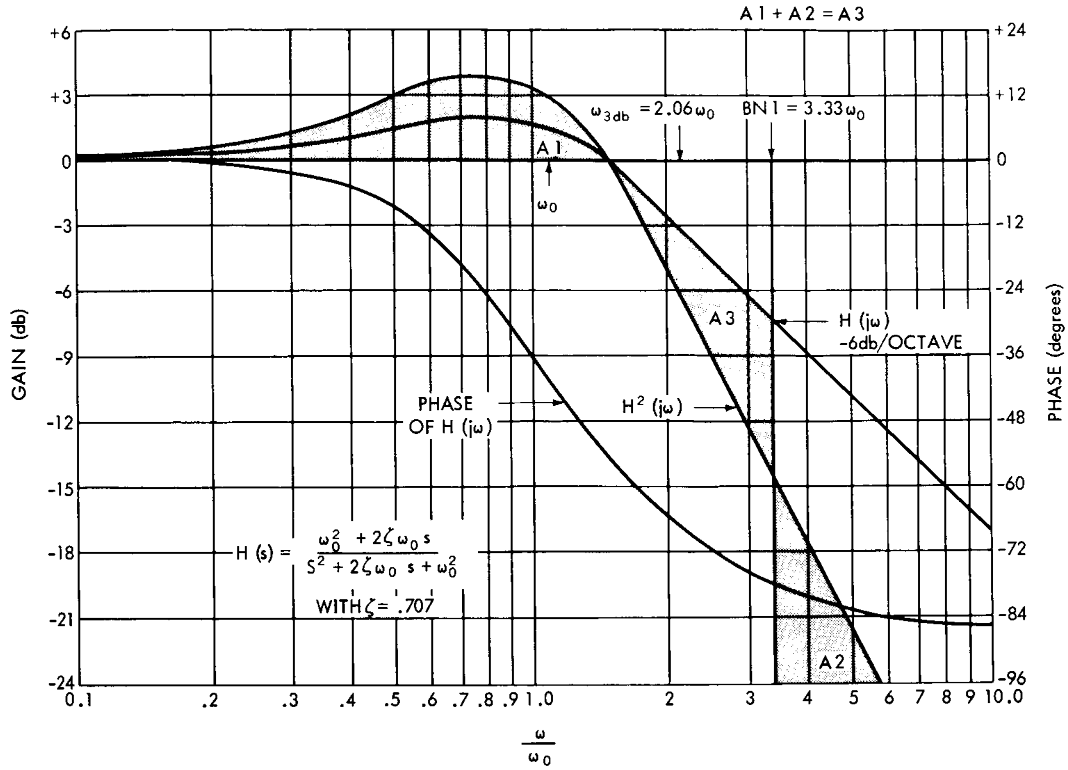


Figure 6—Frequency response of a second order (Type 1) phase lock loop trackir ˆilter showing the ENBW, $\omega_{3db}$, and $\omega_0$ frequencies.

$$\omega_{3db}^4 - 4\omega_0^2\omega_{3db}^2 - \omega_0^4 = 0 . \tag{30}$$

The 3-db bandwidth can be shown to be $\omega_{3db} = 2.06\ \omega_0 \big|_{\zeta=.707}$ . The three other roots are inapplicable.

## Computing the ENBW

*Method 1: ENBW Analysis of the Second Order (Type 1) Phase Lock Loop by Evaluating the Residues*

From Equation 6, given below,

$$BN1 = \int_0^\infty |H(j\omega)|^2\ d\omega\ (\mathrm{rad/sec}) \tag{6}$$

with

$$|H(j\omega)|^2 = \left.\frac{\omega_0^2\left(\omega_0^2 + 2\omega^2\right)}{\omega^4 + \omega_0^4}\right|_{\zeta=.707} \tag{31}$$

the ENBW for $\zeta = 0.707$ is found to be from Appendix B:

$$
\begin{aligned}
\text{BN1} &= \frac{3}{4}\sqrt{2}\,\pi\omega_0 \\
&= 3.33\ \omega_0 \text{ (rad/sec)} \\
&= 3.33\ f_0 \text{ (cps)} \\
&= 0.531\ \omega_0 \text{ (cps)}\,.
\end{aligned}
\tag{32}
$$

*Method 2: ENBW Analysis of the Second Order (Type 1) Phase Lock Loop by use of the Table of Integrals*

From the general relationship found for the ENBW (see Appendix B) where

$$
\text{BN1} = \left(\frac{4\zeta^2 + 1}{8\zeta}\right)\omega_0 \text{ (cps)}\,, \tag{33}
$$

one can plot the ENBW as a function of $\zeta$ for $\omega_0 = 1$ (Figure 7). Analysis of Figure 7 indicates that

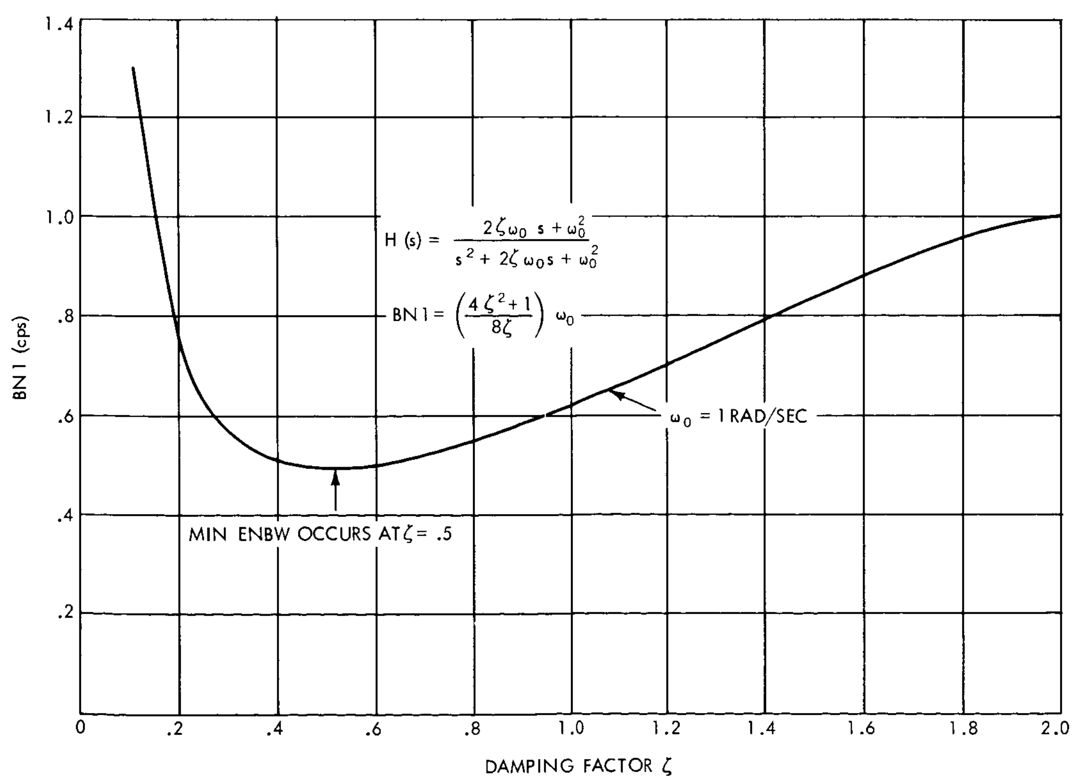


Figure 7—Plot of the equivalent noise bandwidth (one-sided) for the second order (Type 1) phase lock loop tracking filter.

systems with low damping factors have large noise bandwidths. For large damping factors, the ENBW increases in direct proportion to the damping factor (BN1 = $\zeta/2$ for large $\zeta$ ). The minimum ENBW occurs at $\zeta = 0.5$, BN1 min = $0.5\ \omega_0$ (cps).

## A Two Sided ENBW for the Second Order (Type 1) Phase Lock Loop Tracking Filter

It would be appropriate at this time to introduce the concept of the two sided ENBW (BN2) using the tracking filter as an example.

The d-c component or low frequency of the phase lock loop is truly offset to the frequency of the VCO. The VCO can track a frequency in the range $\left(f_{vco_L} \text{ to } f_{vco_H}\right)$ and while resting at any particular frequency in this range, has the frequency response.

$$H(s) = \frac{2\zeta\omega_0 s + \omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \tag{27}$$

centered around the resting frequency. This is shown in Figure 8.

The phase lock loop actually responds to changes of frequencies both above and below the frequency of the VCO (positive and negative frequencies), resulting in a two-sided tracking filter, shown in Figure 9.

The reason for this two sided tracking bandwidth for a phase lock loop can be seen by examining the error signal appearing at the phase detector output, as illustrated by the block diagram in Figure 10.

The phase detector is functioning as a multiplier; hence, the error signal appearing at the output of the phase detector is

$$V_\epsilon(t) = \theta_{in}(t)\ \theta_{out}(t)\ . \tag{34}$$

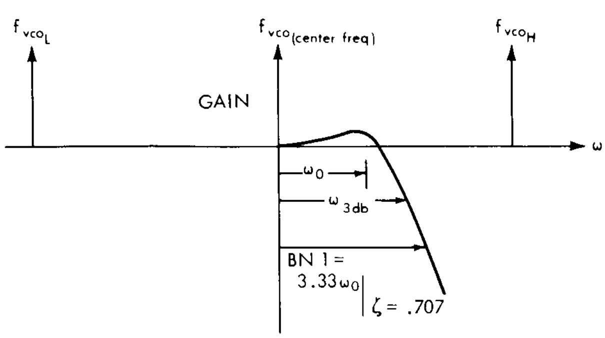


Figure 8—One-sided tracking bandwidth.

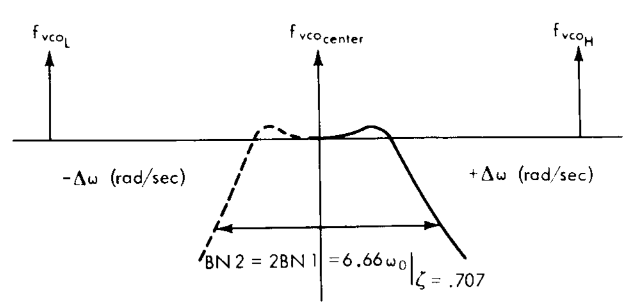


Figure 9—Two-sided tracking bandwidth.

