Fifth Quarterly Report

Submitted to: Jet Propulsion Laboratory
4800 Oak Grove Drive
Pasadena, California

Contract No. 950670

April 1965

School of Electrical Engineering
Purdue University
Lafayette, Indiana 47907

by

R. Sridhar
Principal Investigator

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Introduction

The Annual Report submitted to JPL in January 1965 contained results obtained during the first year of work on this contract. The present report is concerned with extensions of some of the work reported previously.

This report is in two distinct parts. The first is concerned with a statistical approach to the problems of estimation and of combined identification and control. The material presented in this section represents a continuation and, to a certain extent, a summation of the material in chapter 5 of the annual report.

The second section of this report deals with the problem of the worst initial condition. For a particular specific optimal control scheme, it is useful to know which initial condition on the state of the system being controlled will cause the "worst" system behavior, where "worst" implies that the performance index takes on its maximum value. The initial condition within the range of interest which causes the performance index to take on its maximum value is termed the "worst initial condition."

Areas for future investigations are indicated in each section.

SECTION I

A Statistical Formulation of the Estimation and the Combined Identification and Control Problems.

Introduction

This section represents work which is related to that reported in chapter 5 of the annual report. It is not intended that this section be read completely separately from Chapter 5 as:

(1) Answers to several of the questions posed in that chapter will be suggested in this section and

(2) The discussion here in a sense provided the motivation for the work reported in chapter 5.

The purpose of this section is the following: (A) To survey the literature and to outline the results available concerning estimation problems involving continuous nonlinear dynamical systems (B) To survey the literature and to outline the results available concerning combined identification and control problems involving continuous dynamical systems with disturbances (C) To present a critical review of items (A) and (B), and (D) To suggest an alternate approach to the estimation problem and also to the combined identification and control problem.

For the estimation problem of the type under consideration, there are many seemingly related results reported in the literature. Essentially, there are three different viewpoints being presented by the various writers. The different viewpoints are those of the physicist, the mathematician, and the engineer.

The Brownian motion problem in physics has aspects which are similar to the estimation problems under consideration. The papers by Chandrasekhar [2], Kac [8], and Wang and Uhlenbeck [13] are among the classics in this area. Brownian motion is the name given to the motion of a small particle immersed in a fluid. In the past, the physicists have been concerned with constructing models which describe the gross effects of the motion of the immersed particle. One such model is an ordinary differential equation with position as the dependent variable. In order to account for the approximately 10^{21} collisions per second that the particle undergoes, a rapidly fluctuating forcing function is applied to the differential equation. Occasionally, to simplify the analysis, the forcing function is assumed to be gaussian white noise. The success in predicting the gross effects has been the justification of using this model.

The mathematician on the other hand is concerned with the mathematical meaning of the model. For example, see Doob, reference [4]. The question is whether it is mathematically correct to treat a differential equation with a gaussian white noise input by the normal rules of integration. If not, can a consistent set of axioms and rules be developed? Such questions often lead the mathematician into developing an entirely new area of pure mathematics. In this particular instance the new area is that of stochastic or Ito equations, [3, 7]. There is no guarantee that the new mathematics will help in the solution of the original problem of direct physical interest.

The engineer often begins with a specified physical system. In many cases it is known from experience that the physical system may be described, within observational errors, by an ordinary differential equation. It is also known that disturbances from various sources are acting on the system. The object is to describe in a useful manner the effects of the disturbances on the system. It is not the intent here to become involved with philosophical discussions concerning the nature of the disturbances, i.e., whether the terms unknown or unknowable are more appropriate. Rather it will be conceded that a statistical approach to the disturbances may be valid. Some writers, [9, 10, 15], in order to obtain models for the disturbances which are tractable, appeal to the purely mathematical results in the areas of stochastic or Ito equations. In doing this they end up with models for the system which are also stochastic or Ito equations. While all the operations may be mathematically correct, this nevertheless represents a dilemma. Initially the system was described adequately by an ordinary differential equation, which could be simulated; the final result is a mathematical representation of the system in terms of stochastic of Ito equations which cannot be simulated. It would seem that too high a price is being paid for models of the disturbances which are mathematically tractable.

The remaining sections of the report will develop the various ideas briefly mentioned above in more detail. For convenience the scalar case will be considered.

1. Statistical Analysis of Dynamical Systems

Consider the following class of systems; let the equation of motion be given by

$$\dot{x} = g(t, x) + k(t, x)\, u(t) \tag{1-1}$$

and let the output observations be

$$y(t) = h(t, x) + v(t) \tag{1-2}$$

where $u(t)$ and $v(t)$ represent disturbances.

A fundamental problem in the statistical analysis of dynamical systems is the following: Given a statistical description of the disturbances $u(t)$ and $v(t)$, a statistical description of $x(t_o)$, the initial condition for equation (1-1), and a realization of the process y, determine a statistical description of the process x.

In particular, for estimation purposes the statistical description of the process x that would be the most desirable would be the density function of x conditioned on the realization of y, i.e., the function $P(a, t)$ where

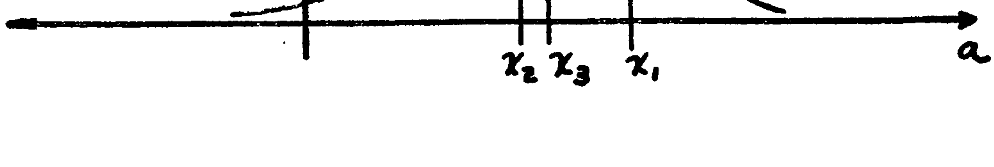
$$P(a, t)\,da = \Pr\{a < x(t) \leq a + da \mid y(\tau),\ t_o \leq \tau \leq t\} \tag{1-3}$$

The a-posteriori density function, $P(a, t)$ of the state contains all the statistical information concerning the value of x at time t. If $P(a, t)$ could be obtained, then in principle the problem of estimating $x(t)$ is solved.

For example let $\hat{x}(t)$ denote an estimate of $x(t)$, and denote the error in this estimate by $e(t)$; thus

$$e(t) = x(t) - \hat{x}(t) \tag{1-4}$$

Suppose $P(a, t)$ is known and is as indicated below.



Then any criterion based upon the error may be minimized. For example, for P(a, t) as shown, the estimate $\hat{x} = x_1$, the maximum of P(a, t), or the mode, would represent the most likely state. The estimate $\hat{x} = x_2$, the conditional mean, would represent the minimum variance estimate of x, i.e.,

$$\min_{\hat{x}} \int (a - \hat{x})^2 \, P(a, t) \, da \qquad (1\text{-}5)$$

Finally the estimate $\hat{x} = x_3$, the median, would represent the minimax estimate of x, i.e., the value which minimizes $|a - x_3|$, the maximum possible error.

With the knowledge of P(a, t), in addition to being able to determine the best estimate to any particular criterion, it is also possible to determine the spread about the particular estimate. Thus a quality of the estimate or confidence level may also be determined.

In principle, many different types of estimation problems may be considered by the present format. In general, the density function of x conditioned on the realization of y may be written as

$$P(a, t_1, t_2) \, da = \Pr\{a < x(t_1) \leq a + da \mid y(\tau), \; t_o \leq \tau \leq t_2\} \qquad (1\text{-}6)$$

For $t_1 > t_2$ the estimation problem becomes the so-called prediction problem, for $t_1 < t_2$ it becomes the so-called smoothing problem, and for $t_1 = t_2$ it becomes the so-called filtering problem.

The estimation problem considered herein will be the filtering problem, i.e., to estimate the value of the state x(t) based on $y(\tau)$, $t_o \leq \tau \leq t$. For real time applications the variable t is the independent variable, the running time variable. The estimation of the current value of the state x(t) based on $y(\tau)$, $t_o \leq \tau \leq t$, for all values of t is defined as the sequential filtering problem.

A complete solution of the sequential filtering problem could be obtained, in principle, if the time evolution of the <u>a-posteriori</u> density function of $x(t)$ were known. Thus it is necessary to determine how the function $P(a, t)$ evolves in time.

At this point it is necessary to appeal to the existing literature on the theory of stochastic processes. The area of stochastic process theory which has received considerable attention as far as applications to physical problems is concerned is the theory of Markov processes. In some sense the great effort in the application of Markov process theory to physical problems has in part been motivated by the desire to make the stochastic problem somewhat analogous to the deterministic problem. Bellman [1] seems to be one of the first writers to point out the application of Markov processes to certain control and filtering problems. A Markov process z of the type of interest here is defined by requiring that

$$\Pr\{a < z(t + \Delta) \leq a + da \,\Big|\, z(\tau),\ t_0 \leq \tau \leq t\} = \Pr\{a < z(t + \Delta) \leq a + da \,\Big|\, z(t)\} \tag{1-7}$$

for all a, t, and Δ. In a sense then the future statistical behavior of the Markov process z depends on the "initial condition" and not on how the process arrived at this "initial condition". This is analogous to certain deterministic processes.

Historically one of the first physical problems to be treated using the Markov process approach was that of Brownian Motion. Even though the basic starting places for applying Markov process theory to Brownian motion and to the filtering problem are considerably different it will be instructive to consider Brownian motion.

2. Brownian Motion

An eloquent discussion of the theory of Brownian Motion is contained in the references [2, 8, 13]. It is the intention here to display the pertinent facts as they relate to the estimation problem.

Let x denote the position of the particle and let $P(a,t \mid y)$ denote the transition probability

$$P(a, t \mid y)\, da = \Pr\{a < x(t_1 + t) \leq a + da \mid x(t_1) = y\} \qquad (2\text{-}1)$$

The basic starting place for the study of Brownian motion [13] is to make the assumptions (1) that x is a Markov process and (2) that only the first and second order moments of the change, in time δt, of the coordinate of the particle are proportional to δt, i.e., if

$$a_n(z, \delta t) = \int (a - z)^n P(a, \delta t \mid z)\, da \qquad (2\text{-}2)$$

then as $\delta t \to 0$ only a_1 and a_2 become proportional to δt. Let

$$\begin{aligned} A(z) &= \lim \frac{1}{\delta t} a_1(z, \delta t) \\ B(z) &= \lim \frac{1}{\delta t} a_2(z, \delta t) \end{aligned} \qquad (2\text{-}3)$$

Under these assumptions it can be shown [13] that P satisfies the Fokker-Planck equation

$$\frac{\partial P(x,t)}{\partial t} = - \frac{\partial}{\partial x} [A(x)\, P(x,t)] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [B(x)\, P(x,t)] \qquad (2\text{-}4)$$

A somewhat different approach to the Brownian Motion problem is to construct a model to describe the motion of the particle. This is usually done by modeling the motion of the particle by Langevin's equation

$$\dot{x} = - \beta x + \xi(t) \qquad (2\text{-}5)$$

where $\xi(t)$ represents a function which varies extremely rapidly compared to the variation of x. In order to consider the slightly more general case as represented by equations (2-3) the generalized Langevin's equation becomes

$$\dot{x} = A(x) + B(x)\,\xi(t) \tag{2-6}$$

For x to be a Markov process with the transition probability obeying the Fokker-Planck equation the mathematics requires that $\xi(t)$ be gaussian white noise. (Assume for convenience that ξ has zero mean and $E[(d\xi)^2] = 1 \cdot dt$.)

