Final Report

Contract NAS 8-11143

ADVANCE RESEARCH ON CONTROL SYSTEMS FOR THE SATURN LAUNCH VEHICLE

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CRA

CONTROL RESEARCH ASSOCIATES
4321 HARTWICK ROAD
COLLEGE PARK, MARYLAND

ADVANCED RESEARCH ON CONTROL SYSTEMS FOR THE SATURN LAUNCH VEHICLE

by

D. C. Lewis
P. Mendelson

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ABSTRACT

Advance Research on Control Systems for the Saturn Launch Vehicles

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Presented are the results of Contract NAS8-11143 on studies of a minimax problem in launch vehicle control. This is the general problem of finding a control which minimizes the maximum peak value of a selected index of performance over a class of admissible disturbances while satisfying a bounded state variable constraint. Under this contract, some rigorous mathematical foundations were established for the minimax problem. These include a precise formulation of the problem and existence theorems for solution of the problem. In connection with the bounded state variable problem, a new concept called the "core of the region of bounded state variables" is introduced. This is roughly defined as the largest set of initial conditions for which solution trajectories remain inside the bounded set of state variables under any admissible disturbance. Existence and uniqueness of the core of a region are discussed, and for some sample second and third order problems, the core is computed.

Author

PREFACE

During the period from March, 1963 through December, 1963 a program of research relating to the synthesis of optimal control techniques for the saturn launch vehicle was carried out. This work was done under Contract NAS 8-5002. The scope of this work comprised three specific areas of endeavor:

(a) time optimal control
(b) time optimal control with constraints
(c) certain minimax problems in control theory

All intended areas of Part (a) were covered. This included a search for the control law in closed form for the fourth order linear control system (single control element) having two zero eigenvalves and two real and equal but oppositely signed non-zero eigenvalves.

Substantial results relative to Part (b) were obtained. The outcome of this preliminary work indicated that useful results could probably be obtained through further research. The formulation of the minimax problem of Part (c) as well some preliminary qualitative results in that area were obtained.

The work of this contract (NAS 8-11143) is a direct follow-on to the work on Contract NAS 8-5002. The main emphasis herein is centered on the minimax problem.

At the outset of this project it was decided that rather than submit informal progress reports with a single detailed report at the end, CRA would prepare detailed progress reports each month with the intent that each of these reports would be used as a chapter in the final report. This approach to the documentation of our work has been followed. The main advantage of this method is that it provides NASA with monthly reports which explain exactly and in detail the nature and direction of our progress. Hence, the technical officer is better able to direct the efforts of the contractor towards the desired NASA goal and also NASA personnel are able to digest our results continuously rather than be burdened with the necessity of having to read one large weighty report after the contract is over to learn the results of our work. The main disadvantage to this approach is that some of the material in later chapters supercedes certain material in the early chapters. The reader is urged to refer to the Appendix of Notes and Errata in order to overcome this difficulty.

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CHAPTER 1

1. Introduction: Rough Statement of the Minimax Problem In Control Theory

Consider the system

$$S: \dot{x} = f(x, w, u), \tag{1.1}$$

where x is an n-vector representing the system's state, $\dot{x}$ represents differentiation with respect to time t, f is an n-vector function of (x, w, u), while $w = w(t)$ and $u = u(t)$ are n-vector functions to be more properly described below. The function f is given. The vector $w = w(t)$ is restricted to a certain class W of allowable "winds", while $u = u(t)$, the control vector, is restricted to a given class U of admissible controls.

The system S is viewed as being subject to various disturbances (winds) belonging to the given class W. It is controlled through the imposition of controls belonging to the given class U. It is assumed hereby that the classes U and W are such that it is always possible to control the system so that its deviation from the desired state is in any case bounded by preassigned bounds, say, through inequalities of the form

$$|x_i| \leq a_i, \qquad i = 1, \dots, n, \tag{1.2}$$

where $a_1, \dots, a_n$ are given positive constants. This assumption may be formulated more generally by the statement that a proper choice of control exists so that $x(t) \epsilon R$ for all t in some interval (say, $0 \leq t \leq T$), where R is a preassigned set in the space of x.

During the course of its controlled motion the system may recede from its desired state, or some function $F(x_1,\ldots,x_n)$ may grow uncomfortably large. The minimax problem may be described briefly as the problem of identifying the control (or controls) such that the maximum deviation of the system (or of F), throughout its motion, from its desired state shall be a minimum. In the event that $|F|$ is our criterion, the constraints embodied in (1.2) must still be satisfied.

It must be apparent to the reader that the question posed above is hopelessly general and hopelessly vague. Our first task will be to attempt a precise formulation of a more tractable problem.

2. Non-Topological Aspects of Minimax Problems

We consider two spaces X and Y and a mapping of the cartesian product space $X \times Y$ into a bounded set S of the real numbers. Thus, with $x \epsilon X$ and $y \epsilon Y$, $F(x, y) \epsilon S$. Since S is bounded,

$$\underset{x \epsilon X}{\text{l.u.b.}} \; F(x, y) = \varphi(y) \tag{2.1}$$

exists for all $y \epsilon Y$,

and

$$\underset{y \epsilon Y}{\text{g.l.b.}} \; F(x, y) = \psi(x) \tag{2.2}$$

exists for all $x \epsilon X$. Moreover $\varphi(y) \epsilon \bar{S}$ for all $y \epsilon Y$ and $\psi(x) \epsilon \bar{S}$ for all $x \epsilon X$, where $\bar{S}$, being the closure of a bounded set S of real numbers, is itself bounded. Hence, there exist real numbers A and $B \epsilon \bar{S}$, such that

$$\underset{y \epsilon Y}{\text{g.l.b.}} \; \varphi(y) = A \tag{2.3}$$

and

$$\underset{x \epsilon X}{\text{l.u.b.}} \; \psi(x) = B. \tag{2.4}$$

Then without introducing any topology at all into the spaces X and Y, we are able to prove the following theorem

Theorem 2.1 $A \geqq B$, that is

$$\underset{y \epsilon Y}{\text{g.l.b.}} \left[\underset{x \epsilon X}{\text{l.u.b.}} \; F(x, y) \right] \geqq \underset{x \epsilon X}{\text{l.u.b.}} \left[\underset{y \epsilon Y}{\text{g.l.b.}} \; F(x, y) \right].$$

Proof. Let ϵ be any positive number. Then since $B-\epsilon$ is not an upper bound for $\psi(x)$ for all x in X even though B is an upper bound (in accordance with (2.4)), there must exist $x_o \epsilon X$ such that

$$B-\epsilon < \psi(x_o) \tag{2.5}$$

By (2.2), for all $y \epsilon Y$,

$$\psi(x_o) \leq F(x_o, y) \tag{2.6}$$

which by (2.1) does not exceed $\varphi(y)$. Hence, we see from (2.5) and (2.6) that

$$B-\epsilon \leq \varphi(y) \tag{2.7}$$

for all $y \epsilon Y$.

Similarly from (2.3), we see that there exists $\bar{y} \epsilon Y$ $\ni$

$$\varphi(\bar{y}) < A + \epsilon \tag{2.8}$$

Hence, from (2.7) and (2.8), on taking $y = \bar{y}$ in (2.7), we find that

$$B-\epsilon < A + \epsilon$$

Since this is true for every positive number ϵ, we conclude, by allowing ϵ to approach 0, that $B \leq A$, as we wished to prove.

In the special case where the various functions mentioned above take on their greatest lower and their least upper bounds the sense of Theorem 2.1 is to the effect that

$$\min_{y \epsilon Y} \left[\max_{x \epsilon X} F(x, y) \right] \geq \max_{x \epsilon X} \left[\min_{y \epsilon Y} F(x, y) \right] \tag{2.9}$$

It is not, in general, possible to prove that the sign $\geqq$ in Theorem 2.1 (or, even in the more special case (2.9)) can be replaced by the sign =. For example, suppose that X and Y contain only two elements 0 and 1 and let

$$F(0, 1) = 0, \qquad F(1, 1) = 1$$
$$F(0, 0) = 1, \qquad F(1, 0) = 0$$

then we see at once that $\varphi(0) = \varphi(1) = 1$ while $\psi(0) = \psi(1) = 0$. Hence $A = \min \varphi(y) = 1$ and $B = \max \psi(x) = 0$, so that we have here $A > B$.

Nevertheless there exist familiar cases in which $A = B$. For instance, if $F(x, y) = y^2 - x^2$ where $-1 \leqq x, y \leqq +1$. Here we have $\psi(x) = -x^2$, $\varphi(y) = y^2$, so that $A = B = 0$. This might suggest to the beginner that when $F(x, y)$ is an analytic function of the two real variables x and y we might expect to have $A = B$. That even this need not be the case is easily shown by the following example:

$$F(x, y) = 1 - (x-y)^2 \qquad 0 \leqq x, y \leqq 1$$

$$\min_y F(x, y) = \psi(x) = \begin{cases} 2x-x^2 & \text{if } 0 \leqq x \leqq 1/2 \\ 1-x^2 & \text{if } 1/2 \leqq x \leqq 1 \end{cases}$$

$$\max_x F(x, y) = \varphi(y) = 1$$

$$\min \varphi(y) = A = 1, \qquad \max \psi(x) = 3/4 = B$$

so that $A > B$.

A further investigation of the question of when $A = B$ leads to Theorems 2.2 and 2.3, which are slight generalizations of corresponding theorems of Von Neumann. We first, however, introduce the following definition

DEFINITION: Let η be any positive number. A point $(x_o, y_o) \in X x Y$ is said to below to an η-saddle set $S_\eta \subset X x Y$ if

$$\underset{x \in X}{\text{l.u.b.}}\ F(x, y_o) \leq F(x_o, y_o) + \eta \tag{2.10}$$

and

$$F(x_o, y_o) - \eta \leq \underset{y \in Y}{\text{g.l.b.}}\ F(x_o, y) \tag{2.11}$$

<u>Theorem 2.2</u> If for every $\eta > 0$, S_η is not empty, then $A = B$. Moreover, for every $\eta > 0$, and for every point $(x_o, y_o) \in S_\eta$,

$$|A - F(x_o, y_o)| < \eta$$

<u>Proof</u> Let $(x_o, y_o) \in S_\eta$ so that (2.10) and (2.11) are valid. From (2.1) (2.2), (2.3), and (2.4), we write the alternative definitions of A and B

$$A = \underset{y \in Y}{\text{g.l.b.}} \left[\underset{x \in X}{\text{l.u.b.}}\ F(x,y) \right] \tag{2.12}$$

$$B = \underset{x \in X}{\text{l.u.b.}} \left[\underset{y \in Y}{\text{g.l.b.}}\ F(x,y) \right] \tag{2.13}$$

From (2.13) $B \geq \underset{y \in Y}{\text{g.l.b.}}\ F(x_o, y)$. Hence, from (2.11), we have

$$B \geq F(x_o, y_o) - \eta \tag{2.14}$$

From (2.12) $A \leq \underset{x \in X}{\text{l.u.b.}} F(x, y_o)$. Hence, from (2.10), we have

$$A \leq F(x_o, y_o) + \eta \tag{2.15}$$

Hence, from (2.14) and (2.15) we see that $A \leq B + 2\eta$. Since this is true for every positive number η, it follows that $A \leq B$. But, by Theorem 2.1, it is already known that $A \geq B$. Hence, $A = B$. Hence (2.14) can be rewritten

$$A \geq F(x_o, y_o) - \eta \tag{2.16}$$

The last assertion of the theorem then follows at once from (2.15) and (2.16).

Theorem 2.3 Conversely, let $A = B$ and let η be any positive number; then the set S_η is not empty.

Proof Let $\epsilon = (1/2)\eta$ Let x_o be any point in X such that

$$B-\epsilon < \psi(x_o) \leq B \tag{2.17}$$

and let y_o be any point in Y such that

$$A \leq \varphi(y_o) < A + \epsilon$$

Since we are assuming that $A = B$, we can write this last formula as

$$B \leq \varphi(y_o) < B + \epsilon \tag{2.18}$$

Evidently then, from (2.18), $\underset{x \in X}{\text{l.u.b.}} F(x, y_o) = \varphi(y_o) < B + \epsilon$, which from (2.17) is less than

$$\psi(x_o) + 2\epsilon = \underset{y \in Y}{\text{g.l.b.}} F(x_o, y) + \eta$$

so that

$$\underset{x \in X}{\text{l.u.b.}} F(x, y_o) < \underset{y \in Y}{\text{g.l.b.}} F(x_o, y) + \eta \tag{2.19}$$

Hence, for every $\bar{x} \in X$,

$$F(\bar{x}, y_o) \leq \underset{x \in X}{\text{l.u.b.}} F(x, y_o) < \underset{y \in Y}{\text{g.l.b.}} F(x_o, y) + \eta \leq F(x_o, y_o) + \eta$$

It follows that $\underset{x \in X}{\text{l.u.b.}} F(x, y_o) \leq F(x_o, y_o) + \eta$ (2.20)

Going back to (2.19), we also see that, for every $\bar{y} \in Y$,

$$F(x_o, \bar{y}) \geq \underset{y \in Y}{\text{g.l.b.}} F(x_o, y) > \underset{x \in X}{\text{l.u.b.}} F(x, y_o) - \eta \geq F(x_o, y_o) - \eta$$

It follows that $\underset{y \in Y}{\text{g.l.b.}} F(x_o, y) \geq F(x_o, y_o) - \eta$ (2.21)

From (2.10), (2.11), (2.20), and (2.21), it is now seen that $(x_o, y_o) \in S_\eta$, which establishes the theorem.

Our next result is somewhat more transparent.

<u>Theorem 2.4</u> $\underset{y \in Y}{\text{l.u.b.}} \left[\underset{x \in X}{\text{l.u.b.}} F(x, y) \right] = \underset{x \in X}{\text{l.u.b.}} \left[\underset{y \in Y}{\text{l.u.b.}} F(x, y) \right]$

$$= \underset{(x,y) \in X \times Y}{\text{l.u.b.}} F(x, y)$$

with a similar result on replacing l.u.b. by g.l.b.

<u>Proof</u> Let $\underset{y \in Y}{\text{l.u.b.}} F(x, y) = \lambda(x)$ (2.22)

$$\underset{x \in X}{\text{l.u.b.}} \lambda(x) = M \tag{2.23}$$

$$\underset{x \in X}{\text{l.u.b.}} F(x, y) = \varphi(y) \tag{2.24}$$

$$\underset{y \in Y}{\text{l.u.b.}}\ \varphi(y) = N \tag{2.25}$$

$$\underset{(x,\ y)\ \in X x Y}{\text{l.u.b.}}\ F(x,\ y) = P \tag{2.26}$$

From the boundedness of the set S into which F maps XxY, it is obvious, as above, that all these least upper bounds exist. We wish to prove that $M = N = P$. From (2.23), we have $\lambda(x) \leqq M$ for all $x \in X$ and from (2.22) we have $F(x, y) \leqq \lambda(x)$ for all $x \in X$ and all $y \in Y$. Hence $F(x, y) \leqq M$ for all $(x, y) \in XxY$. Hence $P \leqq M$.