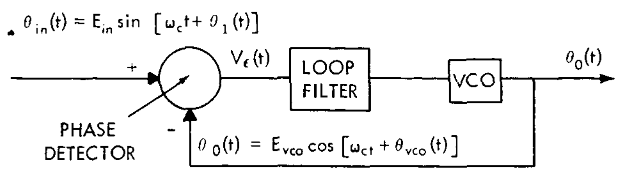


Figure 10—Phase lock loop tracking filter.

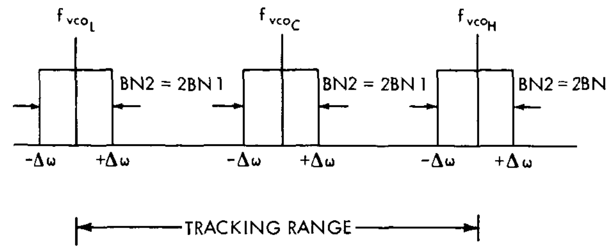


Figure 11—ENBW (two-sided) equivalent response for a phase lock loop tracking filter.

From Appendix C, we see (Equations C5 and C6) that

$$V_\epsilon(t) = \frac{E_{in}\ E_{VCO}}{2} \sin \left[\theta_1(t) - \theta_{VCO}(t)\right], \tag{35}$$

$$= \frac{E_{in}\ E_{VCO}}{2} \left[\Delta\omega\ t\right] \tag{36}$$

where $\Delta\omega t = \theta_1(t) - \theta_{VCO}(t)$, and $\sin \Delta\omega t = \Delta\omega t$ for small $\Delta\omega t$.

If $\theta_1(t) > \theta_{VCO}(t)$, then $\Delta\omega$ is positive, and if $\theta_1(t) < \theta_{VCO}(t)$, then $\Delta\omega$ is negative. When $\Delta\omega$ is positive, it causes the VCO to oscillate at a frequency higher than its center frequency.

Hence, the total equivalent noise bandwidth of a phase lock loop is twice the one sided equivalent noise bandwidth or 6.66 $\omega_0 \mid \zeta = .707$ for the second order (Type 1) phase lock loop tracking filter (Figure 11).

## EQUIVALENT NOISE BANDWIDTH ANALYSIS FOR A THIRD ORDER (TYPE 2) PHASE LOCK LOOP TRACKING FILTER

The transfer function for a typical third order (3 poles), Type 2 (2 zeros) phase lock loop tracking filter (Reference 4) shown in Figure 12 is (from Equation D9),

$$H(s) = \frac{(9/4)\omega_0\ (s + \omega_0/3)^2}{(s + \omega_0)^2 (s + \omega_0/4)}. \tag{37}$$

Appendix D contains the development of the transfer function $H(s)$.

The frequency response of the third order (Type 2) phase lock loop tracking filter is shown in Figure 13.

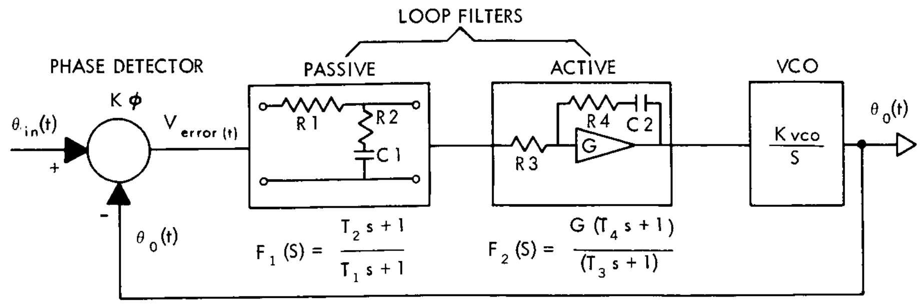


Figure 12—Third order (Type 2) phase lock loop tracking filter.

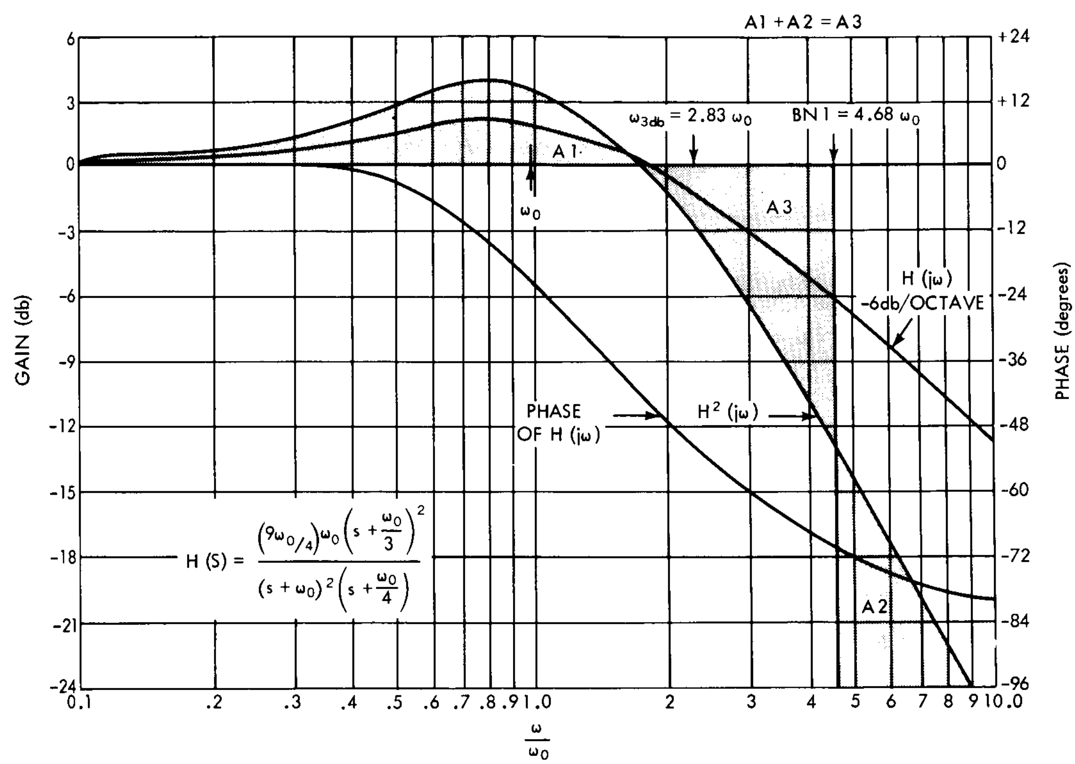


Figure 13—Frequency response of a third order (Type 2) phase lock loop tracking filter showing equivalent noise bandwidth, $\omega_{3\,db}$, and $\omega_0$ frequencies.

## Computing the 3-db Bandwidth

The 3 db bandwidth $(\omega_{3db})$ can be found by finding the value of $\omega$ for which $20 \log |H(j\omega)| = -3\text{db}$.

$$20 \log \left| \frac{\frac{9\omega_0}{4}\left(\frac{\omega_0}{3} + j\omega_{3db}\right)^2}{\left(\frac{\omega_0}{4} + j\omega_{3db}\right)\left(\omega_0 + j\omega_{3db}\right)^2} \right| = -3 \text{ db}. \tag{38}$$

This reduces to:

$$8\left(\frac{\omega_0^2}{16} + \omega_{3db}^2\right)\left(\omega_0^2 + \omega_{3db}^2\right)^2 - 81\,\omega_0^2\left(\frac{\omega_0^2}{9} + \omega_{3db}^2\right)^2 = 0 \tag{39}$$

resulting in $\omega_{3db} = 2.83\,\omega_0$.

## Computing the ENBW

The ENBW, as developed in Appendix E, was found from Equation E11 to be (for $K = 9/4\omega_0$) $BN1 = 4.68\,\omega_0$ (rad/sec) = $0.743\,\omega_0$ (cps).

The general expression computed for the ENBW is (Equation E16)

$$BN1 = 3/4\left[\frac{(2K + \omega_0)}{(6 - \omega_0/K)}\right]. \tag{40}$$

Figure 14 is a plot of the ENBW for the third order (Type 2) phase lock loop of $BN1 = (3/4)\left[(2K + \omega_0)/(6 - \omega_0/K)\right]$ with $\omega_0 = 1$. The minimum ENBW occurs at a value of $K = .5$ (cps), resulting in the minimum ENBW of $0.375\,\omega_0$ (cps).