It is at this point that the basic differences in the starting places for the study of Brownian motion and the study of the estimation problems considered become apparent. In the Brownian motion problem the basic assumptions concern the statistical character of the motion. From these statistical descriptions the Fokker-Planck equation describing the time evolution of the probability density function may be obtained. Knowing the statistical character of the motion, a model in the form of a differential equation may also be determined, and results derived via this model. Thus the differential equation itself is not the basic starting place. For the estimation problems under consideration, the differential equation describing the motion of the system is specified. Thus one starts with the differential equation. The stochastic nature of the problem enters via random inputs to the system, observational errors, and unknown initial conditions.

Certain mathematical difficulties occur when the differential equation is the starting place and one attempts to model the state variables by a Markov process. For example, in equation (2-6), for x to be a Markov process $\xi(t)$ must be gaussian white noise. The mathematical difficulty is that with $\xi(t)$ being gaussian white noise, equation (2-6) cannot be interpreted as an ordinary differential equation but should be treated as a stochastic equation; see for examples references [3,4].

Stochastic equations cannot be treated as ordinary differential equations.

Except for this mathematical difficulty, for estimation problems of the class under consideration, starting with equation (2-6) is appealing. In this case the Fokker-Planck equation describes the time evolution of the a-priori (or open loop) probability density function of x. In order to account for the observations, i.e., the realization of y, one would like to generalize equation (2-4) so that the time evolution of the conditional probability density function of x could be obtained. This conditional probability density function would be the a-posteriori (or closed loop) density function of x.

Since physical systems are never disturbed by infinite bandwidth signals, or equivalently gaussian white noise, initially some approximations must be made. It would seem plausible that if the disturbances acting on the system and the observational errors are white over the effective bandwidth of the system, then these signals could be approximated by infinite bandwidth signals and then the mathematical results from the Theory of Markov Processes applied.

3. An Application of the Theory of Conditional Markov Processes to Filtering Problems

Apparently the first systematic study of conditional Markov Processes was made by Stratonovich [12]. Kushner [9, 10] considered several types of conditional Markov Processes. The results given in this section are contained in reference [10].

The starting place for the study of the conditional Markov Process of interest is the equation

$$\dot{x} = g(t, x) + u(t) \tag{3-1}$$

the output observations are given by

$$y(t) = h(t, x) + v(t) \tag{3-2}$$

where $u(t)$ and $v(t)$ are gaussian white noise. It must be emphasized that there are mathematical difficulties involved in the interpretation of (3-1). Also, regardless of the meaning of (3-1) it only approximates in some manner the actual physical situation. In order to employ notation commonly used, (3-1) and (3-2) will be written as

$$dx = g(t, x)\, dt + d\xi \tag{3-3}$$

$$dz = h(t, x)\, dt + d\zeta \tag{3-4}$$

where it will be assumed that $E(d\xi) = E(d\zeta) = 0$ and $E[(d\xi)^2] = \sigma_2^{\,2}\, dt$ and $E[(d\zeta)^2] = \sigma_1^{\,2}\, dt$. Notice that formally $y = dz/dt$.

Let $Y(t)$ represent all the observed information in the interval $t_o \leq \tau \leq t$, i.e., the entire function $z(\tau)$, $t_o \leq \tau \leq t$ or $y(\tau)$, $t_o \leq \tau \leq t$. The conditional density function of x at time t will be denoted by $P(a, t \mid Y(t))$ or $P(a, t)$ or P. It can be shown [10] that $P(a, t)$ is a Markov process, i.e.,

$$\Pr\{p < P(a, t + \Delta) \leq p + dp \mid P(a, \tau),\ \ t_o \leq \tau \leq t\}$$
$$= \Pr\{p < P(a, t + \Delta) \leq p + dp \mid P(a, t)\} \tag{3-5}$$

Using a method of analysis analogous to that used to derive the Fokker-Planck equation for Brownian Motion, it can be shown [10] that the time evolution of $P(a, t)$ is described by

$$\begin{aligned} dP(a, t) &= P(a, t + dt) - P(a, t) \\ &= P(a, t)\,(dz - Eh\, dt)\, \frac{1}{\sigma_1^{\,2}}\,(h - Eh) \\ &\quad - (g\, P)_a\, dt + \frac{1}{2}\,(\sigma_2^{\,2}\, P)_{aa}\, dt \end{aligned} \tag{3-6}$$

where $(\cdot)_a$ denotes the partial derivative of the quantity $(\cdot)$ with respect to a, and $Eh = E[h(t, a)]$

$$= \int h(t, a)\, P(a, t)\, da$$

There seems to be some confusion in the technical literature as to whether equation (3-6) represents a partial differential equation after formally dividing by dt, or whether (3-6) is a stochastic equation. Notice that formally dividing through by dt yields a term $\frac{dz}{dt}$. The observation $y = dz/dt$ has an additive gaussian white noise term; hence to be consistent it would seem that (3-6) is a stochastic equation.

Since the starting place represented by equations (3-1) and (3-2) only approximated the actual physical sistuation, it appears that equation (3-6) still approximates in some manner the actual time evolution of P(a, t).

Neglecting the possible stochastic aspects of equation (3-6), the equation itself is not easily simulated, as it is a forced, partial, differential, integral equation (the E operator implies integration with respect to P).

An alternate Markov Process representation, suggested in references [9, 10], is to consider the moments of P. Let m represent the mean and m_j the j^{th} central moment of P, i.e.,

$$m(t) = \int a \, P(a, t) \, da \tag{3-7}$$

$$m_j(t) = \int (a - m)^j \, P(a, t) \, da \tag{3-8}$$

The equations that the moments of P satisfy may be derived using the same scheme used when deriving equation (3-6). In the derivation of equation (3-6) it is shown [10] that for an arbitrary but triply differentiable function $\ell(a)$

$$\int \ell(a)[P(a, t + dt) - P(a, t)]\, da$$

$$= \int \ell(a)\, dQ\, da + \int (\ell_a\, g dt + \frac{1}{2}\, \ell_{aa}\, \sigma_2^{\,2}\, dt)\, P(a, t)\, da \quad (3\text{-}9)$$

where

$$dQ = P(a, t)(dz - Eh\, dt)\, \frac{1}{\sigma_1^{\,2}}\, (h - Eh) \qquad (3\text{-}10)$$

For example, (i) the equation that $m(t)$ satisfies may be determined as follows

$$dm(t) = m(t + dt) - m(t)$$

$$= \int a\, P(a, t + dt)\, da - \int a\, P(a, t)\, da$$

$$= \int a\, dP\, da \qquad (3\text{-}11)$$

hence letting $\ell(a) = a$ in (3-9) and (3-10) yields

$$dm(t) = E\, g\, dt + \frac{1}{\sigma_1^{\,2}}\, E[h(a - m)]\, (dz - Eh\, dt) \qquad (3\text{-}12)$$

and (ii), the equation that $m_2(t)$ satisfies may be determined as follows

$$dm_2(t) = m_2(t + dt) - m_2(t)$$

$$= \int (a - m(t + dt))^2\, P(a, t + dt)\, dt$$

$$- \int (a - m(t))^2\, P(a, t)\, dt$$

carrying out this calculation yields

$$dm_2(t) = 2E[g(a - m)]\, dt + \sigma_2^{\,2}\, dt$$

$$- \frac{1}{\sigma_1^{\,2}}\, [E(h\{a - m\})]^2\, dt$$

$$+ \frac{1}{\sigma_1^{\,2}}\, [Ea^2h - EhEa^2 - 2m\{Eh(a - m)\}]\{dz - Eh\, dt\} \qquad (3\text{-}13)$$

In the same fashion equations for m_3, m_4, ... and so on may be determined.

Notice that formally dividing (3-12) and (3-13) by dt yields a term dz/dt in the right hand side. In the literature the answer to the question concerning the stochastic nature of these equations analogous to those concerning equation (3-6) is unclear. However, even if this question is neglected, in general, equations (3-12) and (3-13) and so on do not represent ordinary differential equations since the expectation operator is involved. For example

$$E[g(t, a)(a - m)] = \int g(t, a)\,(a - m)\,P(a, t)\,da \qquad (3\text{-}14)$$

Thus the equations for the central moments of P also involve integrations with respect to P.

The present statistical formulation of the filtering problem leads naturally to one version of the so-called stochastic optimal control problem.

4. Combined Identification and Control

The combined identification and control problem as considered in this report consists in taking noisy observations on the output of the system and, based upon the observations, synthesizing control signals which will control the plant in some predetermined fashion. In general the plant itself may not be completely known, i.e., there may be parameters in the plant equation which are unknown.

Florentin in [5, 6], Kushner in [9, 10, 11], and Wonham in [14] consider some of the various aspects of this and related problems of optimal control in the presence of disturbances.

The statistical approach to this problem consists in using the statistical description of the disturbances and observational errors to determine the *a-posteriori* statistical characteristics of the states.

The problem then becomes one of determining these a-posteriori statistics in a sequential manner and controlling these probabilistic measures in some more desireable fashion.

The features of the combined identification and control problem are more apparent when it is compared with the simpler deterministic optimal control problem. A class of deterministic optimal control problems may be described as follows: The equation of motion of the plant is

$$\dot{x} = g(t, x, u) \tag{4-1}$$

The problem is to select $u(t)$ (or $u(x)$) so as to minimize

$$\int_{t_o}^{T} k(\tau, x, u)\, d\tau \tag{4-2}$$

where $x(t_o)$ and $x(T)$ must be on some appropriate initial and terminal manifolds respectively.

A natural stochastic analog of the above is as follows: The plant is described by

$$\dot{x} = g(t, x, u) + \xi(t) \tag{4-3}$$

and the observations are

$$y(t) = h(t, x) + \zeta(t) \tag{4-4}$$

where $\xi(t)$ and $\zeta(t)$ are gaussian white noise. This assumption is, in general, necessary in order to obtain any results.

Based on the observations y, the problem is to synthesize u so as to minimize some index of performance related to the expression (4-2). Let $P(a, t)$ be the a-posteriori density function of the state x at time t, as described in section 2. An appropriate generalization of expression (4-2) is

$$E \int_{t_o}^{T} \int k(\tau, a, u)\, P(a, \tau)\, da\, d\tau \qquad (4-5)$$

The expression (4-5) requires some interpretation. Let $P(a, t_o)$ the a-priori density function of x be given. Suppose $u(\tau)$, $t_o \leq \tau \leq T$ is specified, then for each realization $y(\tau)$, $t_o \leq \tau \leq T$ the expression

$$\int_{t_o}^{T} \int k(\tau, a, u)\, P(a, \tau)\, da\, d\tau \qquad (4-6)$$

is a number, the cost of control for the particular u and y. With $P(a, t_o)$ given and $u(\tau)$, $t_o \leq \tau \leq T$ specified, expression (4-6) is, in general, a random variable.

Thus, for a given $u(\tau)$, expression (4-5) represents the expected cost of control. The stochastic control problem is to select $u(\tau)$ as an operation on the past observations so as to minimize the expected cost of control.