On the other hand, if ϵ is any positive number, we see from (2.23) that there exists a point $x_o \in X$ such that $M-\epsilon \leqq \lambda(x_o)$. Keeping fixed this point x_o, we see from (2.22) that there exists a point $y_o \in Y$ such that $\lambda(x_o) - \epsilon \leqq F(x_o, y_o)$. Hence we find that $M-2\epsilon \leqq \lambda(x_o) - \epsilon \leqq F(x_o, y_o) \leqq P$. Thus $M-2\epsilon \leqq P$ for every positive number ϵ. Hence $M \leqq P$, which, combined with the result $P \geqq M$ obtained above, shows that $M = P$.

Using (2.24) and (2.25) as we used (2.22) and (2.23), it is also easy to prove that $N = P$. Hence $M = N = P$ as we wished to prove.

The minimax problem is said to have a solution if there exists points $x_o \in X$, and $y_o \in Y$, such that

$$F(x_o,\ y_o) = \min_{y \in Y} \left[\max_{x \in X}\ F(x,\ y)\right]$$

It will be proved later that the minimax problem always has a solution if X and Y are sequentially compact spaces, if $F(x, y)$ is upper semicontinuous in its dependence on x, and if $\varphi(y)$ is lower semicontinuous in y. It will also appear that both these conditions involving semi-continuity are automatically satisfied if $F(x, y)$ is continuous in XxY.

From a practical standpoint, however, one might well question the usefulness of this result. For even without assuming any topological properties of X and Y, the following theorem makes it obvious that there always exist approximate solutions in a quite satisfactory sense.

Theorem 2.5 Let $\underset{y\epsilon Y}{\text{g.l.b.}}\ \underset{x\epsilon X}{\text{l.u.b.}}\ F(x, y) = A$ in accordance with the notation used in Theorem 2.1 Let ϵ be an arbitrary positive number. Then there exist points $x_o \epsilon X$ and $y_o \epsilon Y$ such that

$$|F(x_o, y_o) - A| < \epsilon$$

Proof: As previously, we let $\varphi(y) = \underset{x\epsilon X}{\text{l.u.b.}}\ F(x, y)$ so that (2.3) holds. Since A is thus a lower bound for φ but $A + \epsilon$, is not, we see that there exist $y_o \epsilon Y$ such that $A \leq \varphi(y_o) < A + \epsilon$.

With this y_o fixed, there exists x_o such that $\varphi(y_o) - \epsilon < F(x_o, y_o) \leq \varphi(y_o)$. Hence $A - \epsilon \leq \varphi(y_o) - \epsilon < F(x_o, y_o) \leq \varphi(y_o) < A + \epsilon$. Hence we have $A - \epsilon < F(x_o, y_o) \leq A + \epsilon$ for the x_o and y_o chosen in the way indicated.

3. Sufficient Condition for the Existence of a Solution of a Minimax Problem.

In the previous section we promised to prove a theorem on the existence of a solution to the minimax problem

$$\min_{y\epsilon Y} \left[\max_{x\epsilon X} F(x, y) \right]$$

That is, we wish to prove the existence of a point $x_o \epsilon X$ and a point $y_o \epsilon Y$, such that

$$F(x_o, y_o) = \min_{y \epsilon Y} \left[\max_{x \epsilon X} F(x, y) \right]$$

The hypotheses on which our proof is based are as follows:

H_1: F is bounded on the product space XxY

H_2: Both X and Y are not only topological spaces, but are also sequentially compact.

H_3: F(x, y) is upper semi-continuous in its dependence on x. This means that $\limsup_{x \to \bar{x}} F(x, y) \leq F(\bar{x}, y)$ for every $\bar{x} \epsilon X$.

H_4: $\varphi(y) = \max_{x \epsilon X} F(x, y)$, the existence of which is insured by H_3, is lower semi-continuous. This means that $\liminf_{y \to \bar{y}} \varphi(y) \geq \varphi(\bar{y})$ for every $\bar{y} \epsilon Y$.

Notice that the hypothesis H_4 refers to a function $\varphi(y)$ whose existence must first be established from the other hypotheses. This is done as follows:

We must prove that F(x, y) for each fixed y must take on a maximum value. Since F(x, y) is bounded and since every bounded set of real numbers has a least upper bound, it makes sense to let

$$\varphi(y) = \underset{x \epsilon X}{\text{l.u.b.}} F(x, y),$$

and we now have only to prove that there exists a point $x^* = x^*(y) \epsilon X$ such that

$$\varphi(y) = F(x^*, y).$$

Since $\varphi(y) - \frac{1}{n}$ is not an upper bound there will exist at least one point $x_n \epsilon X$ such that

$$F(x_n, y) > \varphi(y) - \frac{1}{n} \tag{3.1}$$

We may also restrict attention to the case where $F(x_n, y) \neq \varphi(y)$, for each n; for otherwise we could take x^* to be equal to an x_n for which this condition is not fulfilled. Since X is sequentially compact the sequence $x_1, x_2, x_3, \dots,$ has at least one convergent sub-sequence, say, $x_{n_1}, x_{n_2}, x_{n_3}, \dots$ This means that there exists $x^* \epsilon X$ such that

$$\lim_{k \to \infty} x_{n_k} = x^*$$

From (3.1) we also have $F(x_{n_k}, y) > \varphi(y) - 1/n_k$. This shows that $\liminf_{k\to\infty} F(x_{n_k}, y) \geq \varphi(y)$. On the other hand from Hypothesis H_3, we have $\limsup_{k\to\infty} F(x_{n_k}, y) \leq F(x^*, y)$. Hence $\varphi(y) \leq \liminf_{k\to\infty} F(x_{n_k}, y) \leq \limsup_{k\to\infty} F(x_{n_k}, y) \leq F(x^*, y)$, so that $\varphi(y) \leq F(x^*, y)$. But we also know that $\varphi(y) \geq F(x^*, y)$ since $\varphi(y)$ is an upper bound for $F(x,y)$. Hence it is clear that $\varphi(y) = F(x^*, y)$, as we wished to prove.

We next need to prove that there exists a point $y_0 \epsilon Y$ where $\varphi(y)$ takes on a minimum. Let $A = \underset{y \epsilon Y}{\text{g.l.b.}} \varphi(y)$. We can then find a sequence of points in Y, say $y_1, y_2, y_3, \dots$ such that

$$\varphi(y_n) < K + \frac{1}{n} \tag{3.2}$$

and we can limit ourselves to the non-trivial case where $\varphi(y_n) \neq A$ for say n. Otherwise we could take y_o equal to a y_n for which this condition is not fulfilled. Since Y is sequentially compact, the sequence $y_1, y_2, y_3, \ldots$ has at least one convergent subsequence, say, $y_{n_1}, y_{n_2}, y_{n_3}, \ldots$ This means that there exists a point y_o in Y such that

$$\lim_{k \to \infty} y_{n_k} = y_o$$

From (3.2) we also have $\varphi(y_{n_k}) < A + 1/n_k$. This shows that $\limsup_{k \to \infty} \varphi(y_{n_k}) \leq A$ On the other hand from Hypothesis H_4, we have $\liminf_{k \to \infty} \varphi(y_{n_k}) \geq \varphi(y_o)$. Hence $\varphi(y_o) \leq \liminf_{k \to \infty} \varphi(y_{n_k}) \leq A$, so that $\varphi(y_o) \leq A$. But we also know that $\varphi(y_o) \geq A$, since A is a lower bound for $\varphi(y)$, Hence $\varphi(y_o) = A = \underset{y \in Y}{\text{g.l.b.}}\ \varphi(y)$.

Thus we are justified in writing for the y_o thus found

$$\varphi(y_o) = \min_{y \in Y} \varphi(y) = \min_{y \in Y} \left[\max_{x \in X} F(x, y) \right] \tag{3.3}$$

But we have already proved above that there exists $x_o = x^*(y_o)$ such that

$$F(x_o, y_o) = \varphi(y_o) \tag{3.4}$$

Thus, from (3.3) and (3.4), we see that there exists $x_o \epsilon X$ and $y_o \epsilon Y$ such that

$$F(x_o, y_o) = \min_{y \in Y} \left[\max_{x \in X} F(x, y) \right],$$

thus completing the proof of the theorem.

We summarize the existence theorem just proved as follows:

Theorem 3.1 If $F(x, y)$ satisfies hypotheses H_1, H_2, H_3, and H_4 given above, there exists a point $(x_o, y_o) \epsilon X x Y$, such that

$$F(x_o, y_o) = \min_{y \epsilon Y} \left[\max_{x \epsilon X} F(x,y) \right] \qquad (3.5)$$

We next want to show that hypotheses H_3 and H_4 can be replaced by the single hypothesis to the effect that $F(x, y)$ is merely continuous on XxY, at least provided that X and Y are metric spaces as well as being sequentially compact. Thus the semicontinuity requirements of H_3 and H_4 are conceived of as being perhaps unnecessarily general. The main facts are more exactly formulated below in Theorem 3.2. Before proceeding to this theorem, we cite the following lemma:

LEMMA. Let G be a continuous mapping of a metric sequentially compact space Ξ onto a metric space Ω. The distance between two points ξ and $\xi' \epsilon \Xi$ is denoted by $|\xi, \xi'|$ and the distance between two points $\omega, \omega' \epsilon \Omega$ is denoted by $\|\omega, \omega'\|$. Then, corresponding to every positive number ϵ, there can be found a positive number $\delta = \delta_\epsilon$, such that

$$\|G(\xi), G(\xi')\| < \epsilon \text{ as long as } \xi \epsilon \Xi, \ \xi' \epsilon \Xi \text{ and } |\xi, \xi'| < \delta.$$

Although this lemma is probably well known, we submit the following proof. For the reader may not be acquainted with the theory of uniform continuity in the generality here formulated.

If the lemma were false there would be some positive number ϵ such that no matter how small δ is taken, say $\delta = 1/n$, it would be possible to find two points, say ξ_n and ξ'_n such that

$$|\xi_n, \xi'_n| < 1/n \tag{3.6}$$

while at the same time

$$\|G(\xi_n), G(\xi'_n)\| \geq \epsilon \tag{3.7}$$

Since Ξ is sequentially compact, the sequence $\xi_1, \xi_2, \xi_3, \ldots$ has at least one convergent subsequence $\xi_{n_1}, \xi_{n_2}, \xi_{n_3}, \ldots$ We therefore write

$$\lim_{k \to \infty} \xi_{n_k} = \xi^* \tag{3.8}$$

From (3.6) we thus have $|\xi_{n_k}, \xi'_{n_k}| < 1/n_k$, so that we also have

$$\lim_{k \to \infty} \xi'_{n_k} = \xi^* \tag{3.9}$$

From (3.7) we have $\|G(\xi_{n_k}), G(\xi'_{n_k})\| \geq \epsilon$. Hence, on letting $k \to \infty$, we find on using the assumed continuity of G that $\|G(\xi^*), G(\xi^*)\| \geq \epsilon$. But this is absurd since the distance of any point from itself is always zero. Thus the assumption that the lemma is false leads to a contradiction. The lemma has therefore been proved.

Theorem 3.2 Let $F(x, y)$ be continuous in the produce space $X \times Y$, where X and Y are sequentially compact metric spaces. Let $\varphi(y) = \max_{x \in X} F(x,y)$. Then φ is a continuous function of $y \in Y$.

Proof. Of course the existence of $\max_{x \in X} F(x, y)$ is an obvious consequence of the continuity of $F(x, y)$ in x(with y fixed), since X is sequentially compact.

Since both X and Y are sequentially compact, the same is true of the product space. Hence we may use our lemma with $\Xi = X x Y$, $G(\xi) = F(x, y)$. This means that if $\epsilon > 0$ is preassigned we may choose δ in such a manner that

$$|F(x,y) - F(x', y')| < \epsilon \text{ as long as } x, x', \in X, y, y' \in Y,$$

$|x, x'| < \delta$ and $|y, y'| < \delta$.

Here, of course, we use $|x, x'|$ to represent the distance between x and x', $|y, y'|$ the distance between y and y', while $|\xi, \xi'| = \max (|x, x'|, |y,y|)$ is the distance between points $(x,y) = \xi$ and $(x',y') = \xi'$ in the product space.

Now let y' be arbitrary and choose x' so that $\varphi(y') = F(x', y')$. This is possible, since $F(x,y')$ for fixed y' assumes its maximum for some value x' of x. Then

$$F(x', y') - \epsilon < F(x,y) \text{ if } |x,x'| \text{ and } |y,y'| \text{ are both less than } \delta.$$

Since $F(x,y) \leq \varphi(y)$ by definition of $\varphi(y)$ and since $F(x',y') = \varphi(y')$ we have

$$\varphi(y') - \epsilon \leq \varphi(y)$$

for every pair (y,y') such that $|y,y'| < \delta$ and $y, y' \in Y$.

Reversing the roles of y and y' we also have

$$\varphi(y) - \epsilon \leq \varphi(y')$$

under the same conditions. Hence

$$|\varphi(y) - \varphi(y')| < \epsilon$$

as long as $|y\ y'| < \delta$. This completes the proof of the theorem.

It is worth noting that while $\max_{x\epsilon X} F(x, y)$ is thus a continuous function of y, the point $x = x(y)$ where $F(x, y)$ assumes its maximum may be a quite discontinuous function of y. Thus if $F(x, y) = xy$ where X and Y are the closed intervals $[-1, +1]$, we have $\varphi(y) = |y|$ which is continuous, but $x(y) = +1$ for $y > 0$, $x(y) = -1$ for $y < 0$, and $x(0)$ is multiple valued.

W close this section by replacing the hypotheses H_1, H_2, H_3, H_4 in Theorem 3.1 by hypotheses to the effect that F is merely continuous and that X and Y are sequentially compact metric spaces. This is justified by Theorem 3.2. However, in order to have a formulation involving a max-min rather than a min-max, we shall replace F by $-G$, The final theorem then reads as follows:

<u>Theorem 3.3</u> If $G(x,y)$ is a real valued function defined for $x\epsilon X$, $y\epsilon Y$; if X and Y are sequentially compact metric spaces, and if G is continuous at every point of the product space XxY, then there exists a point $(x_o,y_o)\epsilon XxY$ such that $F(x_o,y_o) = \max_{y\epsilon Y} \min_{x\epsilon X} G(x,y)$

4. Some Remarks on Function Spaces

The present section enumerates various definitions and results used in the sequel. No attempt is made to state the most general definitions or the most general results. For further detail the reader is referred to the literature (e.g. Kelley, General Topology).

Let R designate the field of real numbers and let X be a vector space over R. We say that X is <u>normed</u> if there exists a real valued function $\|x\|$ on X having the following properties:

$$\begin{aligned}
&\text{(i)} \quad \|x\| \geq 0; \quad \|x\| = 0 \text{ if and only if } x = 0 \\
&\text{(ii)} \quad \|\alpha x\| = |\alpha|\,\|x\| \text{ for all } \alpha\epsilon R \text{ and all } x\epsilon X \\
&\text{(iii)} \quad \|x+y\| \leq \|x\| + \|y\| \text{ for all } x,y\epsilon X
\end{aligned} \tag{4.1}$$

Every normed space is a metric space. In fact, if x and y are any two points in X, we define the <u>distance</u> from x to y, written $d(x, y)$, to be the non-negative real number $\|x-y\|$. It is well known that $d(x,y)$ satisfies all the requisite axioms of a metric. The topology of X may now be defined in terms of $d(x,y)$. Specifically, a <u>spherical neighborhood</u> of a point $x\epsilon X$ is the set of all points $y\epsilon X$ such that $d(x,y) < r$, $r>0$; a sequence $\{x_n\}$ in X is said to <u>converge</u> to a point $x\epsilon X$ if $d(x_n, x) \to 0$ as $n \to +\infty$.