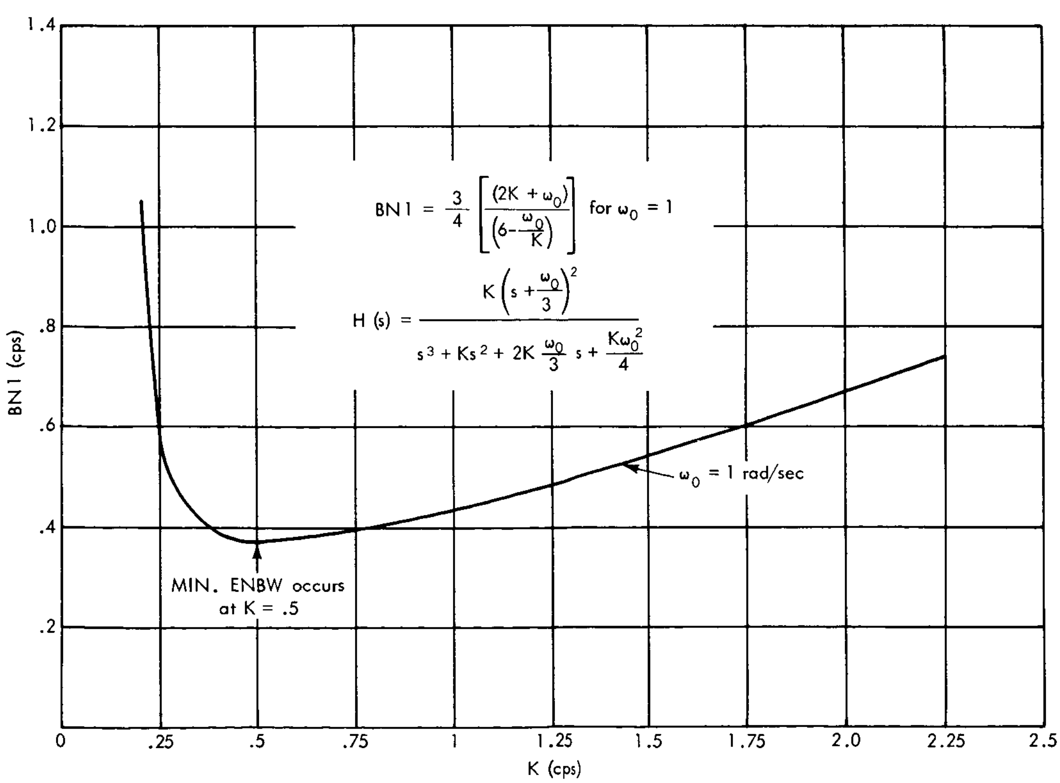


Figure 14—Plot of the equivalent noise bandwidth (one-sided) for the third order (Type 2) phase lock loop tracking filter.

## COMPARISON OF THE ENBW FOR THE SECOND ORDER (TYPE 1) AND THIRD ORDER (TYPE 2) PHASE LOCK LOOP TRACKING FILTERS

By examining the general transfer function of the third order system,

$$H(s) = \frac{K\left(s + \frac{\omega_0}{3}\right)^2}{s^3 + Ks^2 + \frac{2K\omega_0}{3} s + K\left(\frac{\omega_0}{3}\right)^2}, \tag{41}$$

a direct comparison with the second order system, where

$$H(s) = \frac{2\zeta\omega_0 s + \omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \tag{27}$$

is not possible since a damping factor term does not naturally appear in the cubic equation of the third order system.

The ENBW for the second order system was shown to be a function of $\zeta$ and $\omega_0$, while the ENBW for the third order system was shown to be a function of K (cps or rad/sec) and $\omega_0$.

If we let $K = 9\omega_0/4$ in the third order system, the transfer function becomes

$$H(s) = \frac{9\omega_0/4\left(s + \frac{\omega_0}{3}\right)^2}{\left(s + \omega_0\right)^2\left(s + \frac{\omega_0}{4}\right)} = \frac{9\omega_0/4\left(s + \frac{\omega_0}{3}\right)^2}{\left(s^2 + 2\omega_0 s + \omega_0^2\right)\left(s + \frac{\omega_0}{4}\right)}. \tag{42}$$

Examining the denominator term and equating it to the general denominator term for the second order system $\left(s^2 + 2\zeta\omega_0 s + \omega_0^2\right)$, we find that the term in the 2nd order factor which is analogous to $\zeta = 1$ when $K = 9\omega_0/4$ for the third order system. Hence, by allowing K to vary from 0 to $9\omega_0/4$, and computing the roots of the cubic equation to obtain the damping factor $\zeta$, it will then be possible to compare the ENBW for the second and third order systems.

Appendix F is a development of the equations necessary to compute $\zeta$ for each value of K in the third order system. Figure 15 represents a comparison of the ENBW for the second order (Type 1) and third order (Type 2) phase lock loop tracking filters for $\omega_0 = 1$.

## TABULATION OF THE RESULTS FOR ENBW, $\omega_0$, AND $\omega_{3db}$ BANDWIDTHS

Table 1 below summarizes the results for the ENBW, $\omega_0$, and $\omega_{3db}$ relationships for the first order (Type 0), second order (Type 1), and third order (Type 2) filters. Each filter has an attenuation characteristic at $\omega >> \omega_0$ of -6 db/octave.

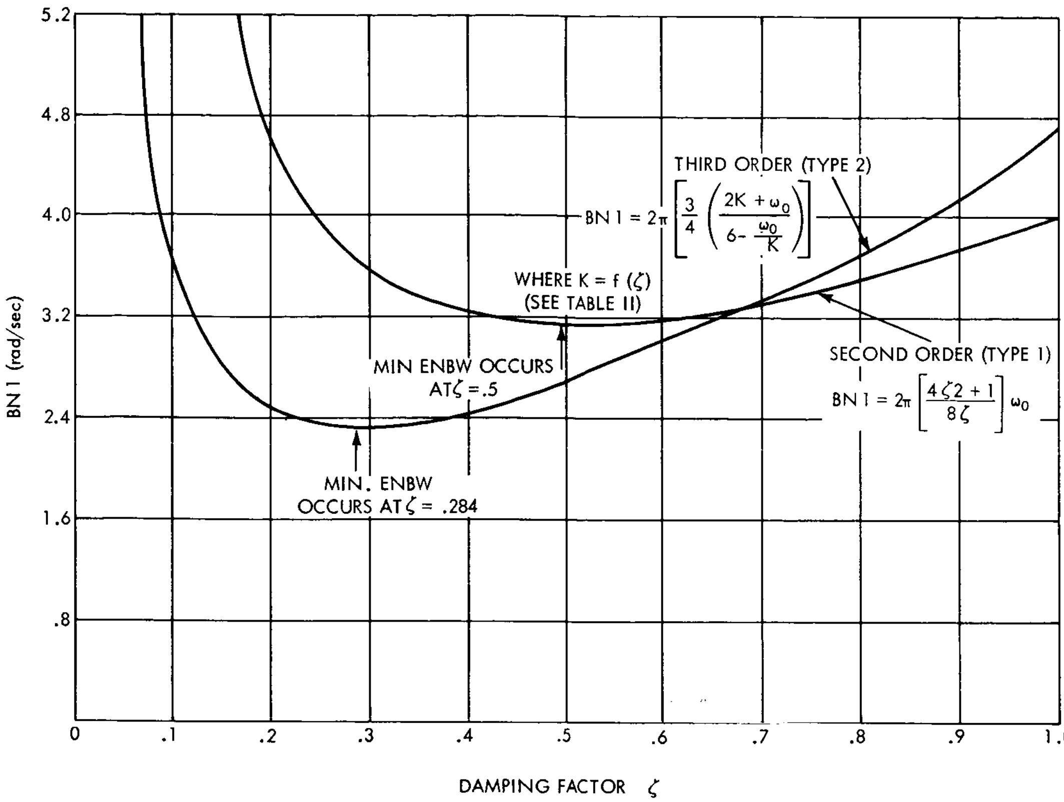


Figure 15—Comparison of the ENBW (one-sided) for the second order (Type 1) and third order (Type 2) phase lock loop tracking filters for $\omega_0 = 1$ rad/sec.

Table 1

Comparison of ENBW and 3-db Bandwidth for First, Second, and Third Order Filters.

| Filter | ENBW (One Sided) | | 3 db bandwidth (rad/sec) |
|---|---|---|---|
| | (rad/sec) | (cps) | |
| First Order (Type 0) | $1.57\omega_0$ | $.25\ \omega_0$ | $\omega_0$ |
| Second Order (Type 1) For $\zeta = .707$ | $3.33\omega_0$ | $.531\ \omega_0$ | $2.06\ \omega_0$ |
| Third Order (Type 2) For $K = 9\omega_0/4$. $\zeta = 1$ | $4.68\omega_0$ | $.743\ \omega_0$ | $2.83\ \omega_0$ |

## SUMMARY

The concept of the equivalent noise bandwidth (ENBW) and its relationship to the natural frequency $(\omega_0)$ and the 3 db bandwidth of several types of filters have been discussed. Computation of filter ENBW by evaluating an integral with special tables of integrals (Method 2) is fast and simple. An alternative method is by computing the sum of the residues of the ENBW integral.

In comparing the ENBW of the second order (Type 1) and third order (Type 2) phase lock loop tracking filters (Figure 15), it is seen that for practical values of the damping factor $(\zeta)$ the ENBW for a third order (Type 2) system is normally greater (for the same $\omega_0$) than the ENBW for the second order (Type 1) phase lock loop tracking filter. Hence, the signal-to-noise improvement will be better when using the second order (Type 1) system. However, the tracking rate will be improved in the case of the third order system (see Appendix H).

## ACKNOWLEDGMENT

The author would like to acknowledge the valuable comments and suggestions made by Mr. A. M. Demmerle of the Goddard Space Flight Center.

(Manuscript received March 16, 1965)

## REFERENCES

1. Rochelle, R. W., "Pulse Frequency Modulation Telemetry," NASA Technical Report R-189, January 1964.
2. Newton, G. C., Gould, L. A., and Kaiser, J. F., "Analytical Design of Linear Feedback Controls," New York: John Wiley and Sons, Inc., 1957, pp. 366-381.
3. Churchill, R. V., "Introduction to Complex Variables and Applications," New York: McGraw Hill Co., 1948.
4. Anon, "Phase-Lock Tracking Filter Model VIII," (Instruction Manual), Anaheim, California, Interstate Electronics Corp., First Rev., December 1960.

Appendix A

## Development of the Transfer Function for the Second Order (Type 1) Phase Lock Loop Tracking Filter

The standard configuration for a servo system is shown in Figure A1 and has a transfer function of

$$H(s) = \frac{G(s)}{1 + G(s)\,H^1(s)} = \frac{\theta_0}{\theta_{in}} . \tag{A1}$$

Hence, for Figure 5, $H^1(s) = 1$ and

$$G(s) = K\phi \frac{K_{vco}}{s} \left(\frac{T_2 s + 1}{T_1 s + 1}\right) = \frac{K(T_2 s + 1)}{s(T_1 s + 1)} , \tag{A2}$$

where $T_1 = (R_1 + R_2)C$, $T_2 = R_2 C$, and $K = K\phi\, K_{vco}$.