With the gaussian white noise assumption on ξ and ζ, $P(a, t)$ is itself a Markov Process. Hence the techniques of Dynamic Programming [1] are applicable. It will be convenient to use the conditional moments of P as the sufficient statistics for the problem. Let $m(t)$, $m_2(t)$, ..., $m_j(t)$, ... represent the conditional moments (a-posteriori) of the process. Define the value function

$$V[m(t_o), m_2(t_o), \ldots; t_o] = \underset{u}{\mathrm{Min}}\; E \int_{t_o}^{T} \int k(\tau, a, u)\, P(a, \tau)\, da\, d\tau \qquad (4-7)$$

Use of the principle of optimality yields

$$V[m(t_o), m_2(t_o), \ldots ; t_o] = \underset{\substack{u \\ [t_o, t_o+\Delta]}}{\text{Min}} \; E\left[\int_{t_o}^{t_o+\Delta} \int k(\tau,a,u)\, P(a,\tau)\, da\, d\tau \right.$$

$$\left. + V[m(t_o+\Delta), m_2(t_o+\Delta), \ldots ; t_o] \right] \qquad (4\text{-}8)$$

Expanding V in a Taylor series and retaining all terms whose expectation is of order Δ yields

$$V[m(t_o), m_2(t_o), \ldots ; t_o] = \underset{\substack{u \\ [t_o, t_o+\Delta]}}{\text{Min}} \; E\left[\int_{t_o}^{t_o+\Delta} \int k(\tau,a,u)\, P(a,\tau)\, da\, d\tau \right.$$

$$+ V[m(t_o), m_2(t_o), \ldots ; t_o] + \left.\frac{\partial V}{\partial t}\right|_{t_o} \Delta + \sum_i \left.\frac{\partial V}{\partial m_i}\right|_{t_o} \{m_i(t_o+\Delta) - m_i(t_o)\}$$

$$\left. + \frac{1}{2}\sum_i \sum_j \left.\frac{\partial^2 V}{\partial m_i \partial m_j}\right|_{t_o} \{m_i(t_o+\Delta) - m_i(t_o)\}\{m_j(t_o+\Delta) - m_j(t_o)\} + o(\Delta)\right] \quad (4\text{-}9)$$

Letting $\Delta \to dt$ and cancelling out the common V term results in

$$0 = \underset{u(t)}{\text{Min}} \; E\left[dt \int k(t,\, a,\, u)\, P(a,\, t)\, da + \frac{\partial V}{\partial t}\, dt + \sum_i \frac{\partial V}{\partial m_i}\, dm_i \right.$$

$$\left. + \frac{1}{2}\sum_i \sum_j \frac{\partial^2 V}{\partial m_i \partial m_j}\, dm_i\, dm_j + o(dt)\right] \qquad (4\text{-}10)$$

where $\dfrac{o(dt)}{dt} \to 0$ as $dt \to 0$

The expressions for the change in the central moments, i.e., the dm_j's appearing in equation (4-10), may be determined by the method outlined in section 3.

Notice that the form of equation (4-10) depends on P(a, t). Clearly equation (4-10) is considerably more complex than the Hamilton-Jacobi equation which arises in deterministic problems. The only case in which a solution to (4-10) seems to be possible is when the plant and observations are linear and the function k in the performance index is quadratic. In this case only m(t) and $m_2(t)$ appear in the equation of dynamic programming. This results in the well known separation of the control and estimation functions as discussed by many writers [5, 9, 14].

5. Critique I - Stochastic Estimation

Since the Markov Process approach to Brownian Motion provides the motivation to determine the effects of the observations on the density function, it is interesting to see what type of comments the physicists themselves make. S. Chandrasekhar in reference [2] page 21 (page 23 in Wax) carefully points out the drastic nature of the assumptions implicit in writing equation (2-5). He also states, "They (the assumptions) are made with the reliance on physical intuition and the a-posteriori justification by the success of the hypothesis." M. Kac in reference [8] page 370 (page 296 in Wax) points out that theories for Brownian Motion based on starting with the diffusion equation for the probability density function are only approximate. He also states, "These limitations of the theory were already recognized by Einstein and Smoluchowski but are often disregarded by writers who stress that in Brownian Motion the velocity of the particle is infinite. This paradoxical conclusion is a result of stretching the theory beyond the bounds of its applicability."

With these comments in mind, for the problem of determining the a-posteriori statistics, in which the observations are incorporated and in view of (i) the possible stochastic aspects of the resulting equation, and of (ii) the drastic assumptions required, it seems that the mathematical rigor required to arrive at the results is seldom justified in many physical problems. It is more plausible to start closer to the actual physical problem, to make assumptions based upon physical intuition, and to experimentally verify the results. This was the motivation for formulating the sequential estimation problem in the annual report where the disturbances and measurement errors were treated as unknown signals and a least squares criterion was used for the estimates.

With this drastic difference in viewpoints in mind, it would still be interesting to compare the results from the two approaches. For simplicity consider the scalar case with linear observations. In this case the equations for the conditional moments become

$$dm = E[g(t, a)]\, dt + \frac{m_2(t)}{\sigma_1^2}\,(dz - m dt)$$

$$dm_2 = 2E[g(t, a)(a - m)]\, dt + dt\sigma_2^2 - \frac{1}{\sigma_1^2}\, m_2^2(t)\, dt + \frac{m_3(t)}{\sigma_1^2}\,(dz - m dt) \qquad (5\text{-}1)$$

$$\vdots$$

The sequential estimator equations become

$$\frac{d\hat{x}}{dt} = g(t, x) + 2R(t)\,(y(t) - \hat{x}(t))$$

$$\frac{dR}{dt} = 2g_x(t, \hat{x})\, R(t) - 2R^2(t) + 1 \qquad (5\text{-}2)$$

These equations were derived in the annual report.*

Formally dividing equations (5-1) through by dt, recalling that $y = dz/dt$, and neglecting $m_3(t)$, there seems to be a certain similarity

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* R(t) here corresponds to P(t) in equation (5.71) of the annual report with $w(T, \hat{x}) = \frac{1}{2}$ and $h(T, \hat{x}) = \hat{x}$. R(t) is used to avoid confusion with P(a, t).

between the two results. Notice however, that the equations for the conditional means would still not be ordinary differential equations due to the expected value operator.

6. An Interpretation of the Sequential Estimator

Consider the term

$$E[g(t, a)(a - m)] \tag{6-1}$$

from the right-hand side of (5-1).

$$E[g(t, a)(a - m)] = \int g(t, a)(a - m)\, P(a, t)\, da \tag{6-2}$$

Expanding $g(t, a)$ in a Taylor series about the conditional mean m, yields

$$E[g(t, a)(a - m)] = \int [g(t, m) + \frac{\partial g}{\partial a}(t, m)(a - m) + \ldots]\,(a - m)\, P(a, t)\, da \tag{6-3}$$

$$= \int g(t, m)(a - m)\, P(a, t)\, da + \int \frac{\partial g}{\partial a}(t, m)(a - m)^2\, P(a, t)\, da + \ldots \tag{6-4}$$

$$= \frac{\partial g}{\partial a}(t, m)\, m_2(t) + \ldots \tag{6-5}$$

Thus approximating the terms in (5-1) containing an expected value operator by the first non-zero term in its Taylor series, formally dividing through by dt, and neglecting $m_3(t)$ yields

$$\frac{d\bar{m}}{dt} = g(t, \bar{m}) + \frac{\bar{m}_2(t)}{\sigma_1^{\,2}}\,(y(t) - \bar{m}(t))$$

$$\frac{d\bar{m}_2}{dt} = 2g_x(t, \bar{m})\, \bar{m}_2(t) + dt\, \sigma_2^{\,2} - \frac{1}{\sigma_1^{\,2}}\, \bar{m}_2^{\,2}(t) \tag{6-6}$$

The bars in equations (6-6) indicate that $\bar{m}$ and $\bar{m}_2$ are only approximations to m and m_2. A comparison of equations (5-2) and (6-6) suggests that if the disturbances and observational errors are gaussian white noise, then the estimate $\hat{x}(t)$ is some approximation to the conditional mean $m(t)$ and that $R(t)$ is some approximation to the conditional variance $m_2(t)$. This is suggestive of the interpretation which could be given to $R(t)$. However, it must be pointed out that the assumptions made and the approach taken with the sequential least squares estimator are valid for a considerably larger class of physical problems.

7. Critique II - Stochastic Control

The statistical formulation of the combined identification and control problem has two inherent difficulties. The first is that the statistical formulation of the estimation problem is a basic portion of the overall formulation. Thus, whatever objections are raised concerning the statistical formulation of the filtering problem may also be raised here. The second is the complexity of the optimality equation itself. Feedback solutions to the deterministic optimal control problem are usually very difficult to obtain. Except in rare cases solutions to equation (4-10) would be impossible to obtain.

Of course ther is always the additional practical objection, perhaps the most valid of all, namely that many control problems are not sufficiently well formulated so as to justify the rigor necessary in this approach.

Intuitively it would seem that in some cases a somewhat simpler, and of course mathematically sub-optimal, scheme would provide satisfactory control of the plant. Hopefully the assumptions necessary to use some sub-optimal scheme would be more in accordance with the actual physical situation.

8. A Suboptimal Control Scheme

A suboptimal control scheme which is quite appealing is to use the sequential state estimator to provide estimates of the current state, and then to use the estimated state for control purposes. In particular, if a suitable feedback control law is known for the deterministic problem then replacing the true state by the estimated state in the control law would provide one useful scheme of controlling the plant based on noisy measurements of its output. If there are unknown parameters in the system which effect the control law, then the estimated values as produced by the state estimator could be used in place of the unknown parameters.

In addition to the examples presented in the previous report a number of additional experiments were performed in order to test further the above control scheme. In all of these examples the controller and state estimator were both turned on simultaneously. Linear plants were selected for these additional studies as it was necessary to know the optimal feedback control law (with respect to some performance index) in order to implement the overall scheme.

Example 1.

The plant equations

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u \qquad |u| \leq 1 \qquad (8\text{-}1)$$

with output observations on x_1. The control law used was $u^*(\hat{x}_1, \hat{x}_2)$ where the performance index for the deterministic problem was minimum time.

Figure 1 displays the results for $x_1(0) = 1.0$, $x_2(0) = 1.0$, $\hat{x}_1(0) = 0.0$, $\hat{x}_2(0) = 0.0$. Figure 2 displays the results for $x_1(0) = 1.0$, $x_2(0) = -0.5$, $\hat{x}_1(0) = 0.0$, $\hat{x}_2(0) = 0.0$.

Example 2.

Plant equations

$$\dot{x}_1 = 0.2x_1 + x_2$$
$$\dot{x}_2 = -0.1x_1 + u \qquad (8\text{-}2)$$

with output observations on x_1. The roots to the open loop characteristic equations are $\lambda_1 = -0.1$ and $\lambda_2 = 0.2$. The control law used was $u = -1.44\hat{x}_1 - 2.1\hat{x}_2$. Hence the performance index for the deterministic problem would be quadratic in x plus cost of control u^2 and infinite time.

Figure 3 displays the results for $x_1(0) = 2.0$, $x_2(0) = 2.0$, $\hat{x}_1(0) = 0.0$, and $\hat{x}_2(0) = 0.0$.

Example 3.

Plant equations

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + u \qquad |u| \leq 1 \qquad (8\text{-}3)$$

with observations on x_1. The control law used was $u^*(\hat{x}_1, \hat{x}_2)$ where the performance index for the deterministic problem was minimum time. The switching boundaries in this case are more complex than those of example 1.

Figure 4 displays the results for $x_1(0) = 2.0$, $x_2(0) = 1.0$, $\hat{x}_1(0) = 0.0$, $\hat{x}_2(0) = 0.0$. Figure 5 displays the results for $x_1(0) = -4.0$, $x_2(0) = -0.5$, $\hat{x}_1(0) = 0.0$, $\hat{x}_2(0) = 0.0$.

Example 4.

Plant equations

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = K\,u \qquad |u| \leq 1 \qquad (8\text{-}4)$$

with observations on x_1. In this example the constant moment of inertia K is unknown and hence must be estimated. The control law used was $u^*(\hat{x}_1, \hat{x}_2, \hat{K})$ where the performance index for the deterministic problem was minimum time. In this case the switching boundaries also depend on K.