A topological space X is said to be <u>sequentially compact</u> if every sequence in X has a convergent subsequence. The cartesian product of a finite collection of sequentially compact spaces is sequentially compact.

Let Y be a real normed vector space and let X be sequentially compact. Let $f: x \to Y$ be continuous. Then it is well known that $\|f(x)\|$ attains both its maximum and minimum values in X.

Let $C(X, Y)$ denote the family of all continuous functions $f: X \to Y$. $C(X,Y)$ is a real vector space. Moreover, the function $\| \quad \|: C(X,Y) \to R$ defined by

$$\|f\| = \max_{x \in X} \|f(x)\| \tag{4.2}$$

is well-defined and furnishes a norm in $C(X,Y)$. The topology induced on $C(X,Y)$ be this norm is called the uniform topology. If $f_n \to f$ in the uniform topology, we say that f_n converges uniformly to f.

In the special case when $Y = R$, one has

$$\|f\| = \max_{x \in X} |f(x)|,$$

where $| \quad |$ denotes the familiar "absolute value", Still assuming $Y = R$, we have: $f_n \to f$ if and only if $\|f_n - f\| \to 0$ or, equivalently, if and only if

$$\max_{x \in X} |f_n(x) - f(x)| \to 0 \text{ as } n \to +\infty.$$

Let a and b be real numbers, with $a < b$. Let E^m denote the m - dimensional Euclidean space. E^m is a normed real vector space. Hence, in accordance with the preceding paragraph, $C([a,b], E^m)$ is also a real normed vector space. In order to simplify our notation we shall write $C(a, b; m)$ for $C([a,b], E^m)$.

Let K be a subset of $C(a, b; m)$. Then K is said to be <u>uniformly bounded</u> if there exists a number $M > 0$ such that $\|f\| \leq M$ for all $f \epsilon K$.

K is said to be <u>uniformly continuous</u> if given any $\epsilon > 0$ there exists a $\delta = \delta(\epsilon)$ such that $|x-y| < \delta$ implies $\|f(x)-f(y)\| < \epsilon$ for all $x, y \epsilon [a,b]$ and all $f \epsilon K$.

In the sequel we shall make use of the following famous theorem.

<u>Theorem</u> (ASCOLI) Let K be a uniformly bounded and uniformly continuous subset of $C(a, b; m)$. Then K is sequentially compact (in the uniform topology).

Otherwise stated, the theorem says that a family K of functions $f: [a,b] \rightarrow E^m$, which is (1) uniformly bounded, and (2) uniformly continuous, has the property that any sequence in it contains a (uniformly) convergent subsequence.

5. Precise Formulation of A Minimax Problem In Control Theory

Let X be the space of x and let R be a closed bounded region in X which contains the origin. For the sake of definiteness the reader may take R to be the set

$$R = \left\{ x \mid |x_i| \le a_i, \quad i = 1, \ldots, n \right\}, \tag{5.2}$$

where a_i are preassigned bounds (cf. § 1). However, other sets R may be of interest in the future.

We shall say that the system S is T - tame with respect to R if given the positive number T there exists a subregion $R_T \subset R$ such that for any $x_o \in R_T$ and any $\omega \in W$ there exists a control $u \in U$ such that $x(t, x_o, \omega, u) \in R$ for all $0 \le t \le T$. We assume hereby that $x(0, x_o, \omega, u) = x_o$.

Let R and T be fixed (given) and assume that S is T-tame with respect to R.

Let $x_o \in R_T$. Let

$$g(x_o, \omega, u) = \max_{0 \le t \le T} \|x(t, x_o, \omega, u)\| \tag{5.3}$$

$$h(x_o, \omega) = \underset{u \in U}{\text{g.l.b.}}\; g(x_o, \omega, u) \tag{5.4}$$

$$m(x_o) = \underset{\omega \in W}{\text{l.u.b.}}\; h(x_o, \omega) \tag{5.5}$$

We shall show that the functions g, h and m are well-defined.

(1) g is well defined. This is so because $x(t, x_o \omega, u)$ is continuous in t, whence $\|x(t)\|$ attains its maximum over the interval $[0, t]$.

(2) h <u>is well defined</u>. S is T-tame, hence given $x_o \epsilon R_T$ and $\omega \epsilon W$ there exists at least one $u \epsilon U$ such that $x(t, x_o, \omega, u^*) \epsilon R$ for all $t \epsilon [0, T]$. R is bounded, hence there exists an $r > 0$ such that R is contained in a sphere of radius r with center at the origin. Hence $g(x_o, \omega, u^*) \leq r$.

Hence, $h(x_o, \omega) \leq r$.

(3) m <u>is well defined</u>. We have $h(x_o, \omega) \leq r$ for all $\omega \epsilon W$, hence

$$\underset{\omega \epsilon W}{\text{l.u.b.}}\ h(x_o, \omega) \leq r,$$

hence $m(x_o)$ is well defined.

We are now in a position to give the first precise formulation of a <u>minimax problem</u>:

(1) For a given $x_o \epsilon R_T$, find $m(x_o)$.

(2) Determine whether there exists a u^* in U such that

$$m(x_o) = \underset{\omega \epsilon W}{\text{l.u.b.}} \ \max_{0 \leq t \leq T} \|x(t, x_o, \omega, u^*)\|$$

(3) Determine whether u^* is unique.

(4) Determine whether there exists a ω^* in W such that

$$m(x_o) = \max_{0 \leq t \leq T} \|x(t, x_o, \omega^*, u^*)\|.$$

Item (4) is of lesser importance than items (1) - (3).

6. Existence of Solutions To The Minimax Problem In Control Theory

Consider the system

$$S: \quad \dot{x} = f(x,\omega,u) \tag{6.1}$$

where x, f, ω and u are as described in § 1. We assume that f satisfies a Lipschitz condition in ω, u, namely that

$$\|f(x,\omega,u) - f(x,\omega_1,u_1)\| \leq c_1 \|\omega-\omega_1\| + c_2 \|u-u_1\|. \tag{6.2}$$

It is well known that these assumptions are sufficient to assure the existence and uniqueness of solutions for S.

Theorem 6.1 If U and W are sequentially compact and S is T-tame then there exists a solution for the minimax problem for S. In other words, given $x_o \epsilon R_T$ there exist $u^* \epsilon U$ and $\omega^* \epsilon W$ such that

$$m(x_o) = \max_{0 \leq t \leq T} \|x(t,x_o,\omega^*, u^*)\|.$$

Proof. The solution to system (6.1) is given by

$$x(t,x_o,\omega,u) = x_o + \int_o^t f(x(\tau), \omega(\tau), u(\tau))\, d\tau \tag{6.3}$$

Let x_o be fixed. We shall show that the mapping

$$\mu: \; W x U \rightarrow [0, +\infty) \tag{6.4}$$

defined by

$$\mu(\omega,u) = g(x_o,\omega,u) \tag{6.5}$$

is continuous.

One has

$$x(t,x_o,\omega,u) = x_o + \int_o^t f(x(\tau),\ \omega(\tau),\ u(\tau))\ d\tau$$

$$x(t,x_o,\omega',u') = x_o + \int_o^t f(x(\tau),\omega'(\tau),\ u'(\tau))\ d\tau,$$

whence

$$x(t,x_o,\omega,u) = x(t,x_o,\omega',u') = \int_o^t \left[f(x,\omega,u) - f(x,\omega',u')\right] d\tau$$

so

$$\|x(t,x_o,\omega,u) - x(t,x_o,\omega',u')\| \leq$$

$$\leq \int_0^t \| f(x,\omega,u) - f(x,\omega',u')\| \ d\tau$$

$$\leq \int_0^T \left\{c_1 \|\omega-\omega'\| + c_2 \|u-u'\|\right\} d\tau$$

$$\leq M\ (\|\omega-\omega'\| + \|u-u'\|),$$

where $M = \max\ (c_1T,\ c_2T)$, and t is any number in the interval $[0,T]$.

It follows that

$$|g(x_o,\omega,u) - g(x_o,\omega',u')| \leq M(\|\omega-\omega'\| + \|u-u'\|)$$

whence μ is continuous.

Theorem 6.1 now follows as an immediate consequence of Theorem 3.3. The proof is complete.

Corollary 1. In the case of a linear system

$$S : \dot{x} = Ax + Bu + \omega,$$

where A and B are nxn constant matrics and c is a constant n-vector, the Lipschitz condition (6.2) is satisfied and hence Theorem 6.1 applies.

Corollary 2. Let S be as in Theorem 6.1. Let U and V be uniformly bounded and uniformly continuous subsets of $C(0, T; n)$ and $C(0, T; 1)$, respectively. Then the conclusion of Theorem 6.1 holds.

Proof. Theorem 6.1 together with Ascoli's theorem.

CHAPTER 2

I. Alternative Formulations Of The First Minimax Problem In Control Theory

In § 5 of our Second Progress Report we formulated our first minimax problem in control theory. This formulation involved a set R defined by inequalities of the form $|x_i| \leq a_i$, $i = 1,\ldots, n$, where the a_i's were given real constants. We repeat this formulation here except that the set R of our Second Progress Report is here replaced by a sphere R of given radius r with center at the origin. The reason for this change is that the parallelopiped set R of our Second Progress Report entails certain non-trivial difficulties which are avoided in the event that R is a sphere. The nature of these difficulties will be brought out in our next progress report. Suffice it to say that the following discussion is strictly limited to the case when R is a sphere and that changing R into a parallelopiped (or some other bounded set) would require non-trivial modifications.

Consider then the system

$$S: \quad \dot{x} = f(x,w,u) \tag{1.1}$$

and the (given) set

$$R = \{x| \quad \|x\| \leq r\} \tag{1.2}$$

Assume that S is T-tame with respect to R (cf. Second Progress Report, p. 21). Let $x_o \epsilon R_T$ (loc. cit.). Define

$$g(x_o,w,u) = \max_{0\leq t\leq T} \|x(t,x_o,w,u)\| \tag{1.3}$$

$$h(x_o,w) = \operatorname*{g.l.b.}_{u \in U} g(x_o,w,u) \tag{1.4}$$

$$m(x_o) = \operatorname*{l.u.b.}_{w \in W} h(x_o,w) \tag{1.5}$$

We have shown (loc. cit., pp. 21-22) that g, h and m are well-defined. Our first minimax problem consists of the following questions;

(1) For a given $x_o \epsilon R_T$, find $m(x_o)$.

(2) Determine whether there exists a $u^* \epsilon U$ such that

$$m(x_o) = \underset{w \in W}{\text{l.u.b.}} \max_{0 \leq t \leq T} \|x(t,x_o,w,u^*)\|.$$

(3) Determine whether u^* is unique.

(4) Determine whether there exists a $w^* \epsilon W$ such that

$$m(x_o) = \max_{0 \leq t \leq T} \|x(t,x_o,w^*,u^*)\|$$

For the sake of intuitive clarity, we now allow ourselves a certain looseness of speech and replace g.l.b. and l.u.b. by min and max, respectively. The problem of determining $m(x_o)$ is then the problem of finding

$$\max_{w \epsilon W} \min_{u \epsilon U} g(x_o,w,u), \tag{1.6}$$

where $g(x_o,w,u)$ is as defined in (1.3). Therefore, strictly speaking, our first minimax problem was not really a min-max problem but a max-min problem. This reversal of the order of min and max is forced in the sense that a simple minded attempt to interchange the min and max in (1.6) must of necessity lead to a meaningless problem.

In fact, let $g(x_o,w,u)$ be as defined in (1.3) and let

$$k(x_o,u) = \underset{w \epsilon W}{\text{l.u.b.}} \ g(x_o,w,u) \tag{1.7}$$

$$\bar{m}(x_o) = \underset{u \in U}{\text{g.l.b.}} \ k(x_o,u). \tag{1.8}$$

Speaking loosely again,

$$\bar{m}(x_o) = \min_{u\epsilon U} \max_{w\epsilon W} g(x_o,w,u) \tag{1.9}$$

so that $\bar{m}(x_o)$ corresponds to the interchange of the order of min and max in (1.6).

Let $x_o \epsilon R_T$ be fixed. A choice of $w\epsilon W$ and $u\epsilon U$ gives rise to a unique trajectory $x(t,x_o,w,u)$ passing through x_o at time $t = 0$. The function $g(x_o,w,u)$ gives the maximum value attained by $\|x(t,x_o,w,u)\|$ along the arc of this trajectory corresponding to the time interval $0 \leq t \leq T$. Now, it has been assumed that S is T-tame. Hence given any $w\epsilon W$ there exists a $u' = u(w)\epsilon U$ such that $x(t,x_o,w,u')\epsilon R$ for all $0\leq t\leq T$. This, however, does not preclude the possibility that given a $u\epsilon U$ there exists a $w'\epsilon W$ such that $x(t,x_o,w',u)$ strays out of R during the time interval $0\leq t\leq T$. As a matter of fact, it appears necessary to assume this very possibility, namely that for any initial point $x_o\epsilon R_T$ and any fixed control $u\epsilon U$ there must always exist a "bad enough" wind $w\epsilon W$ which drives the system out of the set R during the time interval $0\leq t\leq T$. Otherwise, there would exist at least one initial point x_o and one control function $\bar{u} = \bar{u}(x_o)\epsilon U$ such that $x(t,x_o,w,\bar{u})\epsilon R$ for all $0\leq t\leq T$ and all $w\epsilon W$. Since $\bar{u}$ is completely independent of w, the existence of such a control would be nothing short of a panacea. It would be a control which is fixed throughout the motion, unresponsive to any winds encountered in flight (being, as it is, solely dependent on the initial point x_o and nothing else) and yet sufficient to assure the safety of the system (by keeping it neatly tucked within the set R) against all winds! If we assume that the class W is neither empty nor otherwise trivial (for example, it is natural to require that if $w\epsilon W$ then $\lambda w\epsilon W$ for all real λ satisfying $-1\leq\lambda\leq 1$) then the existence of $\bar{u}$, perhaps even for one particular point x_o, would rule out most systems S of practical significance. The only type of system S which is likely to remain would be so inherently stable relative to the whole class W, or the constant T so small or R so large that the safety of S was never jeopardized in the first place.

At any rate, it is not truly necessary at this point to decide whether the existence of $\bar{u}(x_o)$ must be ruled out at every single point of R_T. Suffice it to say that it must be ruled out in a substantial part of R_T. We assume hereby that the initial point x_o under consideration is one for which no $\bar{u}(x_o)$ exists within the class U. This, in turn, means that given any $u \epsilon U$ there exists a $w \epsilon W$ such that $x(t', x_o, w, u) \notin R$ for some $t' \epsilon [0,T]$.