Therefore, the transfer function becomes, successively,

$$H(s) = \frac{\dfrac{K(T_2 s + 1)}{s(T_1 s + 1)}}{1 + \dfrac{K(T_2 s + 1)}{s(T_1 s + 1)}} , \tag{A3}$$

$$H(s) = \frac{T_2\, s + 1}{\left(\dfrac{T_1}{K}\right) s^2 + \left(\dfrac{1}{K} + T_2\right) s + 1} , \tag{A4}$$

and

$$H(s) = \frac{T_2 K}{T_1} \frac{\left(s + \dfrac{1}{T_2}\right)}{\left[s^2 + \left(\dfrac{1 + KT_2}{T_1}\right) s + \dfrac{K}{T_1}\right]} . \tag{A5}$$

In actual practice, $KT_2 >> 1$, so that the transfer function is

$$H(s) = \frac{T_2 K}{T_1} \frac{\left(s + \frac{1}{T_2}\right)}{s^2 + \frac{KT_2}{T_1} s + \frac{K}{T_1}} , \qquad \text{(A6)}$$

for $2\zeta\omega_0 = KT_2/T_1$ and $\omega_0^2 = K/T_1$ . If we combine Equation 27

$$H(s) = \frac{2\zeta\omega_0 \; s + \omega_0^2}{s^2 + 2\zeta\omega_0 \; s + \omega_0^2} , \qquad \text{(27)}$$

and Equation A1,

$$H(s) = \frac{G(s)}{1 + G(s)} \qquad \text{(A1)}$$

we find that the open loop transfer function $G(s)$ becomes (Figure A2)

$$G(s) = \frac{2\zeta\omega_0 \left(s + \frac{\omega_0}{2\zeta}\right)}{s^2} . \qquad \text{(A7)}$$

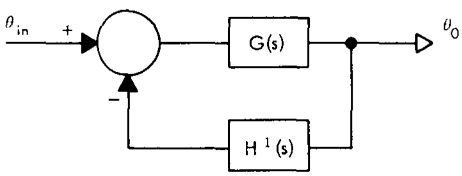


Figure A1—Standard configuration for a servo system.

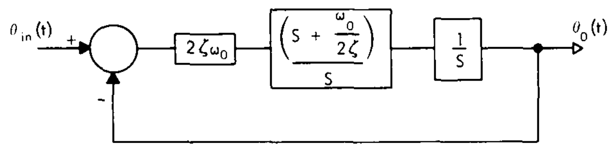


Figure A2—Simplified second order phase lock loop tracking filter.

Appendix B

## Computations for the ENBW for the Second Order (Type 1) Phase Lock Loop Tracking Filter

*Method 1: ENBW Analysis of the Second Order (Type 1) Phase Lock Loop by Evaluating the Residues*

From Equation 6, we see that BN1 is, successively,

$$\mathrm{BN1} = \int_0^\infty |H(j\omega)|^2 \, d\omega \ (\mathrm{rad/sec}) \ , \tag{6}$$

$$\mathrm{BN1} = \int_0^\infty \left| \frac{\omega_0^2 + j2\zeta\omega_0\omega}{\left(\omega_0^2 - \omega^2\right) + j2\zeta\omega_0\omega} \right|^2 d\omega \ , \tag{B1}$$

$$= \int_0^\infty \frac{\omega_0^2\left(\omega_0^2 + 4\zeta^2\omega^2\right) d\omega}{\omega^4 + \left(4\zeta^2\omega_0^2 - 2\omega_0^2\right)\omega^2 + \omega_0^4} \ . \tag{B2}$$

A typical compromise for minimum ENBW and minimum transient time response for a phase lock loop occurs when $\zeta = 0.707$. Therefore,

$$\mathrm{BN1} = \int_0^\infty \frac{\omega_0^2\left(\omega_0^2 + 2\omega^2\right)}{\omega^4 + \omega_0^4} \, d\omega \ , \tag{B3}$$

$$= \frac{1}{2}\int_{-\infty}^\infty \frac{\omega_0^2\left(\omega_0^2 + 2\omega^2\right)}{\omega^4 + \omega_0^4} \, d\omega \ . \tag{B4}$$

The four roots of $\omega$ (poles of $|H(j\omega)|^2$) are (Figure B1):

$$\omega_1 = \omega_0 \, e^{\frac{\pi}{4} j} = \left(\frac{1+j}{\sqrt{2}}\right)\omega_0 \, , \tag{B5}$$

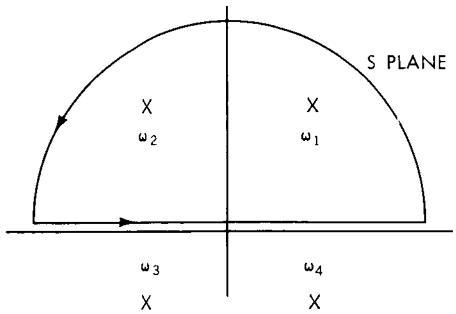


Figure B1—Poles of $1/(\omega^4 + \omega_0^4)$.

$$\omega_2 = \omega_0 e^{\frac{3\pi j}{4}} = \left(\frac{-1 + j}{2}\right) \omega_0 , \tag{B6}$$

$$\omega_3 = \omega_0 e^{\frac{5\pi j}{4}} = \left(\frac{-1 - j}{2}\right) \omega_0 , \tag{B7}$$

$$\omega_4 = \omega_0 e^{-\frac{\pi j}{4}} = \left(\frac{1 - j}{2}\right) \omega_0 . \tag{B8}$$

Since the degree of the denominator of $|H(j\omega)|^2$ is twice the degree of the numerator, each residue can be evaluated by taking the derivative of the denominator (Reference 3) and evaluating the residue for each pole in the upper half plane.

$$\begin{aligned}
BN1 &= \frac{\omega_0^2}{2} \left[ 2\pi i \sum \text{Residues of } \omega_1 \text{ and } \omega_2 \right] , \\
&= \pi i \omega_0^2 \left\{ \left. \frac{\omega_0^2 + 2\omega^2}{4\omega^3} \right|_{\omega=\omega_1} + \left. \frac{\omega_0^2 + 2\omega^2}{4\omega^3} \right|_{\omega=\omega_2} \right\} , \\
&= \frac{3}{4} \sqrt{2}\pi \, \omega_0 , \\
&= 3.33 \, \omega_0 \left(\frac{\text{rad}}{\text{sec}}\right) , \\
&= \frac{3}{8} \sqrt{2} \, \omega_0 = .531 \, \omega_0 \ (\text{cps}) .
\end{aligned} \tag{B9}$$

*Method 2: ENBW Analysis of the Second Order (Type 1) Phase Lock Loop by use of the Table of Integrals*

Since Equation 27 gives the transfer function as

$$H(s) = \frac{c(s)}{d(s)} = \frac{2\zeta\omega_0 s + \omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} , \tag{27}$$

we may set $N = 2$ in the series expression for $c(s)$ in Equation 21b and in the series expression for $d(s)$ in Equation 21c. From Equation 5, we see that BN1 is, successively,

$$\mathrm{BN1} = \frac{1}{2\pi j} \int_{0}^{+j\infty} |H(j\omega)|^2 \, d(j\omega) \ (\mathrm{cps}) \ , \tag{5}$$

$$= \frac{1}{2} \left[ \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} |H(j\omega)|^2 \, d(j\omega) \right] , \tag{B10}$$

$$= \frac{1}{2} (I_2) \ , \tag{B11}$$

$$= \frac{1}{2} \ \frac{c_1^2 d_0 + c_0^2 d_2}{2 \ d_0 \ d_1 \ d_2} \ , \tag{B12}$$

where $c_1 = 2\zeta\omega_0$, $c_0 = \omega_0^2$, $d_2 = 1$, $d_1 = 2\zeta\omega_0$, and $d_0 = \omega_0^2$. Therefore,

$$\mathrm{BN1} = \left( \frac{4\zeta^2 + 1}{8\zeta} \right) \omega_0 \ (\mathrm{cps}) \ . \tag{B13}$$

For a damping factor of $\zeta = .707$, the ENBW may be expressed by

$$\mathrm{BN1} = \frac{3}{8} \sqrt{2} \, \omega_0 = .531 \, \omega_0 \ (\mathrm{cps}) \ , \tag{B14}$$

$$= 3.33 \, \omega_0 \ (\mathrm{rad/sec}) \ . \tag{B15}$$

This agrees with the method of evaluating residues; however, as can be seen, much computation time is saved by using method 2 instead of method 1.

Appendix C

## Analysis of the Two Sided ENBW for a Phase Lock Loop Tracking Filter

The phase detector in Figure 9 is functioning as a multiplier, hence, the error signal appearing at the output of the phase detector is (from Equation 34)

$$V_\epsilon(t) = \theta_{in}(t)\,\theta_{out}(t) , \tag{C1}$$

or

$$V_\epsilon(t) = E_{in} \sin\left[\omega ct + \theta_1(t)\right] E_{vco} \cos\left[\omega ct + \theta_{vco}(t)\right] . \tag{C2}$$

From the trigonometric identity $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$, it follows that

$$\sin x \cos y = \frac{\sin(x+y) + \sin(x-y)}{2} .$$

Letting $x = \omega_c t + \theta_1(t)$ , and $y = \omega t + \theta_{vco}(t)$, we may then express $V_\epsilon(t)$ by

$$V_\epsilon(t) = \frac{E_{in}\,E_{vco}}{2} \sin\left[\theta_1(t) - \theta_{vco}(t)\right] + \frac{E_{in}\,E_{vco}}{2} \sin\left[2\omega_c t + \theta_1(t) + \theta_{vco}(t)\right] . \tag{C3}$$

Assuming that the $2\omega_c t$ term is negligible since it is filtered out by the loop filter, $V_\epsilon(t)$ becomes

$$V_\epsilon(t) = \frac{E_{in}\,E_{vco}}{2} \sin\left[\theta_1(t) - \theta_{vco}(t)\right] . \tag{C4}$$

Again, assuming that the system is tracking near phase, the difference angle is normally small, and since $\sin\theta \approx \theta$ for small angles, we find that

$$V_\epsilon(t) = \frac{E_{in}\,E_{vco}}{2} \left[\theta_1(t) - \theta_{vco}(t)\right] , \tag{C5}$$

$$= \frac{E_{in}\,E_{vco}}{2} \left[\Delta\omega t\right] . \tag{C6}$$

If $\theta_1(t) > \theta_{vco}(t)$, then $\Delta\omega t$ is positive, and if $\theta_1(t) < \theta_{vco}(t)$ , then $\Delta\omega t$ is negative.