Figure 6 displays the results for $x_1(0) = 2.0$, $x_2(0) = 1.0$, $K = 2.0$, $\hat{x}_1(0) = 0.0$, $\hat{x}_2(0) = 0.0$, $\hat{K}(0) = 1.0$.

It is interesting to note that in all the examples considered the proposed control scheme does indeed control the plant.

While certainly no claim is made concerning the optimality of the proposed scheme, i.e., the separation of estimation and control functions, from the examples tested it does seem to present a feasible scheme of obtaining suboptimal controls.

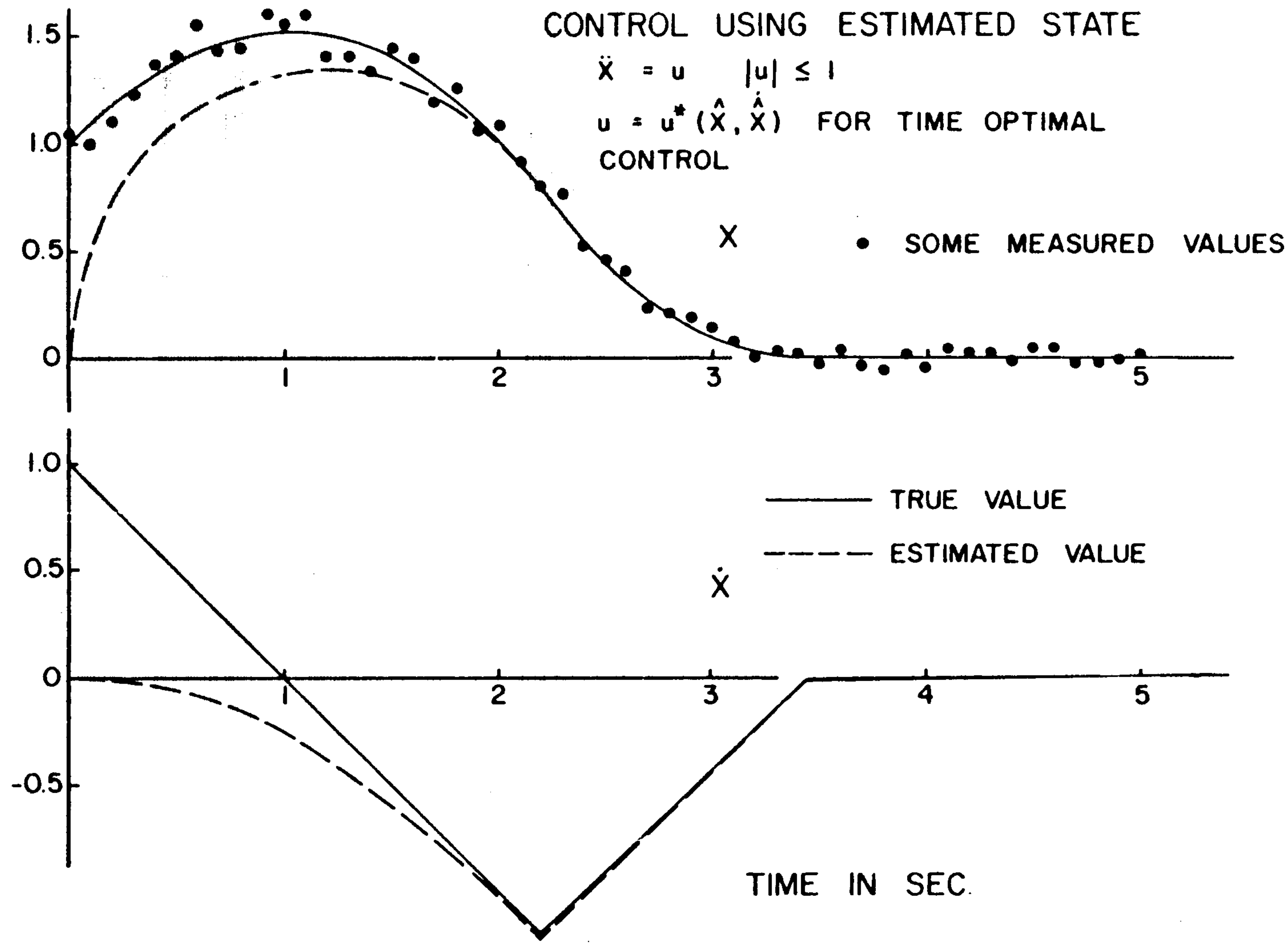
CONTROL USING ESTIMATED STATE
$\ddot{X} = u \quad |u| \leq 1$
$u = u^{*}(\hat{X}, \hat{\dot{X}})$ FOR TIME OPTIMAL CONTROL
X
• SOME MEASURED VALUES
1.5
1.0
0.5
0
1
2
3
5
TRUE VALUE
ESTIMATED VALUE
$\dot{X}$
1.0
0.5
0
-0.5
1
2
3
4
5
TIME IN SEC.

Figure 1

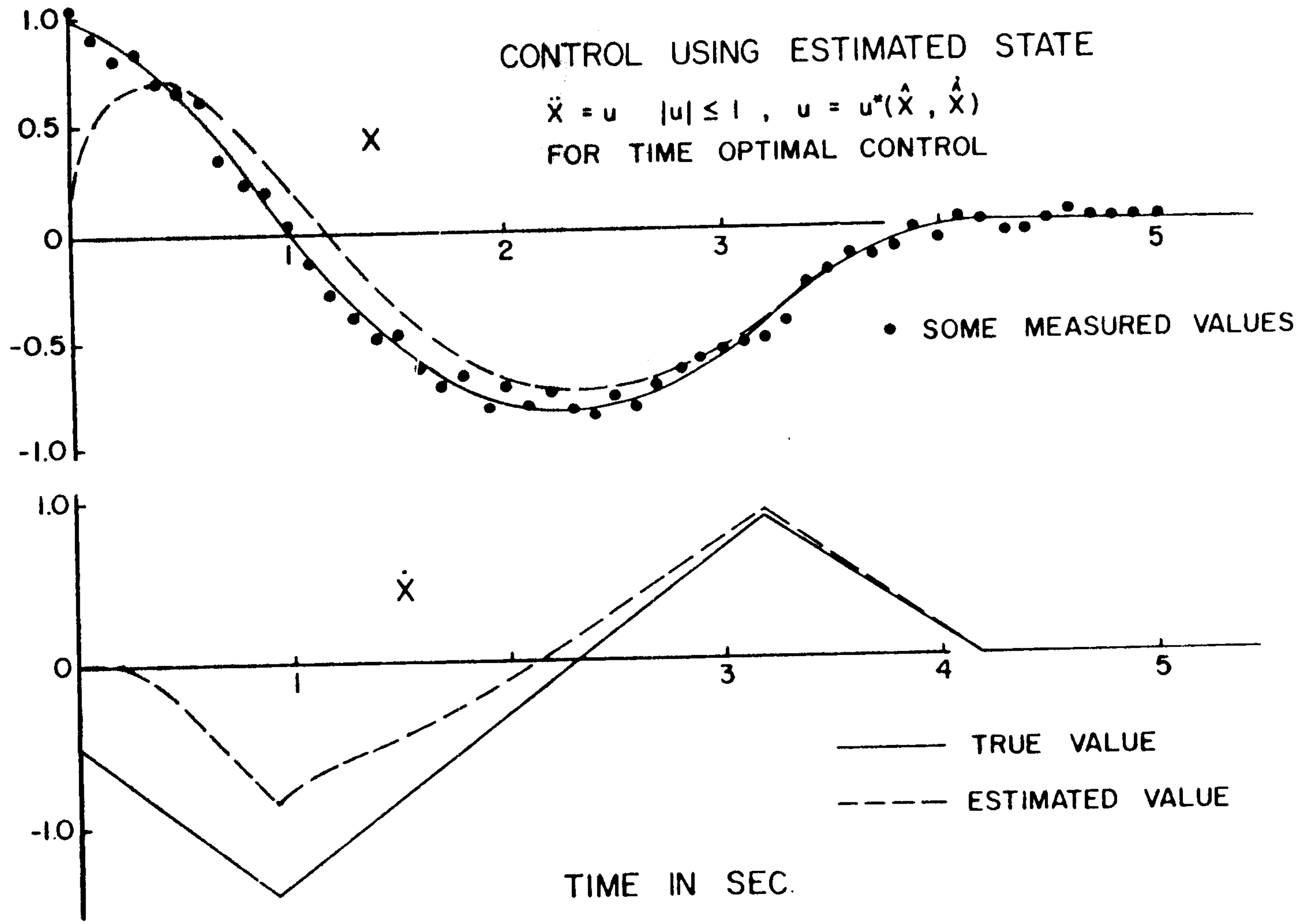
CONTROL USING ESTIMATED STATE
$\ddot{X} = u \quad |u| \leq 1 \;, \quad u = u^*(\hat{X}, \hat{\dot{X}})$
FOR TIME OPTIMAL CONTROL
X
• SOME MEASURED VALUES
1.0
0.5
0
-0.5
-1.0
1
2
3
5
$\dot{X}$
1.0
0
-1.0
1
3
4
5
TRUE VALUE
ESTIMATED VALUE
TIME IN SEC.

Figure 2

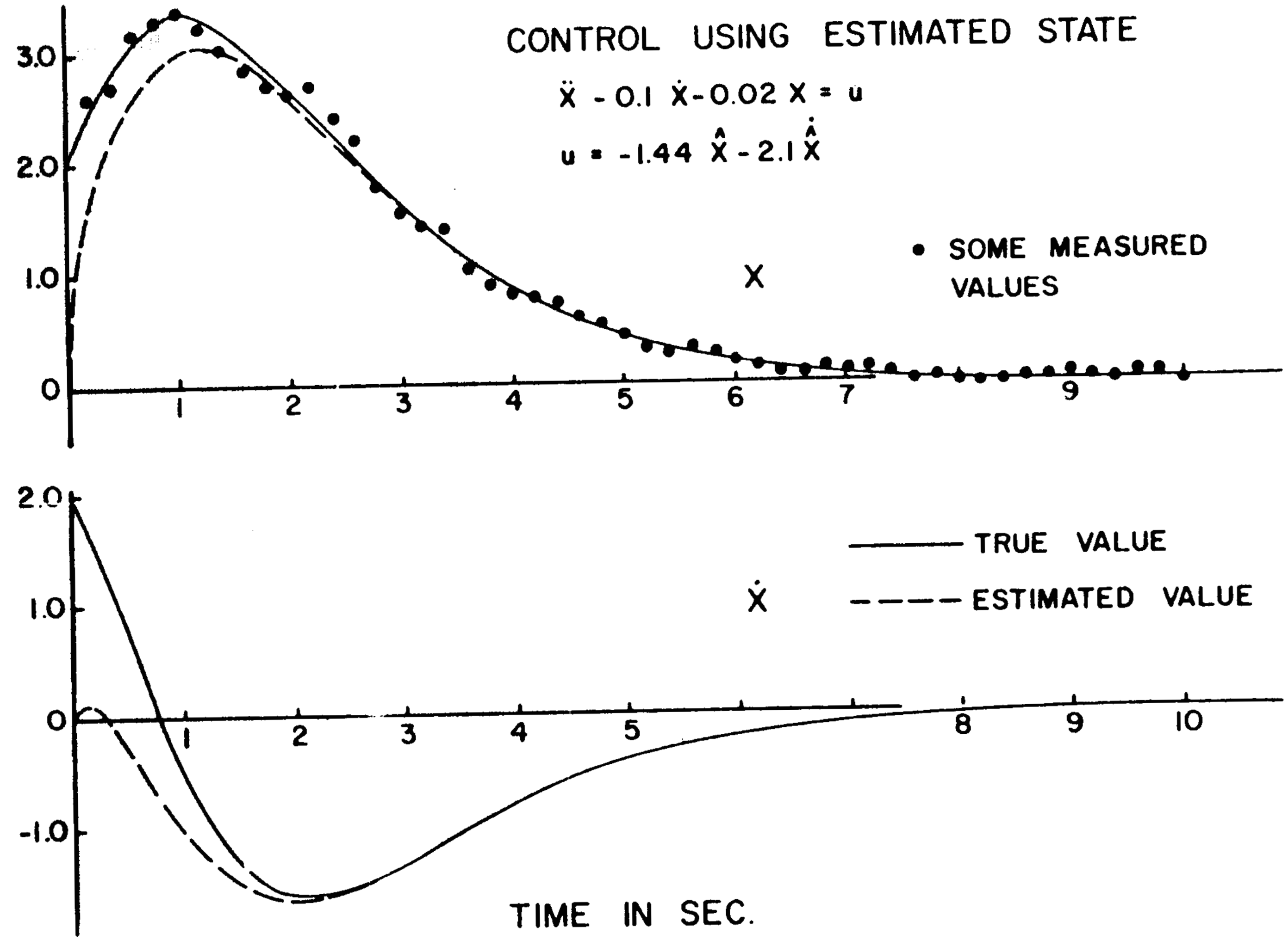
CONTROL USING ESTIMATED STATE
Ẍ - 0.1 Ẋ - 0.02 X = u
u = -1.44 X̂ - 2.1 X̂̇
• SOME MEASURED VALUES
X
Ẋ
TRUE VALUE
ESTIMATED VALUE
TIME IN SEC.
3.0
2.0
1.0
0
-1.0
1
2
3
4
5
6
7
8
9
10