It follows that

$$k(x_o, u) > r \quad \text{for all} \quad u \epsilon U,$$

whence

$$\bar{\bar{m}}(x_o) \geqslant r. \tag{1.10}$$

The number $\bar{\bar{m}}(x_o)$ does not measure what we set out to obtain, namely the "least possible deviation attainable (starting at x_o) under the most adverse circumstances". This is so due to the unfortunate definition (1.7)-(1.8) in which the natural order of things is reversed. Thus (1.4) measures the best possible defense against a given wind (control follows perturbation), whereupon (1.5) determines the least success of this procedure. On the other hand, (1.7) measures the worst possible performance (i.e. failure!) of a given control (control precedes wind), whereupon (1.8) measures the least possible such failure. Equation (1.10) simply states that the least of all failures is itself a failure.

Nevertheless, there is a way to formulate our first problem as a true min-max problem except that this formulation requires a more subtle approach than that embodied in (1.7) - (1.8). We proceed in two steps: First we show that the max-min $m(x_o)$ defined by (1.4) - (1.5) in $W \times H$ may be replaced by an equivalent max-min $m'(x_o)$ defined by (1.12) in a new cross product set $W \times H$ (Theorem 1.1). Next we show that $m'(x_o)$ equals the min-max $v(x_o)$ (cf. (1.13))which is also defined in $W \times H$ (Theorem 1.2). This establishes what we set out to accomplish.

Let $H = H(W,U)$ be the set of <u>all</u> functions h which map the set of winds W <u>into</u> the set of controls U. In symbols

$$H(W,U) = \left\{ h \,|\, h: W \to U \right\} \tag{1.11}$$

Let

$$m'(x_o) = \underset{w \in W}{\text{l.u.b.}} \; \underset{h \in H}{\text{g.l.b.}} \; g(x_o,w,h(w)), \tag{1.12}$$

where g is as defined in (1.3).

<u>Theorem 1.1</u> $m'(x_o) = m(x_o)$

<u>Proof</u>. For every $u \in U$ let $h_u \in H$ be that element of H whose range is $\{u\}$; thus

$$h_u(w) = u \quad \text{for all } w \in W.$$

Let $w' \in W$ be fixed. Then

$$\{h(w') \mid h \in H\} \supset \{h_u(w') \mid u \in U\} = U.$$

On the other hand, one has trivially

$$\{h(w')|\ h\epsilon H\} \subset U,$$

whence

$$\{h(w')|\ h\epsilon H\} = U.$$

Since w' is arbitrary it follows that

$$\{h(w)|h\epsilon H,\ w \text{ fixed}\} = U \quad \text{for every} \quad w\epsilon W.$$

Therefore

$$\underset{h\ \epsilon\ H}{\text{g.l.b.}}\ g(x_o,w,h(w)) = \underset{u\ \epsilon\ U}{\text{g.l.b.}}\ g(x_o,w,u),$$

whence

$$m'(x_o) = m(x_o),$$

and the proof is complete.

Let

$$v(x_o) = \underset{h\ \epsilon\ H}{\text{g.l.b.}}\ \ \underset{w\ \epsilon\ W}{\text{l.u.b.}}\ g(x_o,w,h(w)) \tag{1.13}$$

<u>Theorem 1.2.</u> $v(x_o) = m'(x_o)$.

<u>Proof</u>. We note first (cf. Second Progress Report, p. 3, Theorem 2.1) that

$$\underset{h}{\text{g.l.b.}}\ \ \underset{w}{\text{l.u.b.}}\ \ g(x_o,w,h(w)) \geqslant \underset{w}{\text{l.u.b.}}\ \ \underset{h}{\text{g.l.b.}}\ \ g(x_o,w,h(w)),$$

whence

$$v(x_o) \geqslant m'(x_o) \tag{1.14}$$

We shall now show that $v(x_o) \leqslant m'(x_o)$.

Let $\epsilon>0$ be fixed. For every w, choose a $u_\epsilon(w)$ such that

$$g(x_o,w,u_\epsilon(w)) < \underset{u}{\text{g.l.b.}}\ g(x_o,w,u) + \epsilon \tag{1.15}$$

Let $h_\epsilon: W \to U$ be the element of H defined by

$$h_\epsilon(w) = u_\epsilon(w) \quad \text{for every } w\epsilon W. \tag{1.16}$$

Then, by (1.15) and (1.16), we get

$$\underset{w}{\text{l.u.b.}}\ g(x_o,w,h_\epsilon(w)) \leqslant \underset{w}{\text{l.u.b.}}\ \underset{u}{\text{g.l.b.}}\ g(x_o,w,u) + \epsilon \tag{1.17}$$

so

$$\underset{w}{\text{l.u.b.}}\ g(x_o,w,h_\epsilon(w)) \leqslant m(x_o) + \epsilon \tag{1.18}$$

On the other hand, it is easy to see that

$$\underset{h}{\text{g.l.b.}}\ \underset{w}{\text{l.u.b.}}\ g(x_o,w,h(w) \leqslant \underset{w}{\text{l.u.b.}}\ g(x_o,w,h_\epsilon(w)). \tag{1.19}$$

Hence, by (1.13), (1.18) and (1.19), we get

$$v(x_o) \leqslant m(x_o) + \epsilon$$

But ϵ is arbitrary, so

$$v(x_o) \leqslant m(x_o) = m'(x_o) \tag{1.20}$$

Hence $v(x_o) = m'(x_o)$.

Corollary 1.1 Let g and H be as defined in (1.3) and (1.11), respectively.

Then $m(x_o) = m'(x_o) = v(x_o)$.

2. Existence of Solutions to the first Minimax Problem in Control Theory

In this section we give a corrected proof of Theorem 6.1 of the previous progress report. The proof depends upon an essentially well known lemma on the continuity of solutions of differential equations with respect to small variations in the equations themselves. This lemma is formulated below as Theorem 2.1 in the form convenient for our purposes; and since the theorem in exactly this form may be hard to find in the literature, we give its complete proof using familiar techniques. The main theorem is, however, Theorem 2.2 which is essentially a repetition of Theorem 6.1 of the previous progress report; but it also contains a statement tending to give meaning to the quantity $m(x_o)$ in the case where R is a sphere. In this connection the reader is referred to the first paragraph of Section 1.

In what follows leading up to Theorem 2.1 we have replaced the pair (w,u), i.e. ("wind","control"), by a single vector φ. The purpose of this modification is solely to abbreviate the statement and proof of Theorem 2.1. Once this theorem is proved, we revert to the previous terminology.

We consider a differential system of the form

$$\dot{x} = f(x,\varphi(t)) \tag{2.1}$$

where x is an n-vector, the dot denotes differentiation with respect to the independent variable t (referred to as the time), f is an n-vector function, and φ is a k-vector. We suppose $f(x,\varphi)$ to be defined for all x in n-dimensional vector space or some open subset X thereof and for all φ in a bounded portion V of k-dimensional vector space.

The known functions $\varphi(t)$ inserted in the right hand number of (2.1) are assumed to belong to a space Φ of functions whose domain is the time interval $0 \leq t \leq T^*$ and whose range is in V. We introduce into the space Φ a metric, denoted by $|\varphi - \bar{\varphi}|$ and defined by

$$|\varphi-\bar{\varphi}| = \underset{0 \leq t \leq T^*}{\text{l.u.b.}} \|\varphi(t) - \bar{\varphi}(t)\|$$

We assume that $f(x,\varphi)$ is continuous in $X \times V$ and, denoting by K any bounded subset of X, we assume that there exists a constant L_K such that the following Lipschitz condition

$$\|f(\bar{x},\varphi) - f(x,\varphi)\| \leq L_K \|\bar{x} - x\| \tag{2.2}$$

is valid whenever $\bar{x}$ and x both lie in K and $\varphi \epsilon V$. Finally we assume that $f[x(t), \varphi(t)]$ is Lebesgue measurable whenever $\varphi \epsilon \Phi$ and whenever x is a continuous function of t with values in X. These conditions are sufficient to insure the existence and uniqueness of the solution $x(t,x_o,\varphi)$ of (2.1), corresponding to any $\varphi \epsilon \Phi$, such that $x(0,x_o,\varphi) = x_o$. It is assumed that this solution exists over a time interval $[0,T] \subset [0,T^*]$, regarded as fixed at least for fixed x_o, for any $\varphi \epsilon \Phi$. That is T may depend on x_o but not on φ.

Theorem 2.1. If V is sequentially compact, then $x(t,x_o,\varphi)$ is continuous in t and φ for $t\in[0,T]$ and $\varphi\in\Phi$.

Proof. Let $\bar{t} \in [0,T]$ and $\bar{\varphi}\in\Phi$, and let ϵ be a preassigned positive number. Now $x(t,x_o,\varphi)$ and $x(t,x_o,\bar{\varphi})$ both exist for $0\leq t\leq T$ if φ as well as $\bar{\varphi}$ belong to Φ. It is required to show how to construct a number δ such that

$$\|x(t,x_o,\varphi) - x(\bar{t},x_o,\bar{\varphi})\| < \epsilon \tag{2.3}$$

as long as $|t-\bar{t}|$ and $\|\varphi-\bar{\varphi}\|$ both do not exceed δ.

For this purpose choose a positive number $\alpha < 2^{-1}\epsilon$ and let the set K be defined as follows:

$$x\in K \text{ if and only if } \|x - x(t,x_o,\bar{\varphi})\| \leq \alpha \text{ for some } t \text{ on } [0,T].$$

Since X is open and the set of points x on the trajectory $x = x(t,x_o,\bar{\varphi})$ for $0\leq t\leq T$ is compact, K will be a subset of X if α is sufficiently small. We assume that α has been so chosen. It is also obvious that K is bounded and closed.

Since $K\times V$ is sequentially compact and $f(x,\varphi)$ is continuous, it must also be bounded in $K\times V$. Let us therefore write

$$\|f(x,\varphi)\| < B \text{ for all } x\in K \text{ and } \varphi\in V. \tag{2.4}$$

Moreover the Lipschitz condition (2.2) is available.

Let $\eta = \alpha L(e^{LT}-1)^{-1}$, where $L = L_K$, is the Lipschitz constant appearing in (2.2). Choose σ so that

$$\|f(x,\bar{\varphi}) - f(x,\varphi)\| < \eta \text{ for } \|\bar{\varphi}-\varphi\| < \sigma, \tag{2.5}$$

provided also, of course, that φ, as well as $\bar{\varphi}$, is in V. This is possible since we have uniform continutiy in KxV. Notice that, if $\varphi \epsilon \Phi$ and $\|\bar{\varphi}-\varphi\| < \sigma$, we also have a fortiori (from the definition of $\|\bar{\varphi}-\varphi\|$) that $\|\bar{\varphi}(t) - \varphi(t)\| < \sigma$, so that $\|f(x,\bar{\varphi}(t)) - f(x,\varphi(t))\| < \eta$ for any t on $[0,T^*]$ and $x \epsilon K$.

Corresponding to any $\varphi \epsilon \Phi$ such that $\|\bar{\varphi} - \varphi\| < \sigma$, we consider the sequence

$$x^0(t) = x(t,x_0,\bar{\varphi})$$

$$x^m(t) = x_0 + \int_0^t f\left[x^{m-1}(\tau), \varphi(\tau)\right] d\tau, \quad m = 1,2,3,\ldots \tag{2.6}$$

We first prove by induction that the members of this sequence exist and that for $m = 0, 1, 2, \cdots$

$$\|x^m(t) - x(t,x_0,\bar{\varphi}(t)\| \leq \eta \sum_{s=1}^{m} \frac{L^{s-1}t^s}{s!} \tag{2.7}$$

which, for $0 \leq t \leq T$ does not exceed $\eta L^{-1}(e^{LT}-1) = \alpha$ so that our inductive proof also establishes the fact that $x^m(t) \epsilon K$ for $m=0,1,2,\ldots$ and for $t \epsilon [0,T]$.

The statement is obviously true for $m = 0$. Make the inductive hypothesis that (2.7) holds when m is replaced by $m-1$ and that consequently $x^{m-1}(t) \epsilon K$. Using the abbreviation $\bar{x}(t) = x(t,x_0,\bar{\varphi})$ we evidently have

$$\bar{x}(t) = x_0 + \int_0^t f\left[\bar{x}(\tau), \bar{\varphi}(\tau)\right] d\tau \tag{2.8}$$

This is true, since (2.8) is merely the integrated form of (2.1) with $\varphi=\bar{\varphi}$ and $x=\bar{x}$. Using this same abbreviation our inductive hypothesis appears in the form,

$$\|x^{m-1}(\tau) - \bar{x}(\tau)\| \leq \eta \sum_{s=1}^{m-1} \frac{L^{s-1}\tau^{s}}{s!} \tag{2.9}$$

for $0 \leq \tau \leq T$. Subtacting (2.8) from (2.6) and performing some obvious estimates, we find that

$$\|x^m(t)-\bar{x}(t)\| \leq \int_0^t \|f\left[x^{m-1}(\tau),\varphi(\tau)\right] - f\left[\bar{x}(\tau),\bar{\varphi}(\tau)\right]\| \, d\tau$$

$$\leq \int_0^t \|f\left[x^{m-1}(\tau),\varphi(\tau)\right] - f\left[x^{m-1}(\tau),\bar{\varphi}(\tau)\right]\| \, d\tau + \int_0^t \|f\left[x^{m-1}(\tau),\bar{\varphi}(\tau)\right] - f\left[\bar{x}(\tau),\bar{\varphi}(\tau)\right]\| \, d\tau$$

Since $x^{m-1}(\tau)\epsilon K$, $\bar{x}(\tau)\epsilon K$, $\varphi \epsilon \Phi$ and $\|\bar{\varphi} - \varphi\| < \sigma$, we may use both (2.5) (with x replaced by x^{m-1}) and the Lipschitz condition. We accordingly find that

$$\|x^m(t) - \bar{x}(t)\| \leq \int_0^t \eta d\tau + L\int_0^t \|x^{m-1}(\tau) - \bar{x}(\tau)\| \, d\tau$$

Hence, from our inductive hypothesis (2.9) we deduce that

$$\|x^m(t) - \bar{x}(t)\| \leq \eta t + \eta \sum_{s=1}^{m-1} \frac{L^s t^{s+1}}{(s+1)!} = \eta \sum_{s=1}^{m} \frac{L^{s-1}t^s}{s!}$$

This completes the induction.

Now that $x^m(t)$ is known to lie in K, where the uniform Lipschitz condition is valid, it is easy to see that the well known technique due to Picard and Lindelöf is available to prove that as $m \to \infty$, the $x^m(t)$ tend uniformly to the solution of (2.1), which takes on the initial value x_o. Since this solution is unique, we thus are enabled to write

$$\lim_{m \to \infty} x^m(t) = x(t,x_o,\varphi) \text{ uniformly on } [0,T].$$

Hence, from (2.7), we find that

$$\|x(t,x_o,\varphi) - x(t,x_o,\bar{\varphi})\| \leq \eta L^{-1}(e^{LT}-1) = \alpha < (1/2)\epsilon \qquad (2.10)$$

as long as $\|\bar{\varphi}-\varphi\| < \sigma$. Moreover, from (2.1) and (2.4), it is also clear that

$$\|x(t,x_o,\bar{\varphi}) - x(\bar{t},x_o,\bar{\varphi})\| \leq B|t-\bar{t}| < (1/2)\epsilon \qquad (2.11)$$

if $|t-\bar{t}| < (2B)^{-1}\epsilon$. From (2.10), (2.11), and the triangle inequality, we thus find that

$$\|x(t,x_o,\varphi) - x(\bar{t},x_o,\bar{\varphi})\| < (1/2)\epsilon + (1/2)\epsilon$$

provided that $|\bar{t}-t|$ and $\|\bar{\varphi}-\varphi\|$ are both less than $\delta = \min[\sigma,(2B)^{-1}\epsilon]$. This complets the proof of Theorem 2.1.