The output of the phase detector is a dc voltage that varies with the phase difference between the input and reference signals. The transfer function for the phase detector is shown in Figure C1.

As the phase difference varies from 0 to $\pi$, or $\pi$ to $2\pi$, the voltage will go through its full range. The phase lock loop will only stay in lock between 0 and $\pi$ (or $\pi$ and $2\pi$) phase difference points. Usually the phase lock loop is designed so that the center frequency lies at $\pi/2$ phase difference point. Therefore, the feedback frequency will be lagging the input-signal frequency by 90 degrees.

The transfer function of the VCO is shown in Figure C2. The dc level at the input of the VCO sets the output frequency. With no dc input signal (ground), the oscillator can be designed to operate at the system's center frequency.

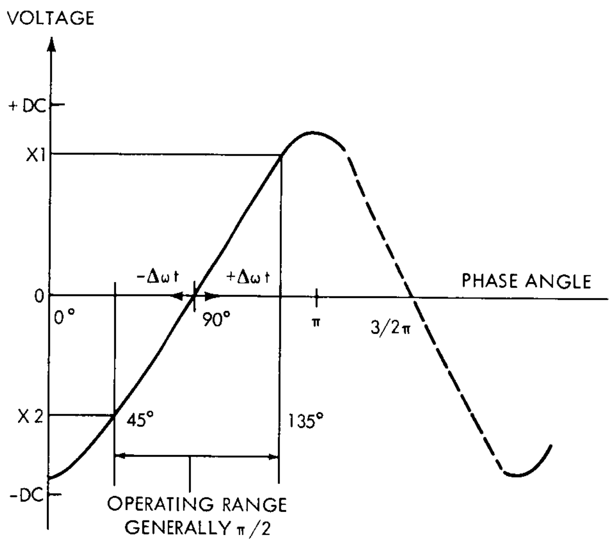


Figure C1–Phase detector transfer function.

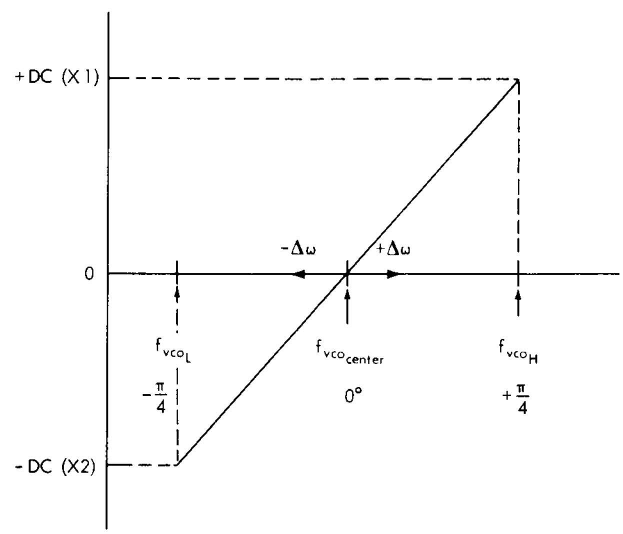


Figure C2–VCO transfer function.

Appendix D

## Development of the Transfer Function for the Third Order (Type 2) Phase Lock Loop Tracking Filter

As in the second order (Type 2) phase lock loop (see Figure A1), the transfer function is

$$H(s) = \frac{G(s)}{1 + G(s)\,H^1(s)} , \tag{D1}$$

where $H^1(s) = 1$. From Figure 12, $G(s)$ becomes

$$G(s) = \frac{GK\phi\, K_{vco}}{s} \left(\frac{T_2 s + 1}{T_1 s + 1}\right)\left(\frac{T_4 s + 1}{T_3 s + 1}\right) , \tag{D2}$$

where $T_1 = (R_1 + R_2) C_4$, $T_2 = R_2\, C_1$, $T_3 = GR_3\, C_2$, and $T_4 = R_4\, C_2$. Hence the transfer function is now

$$H(s) = \frac{\dfrac{GK\phi\, K_{vco}}{s}\left(\dfrac{T_2 s + 1}{T_1 s + 1}\right)\left(\dfrac{T_4 s + 1}{T_3 s + 1}\right)}{1 + \dfrac{GK\phi\, K_{vco}}{s}\left(\dfrac{T_2 s + 1}{T_1 s + 1}\right)\left(\dfrac{T_4 s + 1}{T_3 s + 1}\right)} . \tag{D3}$$

Taking a practical case where $R_2\, C_1 = R_4\, C_2 = 3/\omega_0$, and

$$\alpha_1 = \frac{R_1 + R_2}{R_2} , \quad \alpha_2 = \frac{R_3}{R_4} , \tag{D4}$$

$$\frac{GK\phi\, K_{vco}}{\alpha_1\, \alpha_2} = K\omega_0 , \tag{D5}$$

we find that the transfer function may be now written

$$H(s) = \frac{K\left(s + \frac{\omega_0}{3}\right)^2}{s^3 + Ks^2 + \frac{2K\omega_0}{3}\, s + K\left(\frac{\omega_0}{3}\right)^2} . \tag{D6}$$

Taking the typical case where, $K = 9\omega_0/4$ we may see that

$$H(s) = \frac{9\omega_0/4 \left(s + \frac{\omega_0}{3}\right)^2}{(s + \omega_0)^2 \left(s + \frac{\omega_0}{4}\right)} . \tag{D7}$$

By equating Equations D1 and D7,

$$H(s) = \frac{9\omega_0/4 \left(s + \frac{\omega_0}{3}\right)^2}{(s + \omega_0)^2 \left(s + \frac{\omega_0}{4}\right)} = \frac{G(s)}{1 + G(s)}$$

and the open loop transfer function $G(s)$ becomes (Figure D1):

$$G(s) = \frac{9\omega_0/4 \left(s + \frac{\omega_0}{3}\right)^2}{s^3} . \tag{D8}$$

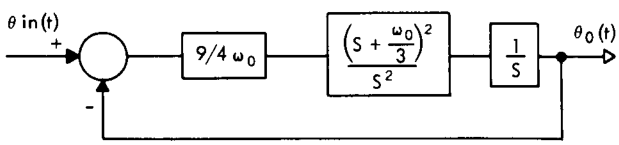


Figure D1–Simplified third order phase lock loop tracking filter.

Appendix E

# Computations for the ENBW for the Third Order (Type 2) Phase Lock Loop Tracking Filter

*Method 1: ENBW Analysis of the Third Order (Type 2) Phase Lock Loop Tracking Filter by Evaluating the Residues*

From Equation 6, we saw that BN1 is, successively,

$$\mathrm{BN1} = \int_0^{\infty} |H(j\omega)|^2 \, d\omega \ (\mathrm{rad/sec}) \ , \tag{6}$$

$$\mathrm{BN1} = \frac{1}{2} \int_{-\infty}^{\infty} |H(j\omega)|^2 \, d\omega \ , \tag{E1}$$

$$= \frac{81}{32} \omega_0^2 \int_{-\infty}^{\infty} \frac{\left(\frac{\omega_0^2}{9} + \omega^2\right)^2}{\left(\frac{\omega_0^2}{16} + \omega^2\right)\left(\omega_0^2 + \omega^2\right)^2} \, d\omega \ , \tag{E2}$$

$$= \frac{81 \ \omega_0^2}{32} \left[2\pi i \sum \mathrm{Residues}\right] . \tag{E3}$$

The six roots of $\omega$ (poles of $|H(j\omega)|^2$) are (Figure E1):

$$\left.\begin{aligned} \omega_1 &= j \frac{\omega_0}{4} \ , \quad \omega_2 = -j \frac{\omega_0}{4} \\ \omega_3 &= +j\omega_0 \ (\text{multiple poles}) \ , \\ \omega_4 &= -j\omega_0 \ (\text{multiple poles}) \ . \end{aligned}\right\} \tag{E4}$$

Therefore, the ENBW is

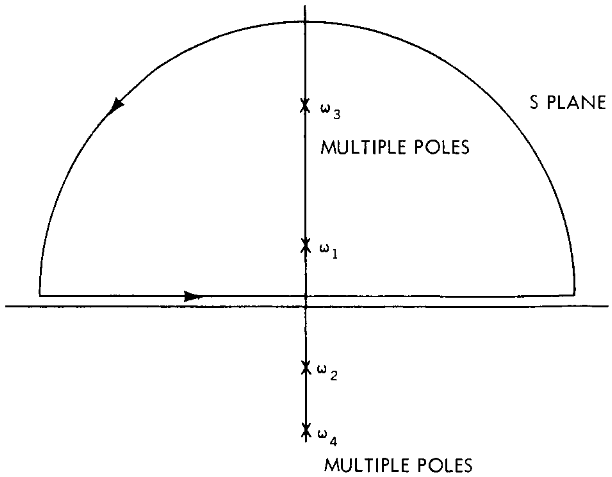


Figure E1—Poles of $\left[\left(\frac{\omega_0^2}{16} + \omega^2\right)\left(\omega_0^2 + \omega^2\right)^2\right]^{-1}$.