Figure 3

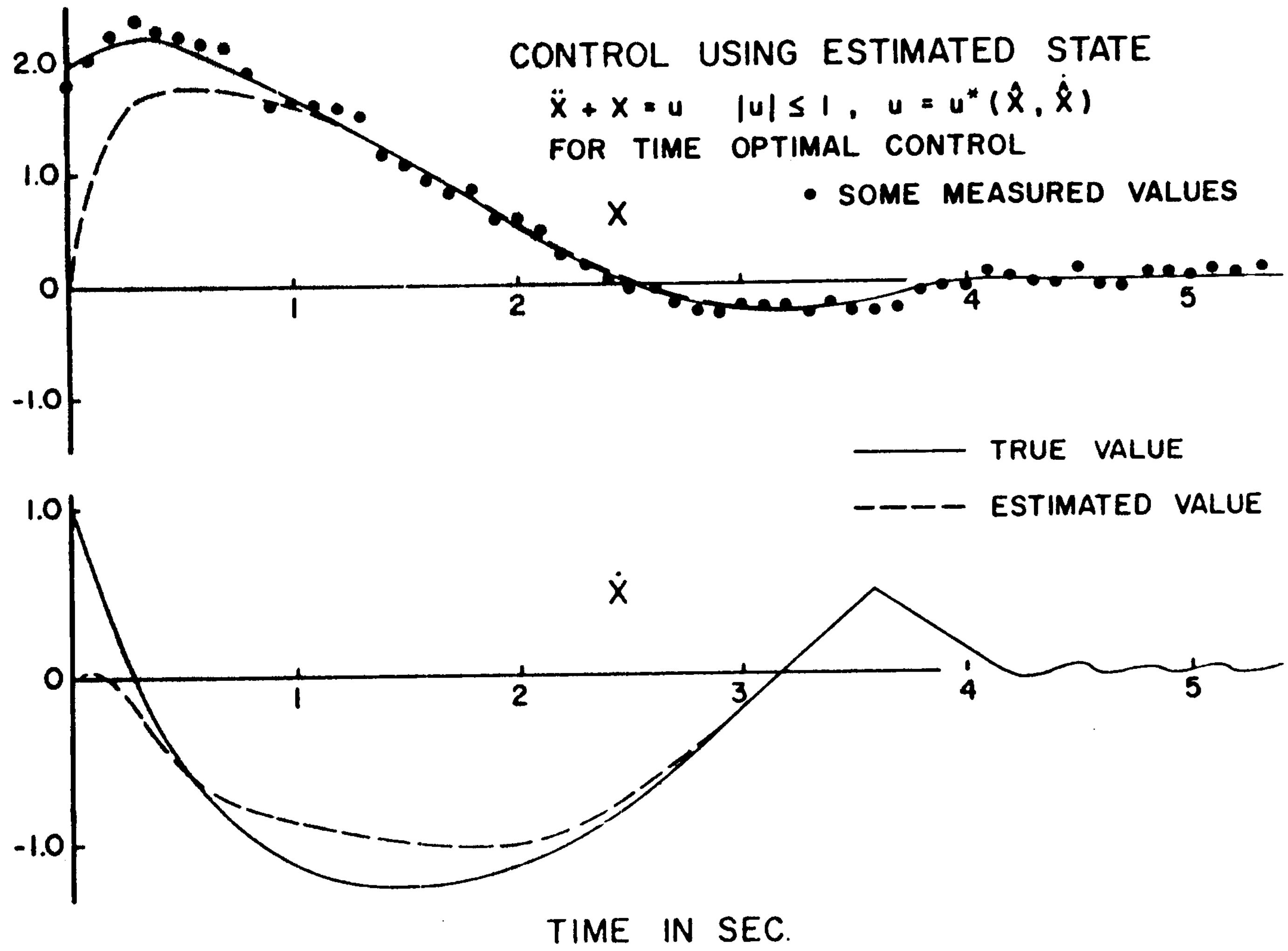
CONTROL USING ESTIMATED STATE
$\ddot{X} + X = u \quad |u| \leq 1, \quad u = u^*(\hat{X}, \hat{\dot{X}})$
FOR TIME OPTIMAL CONTROL
• SOME MEASURED VALUES
X
TRUE VALUE
ESTIMATED VALUE
$\dot{X}$
2.0
1.0
0
-1.0
1
2
3
4
5
TIME IN SEC.

Figure 4

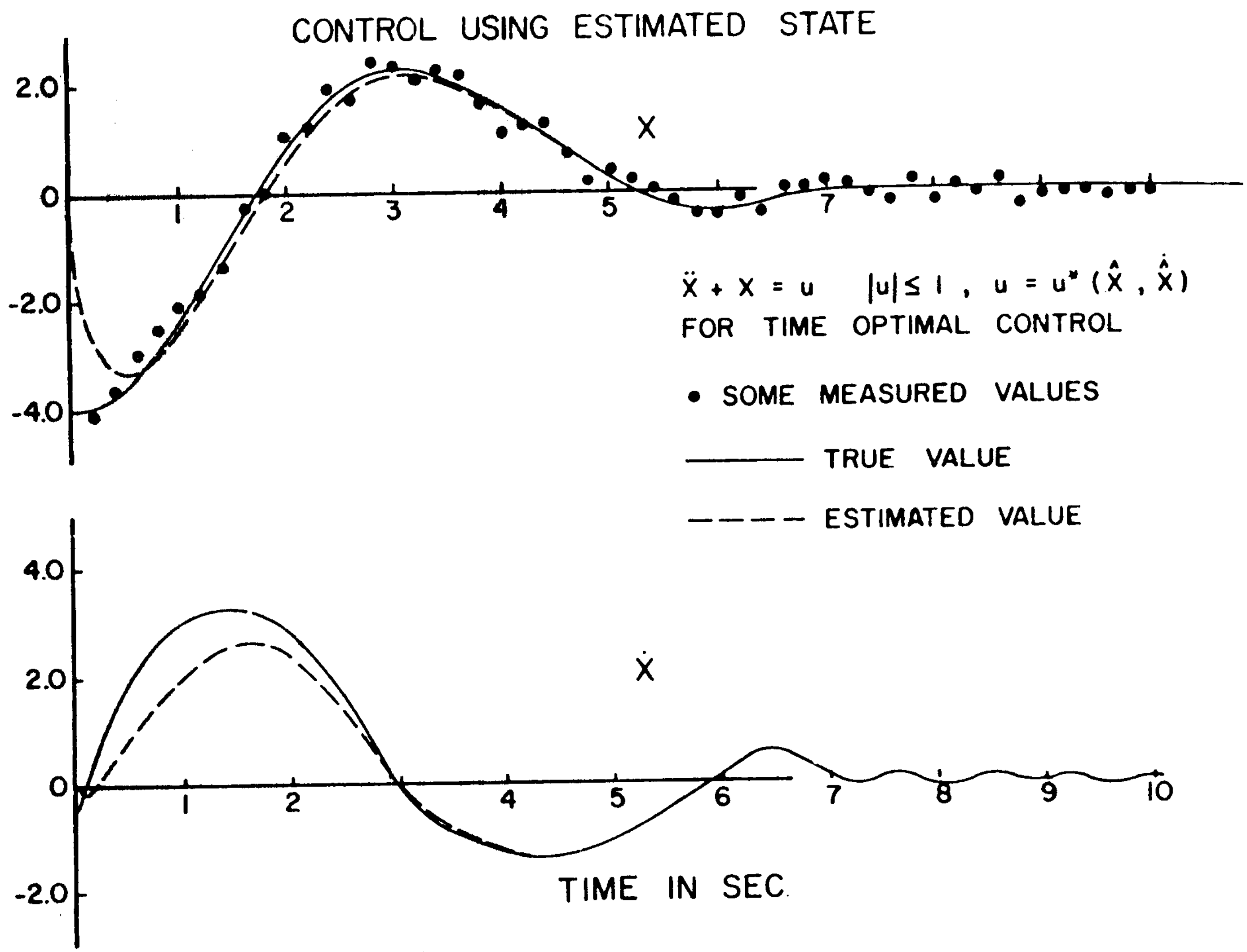
CONTROL USING ESTIMATED STATE
2.0
0
-2.0
-4.0
X
1
2
3
4
5
7
$\ddot{X} + X = u \quad |u| \leq 1 \,, \quad u = u^* (\hat{X}, \hat{\dot{X}})$
FOR TIME OPTIMAL CONTROL
• SOME MEASURED VALUES
TRUE VALUE
ESTIMATED VALUE
4.0
2.0
0
-2.0
$\dot{X}$
1
2
4
5
6
7
8
9
10
TIME IN SEC