We now revert to the system

$$S: \quad \dot{x} = f(x,w,u) \qquad (2.12)$$

where $x \epsilon X$, f, w and u are as described in § 1 of the Second Progress Report. We assume that f is continuous in x, w, and u, and moreover satisfies a Lipschitz condition in x,

$$\|f(x',w,u) - f(x,w,u)\| \leq L_K\| x' - x\|$$

when x and x' belong to any fixed bounded region $K \subset X$ and w and u take on any values in the ranges of functions in the sets W and U respectively. It is well known that these assumptions are sufficient to insure the existence and uniqueness of solutions for S, at least if $f[x(t),w(t),u(t)]$ is measurable when $x(t)$ is continuous, and $w \in W$ and $u \in U$.

Theorem 2.2 If U and W are sequentially compact and S is T-tame, then there exists a solution for the first minimax problem for S. In other words, given $x_0 \in R_T$, there exist $u^* \in U$ and $w^* \in W$ such that

$$m(x_0) = \max_{0 \leq t \leq T} \|x(t,x_0,w^*,u^*)\|,$$

where $x(t,x_0,w,u)$ is the solution of S reducing to x_0 when $t = 0$. Moreover, the trajectory $x(t,x_0,w^*,u^*)$ does not leave the sphere R for $0 \leq t \leq T$ and therefore the number $m(x_0)$ is of significance in the sense that it measures the worst possible result which might be obtained when the best control is used to counteract any wind.

Proof. We invoke Theorem 2.1 to insure us that $x(t,x_0,w,u)$ is continuous in t, w and u. In doing this, of course, we interpret the φ of Theorem 2.1 to stand for the pair (w, u) and the space Φ then consists of all pairs of functions (w, u) in which $w \in W$ and $u \in U$. It is seen that the hypotheses, both those stated explicitly in Theorem 2.1 and those contained in the preamble, are satisfied.

Since $x(t,x_0,w,u)$ is continuous, as stated, the same is true of $\|x(t,x_0,w,u)\|$.

From Theorem 3.2, of the Second Progress Report, it now follows that

$g(x_o,w,u) = \max_{0\leq t\leq T} \|x(t,x_o,w,u)\|$ is a continuous function of w and u.

From Theorem 3.3, of the Second Progress Report, we now conclude that there exist $w^*\in W$ and $u^*\in U$ such that

$$g(x_o,w^*,u^*) = \max_{w\in W} \left[\min_{u\in U} g(x_o,w,u)\right] \tag{2.13}$$

Using equations (1.5), (1.4), (2.13) and (1.3) in this sequence we thus obtain

$$m(x_o) = \max_{w\in W} h(x_o,w) = \max_{w\in W} \left[\min_{u\in U} g(x_o,w,u)\right] = g(x_o,w^*,u^*)$$

$$= \max_{0\leq t\leq T} \|x(t,x_o,w^*,u^*)\|,$$

thus completing the proof of Theorem 2.2

CHAPTER 3

In previous progress reports we dealt with a certain minimax problem which appeared to be of limited significance in regard to control systems for launch vehicles. We now propose another problem, which at first sight might seem only trivially different from the previously considered problem. Yet, the difference is just enough so that we are led to a significant minimax problem, whereas previously we had decided that the only immediately significant problem was a maximin problem, even though at the end of § 1 of the Third Progress Report, we managed, by fairly sophisticated ideas, to rephrase it as an equivalent minimax problem.

We now consider the system

$$S: \quad \dot{x} = f(x,w,u)$$

where x is an n-vector representing the system's state, the dot represents differentiation with respect to time t, w and u are vectors (of any dimensionality, not necessarily n), and where f is an n-vector function of x,w,u, defined, say, for $x \in X$, $w \in W^*$, and $u \in U^*$.

We consider a class W of functions w (referred to as "winds") which map the time interval $[0,T]$ into the set W^* and a class U of functions u (referred to as controls) which map X into U^*.

Under suitable conditions on f and the class W and U, the system S, which is now thought of as taking the form,

$$\dot{x} = f(x,w(t),u(x)),$$

admits through each initial point x_o, a unique solution

$$x = x(t,x_o,w,u), \quad \text{with} \quad x(0,x_o,w,u) = x_o.$$

Moreover, the system is supposed to disintegrate whenever the solution emerges from some given region $R \subset X$. We find it therefore useful to say that the system S is uniformly T-tame with respect to R, if there exists a subregion $R_T(\subset R)$ and a non-vacuous sub-class $V(\subset U)$ of controls, such that $x(t,x_o,w,u) \in R$ as long as $x_o \in R_T$, $0 \leq t \leq T$, $w \in W$, and $u \in V$. We henceforth assume that S is uniformly T-tame in the sense of this definition.

Suppose next that it is desirable that a given function $F(x)$ be kept as small as possible during the motion. More precisely, we are interested in minimizing by proper choice of the control u the maximum value of $F[x(t,x_o,w,u)]$ for $x_o \in R_T$, $0 \leq t \leq T$, and for $w \in W$. Thus, letting

$$g(w,u) = \max_{x_o \in R_T} \left\{ \max_{0 \leq t \leq T} F\,[x(t,x_o,w,u)] \right\} ,$$

we pose the question as to the existence of a "bad" wind $w^* \in W$ and a "good" control $u^* \in V$, such that

$$\underset{u \in V}{\text{g.l.b.}} \left[\underset{w \in W}{\text{l.u.b.}} \; g(w,u) \right] = g(w^*, u^*) .$$

If we prefer not to make our results independent of x_o, we could let

$$g(x_o,w,u) = \max_{0 \leq t \leq T} F\left[x(t,x_o,w,u)\right]$$

and pose the question as to the existence of a "bad" wind $w^* \in W$ (relative to this particular initial point) and a "good" control $u^* \in V$ (also relative to this particular initial point), such that

$$\underset{u \in V}{\text{g.l.b.}} \left[\underset{w \in W}{\text{l.u.b.}} \; g(x_o,w,u) \right] = g(w^*, u^*).$$

The material given in sections 2 and 3 of the Second Progress Report is probably sufficiently general to apply to such questions of existence in the present problem. However, it is clear that considerable care must be exercised in specifying the class U. If this class is not Lipschitzian, for instance, we could be faced with the loss of the uniqueness of solutions of S.

In comparing the present problem with the one formulated previously in § 5 of the Second Progress Report and treated more fully in the Third Progress Report, it is seen that in the previous work the class U was a class of functions $u(t)$ defined on the interval $0 \le t \le T$, whereas now the class U is a class of functions $u(x)$ defined for $x \epsilon X$. Of course, since x is eventually thought of as a function of t, it appeared at first sight that this difference was only superficial, $u(t) = u(x(t))$. But this appearance, itself, is superficial. Our previous fundamental difficulty was connected with the fact that it did not seem sensible to select a good control $u(t)$ independently of the wind w; whereas, now, even though $u(x)$ may be selected independently of the wind, we are led inevitably to a $u(t) = u[x(t)]$, which is indeed <u>not</u> independent of the wind w, since the $x(t)$ written above is just an abbreviation for $x(t,x_0,w,u)$, which, of course, depends explicitly on both the wind and the control.

It may have been noticed by the reader that in the above formulation we have used and defined the expression "uniformly T-tame". Evidently, it is up to us to explain the significance of this term and its relationship to the (non-uniform) "T-tameness" first introduced in the Second Progress Report, p. 21. For this purpose consider again the system

$$S: \quad \dot{x} = f(x,w,u). \tag{1}$$

We say that a set $A(\subset R)$ is T-tame with respect to R if for any $x_o \epsilon A$ and any $w \epsilon W$ there exists a control $u(x) = u(x,x_o,w)$ in U such that

$$x(t,x_o,w,u) \epsilon R \tag{2}$$

for all $0 \leqq t \leqq T$. Thus, for any $x_o \epsilon A$ the set

$$U(x_o,w) = \left\{ u \epsilon U \;\middle|\; \text{condition (2) is satisfied} \right\} \tag{3}$$

is not empty.

We say that the system S is T-tame with respect to R if there exists a non-empty subset R_T of R which is T-tame with respect to R.

Let A be T-tame with respect to R. If $x_o \epsilon A$ we know that $U(x_o,w) \neq \emptyset$ for every $w \epsilon W$. However, the set

$$V(x_o) = \bigcap_{w \epsilon W} U(x_o,w) \tag{4}$$

may be empty.

Now let A be T-tame with respect to R. We shall say that a point $x_o \epsilon R_T$ is uniformly T-tame with respect to R if $V(x_o) \neq \emptyset$. This is equivalent to the requirement that there exists at least one $v^*(x) \epsilon U$ such that

$$x(t,x_o,w,v^*) \epsilon R \quad \text{for all } 0 \leqq t \leqq T \text{ and all } w \epsilon W \tag{5}$$

Here v^* is, of course, independent of w.

<u>A set A is uniformly T-tame with respect to R</u> if every point in A is uniformly T-tame with respect to R and, in addition, the set

$$\bigcap_{x_0 \in A} V(x_0) \tag{6}$$

is not empty. Thus, a set $A \subset R$ is uniformly T-tame with respect to R iff there exists at least one (fixed) $v^* \in U$ such that

$$x(t,x_0,w(t),v^*(x)) \in R \text{ for } 0 \leq t \leq T, \text{ all } w \in W \text{ and all } x_0 \in A. \tag{7}$$

Here v^* is independent both of w and x_0.

If A is uniformly T-tame with respect to R we denote the (non empty) set (6) by $V(A)$.

We say that the system S is uniformly T-tame with respect to R if there exists a non-empty subset $R_1 \subset R$ such that $V(R_1) \neq \emptyset$.

Let $A \ (\subset R)$ be uniformly T-tame with respect to R. If V_1 is any non-empty subset of $V(A)$ then clearly

$$x(t,x_0,w,u) \in R \text{ for all } 0 \leq t \leq T, \text{ all } x_0 \in A, \text{ all } w \in W \text{ and all } v \in V_1 \tag{8}$$

The converse also holds: If V_1 satisfies (8) then $V_1 \subset V(A)$.

Given a non-empty subset V in U, the set

$$\left\{ x_0 \in R \;\middle|\; x(t,x_0,w,u) \in R \text{ for all } 0 \leq t \leq T, \text{ all } w \in W, \text{ all } v \in V \right\} \tag{9}$$

will be denoted by $R(V)$.

The following relations clearly hold:

(1) $A \subset B (\subset R) \rightarrow V(B) \subset V(A)$

(2) $V_1 \subset V_2 (\subset U) \rightarrow R(V_2) \subset R(V_1)$.

(3) $R(V(A)) \supset A$

(4) $V(R(V_o)) \supset V_o$.

CHAPTER 4

1. INTRODUCTION AND FORMULATION OF THE EXISTENCE PROBLEM

In the fourth progress report, we indicated a desirable modification of the minimax problem as previously understood. While this modification permits the application of certain general results contained in the second progress report, it was found necessary to generalize Theorem 2.1 of the third progress report in order to obtain a suitable tool for establishing the desired existence theorem. The generalized Theorem 2.1 is likewise denoted as Theorem 2.1 and appears in the next section of the present report with complete proof. The method of proceeding to the proof of the existence theorem is given in Section 3. The problem itself has already been formulated in the fourth progress report. For the sake of completeness, we reproduce the formulation here.

We consider the system

S: $\dot{x} = f(x,w,u)$

where x is an n-vector representing the system's state, the dot represents differentiation with respect to time t, w and u are vectors (of any dimensionality, not necessarily n), and where f is an n-vector function of x, w, u, defined, say, for $x \in X$, $w \in W^*$, and $u \in U^*$.

We consider a class W of functions w(referred to as "winds") which map the time interval $[0,T]$ into the set W^* and a class U of functions u(referred to as "controls") which map X into U^*.

Under suitable conditions on f and the class W and U, the system S, which is now thought of as taking the form

$$\dot{x} = f(x,w(t), u(x)),$$

admits through each initial point x_0, a unique solution $x = x(t,x_0,w,u)$

such that $x(0,x_o,w,u) = x_o$.

Moreover, the system is supposed to disintegrate whenever the solution emerges from some given region $R \subset X$. We find it therefore useful to say that the system S is uniformly T-tame with respect to R, if there exists a subregion R_T $(\subset R)$ and a non-vacuous sub-class $V(\subset U)$ of controls, such that $x(t,x_o,w,u) \in R$ as long as $x_o \in R_T$, $0 \leq t \leq T$, $w \in W$ and $u \in V$. We henceforth assume that S is uniformly T-tame in the sense of this definition.

Suppose next that it is desirable that a given continuous function $F(x)$ be kept as small as possible during the motion. More precisely, we are interested in minimizing by proper choice of the control u the maximum value of $F[x(t,x_o,w,u)]$ for $x_o \in R_T$, $0 \leq t \leq T$, and for $w \in W$. Thus, letting

$$g(w,u) = \max_{x_o \in R_T} \left\{ \max_{0 \leq t \leq T} F[x(t,x_o,w,u)] \right\},$$

we pose the question as to the existence of a "bad" wind $w^* \in W$ and a "good" control $u^* \in V$, such that

$$\underset{u \in V}{\text{g.l.b.}} \left[\underset{w \in W}{\text{l.u.b.}} g(w,u) \right] = g(w^*,u^*).$$

In section 3 we answer the question in the affirmative, at least if $f[x,w(t), u(x)]$ satisfies a certain reasonable Lipschitz condition and if R_T, W, and V are compact.

There remains a lot to do in examining the usefullness and significance of this result. For instance, both the Lipschitz condition on $f[x,w(t), u(x)]$ and the hypothesis on the compactness of W preclude the possibility of using discontinuous controls. Also the limitation of u to the set V is awkward. One might expect a better result if we were to contemplate

$$\underset{u \in U}{\text{g.l.b.}} \left[\underset{w \in W}{\text{l.u.b.}} \; g(w,u) \right],$$

but this leads to technical difficulties, since $g(w,u)$ may fail to be defined throughout the product space $W \times U$. Other unanswered questions concern the properties of the maximal controllable region R_T. Is it connected? Does it contain the origin?

We hope to answer questions of this type in the next progress report.

2. ON THE CONTINUITY OF SOLUTIONS OF DIFFERENTIAL SYSTEMS.

We consider a differential system of the form

$$\dot{x} = f[x,\varphi(x,t)] \tag{2.1}$$

where x is an n-vector, the dot denotes differentiation with respect to the independent variable t (referred to as the time), f is an n-vector function, and φ is a k-vector. We suppose $f(x,\varphi)$ to be defined for all x in n-dimensional vector space or some open subset X thereof and for all φ in a bounded portion V of k-dimensional vector space.