$$\mathrm{BN1} = \frac{81}{16}\pi j \omega_0^2\left(\sum \text{Residues of R1 and R3}\right), \tag{E5}$$

where for a simple pole R1 is,

$$\mathrm{R1} = \left.\frac{\left(\frac{\omega_0^2}{9} + \omega^2\right)^2 (\omega - \omega_1)}{(\omega - \omega_1)(\omega - \omega_2)(\omega_0^2 + \omega^2)^2}\right|_{\omega_1 = j\frac{\omega_0}{4}} \tag{E6}$$

$$= \frac{+.00538}{j\omega_0}. \tag{E7}$$

Since R3 contains multiple poles, it is found by the following analysis (Reference 3). Its first derivative is

$$\mathrm{R'3} = \frac{\left(\frac{\omega_0^2}{9} + \omega^2\right)^2 (\omega - \omega_3)^2}{\left(\frac{\omega_0^2}{16} + \omega^2\right)(\omega - \omega_4)^2(\omega - \omega_3)^2}, \tag{E8}$$

where $\mathrm{R3} = d/d\omega\,(\mathrm{R'3})|_{\omega = j\omega_0}$ for multiple poles. Thus, we have

$$\mathrm{R3} = \left.\frac{\left(\frac{\omega_0^2}{16} + \omega^2\right)(\omega + j\omega_0)^2\, 4\omega\left(\frac{\omega_0^2}{9} + \omega^2\right) - \left(\frac{\omega_0^2}{9} + \omega^2\right)^2\left[\left(\frac{\omega_0^2}{16} + \omega^2\right)(2)(\omega + j\omega_0) + (\omega + j\omega_0)^2\, 2\omega\right]}{\left(\frac{\omega_0^2}{16} + \omega^2\right)^2 (\omega + j\omega_0)^4}\right|_{\omega = j\omega_0} \tag{E9}$$

which reduces to $\mathrm{R3} = +.289/j\omega_0$. Therefore the ENBW becomes, after evaluating residues,

$$\therefore \mathrm{BN1} = \frac{81\,\pi\omega_0^2\, j}{16}\left(\frac{.00538}{j\omega_0} + \frac{.289}{j\omega_0}\right) \tag{E10}$$

$$\mathrm{BN1} = 4.68\,\omega_0\ (\text{rad/sec}) = .743\,\omega_0\,(\text{cps}). \tag{E11}$$

Hence, the two sided equivalent noise bandwidth for the third order (Type 2) phase lock loop tracking filter is

$$\mathrm{BN2} = 2\,\mathrm{BN1} = 9.36\,\omega_0\ (\text{rad/sec}). \tag{E12}$$

*Method 2: ENBW Analysis of the Third Order (Type 2) Phase Lock Loop Tracking Filter by Use of the Table of Integrals*

The general form of the transfer function for the third order (Type 2) system was shown to be (Equation 41)

$$H(s) = \frac{Ks\left(+\frac{\omega_0}{3}\right)^2}{s^3 + Ks^2 + 2K\left(\frac{\omega_0}{3}\right)s + K\left(\frac{\omega_0}{3}\right)^2} .$$

From Equations 5, 23, and 25, we find that BN1 is, successively,

$$BN1 = \frac{1}{2}\left[\frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} |H(j\omega)|^2 \, d(j\omega)\right] \text{ cps}, \tag{25}$$

$$= \frac{K^2}{2}(I_3), \tag{E13}$$

$$= \frac{K^2}{2}\left[\frac{c_2^2 d_0 d_1 + (c_1^2 - 2c_0 c_2) d_0 d_3 + c_0^2 d_2 d_3}{2 d_0 d_3 (-d_0 d_3 + d_1 d_2)}\right], \tag{E14}$$

since $N = 3$ for $H(j\omega)$, and where $c_2 = 1$, $c_1 = 2/3\ \omega_0$, $c_0 = \omega_0^2/9$, $d_3 = 1$, $d_2 = K$, $d_1 = 2K\omega_0/3$, and $d_0 = K\omega_0^2/9$. From this we see that

$$BN1 = \frac{K^2}{2}\left[\frac{\frac{2K^2\omega_0^3}{27} + \left(\frac{4\omega_0^2}{9} - \frac{2\omega_0^2}{9}\right)\frac{K\omega_0^2}{9} + \frac{\omega_0^4}{81}K}{\frac{2K\omega_0^2}{9}\left(-\frac{K\omega_0^2}{9} + \frac{2K^2\omega_0}{3}\right)}\right], \tag{E15}$$

$$= \frac{3}{4}\,\frac{(2K + \omega_0)}{\left(6 - \frac{\omega_0}{K}\right)}, \tag{E16}$$

and for $K = 9\omega_0/4$, the ENBW becomes, finally, $BN1 = .743\ \omega_0$ (cps) $= 4.68\ \omega_0$ (rad/sec). This agrees with the results computed by Method 1.

Appendix F

# Determining the Relationship Between the Gain Factor (K) and the Damping Factor ($\zeta$) for the Third Order (Type 2) Phase Lock Loop Tracking Filter

By examining the factors of the quadratic equation $s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0$, one finds that $s = -\zeta\omega_0 \pm \omega_0 \sqrt{\zeta^2 - 1}$ or its factors are $\left(s + \zeta\omega_0 + \omega_0 \sqrt{\zeta^2 - 1}\right)$ and $\left(s + \zeta\omega_0 - \omega_0 \sqrt{\zeta^2 - 1}\right)$. Plotting the roots of the quadratic equation in the S plane (see Figure F1) allows one to describe the damping factor $\zeta$ $\left(\text{for } 0 \le \zeta \le 1,\ \omega_0 \sqrt{\zeta^2 - 1} \text{ becomes } j\omega_0 \sqrt{1 - \zeta^2}\right)$. The damping factor is seen to be $\zeta = \cos\theta$. Hence when $\theta = 0°$, maximum damping results since $\zeta = 1$, and when $\theta = 90°$, no damping results since $\zeta = 0$, and oscillatory motion occurs.

The physical significance of the damping factor $\zeta$ can be seen by taking the inverse LaPlace transform $(L^{-1})$ of the quadratic equation

$$\frac{1}{s^2 + 2\zeta\omega_0 s + \omega_0^2} = 0. \qquad \text{(F1)}$$

This results in an exponential damped sinusoidal function with the damping controlled by $\zeta$ and $\omega_0$, so that

$$L^{-1}\left[\frac{1}{\left(s^2 + 2\zeta\omega_0 s + \omega_0^2\right)}\right] = \frac{\omega_0}{\sqrt{1 - \zeta^2}}\, \epsilon^{-\zeta\omega_0 t} \sin \omega_0 \sqrt{1 - \zeta^2}\, t. \qquad \text{(F2)}$$

When $\zeta = 0$,

$$L^{-1}\left(\frac{1}{s^2 + \omega_0^2}\right) = \omega_0 \sin \omega_0 t \qquad \text{(F3)}$$

with no damping of the sinusoidal function.

To determine the value of $\zeta$ for each K value of the third order system, the denominator term of $H(s)$ can be solved in the general cubic equation of the form

$$y^3 + py^2 + qy + r = 0. \qquad \text{(F4)}$$

The general cubic equation may be reduced to

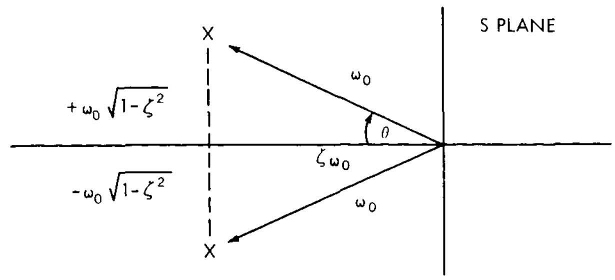


Figure F1—Poles of $1/(s^2 + 2\zeta\omega_0 s + \omega_0^2)$.

the form $x^3 + ax + b = 0$ by substituting for $y$ the value, $x - P/3$. Here, $a = (1/3)(3q - p^2)$ and, $b = (1/27)(2p^3 - 9pq + 27r)$. For solution,

$$A = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}\ , \tag{F5}$$

and

$$B = \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}\ . \tag{F6}$$

The values of $y$ will then be given by Equations F7, F8 and F9 if $b^2/4 + a^3/27 \geq 0$ :

$$y_1 = A + B - \frac{p}{3}\ , \tag{F7}$$

$$y_2 = -\left(\frac{A + B}{2} + \frac{p}{3}\right) + \left(\frac{A - B}{2}\right)\sqrt{-3}\ , \tag{F8}$$

$$y_3 = -\left(\frac{A + B}{2} + \frac{p}{3}\right) - \left(\frac{A - B}{2}\right)\sqrt{-3}\ . \tag{F9}$$

If $b^2/4 + a^3/27 < 0$, compute the value of the angle $\phi$ in the expression

$$\cos\phi = \frac{-b/2}{\sqrt{-a^3/27}}\ . \tag{F10}$$

Then, $y$ will have the following values:

$$y_1 = 2\sqrt{\frac{-a}{3}}\cos\frac{a}{3} - \frac{P}{3} \tag{F11}$$

$$y_2 = 2\sqrt{\frac{-a}{3}}\cos\left(\frac{a}{3} + 120°\right) - \frac{P}{3} \tag{F12}$$

$$y_3 = 2\sqrt{\frac{-a}{3}}\cos\left(\frac{a}{3} + 240°\right) - \frac{P}{3} \tag{F13}$$

The denominator term of $H(s)$ for the third order system, with $\omega_0 = 1$, is

$$s^3 + Ks^2 + \frac{2}{3}Ks + \frac{K}{9} = 0\ , \tag{F14}$$

where $p = K$, $q = 2/3\ K$, and $r = K/9$ in the cubic form.

Therefore, $a = (K/3)(2 - K)$, and $b = (K/27)(2K^2 - 6K + 3)$.

For the case when $K = 9/4$, the cubic equation reduces to $(s^2 + 2s + 1)(s + 1/4) = 0$ where $\zeta = 1$ when $\omega_0 = 1$. For values of K less than 9/4, there will be a corresponding value for $\zeta$. Each value of $\zeta$ was found by programming the Equations F15, F16 and F17 on a computer and computing $\zeta$ for a range of K values. Figure 15 represents a comparison of the ENBW for the second order (Type 1) and third order (Type 2) phase lock loop tracking filter for $\omega_0 = 1$. Table F1 contains the results of solving the equivalent damping factor $\zeta$ for different values of K.