Figure 5

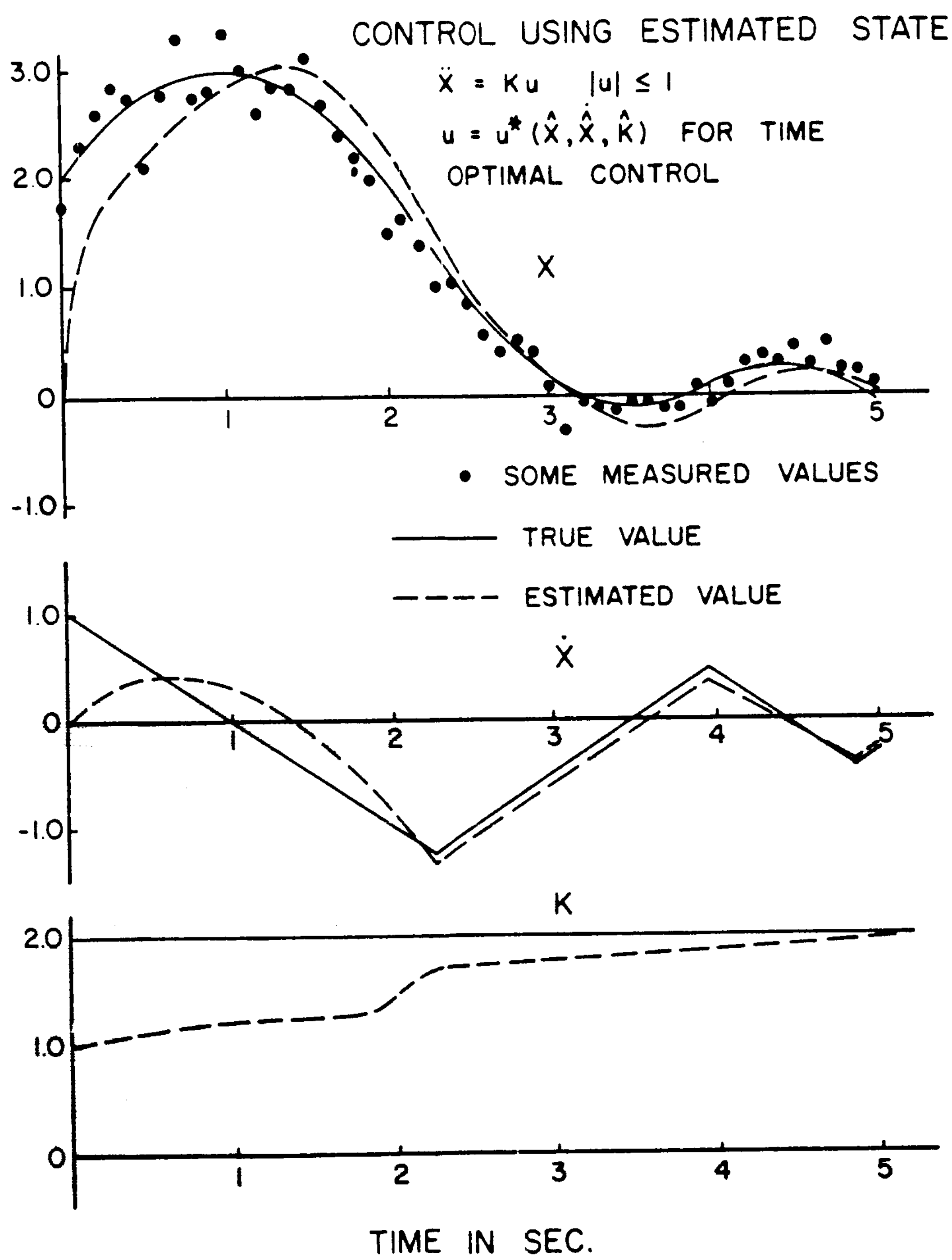
CONTROL USING ESTIMATED STATE
$\ddot{X} = Ku \quad |u| \leq 1$
$u = u^*(\hat{X}, \hat{\dot{X}}, \hat{K})$ FOR TIME
OPTIMAL CONTROL
X
3.0
2.0
1.0
0
-1.0
1
2
3
4
5
SOME MEASURED VALUES
TRUE VALUE
ESTIMATED VALUE
$\dot{X}$
K
TIME IN SEC.

Figure 6

Conclusions

Some general results for the estimation of conditional Markov Processes have been reviewed. The drastic assumptions that this approach requires partly provided the motivation for the more intuitive approach based on a least squares criterion. The latter method with experimental results was discussed in the annual report. The Markov Process approach, however, does provide a possible stochastic interpretation of the sequential state estimator. In particular, it is conjectured that the $R(t)$ in the sequential scheme is some approximation to the conditional error covariance matrix. In the experiments performed the initial condition $R(0)$ was always selected to be symmetric and positive definite.

A possible formulation of the combined identification and control problem was presented using the results from the estimation of conditional Markov Processes. Motivated by the complexity of this approach, a suboptimal combined identification and control scheme was suggested. The additional experimental results indicate that using the estimated state as produced by the sequential state estimator for control purposes may be a feasible method of controlling systems based on partial information.

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SECTION II

Worst Case Initial Condition

Introduction

Once a feedback control system has been designed, either on paper or in actual hardware, the design must be tested to determine whether or not it meets the performance specifications. Often, it is of interest to find the maximum value of the specified value function or an index of performance over the range of the initial conditions possible. The worst case initial condition problem is considered for a class of dynamical systems which may be described by a set of first-order differential equations.

1. Motivation

A typical optimal control problem is the following:

The plant is described by a set of n first order differential equations

$$\dot{\underline{x}} = \underline{f}(t, \underline{x}, u),$$
$$\underline{x}(0) = \underline{C} \tag{1-1}$$

where the initial condition $\underline{C}$ lies in a set Ω and $u(t)$ is the scalar control function.

The control problem is to find $u(t)$ such that a given functional of $\underline{x}(t)$ and $u(t)$, called the index of performance, of the form

$$I_1(u) = \int_0^T g(t, \underline{x}, u)\, dt \tag{1-2}$$

is minimized.

The f_i, $i = 1, 2, \ldots, n$ are assumed to possess piecewise continuous second partial derivatives with respect to all their arguments. $g(t, \underline{x}, u)$

is a scalar-valued function of its arguments, and $g(t, \underline{x}, u) \geq 0$ for all $\underline{x}$ and u in the domain of interest. The function $g(t, \underline{x}, u)$ is assumed to possess piecewise continuous second partial derivatives with respect to all its arguments.

It has been pointed out [1] that even if an optimal control law, i.e., $u^*(t) = u(t, \underline{x})$ can be obtained, it will not be a satisfactory practical solution in general because of the complexity of the dependence of the optimal control law on the current state of the system and the current time. Also it is necessary to have knowledge of all the state variables at every instant of time. It is desirable to incorporate the physical constraints dictated by the application while formulating the optimal control problem. Incorporation of some of these factors in the optimal control problem leads to the Specific Optimal Control problem which will henceforth be referred to as the SOC problem.

The SOC problem is defined as follows:

Given a plant with dynamical equations of the form

$$\dot{\underline{x}} = \underline{f}(t, \underline{x}, u)$$

$$\underline{x}(0) = \underline{C} \in \Omega$$

The specific control law is of the form

$$u = h(\underline{y}, \underline{b}) \qquad (1\text{-}3)$$

In equation (1-3) $\underline{y}$ is a p-dimensional vector ($p \leq n$) which is a known function of the state $\underline{x}$; h is a pre-specified function of its arguments. The vector $\underline{b}$ is a q-dimensional constant vector of unknown parameters.

It is desired to determine the unknown parameters $\underline{b}$ such that an index of performance of the form

$$I_1(u) = \int_0^T g(t, \underline{x}, u)\, dt$$

is minimized.

This problem can be approached by various methods [1]. It is known that the feedback coefficients $\underline{b}$ depend in general on the initial condition $\underline{C}$ and the duration of the process T. For satisfactory implementation of the SOC, it is important that this dependence be the least possible.

The natural question one may ask is: Suppose $\underline{b}$ is given for various $\underline{C} \in \Omega$, how should one choose a particular set of feedback coefficients $\underline{b}$ such that the controller will be satisfactory for all initial conditions $\underline{C} \in \Omega$? The problem as posed above is very imprecisely defined. At present, no analytical techniques exist for solving this problem. The practical limitations and the requirements of the system should give enough insight to choose a feedback coefficient vector $\underline{b}$ for a given set of initial conditions Ω.

Once the design of the controller has been accomplished, it is important to know the maximum value of the index of performance and the initial condition for which the maximum occurs in the set Ω. This initial condition will be called the worst initial condition.

2. Problem Statement

For a given feedback coefficient vector $\underline{b}$, equations (1-1) and (1-2) become

$$\dot{\underline{x}} = \underline{F}(t, \underline{x})$$

$$\underline{x}(0) = \underline{C} \in \Omega \qquad (2\text{-}1)$$

and

$$I = \int_0^T G(t, \underline{x})\, dt \qquad (2\text{-}2)$$

where

$$\underline{F}(t, \underline{x}) = \underline{f}(t, \underline{x}, u) \Big|_{u = h(\underline{y}, \underline{b})}$$

$$G(t, \underline{x}) = g(t, \underline{x}, u) \Big|_{u = H(\underline{y}, \underline{b})}$$

The problem then is to find the worst initial condition $\underline{C} \in \Omega$ which corresponds to the largest value of the performance index.

The F_i, $i = 1, 2, \ldots, n$ are assumed to possess piecewise continuous second partial derivatives with respect to all their arguments. $G(t, \underline{x})$ is a scalar valued function of its arguments, and $G(t, \underline{x}) \geq 0$ for all t and $\underline{x}$ in the domain of interest. It is assumed that $G(t, \underline{x})$ possesses piecewise continuous second partial derivatives with respect to all its arguments.

3. Necessary Conditions

The necessary conditions for this maximization problem are obtained via the Calculus of Variations [1].

The Hamiltonian is defined as

$$H(t, \underline{x}, \underline{\lambda}) = G(t, \underline{x}) + \langle \underline{\lambda}, \underline{F}(t, \underline{x}) \rangle \tag{3-1}$$

where $\underline{\lambda}$ is an n-dimensional Lagrange multiplier vector and the symbol $\langle \, , \, \rangle$ denotes the Euclidean inner product.

The Canonic equations satisfied along an optimal arc are

$$\begin{aligned} \dot{\underline{x}} &= \underline{F}(t, \underline{x}) \\ -\dot{\underline{\lambda}} &= \frac{\partial G}{\partial \underline{x}} + \left[\frac{\partial \underline{F}}{\partial \underline{x}} \right]' \underline{\lambda} \end{aligned} \tag{3-2}$$

In equation (3-2) the prime denotes the transpose. $\frac{\partial G}{\partial \underline{x}}$ is an n-dminesional vector whose i^{th} term is $\frac{\partial G}{\partial x_1}$. $\frac{\partial F}{\partial \underline{x}}$ is an n x n matrix whose i-j^{th} term is $\frac{\partial F_i}{\partial x_j}$.

The transversality condition at the initial and terminal points yields enough boundary conditions for the solution of the set of 2n-differential equations (3-2). At the terminal time $t = T$, the transversality condition is

$$H(t, \underline{x}, \underline{\lambda}) \Big|_{t=T} dt - < \underline{\lambda}(T), d\underline{x} > = 0 \qquad (3\text{-}3)$$

where dt and $d\underline{x}$ are the differentials on the terminal manifold.

Since the terminal time is fixed and the terminal state is free, equation (3-3) requires that

$$\underline{\lambda}(T) = \underline{0} \qquad (3\text{-}4)$$

At the initial time $t = 0$, the transversality condition is

$$H(t, \underline{x}, \underline{\lambda}) \Big|_{t=0} dt - < \underline{\lambda}(0), d\underline{x} > = 0 \qquad (3\text{-}5)$$

where dt and $d\underline{x}$ are the differentials on the initial manifold Ω .

Since the initial time is fixed, $dt = 0$ and equation (3-5) requires that

$$< \underline{\lambda}(0), d\underline{x} > = 0 \qquad (3\text{-}6)$$

should hold on the initial manifold.

Equations (3-4) and (3-6) will, in general, yield the 2n boundary conditions needed for the solution of the set of differential equations (3-2). However, equation (3-6) usually leads to extremely complicated boundary conditions. The solution of this two point boundary value problem (TPBVP) is very difficult to obtain, even numerically.

To illustrate these points, several special cases are considered below.