The functions $\varphi(x,t)$ inserted in the right hand member of (2.1) are assumed to belong to a space Φ of functions whose domain is the Cartesian product space of a fixed open time interval I and the set X; and the range of the functions φ in Φ is assumed to lie in V. We make Φ a normed metric space by defining the norm of φ as follows:

$$|\varphi| = \underset{t\in I, x\in X}{\text{l.u.b.}} \|\varphi(x,t)\| .$$

Contrariwise $\|\varphi\|$ refers to a norm for the k-dimensional vector space of which V is a subset. Likewise $\|x\|$ refers to a norm for n-dimensional vector space. There is never any possiblity of confusion regarding the use of $\|\ldots\|$ to denote a norm in k- or n-dimensional space.

$f(x,\varphi)$ is assumed to be continuous in $X \times V$.

Denoting by K a bounded subset of X, we assume that there is a "Lipschitz" constant L_K such that

$$\|f[\bar{x}, \varphi(\bar{x},t)] - f[x,\varphi(x,t)]\| \leq L_K \|\bar{x} - x\| \tag{2.2}$$

whenever $\bar{x}$ and x both lie in K and $\varphi\in\Phi$.

We also assume that $f[x(t), \varphi(x(t),t)]$ is Lebesque measurable whenever $\varphi \in \Phi$ and x is a continuous function of t with values in X.

Let $x(t,x_o,\varphi)$ denote the solution of (2.1) which reduces to x_o when $t = 0 \in I$. Here φ is a fixed element in Φ. The classical existence theorems for systems of differential equations assure us that such a solution exists and is unique for $0 \leq t \leq T$ for some sufficiently small T. Given $x_o \in K^* \subset X$, we further assume that $x(t,x_o,\varphi)$ exists over the time interval $[0,T] \subset I$, where T is independent of x_o, and φ, as long as $x_o \in K^*$ and $\varphi \in \Phi$.

Theorem **2.1** If V is sequentially compact, then $x(t,x_o,\varphi)$ is continuous simultaneously in t, x_o, and φ, for $t \in [0,T]$, $x_o \in K^*$, and $\varphi \in \Phi$.

Proof Let $\bar{t} \in [0,T]$, $\bar{\varphi} \in \Phi$, $\bar{x}_o \in K^*$, and let ϵ be a preassigned positive number. Now $x(t,\bar{x}_o,\bar{\varphi})$ and $x(t,x_o,\varphi)$ both exist for $0 \leq t \leq T$, if x_o and φ (as well as $\bar{x}_o$ and $\bar{\varphi}$) belong to K^* and Φ respectively. It is required to show how to construct a number δ such that

$$\| x(t,x_o,\varphi) - x(\bar{t},\bar{x}_o,\bar{\varphi}) \| < \epsilon \tag{2.3}$$

as long as $|t - \bar{t}|$, $\| x_o - \bar{x}_o \|$ and $\| \varphi - \bar{\varphi} \| \leq \delta$.

For this purpose chooose a positive number $\alpha < 2^{-1} \epsilon$ and let the set K be defined as follows:

$x \in K$ if and only if $\| x - x(t,\bar{x}_o,\bar{\varphi}) \| \leq \alpha$ for some t on $[0,T]$.

Since X is ppen and the set of points x on the trajectory $x = x(t,\bar{x}_o,\bar{\varphi})$ for $0 \leq t \leq T$ is compact, K will be a subset of X if α is sufficiently small. We assume that α has been so chosen. It is also obvious that K is bounded and closed.

Since $K x V$ is sequentially compact and $f(x,\varphi)$ is continuous, it must also be bounded in KxV. Let us therefore write

$$\|f(x,\varphi)\| < B \quad \text{for all} \quad x \epsilon K \quad \text{and} \quad \varphi \epsilon V. \tag{2.4}$$

Moreover, the Lipschitz condition (2.2) is available. Let $\eta = \alpha\, L(2e^{LT}-1)^{-1}$, where $L = L_K$ is the Lipschitz constant appearing in (2.2). Choose σ so that

$$\| f(x,\bar{\varphi}) - f(x,\varphi)\| < \eta \qquad \text{for} \quad \|\bar{\varphi} - \varphi\| < \sigma, \tag{2.5}$$

provided also, of course, that φ, as well as $\bar{\varphi}$, is in V. This is possible since we have uniform continuity in KxV. Notice also that, if $\varphi \epsilon \Phi$ and $|\bar{\varphi} - \varphi| < \sigma$, we also have a fortiori (from the definition of $|\bar{\varphi} - \varphi|$) that $\|\bar{\varphi}(t) - \varphi(t)\| < \sigma$, so that

$$\|f(x,\bar{\varphi}(t)) - f(x,\varphi(t))\| < \eta \quad \text{for any} \quad t \in I \quad \text{and} \quad x \in K.$$

Corresponding to any $\varphi \epsilon \Phi$ such that $|\bar{\varphi} - \varphi| < \sigma$, we consider the sequence,

$$x^{o}(t) = x(t,\bar{x}_o,\bar{\varphi})$$

....

$$x^{m}(t) = x_o + \int_0^t f[x^{m-1}(\tau), \varphi(x^{m-1}(\tau),\tau)]d\tau, \qquad m = 1,2,3,\ldots \tag{2.6}$$

We first prove by induction that, if $\|x_o - \bar{x}_o\| < \beta = \eta\, L^{-1}$, the members of this sequence exist and that for m = 0,1,2,...

$$\|x^{m}(t) - x(t,\bar{x}_o,\bar{\varphi}(t))\| \leq \sum_{k=o}^{m-1} \beta \frac{L^k t^k}{k!} + \sum_{k=o}^{m-1} \eta \frac{L^k t^{k+1}}{(k+1)!} \tag{2.7}$$

$$\leq \beta e^{LT} + \eta L^{-1}(e^{LT} -1) = \alpha < 2^{-1}\epsilon.$$

Our inductive proof thus also establishes the fact that $x^m(t) \epsilon K$ for m = 0,1,2,...

The statement is obviously true for $m = 0$. Make the inductive hypothesis that (2.7) holds when m is replaced by $m-1$ and that consequently $x^{m-1}(t) \epsilon K$. Using the abbreviation $\bar{x}(t) = x(t,\bar{x}_o,\bar{\varphi})$, we evidently have

$$\bar{x}(t) = \bar{x}_o + \int_o^t f[\bar{x}(\tau), \bar{\varphi}(\bar{x}(\tau),\tau)]\, d\tau \tag{2.8}$$

This is true, since (2.8) is merely the integrated form of (2.1) with $\varphi = \bar{\varphi}$ and $x = \bar{x}$. Using this same abbreviation our inductive hypothesis appears in the form,

$$\left\| x^{m-1}(\tau) - \bar{x}(\tau) \right\| \leq \sum_{k=o}^{m-2} \beta \frac{L^k \tau^k}{k!} + \sum_{k=o}^{m-2} \eta \frac{L^k \tau^{k+1}}{(k+1)!} \tag{2.9}$$

for $0 \leq \tau \leq T$. Subtracting (2.8) from (2.6), we find that

$$x^m(t) - \bar{x}(t) = x_o - \bar{x}_o + \int_o^t \left\{ f[x^{m-1}(\tau), \varphi(x^{m-1}(\tau),t)] - f[\bar{x}(\tau), \bar{\varphi}(\bar{x}(\tau),\tau)] \right\} d\tau$$

$$= x_o - \bar{x}_o + \int_o^t \left\{ f[x^{m-1}(\tau),\varphi(x^{m-1}(\tau),\tau)] - f[x^{m-1}(\tau),\bar{\varphi}(x^{m-1}(\tau),\tau)] \right\} d\tau$$

$$+ \int_o^t \left\{ f[x^{m-1}(\tau),\bar{\varphi}(x^{m-1}(\tau),\tau)] - f[\bar{x}(\tau),\bar{\varphi}(\bar{x}(\tau),\tau)] \right\} d\tau.$$

Hence

$$\left\| x^m(t) - x(t) \right\| \leq \left\| x_o - \bar{x}_o \right\| + \int_o^t \left\| f[x^{m-1}(\tau),\varphi(x^{m-1}(\tau),\tau)] - f[x^{m-1}(\tau),\bar{\varphi}(x^{m-1}(\tau),\tau)] \right\| d\tau$$

$$+ \int_o^t \left\| f[x^{m-1}(\tau),\bar{\varphi}(x^{m-1}(\tau),\tau)] - f[\bar{x}(\tau),\bar{\varphi}(\bar{x}(\tau),\tau)] \right\| d\tau.$$

We next use the fact that $\| x_o - \bar{x}_o \| < \beta$, as well as (2.5) and (2.2). We thus find that

$$\|x^m(t) - \bar{x}(t)\| < \beta + \int_o^t \eta d\tau + L \int_o^t \|x^{m-1}(\tau) - \bar{x}(\tau)\| d\tau.$$

Hence, from our inductive hypothesis (2.9) we deduce that

$$\|x^m(t) - \bar{x}(t)\| < \beta + \eta t + \sum_{k=o}^{m-2} \beta \frac{L^{k+1} t^{k+1}}{(k+1)!} + \sum_{k=o}^{m-2} \eta \frac{L^{k+1} t^{k+2}}{(k+2)!}$$

$$= \sum_{\ell=o}^{m-1} \beta \frac{L^{\ell} t^{\ell}}{\ell!} + \sum_{\ell=o}^{m-1} \eta \frac{L^{\ell} t^{\ell+1}}{(\ell+1)!}$$

This completes the induction.

Now that $x^m(t)$ is known to lie in K, where the uniform Lipschitz condition is valid, it is easy to see that the well known technique due to Picard and Lindelöf is available to prove that as $m \to \infty$, the $x^m(t)$ tend uniformly to the solution of (2.1), which takes on the initial value x_o, Since this solution is unique, we thus are enabled to write

$$\lim_{m \to \infty} x^m(t) = x(t,x_o,\varphi) \text{ uniformly on } [0,T].$$

Hence, from (2.7), we find that

$$\|x(t,x_o,\varphi) - x(t,\bar{x}_o,\bar{\varphi}(t))\| < 2^{-1} \epsilon \tag{2.10}$$

as long as $\|\bar{\varphi} - \varphi\| < \sigma$ and $\|\bar{x}_o - x_o\| < \beta$. Moreover, from (2.1) and (2.4), it is also clear that

$$\|x(t,\bar{x}_o,\bar{\varphi}), -x(\bar{t},\bar{x}_o,\bar{\varphi})\| \leq B|t - \bar{t}| < 2^{-1}\epsilon \tag{2.11}$$

if $|t - \bar{t}| < (2B)^{-1} \epsilon$. From (2.10), (2.11), and the triangle inequality, we thus find that

$$\|x(t,x_o,\varphi) - x(\bar{t},\bar{x}_o,\bar{\varphi})\| < (1/2)\epsilon + (1/2)\epsilon = \epsilon$$

provided that $|\bar{t} - t|$, $\|\varphi - \bar{\varphi}\|$ and $\|\bar{x}_o - x_o\|$ are all less than $\delta = \min[\sigma,(2B)^{-1}\epsilon,\beta]$. This completes the proof of the Theorem.

3. EXISTENCE OF SOLUTIONS TO A MINIMAX PROBLEM IN CONTROL THEORY.

We recall the problem introduced on p.2 of the Fourth Progress Report, where we set

$$g(w,u) = \max_{x_o \in R_T} \left\{ \max_{0 \leq t \leq T} F[x(t,x_o,w,u)] \right\},$$

F being a continuous real valued function of the vector x. We posed the question as to the existence of a "bad" wind $w^* \epsilon W$ and a "good" control $u^* \epsilon V$, such that

$$\underset{u \in V}{\text{g.l.b.}} \left\{ \underset{w \in W}{\text{l.u.b.}} \quad g(w_1,u) \right\} = g(w^*,u^*).$$

The answer to this question is affirmative if W and V are sequentially compact and if R_T is closed as well as bounded. We support this statement with the following proof:

According to Theorem 2.1, $x(t,x_o,w,u)$ is continuous in t,x_o,w, and u. In reaching this conclusion, we, of course, interpret the $\varphi(x,t)$ of Theorem 2.1, to stand for the pair $(w(t), u(x))$ and the space Φ then consists of all pairs of functions (w,u) in which $w \epsilon W$ and $u \epsilon V$. It is seen that the hypotheses, both those stated explicitly in Theorem 2.1 and those contained in the preamble, are satisfied.

Hence $F(x(t,x_o,w,u))$ is continuous in t,x_o,w, and u. According to Theorem 3.2, p. 15 of the second progress report, it then follows from the compactness of R_T and the interval $0 \leq t \leq T$ as well as of W and V, that $g(w,u)$ is continuous in w and u. Finally, according to Theorem 3.3, p. 17 of the second progress report, there exists $w^* \epsilon W$ and $u^* \epsilon V$ such that

$$g(w^*,u^*) = \min_{u \in V} \left[\max_{w \in W} g(w,u) \right]$$

$$= \underset{u \in V}{\text{g.l.b.}} \left\{ \underset{w \in W}{\text{l.u.b.}} \; g(w,u) \right\},$$

as we wished to prove.

CHAPTER 5

1. SOME ELEMENTARY THEOREMS CONCERNING UNIFORMELY T-TAME SETS

As in the previous progress report, we consider the system

$$S: \quad \dot{x} = f(x,w,u),$$

where x is an n-vector, the dot represents differentiation with respect to time t, w and u are vectors (of any dimensionality, not necessarily n), and where f is an n-vector function of x, w, u, defined, say, for $x \epsilon X$, $w \epsilon W^*$, $u \epsilon U^*$.

We consider a class W of functions w which map the time interval $[0,T]$ into the set W^* and a class U of functions u, which map X into U^*.

Under suitable conditions on f and the classes W and V, the system S_1 which is now thought of as taking the form

$$\dot{x} = f(x,w(t), u(x)),$$

admits through each initial point, a unique solution $x = x(t,x_o,w,u)$, such that $x(0,x_o,w,u) = x_o$.

According to Theorem 2.1 of the previous report $x(t,x_o,w,u)$ is continuous in all its arguments if W and V are sequentially compact metric spaces. This continuity is used in some parts of the present section and in Section 2. It is not used in Section 3. Naturally, wherever the continuity of $x(t,x_o,w,u)$ is not assumed, we are concerned with results which hold even when W and V are not sequentially compact.

For convenience we begin with a reformulation of some definitions already introduced (cf. fourth progress report p. 5) and add thereto a few simple theorems.

<u>Definition 1.1</u> A non-empty set $A \subset R$ is said to be uniformly T-tame with respect to R under the (non-empty) set $V (\subset \mathcal{V})$ of controls if

$$x(t,x_o,w,u) \; \epsilon R$$

for all $x_o \epsilon A$, $t \epsilon [0,T]$, $u \epsilon V$, and $w \epsilon W$.

<u>Definition 1.2</u> Let A be an arbitrary (non-empty) subset of R. Then let

$V(A)$ be the largest subset of $\mathcal{U}$ under which A is uniformly T-tame with respect to R. That is, $V(A)$ consists of all $u \in \mathcal{U}$ such that $x(t,x_o,w,u) \in R$ for all $x_o \in A$, $t \in [0,T]$ and $w \in W$.

Of course, if $V(A)$ is vacuous, A is not uniformly T-tame under any subset of $\mathcal{U}$.