For the cubic equation $s^3 + Ks^2 + 2Ks/3 + K/9 = 0$, the roots are

$$s_1 = A + B - K/3\ , \tag{F15}$$

$$s_2 = -\left(\frac{A + B}{2} + \frac{K}{3}\right) + \left(\frac{A - B}{2}\right)\sqrt{3}j\ , \tag{F16}$$

$$s_3 = -\left(\frac{A + B}{2} + \frac{K}{3}\right) - \left(\frac{A - B}{2}\right)\sqrt{3}j\ . \tag{F17}$$

By letting $C = A+B/2 + K/3$, $F = A + B - K/3$, and $D = (A - B/2)\sqrt{3}$, the cubic equation was solved and can be put in the form of

$$\left[(s_3 + C + Dj)(s_2 + C - Dj)(s_1 + F)\right] = 0. \tag{F18}$$

Figure F2 shows the equivalent value for the damping factor $\zeta$ (where $\zeta = \cos\theta$).

Table F1*

Tabulation of Values for K and $\zeta$ for the Third Order (Type 2) Phase Lock Loop Tracking Filter

| K | $\zeta$ |
|---|---|
| .1 | .09157 |
| .2 | .03757 |
| .3 | .13432 |
| .4 | .21417 |
| .5 | .28373 |
| .6 | .34612 |
| .7 | .40314 |
| .8 | .45594 |
| .9 | .50534 |
| 1.0 | .55190 |
| 1.1 | .59606 |
| 1.2 | .63815 |
| 1.3 | .67843 |
| 1.4 | .71711 |
| 1.5 | .75437 |
| 1.6 | .79035 |
| 1.7 | .82517 |
| 1.8 | .85894 |
| 1.9 | .89174 |
| 2.0 | .92397 |
| 2.1 | .95476 |
| 2.2 | .98510 |
| 2.25 | 1.00000 |

*By the use of this table and Figure 14, Figure 15 was plotted for the 3rd order system.

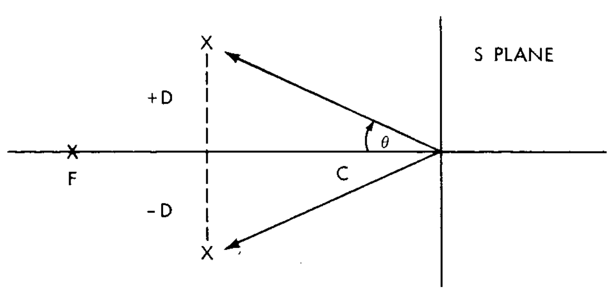


Figure F2—Poles of $1/(s^3 + Ks^2 + 2Ks/3 + K/9)$.

Appendix G

## Computations for the ENBW of a Single Pole Band Pass Filter

The transfer function for a single pole band pass filter (in Figure G1) can be shown to be

$$H(s) = \frac{E_0(s)}{E_{in}(s)} = \frac{(R_2/L)\ s}{s^2 + \left(\frac{R_1 + R_2}{L}\right) s + \frac{1}{LC}} . \tag{G1}$$

By making the following substitutions,

$$Q = \frac{\omega L}{R_1 + R_2} = \frac{\omega_0}{2\pi\ f_{3db}} , \quad \omega_0^2 = \frac{1}{LC} , \quad K = \frac{R_2}{R_1 + R_2} \tag{G2}$$

and by defining $f_{3db}$ as the 3 db down bandwidth (cps), the transfer function becomes

$$H(s) = \frac{\left(K/2\pi\ f_{3db}\ Q^2\right) s}{\left(s/2\pi\ f_{3db}\ Q\right)^2 + \left(1/2\pi\ f_{3db}\ Q^2\right)\ s + 1} . \tag{G3}$$

A frequency response for this filter is shown in Figure G2.

The ENBW is computed by Method 2 of this report, and we find that

$$\text{ENBW} = \frac{\int_0^\infty |H(j\omega)|^2\ d\omega}{|H(j\omega)|^2_{max}} = \frac{\frac{1}{2}\int_{-\infty}^\infty |H(j\omega)|^2\ d\omega}{|H(j\omega)|^2_{max}} \tag{G4}$$

or

$$\text{ENBW} = \frac{1}{2} \frac{I_2}{|H(j\omega)|^2_{max}} , \tag{G5}$$

where

$$I_2 = \frac{c_1^2 d_0 + c_0^2 d_2}{d_0 d_1 d_2} ; \tag{G6}$$

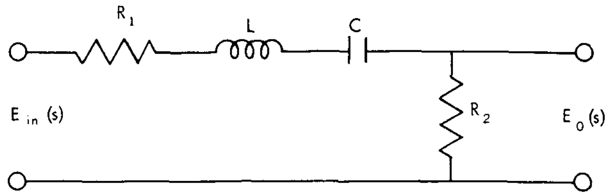


Figure G1–Single pole band-pass filter.

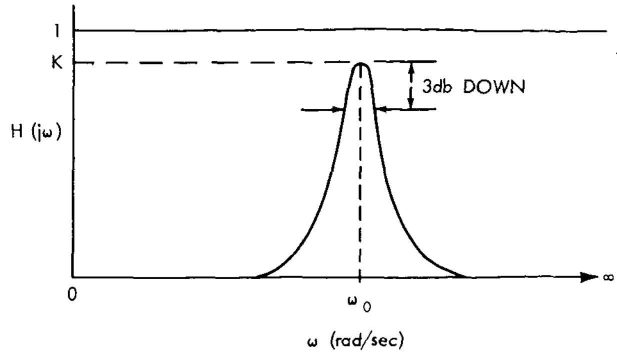


Figure G2–Frequency response of a single pole band pass filter.

with

$$\left.\begin{array}{lll} c_1 = \dfrac{K}{2\pi\ f_{3db}\ Q^2}, & c_0 = 0 & \\ d_2 = \dfrac{1}{4\pi^2\ f_{3db}^2\ Q^2}, & d_1 = \dfrac{1}{2\pi\ f_{3db}\ Q^2}, & d_0 = 1. \end{array}\right\} \tag{G7}$$

Therefore, $I_2$ reduces to the simple expression

$$I_2 = \pi K^2\ f_{3db}\ , \tag{G8}$$

and $|H(j\omega)|_{max}$ occurs when $\omega = 1/\sqrt{LC}$. One can show, therefore, that

$$|H(j\omega)|^2_{max} = K^2\ . \tag{G9}$$

Finally, the ENBW becomes upon substitution and reduction,

$$\mathrm{ENBW} = \frac{\pi K^2\ f_{3db}}{2K^2} = \frac{\pi}{2}\ f_{3db}\ (\mathrm{cps})\ . \tag{G10}$$

Appendix H

# Comparison of the Tracking Rate for the Second Order (Type 1) and Third Order (Type 2) Phase Lock Loop Tracking Filters

From the standard configuration for a servo system (shown in Figure H1), an expression for the phase error signal $\theta_\epsilon(s)$ is found by the following equations:

$$\theta_\epsilon(s) = \theta_{in}(s) - \theta_0(s) \, , \tag{H1}$$

$$\frac{\theta_\epsilon(s)}{\theta_{in}(s)} = \frac{\theta_{in}(s) - \theta_0(s)}{\theta_{in}(s)} \, , \tag{H2}$$

$$\frac{\theta_\epsilon(s)}{\theta_{in}(s)} = 1 - \frac{\theta_0(s)}{\theta_{in}(s)} = 1 - H(s) \, . \tag{H3}$$

Since $\theta_\epsilon(s)$ represents the phase error corresponding to the phase difference (radians) between the input and output frequencies of the phase lock loop, it is desirable to determine a relationship between the phase error and the change of input frequency (radians/second). Since $\theta(\text{radians}) = \int \omega dt$ or $\omega = d\theta/dt$ , Figure H1 can be drawn as shown in Figure H2.

From this relationship we may see that

$$\theta_{in}(s) = \frac{\Delta\omega(s)}{s} \, , \tag{H4}$$

and

$$\theta_\epsilon(s) \quad \theta_{in}(s) \, [1 - H(s)] \, , \tag{H5}$$

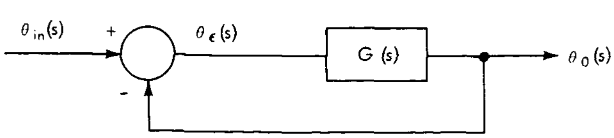


Figure H1—Standard configuration for a servo system with unity feedback.

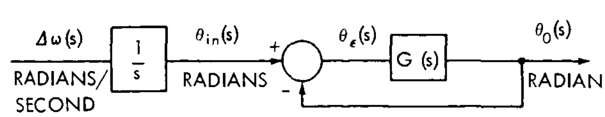


Figure H2—Conversion of a phase input to a frequency input for the phase lock loop.

so we may make one further substitution so that

$$\theta_\epsilon(s) = \frac{\Delta\omega(s)}{s}\left[1 - H(s)\right]. \tag{H6}$$

It is now desirable to compute the tracking rate of a phase lock loop by assuming a ramp input frequency (acceleration in phase). Hence, let

$$\Delta\omega(s) = \frac{C_\omega}{s^2}, \tag{H7}$$

where $C_\omega$ is the slope of the ramp input frequency in radians/second/second. Therefore, a general expression for the phase error in terms of the input tracking rate $C_\omega$ becomes

$$\theta_\epsilon(s) = \frac{C_\omega}{s^3}\left[1 - H(s)\right]. \tag{H8}$$

It will be assumed that the maximum phase error which the phase lock loop can tolerate before "dropping-out-of-lock" will be $\pi/4$ radians, shown in the following analysis.