4. General Second Order System

Plant:

$$\dot{x}_1 = f_1(t,\ x_1,\ x_2)\ ,\ x_1(0) = C_1$$
$$\dot{x}_2 = f_2(t,\ x_1,\ x_2)\ ,\ x_2(0) = C_2 \qquad (4\text{-}1)$$

Index of performance:

$$I = \int_0^T g(t,\ x_1,\ x_2)\ dt \qquad (4\text{-}2)$$

Let the initial manifold Ω be

$$C_1{}^2 + C_2{}^2 \leq 1 \qquad (4\text{-}3)$$

Find $\underline{C} \in \Omega$ such that I is maximum.

The Hamiltonian is

$$H(t,\ x_1,\ x_2,\ \lambda_1,\ \lambda_2) = g(t,\ x_1,\ x_2) + \lambda_1\ f_1(t,\ x_1,\ x_2) + \lambda_2\ f_2(t,\ x_1,\ x_2)$$

The canonic equations are:

$$\dot{x}_1 = f_1(t,\ x_1,\ x_2)$$
$$\dot{x}_2 = f_2(t,\ x_1,\ x_2)$$
$$-\dot{\lambda}_1 = \frac{\partial g}{\partial x_1} + \frac{\partial f_1}{\partial x_1}\lambda_1 + \frac{\partial f_2}{\partial x_1}\lambda_2 \qquad (4\text{-}4)$$
$$-\dot{\lambda}_2 = \frac{\partial g}{\partial x_2} + \frac{\partial f_1}{\partial x_2}\lambda_1 + \frac{\partial f_2}{\partial x_2}\lambda_2$$

At the terminal time (from equation (3-4)),

$$\lambda_1(T) = \lambda_2(T) = 0 \qquad (4\text{-}5)$$

On the initial manifold $C_1{}^2 + C_2{}^2 \leq 1$,

$$<\underline{\lambda}(0),\ d\underline{x}> = 0 \qquad (4\text{-}6)$$

i.e.,

$$\lambda_1(0)\ dC_1 + \lambda_2(0)\ dC_2 = 0 \qquad (4\text{-}7)$$

The initial manifold Ω may be represented in parametric form as follows:

$$C_1 = \tanh \rho_1 \cos \rho_2$$
$$C_2 = \tanh \rho_1 \sin \rho_2 \qquad (4\text{-}8)$$

Thus

$$dC_1 = \operatorname{sech}^2 \rho_1 \cos \rho_2 \, d\rho_1 - \tanh \rho_1 \sin \rho_2 \, d\rho_2$$
$$dC_2 = \operatorname{sech}^2 \rho_1 \sin \rho_2 \, d\rho_1 + \tanh \rho_1 \cos \rho_2 \, d\rho_2$$

Equation (4-7) becomes

$$\operatorname{sech}^2 \rho_1 [\lambda_1(0) \cos \rho_2 + \lambda_2(0) \sin \rho_2] \, d\rho_1$$
$$+ \tanh \rho_1 [-\lambda_1(0) \sin \rho_2 + \lambda_2(0) \cos \rho_2] \, d\rho_2 = 0 \qquad (4\text{-}9)$$

Since ρ_1 and ρ_2 may take on any values, the differentials $d\rho_1$ and $d\rho_2$ are arbitrary and equation (4-9) requires

$$\operatorname{sech}^2 \rho_1 [\lambda_1(0) \cos \rho_2 + \lambda_2(0) \sin \rho_2] = 0 \qquad (4\text{-}10)$$
$$\tanh \rho_1 [-\lambda_1(0) \sin \rho_2 + \lambda_2(0) \cos \rho_2] = 0 \qquad (4\text{-}11)$$

The various possible solutions of (4-10) and (4-11) are analyzed next.

I. In (4-10), let $\operatorname{sech}^2 \rho_1 = 0$, therefore, $\rho_1 = \pm \infty$ and $\tanh \rho_1 = \pm 1$. Then equation (4-11) requires

$$-\lambda_1(0) \sin \rho_2 + \lambda_2(0) \cos \rho_2 = 0 \qquad (4\text{-}12)$$

From (4-12) and (4-8), the following relations at time $t = 0$ should hold

$$\lambda_1(0)\, C_2 = \lambda_2(0)\, C_1 \qquad (4\text{-}13)$$
$$C_1^{\,2} + C_2^{\,2} = 1$$

Equation (4-13) indicates that the extremal points will lie on the boundary of the initial condition manifold $C_1^{\,2} + C_2^{\,2} \leq 1$. Also, it is obvious that there is a possibility of many solutions for equation (4-13) indicating several maximums and minimums of I.

II. In (4-11), let $\tanh \rho_1 = 0$, therefore, $\rho_1 = 0$ and $\operatorname{sech} \rho_1 = 1$. Then equation (4-10) requires

$$\lambda_1(0) \cos \rho_2 + \lambda_2(0) \sin \rho_2 = 0 \tag{4-14}$$

From (4-8),

$$C_1 = 0$$
$$C_2 = 0 \tag{4-15}$$

This indicates that the origin is a possible extremal point. In this case, the adjoint equations are not important and one may just integrate the plant equations (4-1) with the initial conditions (4-15).

If time does not appear explicitly in f_1, f_2 and g and if for $\underline{x} = 0$, $f_1(0) = 0$, $f_2(0) = 0$ and $g(0) = 0$, then $\underline{x}(t) \equiv \underline{0}$ and the index of performance $I = 0$. This will be the trivial solution and indicates the minimum point.

III. If $\tanh \rho_1 \neq 0$ and $\operatorname{sech}^2 \rho_1 \neq 0$, then (4-10) and (4-11) require

$$\lambda_1(0) \cos \rho_2 + \lambda_2(0) \sin \rho_2 = 0$$
$$- \lambda_1(0) \sin \rho_2 + \lambda_2(0) \cos \rho_2 = 0 \tag{4-16}$$

From (4-16) the following relation is obvious

$$\lambda_1^2(0) + \lambda_2^2(0) = 0 \tag{4-17}$$

Since $\lambda_1(0)$ and $\lambda_2(0)$ are real numbers, equation (4-17) requires

$$\lambda_1(0) = \lambda_2(0) = 0 \tag{4-18}$$

For this case, the maximization problem reduces to the following TPBVP:

$$\dot{\underline{x}} = \underline{f}(t, \underline{x})$$
$$-\dot{\underline{\lambda}} = \frac{\partial g}{\partial \underline{x}} + \left(\frac{\partial \underline{f}}{\partial \underline{x}} \right)' \underline{\lambda} \tag{4-19}$$

with boundary conditions

$$\underline{\lambda}(0) = 0 \quad , \underline{\lambda}(T) = 0 \tag{4-20}$$

The boundary condition (4-20) implies that the state $\underline{x}$ is free both at the initial and the terminal ends.

The TPBVP as posed by equations (4-19) and (4-20) contains no information about the initial condition manifold Ω. In general, as the Euclidean norm of the initial condition vector $\underline{C}$, i.e., $C_1^2 + C_2^2$, approaches infinity, the index of performance will increase without bounds. One can intuitively say that in finding the solution of (4-19) and (4-20) by any iterative technique [1], unless there is a local maximum in Ω and the initial approximation is close to the local maximum, the scheme will try to make some suitably defined norm of the initial condition vector as large as possible. Thus, the numerical solution of this TPBVP, except in some special cases, will be very difficult.

To find the local maximum in Ω, one can resort to hill-climbing techniques, like the gradient method [3, 4, 5]. It should be mentioned that none of the available techniques are satisfactory in the multidimensional case.

5. Linear Maximum Problem

Consider an autonomous linear n^{th}-order system with dynamical equations

$$\dot{\underline{x}} = A\,\underline{x} \qquad , \qquad \underline{x}(0) = \underline{C} \tag{5-1}$$

where A is an n x n system matrix and the initial condition $\underline{C}$ lies in the set Ω. Let the index of performance be

$$I = \int_0^T (\underline{x}'\, Q\, \underline{x})\, dt \tag{5-2}$$

where Q is at least a positive semidefinite symmetric matrix.

The state transition matrix $\phi(t)$ is the solution of

$$\dot{\phi} = A\,\phi$$

with initial condition

$$\phi(0) = I \text{ (Identity matrix)}$$

The solution of (5-1) may be written in terms of the state transition matrix $\phi(t)$ as

$$\underline{x}(t) = \phi(t)\,\underline{C} \qquad (5\text{-}3)$$

Substituting (5-3) in (5-2)

$$I = \underline{C}'\left[\int_0^T \phi'(t)\,Q\,\phi(t)\,dt\right]\underline{C}$$

Let

$$R = \int_0^T \phi'(t)\,Q\,\phi(t)\,dt$$

The index of performance I, which is now a function of the initial condition $\underline{C}$, may be written as

$$I(\underline{C}) = \underline{C}'\,R\,\underline{C} \qquad (5\text{-}4)$$

Since the integrand in the equation (5-2) is at least positive semidefinite, the matrix R is clearly at least positive semidefinite. Since Q is symmetric, R is also symmetric.

It is known that a quadratic form $\underline{C}'\,R\,\underline{C}$ is a convex function for all $\underline{C}$, if $\underline{C}'\,R\,\underline{C} \geq 0$ for all $\underline{C}$ (that is, if the form is positive semidefinite) [2].

Since $I(\underline{C})$ is a convex function on the set $\underline{C} \in \Omega$, it is obvious that the maximum of $I(\underline{C})$ will be on the boundary of Ω. Also, if the set Ω is unbounded, then the maximum of $I(\underline{C})$ will, in general be unbounded.

6. Autonomous Nonlinear System

Consider the following problem.

System:

$$\dot{\underline{y}} = \underline{h}(\underline{y}) \quad , \quad \underline{y}(0) = \underline{d} \tag{6-1}$$

Index of performance:

$$I_1 = \int_0^T \ell(\underline{y})\, dt \tag{6-2}$$

Assume: $\underline{h}(0) = \underline{0}$

$$\ell(\underline{0}) = 0 \quad \text{and} \quad \ell\,(\underline{y}) \geq 0 \tag{6-3}$$

At $t = 0$, $\underline{y}$ satisfies

$$\underline{y}'\, Q\, \underline{y} \leq 1 \tag{6-4}$$

where the matrix Q is symmetric and positive definite.

Since Q is symmetric and positive definite, it is easy to find a nonsingular transformation $\underline{X} = P\,\underline{y}$ which will transform this problem to the following:

System:

$$\dot{\underline{x}} = \underline{f}(\underline{x}) \quad , \quad \underline{x}(0) = \underline{C} \tag{6-5}$$

Index of performance:

$$I = \int_0^T g(\underline{x})\, dt \tag{6-6}$$

and $\underline{f}(\underline{0}) = \underline{0}$

$$g(\underline{0}) = 0 \quad \text{and} \quad g(\underline{x}) \geq 0 \tag{6-7}$$

At $t = 0$, $\underline{x}$ satisfies

$$< \underline{x}, \underline{x} > \leq 1 \tag{6-8}$$

The matrix P is the solution of

$$P'\, Q\, P = I \tag{6-9}$$

Proceeding as in section (4),

$$-\dot{\underline{\lambda}} = \frac{\partial g}{\partial \underline{x}} + \left(\frac{\partial \underline{f}}{\partial \underline{x}} \right)' \underline{\lambda} \tag{6-10}$$

The boundary conditions on (6-5) and (6-10) are

(i) at $t = T$, $\underline{\lambda}(T) = 0$ (6-11)

(ii) at $t = 0$, $< \underline{\lambda}(0), d\underline{x} > = 0$ (6-12)

The initial manifold may be written in parametric form as follows:

$$\left.\begin{aligned}
x_1 &= \tanh \rho_1 \sin \rho_2 \sin \rho_3 \cdots\cdots \sin \rho_n \\
x_2 &= \tanh \rho_1 \cos \rho_2 \sin \rho_3 \cdots\cdots \sin \rho_n \\
x_3 &= \tanh \rho_1 \cos \rho_3 \sin \rho_4 \cdots\cdots \sin \rho_n \\
x_4 &= \tanh \rho_1 \cos \rho_4 \sin \rho_5 \cdots\cdots \sin \rho_n \\
&\cdot \\
&\cdot \\
&\cdot \\
x_{n-1} &= \tanh \rho_1 \cos \rho_{n-1} \sin \rho_n \\
x_n &= \tanh \rho_1 \cos \rho_n
\end{aligned}\right] \tag{6-13}$$

Equations (6-12) and (6-13) lead to three possible cases.