Theorem 1.1 Let A and B be subsets of R. Then

$$V(A \cup B) = V(A) \cap V(B) \tag{1.1}$$

$$V(A \cap B) \supset V(A) \cup V(B). \tag{1.2}$$

Proof of (1.1) Let $u \in V(A \cup B)$. Then by Definitions 1 and 2, $u \in V(A)$ and $u \in V(B)$. Hence

$$V(A \cup B) \subset V(A) \cap V(B). \tag{1.3}$$

On the other hand, if $u \in V(A) \cap V(B)$, then $u \in V(A)$ and $u \in V(B)$, so that $x(t,x_o,w,u) \in R$ as long as $0 \le t \le T$ and x_o is in A or B. In other words, as long as $x_o \in A \cup B$. This means that $u \in V(A \cup B)$. Hence we conclude that

$$V(A \cup B) \supset V(A) \cap V(B). \tag{1.4}$$

and then (1.1) follows immediately from (1.3) and (1.4).

Proof of (1.2). Let $u \in V(A) \cup V(B)$. Then $x(t,x_o,w,u) \in R$ for $0 \le t \le T$ and for all $x_o \in A \cap B$, so that $u \in V(A \cap B)$. Hence (1.2) is seen to be valid.

Incidentally we notice from Theorem 1.1 that the union of two sets A and B, each of which is uniformly T-tame under non-vacuous subsets of $\mathcal{U}$, need not be T-tame under any subset of $\mathcal{U}$. For $V(A)$ and $V(B)$ may have no common element.

We also notice the following obvious

Corollary 1.1 If $A \subset C \subset U$, then $V(C) \subset V(A)$. (cf. Fourth Progress Report p. 6(2)).

Proof Let $B = C-A$; so that $C = A \cup B$. Then from Theorem 1.1, we find that $V(C) = V(A \cup B) = V(A) \cap V(B) \subset V(A)$.

Definition 1.3 If V is any non-empty subset of U, we denote, by $Q(V)$, the set of points $x_o \in R$, such that $x(t,x_o,w,u) \in R$ for all $t \in [0,T]$, all $w \in W$ and all $u \epsilon V$. In other words, $Q(V)$ is the largest subset of R that is uniformly T-tame with respect to R under V. We shall refer to it briefly as a maximal uniformly T-tame set (under V with respect to R).

Notice that $Q(V)$ was called $R(V)$ in the Fourth Progress Report, pp. 5-6. Since the letter R is used also for another purpose, it seems wise to change the

Theorem 1.2 Let V_1 and V_2 be subsets of U. Then

$$Q(V_1 \cup V_2) = Q(V_1) \cap Q(V_2) \tag{1.5}$$

Proof of (1.5). Let $x_o \epsilon Q(V_1 \cup V_2)$. Then $x(t,x_o,w,u) \epsilon R$ for all $t \epsilon [0,T]$, all $w \epsilon W$ and for $u \epsilon V_i$ ($i = 1$ or 2). Hence $x_o \epsilon Q(V_i)$ both for $i = 1$ and $i = 2$. In other words $x_o \epsilon Q(V_1) \cap Q(V_2)$, which establishes (1.5) within the sign $=$ replaced by $\subset$.

Next let $x_o \epsilon Q(V_1) \cap Q(V_2)$. This means that $x_o \epsilon Q(V_i)$ for $i =$ both 1 and 2. By definition 1.3 we now have $x(t,x_o,w,u) \epsilon R$ for all $t \epsilon [0,T]$, all $w \epsilon W$, and all $u \epsilon V_i$ ($i = 1$ or 2); that is, as long as $u \epsilon V_1$ or $u \epsilon V_2$; that is, as long as $u \epsilon V_1 \cup V_2$. But by Definition 1.3, this means that $x_o \epsilon Q(V_1 \cup V_2)$, which establishes (1.5) with the sign $=$ replaced by $\supset$.

This result combined with the previous result proves (1.5) completely.

Corollary 1.2 If $V_1 \subset V_3 \subset U$, then $Q(V_3) \subset Q(V_1)$.

Proof Let $V_2 = V_3 - V_1$, so that $V_3 = V_1 \cup V_2$. Then from Theorem 1.2 we find that

$$Q(V_3) = Q(V_1 \cup V_2) = Q(V_1) \cap Q(V_2) \subset Q(V_1).$$

Theorem 1.3 Let R be closed and let A be a subset of R. If U is sequentially compact, then so is $V(A)$.

Proof Let $u_1, u_2, u_3, \ldots$ be an arbitrary sequence of controls in $V(A)$. Then, since U is sequentially compact, there exists a convergent subsequence $u_1^*, u_2^*, u_3^*, \ldots$, whose limit $u^* = \lim_{n \to \infty}$ is an element of $\mathcal{U}$. It suffices to show that $u^* \epsilon V(A)$. Since $u_n^* \epsilon V(A)$, we have $x(t,x_o,t,w,u_n^*) \epsilon R$ for all $x_o \epsilon A$, $t\epsilon[0,T]$, $w\epsilon W$. Since x is continuous, we have

$$\lim_{n \to \infty} x(t,x_o,w,u_n^*) = x(t,x_o,w,u^*).$$

Since R is closed $x(t,x_o,w,u^*) \epsilon R$ for all $x_o \epsilon A$, $t\epsilon[0,T]$ and $w\epsilon W$. Since $V(A)$ consists of all u for which $x(t,x_o,w,u) \epsilon R$ for all $x_o \epsilon A$, $t\epsilon[0,T]$ and $w\epsilon W$, it follows that $u^* \epsilon V(A)$, as we desired to prove.

Let L be a class of controls contained in $\mathcal{U}$ under which it is known that some set $\subset R$ is uniformly T-tame (with respect to R). The largest such set, we call R_T. According to Definition 1.3, $R_T = Q(L)$. Now let

$$V = V(R_T), \tag{1.6}$$

this being the largest class of controls contained in $\mathcal{U}$, under which R_T is uniformly T-tame with respect to R.

<u>Theorem 1.4.</u> $R_T = Q(V)$.

<u>Proof</u> Let $R^* = Q(V)$ (1.7)

Since $L \subset V \subset \mathcal{U}$, we have by Corollary 2 (of Theorem 2)

$$Q(V) \subset Q(L).$$

In other words,

$$R^* \subset R_T \tag{1.8}$$

By (1.6) R_T is uniformly T-tame under V. But by (1.7), R^* is the largest uniformly T-tame set under V. Hence $R^* \supset R_T$. Combining this result with (1.8), we have $R^* = R_T$, as we wished to prove.

2. ON THE INTERPRETATION OF THE SOLUTION OF THE MINIMAX PROBLEM

In what follows the V and the R_T may have been introduced in the manner indicated at the end of Section 1. However all results hold as long as $R_T = Q(V)$, that is, as long as R_T is the largest subset of R, which is T-tame under V (with respect to R).

Let $F(x)$ be continuous in R, and let

$$g(w,u) = \underset{x_0 \in R_T,\ 0 \leq t \leq T}{\text{l. u. b.}} F[x(t,x_0,w,u)].$$

Then, as proved in the last progress report, if V and W are sequentially compact, there exists $u^* \in V$ and $w^* \in W$, such that

$$g(w^*,u^*) = \min_{u \in V} \left[\max_{w \in W} g(w,u) \right].$$

If, however, V or W are not sequentially compact but are merely such that $g(w,u)$ exists, we still have theoretically a result almost as good. In fact, from Theorem 2.5 of the Second Progress Report, we can say that, corresponding to any positive number ϵ, there exists $u^* \in V$ and $w^* \in W$, such that

$$\left| g(w^*,u^*) - \underset{u \in V}{\text{g.l.b.}} \left[\underset{w \in W}{\text{l.u.b.}} g(w,u) \right] \right| < \epsilon$$

We wish to examine the significance of the number $g(w^*,u^*)$ in case R is represented by a formula of the type $H(x) \leq k$, where H is a continuous real valued function of x and where k is a constant. R is thus a closed set. Among other things we wish to examine the connection between $g(w^*,u^*)$ and k in the special case where $F \equiv H$. It will appear that $g(w^*,u^*) \leq k$. Whence, if $g(w^*,u^*)$ is actually less than k, the advantage of using the control u^* instead of any $u \in V$, is that the system may be controlled in a smaller region than R, namely $H(x) \leq g(w^*,u^*)$. This leads to an increased factor of safety but gives less indication of how a larger region than R_T might be controlled within the original R (that is, with an unchanged factor of safety).

<u>Theorem 2.1</u> R_T is closed.

<u>Proof</u> Let $x_o \epsilon R - R_T$. Since R_T is the maximal uniformly T-tame set in R, the point $x(t,x_o,w,u)$ will leave the region R for some t on the interval [0,T], at least, for a suitably chosen $w \epsilon W$ and $u \epsilon V$. That is, for this w, u, and t, we shall have

$$H[x(t,x_o,w,u)] > k.$$

From the continuity of $x(t,\bar{x}_o,w,u)$, considered as a function of its second argument, we see that we must also have (for the same t,w, and u) $H[x(t,\bar{x}_o,w,u)] > k$ for all $\bar{x}_o$ in R sufficiently close to x_o. Hence, by defintion of R_T and R we conclude that $\bar{x}_o \epsilon R - R_T$. In other words $R - R_T$ is open in R. Hence R_T is closed.

In the sequel we suppose that H is of class C' and that $\partial H/\partial x \neq 0$ at any point where $H = k$. This insures that such points are true boundary points of R.

<u>Theorem 2.2</u> If $x_o \epsilon R_T - \partial R_T$, then $H[x(t,x_o,w,u)] < k$ for all $t \epsilon [0,T]$, $w \epsilon W$, and $u \epsilon V$. That is $x(t,x_o,w,u)$ is an interior point of R.

<u>Proof</u> By definition of R_T, we already know that

$$H[x(t,x_o,w,u)] \leq k. \tag{2.1}$$

Suppose, now it were possible to find $t^* \epsilon [0,T]$, $w^* \epsilon W$, and $u^* \epsilon V$, such that

$$H[x(t^*,x_o,w^*,u^*)] = k \tag{2.2}$$

Then, since we assume that $\partial H/\partial x \neq 0$, there are points in every neighborhood of

$x(t^*,x_o,w^*,u^*)$ for which $H > 0$, that is, points outside of R. Now the homeomorphism $\bar{\bar{x}} = x(t^*,\bar{x},w^*,u^*)$ between $\bar{x}$ and $\bar{\bar{x}}$ maps every neighborhood $\bar{N}$ of x_o onto a neighborhood $\bar{\bar{N}}$ of $x(t^*,x_o,w^*,u^*)$. Since x_o is interior to R_T, $\bar{N}$ may be chosen so small that $\bar{N} \subset R_T$. But then $\bar{\bar{N}}$ still contains points outside of R. Thus there are points $\bar{x}$ in $\bar{N} \subset R_T$ which are taken by the transformation $\bar{\bar{x}} = x(t^*,\bar{x},w^*,u^*)$ into points outside of R. Hence R_T is not uniformly T-tame, contrary to assumption. Thus, since (2.2) leads to a contradiction, we see that the equality sign must be excluded in (2.1), and this establishes the theorem.

Assuming that H is continuous over all x-space and that for every x_o in this space $H[x(t,x_o,w,u)]$ is bounded for $t\epsilon[0,T]$, $w\epsilon W$, $u\epsilon V$, we let

$$m(x_o) = \underset{t,w,u}{\text{l.u.b.}} \; H[x(t,x_o,w,u)]$$

By definition of least upper bound, we know that, corresponding to any positive number ϵ, we can find $t^* \epsilon[0,T]$, $w^*\epsilon W$, $u^*\epsilon V$, in such wise that

$$m(x_o) - \epsilon < H]x(t^*,x_o,w^*,u^*)] \leq m(x_o). \tag{2.3}$$

If, in addition, W and V are sequentially compact the t^*, w^*,u^* can always be found in such wise that

$$H[x(t^*,x_o,w^*,u^*)] = m(x_o) = \underset{t,w,u}{\max} \; H[x(t,x_o,w,u)]. \tag{2.4}$$

Moreover, by Theorem 3.2 of the second progress report, we also know that $m(x_o)$ is continuous in x_o in the case of sequential compactness. In general we can prove much more in the case of sequential compactness than in the more general case where compactness is not present. In Theorems 2.3 and 2.4 we summarize some further facts about $m(x_o)$. In Theorem 2.3 we do not hypothesize compactness, while in Theorem 2.4, we do.

<u>Theorem 2.3</u> I If $x_o \epsilon R_T - \partial R_T$, $m(x_o) \leq k$

II If $x_o \epsilon R - R_T$, $m(x_o) > k$

<u>Proof of I</u> By Theorem 2.2, if $x_o \epsilon R_T - \partial R_T$, then $H[x(t,x_o,w,u)] < k$ for all $t\epsilon[0,T]$, $w\epsilon W$, and $u\epsilon V$. Hence k is an upper bound. Hence the <u>least</u> upper bound $m(x_o)$ can not exceed k as stated.

<u>Proof of II</u> Since R_T is maximally uniformly T-tame under V, we can say that, when $x_o\epsilon R - R_T$, there exists some $t\epsilon[0,T]$, some $w\epsilon W$, and some $u\epsilon V$, such that

$H[x(t,x_o,w,u)] > k$. Hence, any upper bound, including in particular the least upper bound $m(x_o)$, must exceed k. Thus we have $m(x_o) > k$ as stated.

Theorem 2.4 If W and V are sequentially compact, then

I If $x_o \epsilon R_T - \partial R_T$, $m(x_o) < k$

II If $x_o \epsilon \partial R_T$, $m(x_o) = k$

Proof of I By (2.4), we have, since W and V are sequentially compact,

$$m(x_o) = H[X(t^*,x_o,w^*,u^*)] \qquad (2.5)$$

for a suitably chosen t^*, w^*, and u^*. But, since $x_o \epsilon R_T - \partial R_T$, we know from Theorem 2.2, that the right member of (2.5) is less than k. Hence the left member is also less than k.

Proof of II If $x_o \epsilon \partial R$, every neighborhood of x_o has points in $R - R_T$. Let $x_1, x_2, x_3, \ldots$ be a sequence of points in $R - R_T$ such that

$$\lim_{n \to \infty} x_n = x.$$

Since x_n is not in the maximally uniformly T-tame set R_T, there exist $t_n \epsilon [0,T]$, $w_n \epsilon W$, and $u_n \epsilon V$, such that

$$H[x(t_n,x_n,w_n,u_n)] > k \qquad (2.6)$$

Since $[0,T]$, W, and V are all sequentially compact, we may (by extracting a suitable subsequence) assume that

$$\lim_{n \to t_n} = t^* \epsilon [0,T], \quad \lim_{n \to \infty} w_n = w^* \epsilon W, \quad \lim_{n \to \infty} u_n = u^* \epsilon V$$

Hence, passing to the limit in (2.6), we see from the continuity of H and $x(t,x_o,w,u)$ that

$$H[x(t^*,x_o,w^*,u^*)] \geq k$$

On the other hand, since $x_o \epsilon \partial R_T \subset R_T$, we know that

$$H[x(t,x_o w,u)] \leq k$$

for all $t \in [0,T]$, $w \epsilon W$, and $u \epsilon V$. Hence, we find that

$$H[x(t^*,x_o,w^*,u^*)] = k,$$

which, of course, is the maximum possible value for $H[x(t,x_o,w,u)]$ for $t \epsilon [0,T]$, $x_o \epsilon R_T$, $w \epsilon W$, and $u \epsilon W$, and $u \epsilon V$. Hence, for such x_o, we must have $m(x_o) = k$, as we wished to prove.