*Second Order (Type 1) Tracking Rate Analysis For a Ramp Input Frequency*

The transfer function for the second order system was shown to be (from Equation 27)

$$H(s) = \frac{2\zeta\omega_0 s + \omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}. \tag{27}$$

Therefore the phase error for a ramp input frequency becomes, successively,

$$\theta_\epsilon(s) = \frac{C_\omega}{s^3}\left[1 - \frac{2\zeta\omega_0 s + \omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}\right], \tag{H9}$$

$$\theta_\epsilon(s) = \frac{C_\omega}{s^3}\left[\frac{s^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}\right], \tag{H10}$$

$$\theta_\epsilon(s) = \frac{C_\omega}{s\left(s^2 + 2\zeta\omega_0 s + \omega_0^2\right)}, \tag{H11}$$

$$\theta_\epsilon(t) = \frac{C_\omega}{\omega_0^2}\left[1 + \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_0 t} \sin\left(\omega_0 \sqrt{1 - \zeta^2}\, t - \psi\right)\right]. \tag{H12}$$

where

$$\psi = \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{-\zeta} . \qquad \text{(H13)}$$

It can be shown that $\theta_\epsilon(t)_{max}$ occurs at $t = 0$, so that for $\zeta = .707$.

$$\theta_\epsilon(t)_{max} = \frac{2C_\omega}{\omega_0^2} , \qquad \text{(H14)}$$

$$\frac{\pi}{4} = \frac{2C_\omega}{\omega_0^2} \qquad \text{(H15)}$$

and $C_\omega$ becomes

$$C_\omega = \frac{\pi}{8} \omega_0^2 \text{ radians/second/second} . \qquad \text{(H16)}$$

Since $C_\omega = 2\pi\, C_f$, we now find that

$$C_f = \frac{1}{16} \omega_0^2 \text{ cycles/second/second} , \qquad \text{(H17)}$$

and since for $\zeta = .707$,

$$BN1 = .531\, \omega_0 \text{ (cps)} , \qquad \text{(H18)}$$

the tracking rate becomes

$$C_f = .222\, BN1^2 \text{ cycles/second/second.} \qquad \text{(H19)}$$

*Third Order (Type 2) Tracking Rate Analysis for a Ramp Input Frequency*

The transfer function for the third order system was shown to be (from Equation 37)

$$H(s) = \frac{\frac{9}{4} \omega_0 \left(s + \frac{\omega_0}{3}\right)^2}{\left(s + \frac{\omega_0}{4}\right)(s + \omega_0)^2} . \qquad \text{(37)}$$

Therefore, the phase error for a ramp input frequency becomes, in succession,

$$\theta_\epsilon(s) = \frac{C_\omega}{s^3}\left[1 - \frac{\frac{9}{4}\omega_0\left(s + \frac{\omega_0}{3}\right)^2}{\left(s + \frac{\omega_0}{4}\right)\left(s + \omega_0\right)^2}\right], \tag{H20}$$

$$\theta_\epsilon(s) = \frac{C_\omega}{s^3}\left[\frac{s^3}{\left(s + \frac{\omega_0}{4}\right)\left(s + \omega_0\right)^2}\right], \tag{H21}$$

$$\theta_\epsilon(s) = \frac{C_\omega}{\left(s + \frac{\omega_0}{4}\right)\left(s + \omega_0\right)^2}, \tag{H22}$$

$$\theta_\epsilon(t) = \frac{4C_\omega}{\omega_0^3}\left[\frac{T\omega_0^2}{1 + T^2\omega_0^2} e^{-t/T} + \frac{\omega_0 \sin\left(\omega_0 t - \psi\right)}{\sqrt{1 + T^2\omega_0^2}}\right], \tag{H23}$$

where

$$T = \frac{\omega_0}{4} \text{ and } \psi = \tan^{-1}\left(T\omega_0\right). \tag{H24}$$

By differentiating the expression for $\theta_\epsilon(t)$ and equating the derivative to zero results in

$$\theta_\epsilon(t)_{max} = \frac{.552C_\omega}{\omega_0^2}. \tag{H25}$$

Hence, equating this maximum phase error to the maximum tolerable phase error of $\pi/4$ radians, the tracking rate becomes

$$C_\omega = \frac{\pi\,\omega_0^2}{4(.552)} \text{ radians/sec}^2 \tag{H26}$$

or

$$C_f = \frac{\omega_0^2}{8(.552)} \text{ cycles/sec}^2 . \tag{H27}$$

Since $BN1 = .743\,\omega_0$ (cps) the tracking rate becomes

$$C_f = .41\ BN1^2 \text{ cycles/second/second}. \tag{H28}$$

Hence, the third order (Type 2) system can track an input ramp frequency 1.85 times that of a second order (Type 1) system having the same ENBW and a $\zeta = .707$.

Appendix I

## List of Symbols and Definitions

| | |
|---|---|
| BN1 | ENBW for positive frequencies $(\omega > \omega_0)$ . |
| BN2 | ENBW for positive and negative frequencies $(-\omega \leq \omega_0 \leq +\omega)$ . |
| C | Capacitance of first order filter. |
| $C_f$ | Tracking rate in cycles/second per second. |
| $C_\omega$ | Tracking rate in radians/second per second. |
| $C_1$ | A capacitor in the passive loop filter of a second or third order phase lock loop tracking filter. |
| $C_2$ | A capacitor in the active loop filter of a third order phase lock loop tracking filter. |
| $c_0, c_1, c_2$ | Constant coefficients of the numerator of the transfer function H(s). |
| c(s) | Series of the form $c_{N-1} S^{N-1} + \cdots\cdots + c_0$. |
| $d_0, d_1, d_2, d_3$ | Constant coefficients of the denominator of the transfer function H(s). |
| d(s) | Series of the form $d_{N-1} s^N + \cdots\cdots\cdots + d_0$. |
| ENBW | Equivalent noise bandwidth - the bandwidth of an ideal rectangular filter which passes the same average noise power from a white noise source as the single pole filter. |
| $E_{in}(s)$ | LaPlace transform of the voltage input to the phase lock loop. |
| $E_0(s)$ | LaPlace transform of the voltage output from the phase lock loop. |
| G | DC gain of the amplifier in the passive loop filter. |
| G(s) | Open loop transfer function. |
| H(j$\omega$) | Network transfer function. |
| H(s) | Closed loop transfer function. |
| $H^1(s)$ | Transfer function of the feedback network. |
| I | Integral which is evaluated to determine ENBW. |
| $\pm j\omega_0$ | Poles of $\lvert H(j\omega)\rvert^2$, equal to $\omega_1$ and $\omega_2$. |
| K | Filter gain. |
| $K_{vco}$ | Transfer function of the voltage controlled oscillator. |
| $N_0$ | Single sided noise power density spectrum of the noise source in volts$^2$/cycle. |

| | |
|---|---|
| Order | The highest degree of the denominator term of the closed loop transfer function $H(s)$. |
| Pole | The root of the denominator term of $H(s) = c(s)/d(s)$. |
| R | Series resistance of first order filter. |
| R1 | Residue of $\omega_1$ (Pole of $\lvert H(j\omega)\rvert^2$). |
| R3 | Residue of $\omega_3$ for multiple poles of $\lvert H(j\omega)\rvert^2 = d/d\omega\ (R'3)\rvert_{\omega=j\omega_3}$. |
| R'3 | Residue of $\omega_3$ (pole of $\lvert H(j\omega)\rvert^2$). |
| $R_1$, $R_2$ | Resistors in the passive loop filter of a second or third order phase lock loop tracking filter. |
| $R_3$, $R_4$ | Resistors in the active loop filter of a third order phase lock loop tracking filter. |
| s | Frequency in radians/second, where $s = j\omega$. |
| S/N Improvement | The ratio of $(S/N)_{out}$ to $(S/N)_{in}$, equal to the log of the ratio between the noise power input to the comb filter and the noise power passing through a single pole filter, multiplied by 10. |
| $T_1$, $T_2$ | Time constants of the passive loop filter. $T_1 = C_1(R_1 + R_2)$, $T_2 = R_2C_1$. |
| $T_3$, $T_4$ | Time constants of the active loop filter. $T_3 = GR_3C_2$, $T_4 = R_4C_2$. |
| Type | The highest degree of the numerator term of the closed loop transfer function $H(s)$. |
| VCO | Voltage controlled oscillator. |
| $V_\epsilon(t)$ | Error signal at the output of the phase detector. |
| Zero | The root of the numerator term of $H(s) = c(s)/d(s)$. |
| $\alpha_1$ | Gain constant of the passive loop filter, where $\alpha_1 = (R_1 + R_2)/R_2$. |
| $\alpha_2$ | Gain constant of the active loop filter, where $\alpha_2 = R_3/R_4$. |
| $\Delta\omega$ | Difference between the input and output frequencies of the phase lock loop in radians/second. |
| $\Delta\omega(s)$ | Input signal to the phase lock loop in radians/second. |
| $\zeta$ | Natural damping factor of the phase lock loop. |
| $\theta$ | The inverse cosine of the damping factor $\theta = \cos^{-1}\zeta$. |
| $\theta_{in}(s)$ | LaPlace transform of the input signal to the phase lock loop (radians). |
| $\theta_0(s)$ | LaPlace transform of the output signal to the phase lock loop (radians). |
| $\theta_\epsilon(s)$ | LaPlace transform of the phase error corresponding to the phase difference in radians between the input and output frequencies of the phase lock loop. |
| $\theta_{in}(t)$ | Input signal to the phase lock loop, in radians. |
| $\theta_{out}(t)$ | Output signal from the phase lock loop, in radians. |
| $\theta_1(t)$ | Input phase deviation signal to the phase lock loop. |
| $\theta_{vco}(t)$ | Output phase deviation signal of the phase lock loop. |

| | |
|---|---|
| $\overline{\sigma^2}$ | Mean-square output noise voltage equal to $N_0/\pi \int_0^\infty \lvert H(j\omega)\rvert^2 \, d\omega$. |
| $\psi$ | Static phase shift (radians), equal to $\tan^{-1}(1 - \zeta^2)^{1/2}/-\zeta$. |
| $\omega$ | Signal frequency. |
| $\omega_0$ | Filter natural resonant frequency. |
| $\omega_{3db}$ | Filter bandwidth measured at 3 db points. |
| $\omega_1, \omega_2, \omega_3, \omega_4$ | Roots of $\omega$ (Poles of $\lvert H(j\omega)\rvert^2$ ). |