(i) $\rho_1 = \pm \infty$, i.e., at $t = 0$, $< \underline{x}, \underline{x} > = 1$.

This implies that the extremal point lies on the boundary of Ω.

(ii) $\rho_1 = 0$, i.e., at $t = 0$, $\underline{x}(0) = 0$.

From (6-7), it is evident that $\underline{x}(t) \equiv 0$ and $I = 0$. This will be the minimum point.

(iii) If $\rho_1 \neq 0$ or $\pm \infty$, then

$$\underline{\lambda}(0) = 0 \tag{6-14}$$

Equations (6-11) and (6-14) imply that $\underline{x}$ is free both at the initial and terminal ends. In the general case, there is the possibility of a local maximum in Ω.

7. Example 1

Consider the system of equations which governs the rotational motion of a rigid body such as a space vehicle about its center of mass. The equations of motion are

$$
\begin{aligned}
I_1 \dot{\omega}_1 &= (I_2 - I_3)\, \omega_2 \omega_3 + u_1 \\
I_2 \dot{\omega}_2 &= (I_3 - I_1)\, \omega_3 \omega_1 + u_2 \\
I_3 \dot{\omega}_3 &= (I_1 - I_2)\, \omega_1 \omega_2 + u_3
\end{aligned}
\tag{7-1}
$$

where ω_i = angular velocity about the i^{th} principal axis

I_i = the moment of inertia about the i^{th} principal axis

u_i = control torque on the i^{th} principal axis.

$$i = 1, 2, 3$$

In terms of the angular momenta

$$x_i(t) = I_i\, \omega_i(t)\ ; \quad i = 1, 2, 3$$

equation (7-1) becomes

$$
\begin{aligned}
\dot{x}_1 &= a_1 x_2 x_3 + u_1 \\
\dot{x}_2 &= a_2 x_3 x_1 + u_2 \\
\dot{x}_3 &= a_3 x_1 x_2 + u_3
\end{aligned}
\tag{7-2}
$$

where

$$a_1 = \left(\frac{1}{I_3} - \frac{1}{I_2}\right),\ a_2 = \left(\frac{1}{I_1} - \frac{1}{I_3}\right) \text{ and } a_3 = \left(\frac{1}{I_2} - \frac{1}{I_1}\right)$$

The initial condition on (7-2) is

$$x_i(0) = C_i \quad , \quad i = 1, 2, 3$$

A suitable performance index for the problem is

$$I = \underset{\underline{u}}{\mathrm{Min}} \int_0^T (<\underline{x}, \underline{x}> + <\underline{u}, \underline{u}>)\, dt \tag{7-3}$$

For this problem, the optimal control law can be obtained using the Hamilton-Jacobi Equation [1].

The Hamiltonian is

$$H(\underline{x}, \underline{u}, \underline{\lambda}) = < \underline{x}, \underline{x} > + < \underline{u}, \underline{u} > + \lambda_1(a_1 x_2 x_3 + u_1) + \lambda_2(a_2 x_3 x_1 + u_2) + \lambda_3(a_3 x_1 x_2 + u_3) \tag{7-4}$$

The optimal control $\underline{u}(t) = \underline{u}^*(t)$ is obtained by minimizing H with respect to $\underline{u}$, i.e., by setting the partials of H with respect to u_i equal to zero.

$$\frac{\partial H}{\partial u_i} = 0 \quad , \quad i = 1, 2, 3$$

This yields

$$u_i^* = - \frac{\lambda_i}{2} \quad , \quad i = 1, 2, 3 \tag{7-5}$$

Define the minimum value of the Hamiltonian as

$$H^*(\underline{x}, \underline{\lambda}) = \underset{\underline{u}}{\text{Min}} \; H(\underline{x}, \underline{\lambda}, \underline{u})$$

$$= H(\underline{x}, \underline{\lambda}, \underline{u}) \Big|_{\underline{u} = -\frac{1}{2}\underline{\lambda}}$$

Thus

$$H^*(\underline{x}, \underline{\lambda}) = < \underline{x}, \underline{x} > + < \underline{u}, \underline{u} > + a_1 \lambda_1 x_2 x_3 + a_2 \lambda_2 x_3 x_1 + a_3 \lambda_3 x_1 x_2 - \frac{1}{4} < \underline{\lambda}, \underline{\lambda} > \tag{7-6}$$

Since the minimum value of the performance index depends on the initial state $\underline{C}$ and the starting instant τ, define the return function $J(\tau, \underline{C})$ as

$$J(\tau, \underline{C}) = \underset{\underline{u}(t)}{\text{Min}} \int_{\tau}^{T} (< \underline{x}, \underline{x} > + < \underline{u}, \underline{u} >) \, dt \tag{7-7}$$

Then $J(t, \underline{x})$ satisfies the following Hamilton-Jacobi partial differential equation [1]:

$$\frac{\partial J}{\partial t} + H^*(\underline{x}, \underline{\lambda}) = 0 \qquad (7\text{-}8)$$

where
$$\lambda_i = \frac{\partial J}{\partial x_i} \qquad (7\text{-}9)$$

The boundary condition on (7-8) is

$$J(T, \underline{x}) = 0$$

To solve (7-8), assume a solution of the form

$$J(t, \underline{x}) = g(t)\,(<\underline{x}, \underline{x}>) \qquad (7\text{-}10)$$

Substituting (7-10) in (7-8) yields

$$\dot{g} - g^2 + 1 = 0 \quad , \qquad g(T) = 0 \qquad (7\text{-}11)$$

The solution of (7-11) is

$$g(t) = \tanh\,(T - t) \qquad (7\text{-}12)$$

From (7-10)

$$J(t, \underline{x}) = \tanh\,(T\text{-}t)\,(<\underline{x}, \underline{x}> \qquad (7\text{-}13)$$

From (7-7) and (7-13)

$$I(\underline{C}) = J(0, \underline{C})$$
$$= \tanh\,(T)\,(<\underline{C}, \underline{C}>) \qquad (7\text{-}14)$$

Thus, the index of performance is equal to a constant ($\tanh T$) multiplied by the norm $(C_1^2 + C_2^2 + C_3^2)$. For any set of initial conditions ($\underline{C} \in \Omega$) the maximum of I will obviously be on the boundary.

From (7-5), (7-9) and (7-13),

$$u_i^* = -\tanh\,(T\text{-}t)\,x_i \quad , \quad i = 1, 2, 3 \qquad (7\text{-}15)$$

that is, the optimal control law is a simple feedback solution where the feedback coefficient $b(t) = \tanh\,(T\text{-}t)$ is time varying. At $t = 0$, $b(0) = \tanh\,(T)$ and at $t = T$, $b(T) = 0$; for all t, $b(t) \geq 0$. Since the time varying gain is normally difficult to implement, it may be desirable to replace $b(t)$ by a constant gain b. Then, the control law is

$$u_i = - b\, x_i \quad , i = 1, 2, 3 \tag{7-16}$$

Using (7-2), (7-3) and (7-16), the index of performance for this problem is obtained via the Hamilton-Jacobi partial differential equation and is given by

$$I_1(\underline{C}) = \frac{1+b^2}{2b} (1-e^{-2bT})(< \underline{C}, \underline{C} >) \tag{7-17}$$

From (7-17) it is evident that for any set of initial conditions ($\underline{C} \in \Omega$) the maximum of I_1 will be on the boundary.

<u>Example 2</u>: A simple second order linear example is considered here to indicate some of the possible difficulties in the numerical solution of the worst case initial condition problem.

Let the system be described by the differential equations

$$\begin{aligned} \dot{x}_1 &= x_2 &, \quad x_1(0) &= C_1 \\ \dot{x}_2 &= - 2x_1 - 3x_2 &, \quad x_2(0) &= C_2 \end{aligned} \tag{7-18}$$

Index of performance:

$$I = \int_0^1 (x_1^{\,2} + x_2^{\,2})\, dt \tag{7-19}$$

Let the set of initial conditions Ω be

$$C_1^{\,2} + C_2^{\,2} \leq 1 \tag{7-20}$$

The Hamiltonian is,

$$H = x_1^{\,2} + x_2^{\,2} + \lambda_1(x_2) + \lambda_2(-2x_1 - 3x_2) \tag{7-21}$$

The canonic equations are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & -3 & 0 & 0 \\ -2 & 0 & 0 & 2 \\ 0 & -2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} \tag{7-22}$$

The transversality condition at the terminal time T = 1 yields

$$\lambda_1(T) = \lambda_2(T) = 0 \tag{7-23}$$

From Section (4), the extremum lies on the boundary, i.e., at the initial time.

$$< \underline{\lambda}(0), \ d\underline{x} > = 0 \tag{7-24}$$

$$C_1^{\ 2} + C_2^{\ 2} = 1$$

For this problem, the analytical solution is possible and the index of performance is given by

$$I = 0.887 \ C_1^{\ 2} + 0.192 \ C_2^{\ 2} + 0.214 \ C_1 C_2 \tag{7-25}$$

The extremal points of (7-25) subject to the condition $C_1^{\ 2} + C_2^{\ 2} = 1$ are (i) Maximum: $C_1 = 0.91$, $C_2 = 0.417$; $C_1 = -0.91$, $C_2 = -0.417$;

$$I = 0.849$$

(ii) Minimum: $C_1 = -0.143$, $C_2 = 0.99$; $C_1 = 0.143$, $C_2 = -0.99$;

$$I = 0.177$$

The TPBVP given by equation (7-22) with boundary conditions (7-23) and (7-24) will have four possible solutions out of which two will be maxima and two minima. In the general case, it is difficult to obtain a numerical solution of a TPBVP which has more than one solution by iterative techniques. It seems that one may have to search for the optimal points on the boundary using some hill-climbing or optimum seeking techniques [3, 4, 5], as was explained earlier.

Conclusion

It is shown in sections 3, 4, and 5 that the worst case initial condition problem in the set Ω of the form $\underline{C}'\, Q\, \underline{C} \leq 1$ has three possible solutions,

(i) $\underline{C} = 0$, which normally will lead to the trivial solution,

(ii) $\underline{C}'\, Q\, \underline{C} = 1$, i.e., $\underline{C}$ lies on the boundary, and

(iii) $\underline{C}$ lies inside the set Ω.

Since the direct numerical solution of case (iii), i.e., solution of TPBVP will, in general, be formidable, it will be of interest to obtain suitable conditions on the system equations and the index of performance for which the worst case initial condition lies on the boundary.

It is shown that for linear systems with quadratic indices of performance, the worst case initial condition always lies on the boundary.

In this discussion, only the hypersphere-type ($\underline{C}'\, Q\, \underline{C} \leq 1$) initial condition manifold Ω is considered. It should be of interest to obtain similar results for any closed type of initial manifold Ω.

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