In the special case where $F = H$, we have

$$g(w,u) = \max_{x_o \epsilon R_T,\ 0 \leq t \leq T} H[x(t,x_o,w,u)]$$

at least, if we restrict attention to the case where we have sequential compactness, so that we can write "max" instead of "l.u.b.". Evidently from Theorems 2.3 and 2.4

$$\max_{x_o \epsilon R_T} m(x_o) = \max_{x_o \epsilon \partial R_T} m(x_o) = k$$

Hence $$k = \max_{x_o \epsilon R_T} \left\{ \max_{0 \leq t \leq T} \left\{ \max_{w \epsilon W} \left\{ \max_{u \epsilon V} H[x(t,x_o,w,u)] \right\} \right\} \right\}$$

$$= \max_{u \epsilon V} \left\{ \max_{w \epsilon W} \left\{ \max_{x_o \epsilon R_T,\, 0 \leq t \leq T} H[x(t,x_o,w,u)] \right\} \right\}$$

according to Theorem 2.4 of the second progress report. Hence, we find that

$$k = \max_{u \epsilon V} \left\{ \max_{w \epsilon W} g(w,u) \right\}.$$

On the other hand, if we let

$$\ell = \min_{u \epsilon V} \left\{ \max_{w \epsilon W} g(w,u) \right\},$$

we have $\ell \leq k$ and ℓ is equal to k if and only if $\max_{w \epsilon W} g(w,u)$

is the same for every $u \in V$. This would mean that the class V contains only the set V^* of controls for which the minimax is attained. Otherwise V^* is a proper subset of V and $\ell < k$. And letting

$$m^*(x_o) = \underset{\substack{t' \in [0,T] \\ w \in W, u \in V^*}}{\text{l. u. b.}} H[x(t,x_o,w,u)]$$

we find that $m^*(x_o) = \ell$ for $x_o \in \partial R_T$ and that this

$$\ell = \max_{u \in V^*} \left\{ \max_{w \in W} g(w,u) \right\} = \min_{u \in V^*} \left\{ \max_{w \in W} g(w,u) \right\}.$$

The region with respect to which we have uniform T-tameness under V^* is the region R^* represented by $H(x) \leq \ell$ instead of the region R represented by $H(x) \leq k$. Since $\ell < k$, it is clear that $R^* \subset R$.

Thus our analysis shows how, by limiting attention to the class V^* of controls which solve the minimax problem, we reduce the size of the region R with respect to which we can control R_T. This is what we referred to previously as an increased factor of safety.

If we wish to retain the original R, we could replace $R_T = Q(V)$ by $R_T^* = Q(V^*)$, which by Corollary 1.2 is a larger set than the original R_T. If V^* were to contain more than a single element, we could then repeat the entire process, using R_T^*, V^*, and R, instead of R_T, V, and R respectively.

3. FIRST COMPUTATIONAL ASPECT OF THE MINIMAX PROBLEM: THE CORE OF R.

3.1 Preliminaries

In previous sections of the present and past Progress Reports we discussed in great length the question of the existence of an optimum minimax control. It was shown that such a control (or controls) does indeed exist under the assumption that the function spaces U and W of admissible controls and admissible winds are sequentially compact. This assumption is rather severe. It excludes, among others, the rather important case of piecewise continuous controls. However, it was remarked above, and is reiterated here, that in the case of non-compact sets U and W one may interchange min and max by g.l.b. and l.u.b., respectively, without affecting the results significantly. Instead of arriving at a distinct element $u^* \epsilon U$ which attains the minimax, we would obtain elements in U which come arbitrarily close to the g.l.b. of the l.u.b.. In order to hue closely to the line of greatest practical interest, we hereby adopt the second point of view. Thus, when we speak of an "optimum minimax control" we no longer imply the existence of a distinct element in U which attains the minimax. Instead we refer to an element which may or may not attain the minimax, but at any rate, comes comfortably close to it.

It is felt that a meaningful class U must contain at least the class of all piecewise constant and all piecewise linear controls. However, for the time being we shall adopt an even larger class. Specifically, we shall assume that the class U is the class of all uniformly bounded, piecewise continuous controllers. The class W will be introduced in the examples.

The proofs contained in our previous work concerning the existence of optimal minimax controls were not constructive in nature. They contain no hint as to the nature of these controls, nor do they yield an algorithm for computing them. The results are admittedly of some theoretical significance, yet they leave the practical problem unresolved.

It is the object of the present section to initiate the search for the actual computation of optimal minimax controls. The reader will recall that the words "optimal minimax controls" are now being used in the sense of our remarks in the first paragraph above.

The main results are contained in the following subsection (3.2). Section 3.3 illustrates these ideas by means of two elementary examples.

3.2 THE CORE OF R.

Consider the system

$$\dot{x} = f(x,w,u), \tag{3.1}$$

where x is a real n-vector, f is a real n-vector function of (x,w,u); $w = w(t)$ and $u = u(x)$ are elements of the (given) function spaces W and U, respectively, and the whole system is well-behaved in the sense that it satisfies sufficient conditions for the existence and uniqueness of solutions.

In the space X of the vector x we are given a closed bounded set R (the set of constraints). We are also given a positive constant T and a function $F(x)$.

Let now V be a subset of U. We recall the following definition: A non-empty set A is uniformly T-tame with respect to R under the (non-empty) set of controls V if for every $x_0 \epsilon A$ and every $w \epsilon W$, the arc

$$\left\{ x(t,x_0,w,u) \;\middle|\; 0 \leq t \leq T \right\}$$

is contained in R for every element $u \epsilon V$. Whenever there is no ambiguity concerning the definition of the set R we shall simply say that A is <u>uniformly T-tame under V</u>.

If V_1 is any subset of U, and A is any subset of R, then the sets $Q(V_1)$, $V(A)$ are well-defined. The reader is referred to section 1 of the present Progress Report for the precise definitions.

A first attempt at a formulation of the "practical" (i.e. computational) aspect of the minimax control problem must proceed essentially as follows:

Let a non-empty set $V \subset U$ be given. Suppose that $Q(V)$ is not empty and let $V' = V(Q(V))$. Find a control $u^* \epsilon V'$ which minimizes (over V') the quantity

$$\max_{w \in W} \; \max_{x_0 \in Q(V)} \; \max_{0 \leq t \leq T} \; F(x).$$

The reader is again reminded of the convention established in the first paragraph of the Preliminaries to this section.

A major question arises: Which class V (in U) should we start with? Our first impulse may be to take $V = U$. However, on second thought this impulse appears to be misguided. The class U is rather large; it is the class of all uniformly bounded (with bound 1) piecewise continuous control functions. It has already been shown that if $V_1 \supset V_2$ then $Q(V_1) \subset Q(V_2)$. Since U is the largest subset of itself, Q (U) is the smallest uniformly T-tame set available in R. In fact, Q(U) may well be empty. For if $x_0 \in Q(U)$ then <u>every</u> (!) piecewise continuous controller constricts the trajectory through x_0, against any wind, for all $0 \leq t \leq T$, within the set R. The system may not even have such points. But be that as it may, the fact remains that choosing our initial V to be the whole set U would result in a uniformly T-tame set Q(U) which is the <u>very smallest possible</u> (This is our "controllable" set!). If we were to proceed from here and actually obtain an optimal minimax control for this choice of V, we would have found the <u>"best" control for the smallest possible "controllable" set!</u> But surely one of our main concerns is to make the "controllable" set as large as possible, for this is the set of initial values in R for which an eventual optimal minimax control will at least insure the safety of the system. We may summarize our dilemma as follows: If we maximize the initial set V, we minimize the "controllable" set Q(V) corresponding to it. How then is one to proceed in order to arrive at the best (i.e maximal) uniformly T-tame set within R? In fact, is there such a maximal uniformly T-tame set? And if the answer to the last question is in the affirmative, what is the set V of controls under which it is uniformly T-tame?

The remainder of this section will provide the answer to the last three questions.

Let $\{u\}$ be the set whose sole element is u. Denote $Q(\{u\})$ by $Q(u)$. We shall make use of the following definition.

Definition 3.1 Let u_1, u_2 be elements of U. We say that u_2 contains u_1, written $u_2 \supset u_1$ (or equivalently $u_1 \subset u_2$) iff the following condition holds: Whenever an arc

$$\{x(t, x_0, w, u_1) \mid 0 \leq t \leq T, \quad x_0 \text{ and } w \text{ fixed}, \quad x_0 \in Q(u_1)\}$$

is contained in R, then also the arc

$$\{x(t, x_0, w, u_2) \mid 0 \leq t \leq T\}$$

is contained in R.

The reader will note that if u is a member of U such that $Q(u)$ is empty, then u is (vacuously) contained in any other member of U.

Proposition 3.1 $u_1 \subset u_2$ iff $Q(u_1) \subset Q(u_2)$.

The proof is trivial. This Proposition furnishes an alternative definition of $\subset$.

Proposition 3.2 If $u_1 \subset u_2$ and $u_2 \subset u_1$ then $Q(u_1) = Q(u_2)$.

Proof Proposition 3.1.

Proposition 3.3 The relation $\subset$ is transitive.

Proof Let $u_1 \subset u_2$ and let $u_2 \subset u_3$. We must show that $u_1 \subset u_3$. This follows directly from Proposition 3.1, for one has

$$Q(u_1) \subset Q(u_2) \subset Q(u_3),$$

whence $u_1 \subset u_3$.

Defintion 3.2 An admissible control $\bar{u} \in U$ is said to be an upper bound in U iff $u \subset \bar{u}$ for every $u \in U$.

It should be noted that it is possible that $u_1 \subset u_2$ and $u_2 \subset u_1$ and yet $u_1 \neq u_2$. Therefore an upper bound in U is not necessarily unique.

Definition 3.3 A set $B \subset R$ is said to be a core of R iff

(1) B is uniformly T-tame

(2) If A is any uniformly T-tame subset of R, then $A \subset B$.

The set R may or may not have a core. However, if it does, then the core is clearly unique. In fact, if B_1 and B_2 are cores of R then $B_1 \subset B_2$ and on the other hand $B_2 \subset B_1$. Hence $B_1 = B_2$. We may thus speak of the core of R. If R has a core we designate it by R_T. It is the largest uniformly T-tame set in R.

We are naturally led to the following major decision: If R_T exists, we shall require that an optimal minimax control must be effective throughout the full core of R.

Proposition 3.4 If $\bar{u}_1$ and $\bar{u}_2$ are upper bounds in U, then $Q(\bar{u}_1) = Q(\bar{u}_2)$.

The proof follows trivially from Proposition 3.2.

The reader will recall that if $Q(u)$ is empty for every $u \epsilon U$, then the optimal minimax problem is vacuous (no controls are available to choose from). Therefore we must assume that there esists at least one control u such that $Q(u) \neq 0$.

We are now in a position to state the main result of this section.

Theorem 3.1 The core of R exists iff U has an upper bound. If $\bar{u}$ is an upper bound in U then $R_T = Q(\bar{u})$. If R_T exists then $V(R_T)$ is the set of all upper bounds in U.

Proof Let $\bar{u}$ be an upper bound in U. Then, by definition, $Q(\bar{u})$ is uniformly T-tame. Let A be an arbitrary uniformly T-tame subset of R. Suppose A is uniformly T-tame under V. Since $V \subset U$ and $\bar{u}$ is an upper bound, we have

$$v \subset \bar{u} \quad \text{for all} \quad v \in V.$$

Hence

$$Q(v) \subset Q(\bar{u}) \quad \text{for all } v \in V.$$

Thus

$$A \subset Q(V) = \bigcap_{v \in V} Q(v) \subset Q(\bar{u}).$$

It follows that $Q(\bar{u})$ is a core, hence the core, of R.

Conversely, let R_T be the core of R and let $V = V(R_T)$. Let $\bar{u} \in V$. Then $Q(\bar{u}) \supset R_T$. Since R_T is the core of R, $Q(\bar{u}) = R_T$. Let now u be an arbitrary element of U. The set $Q(u)$ is uniformly T-tame and hence $Q(u) \subset R_T$. It follows by Proposition 3.1 that $u \subset \bar{u}$. Hence $\bar{u}$ is an upper bound in U. This complete the proof.

Theorem 3.1 provides us with a key to the first computational aspect of the "practical" optimum minimax problem. It may be summarized as follows:

(1) The most desirable set to control is the full core of R (if it exists).

(2) The core of R exists iff U has an upper bound.

(3) The core of R is uniformly T-tame under the set of all upper bounds in U.

(4) The core of R may be obtained by finding an upper bound $\bar{u}$ in U, then computing $Q(\bar{u})$.

3.3 TWO ELEMENTARY EXAMPLES

(i) The System $\dot{x} = w + u$.

Let x be a real variable and consider the system

$$\dot{x} = w(t) + u(x) \tag{3.2}$$

Here $w = w(t)$ is restricted to a certain class W of admissible winds while $u = u(x)$ is restricted to the class U of admissible controls. The class U is the class of all real piecewise continuous functions of the real variable x which map some given interval R containing the origin (the constraint set) into the interval $[-1,1]$. The class W is a given collection of admissible winds whose domain is $[0,T]$ and whose range is $[-\alpha,\alpha]$. It is assumed that W contains all piecewise constant winds (whose domain and range are restricted as above) and that it is, moreover, reasonable enough to insure that system (3.2) satisfies sufficient conditions for the existence and uniqueness of solutions.

Let R be the interval $-A \leq x \leq A$. We shall show that the core of R exists. This will be done by showing that U has an upper bound. The core of R will also be computed.

Let

$$\bar{u}(x) = \begin{cases} -1 & \text{for all } 0 < x \leq 1 \\ 0 & \text{for } x = 0 \\ +1 & \text{for all } -1 \leq x < 0. \end{cases}$$

We shall show that $\bar{u}$ is an upper bound in U.

Suppose first that $\alpha \leq 1$. Consider the system $\dot{x} = w(t) + \bar{u}(x)$. Let $x_0 > 0$, $x_0 \in R$. Then $w(t) + \bar{u}(x(t,x_0,w,\bar{u})) \leq 0$ for any $w \in W$ so long as

$x(t,x_0,w,\bar{u}) > 0$. Moreover, the trajectory $x(t,x_0,w,\bar{u})$ cannot cross over into the negative half of R, since if it were to do so the direction of the vector field would be reversed. Hence

$$0 \leq x(t) \leq x_0 \quad \text{for all} \quad 0 \leq t \leq T \quad \text{and every} \quad w \in W.$$

Therefore, $(o,A] \subset Q(\bar{u})$. Similarly $[-A,0) \subset Q(\bar{u})$. We also have $x(t,0,w,\bar{u}) = 0$ for all $0 \leq t \leq T$ and all $w \in W$. Hence, $0 \in Q(\bar{u})$. Thus $R \subset Q(\bar{u})$ whence $R = Q(\bar{u})$. It follows that $Q(\bar{u})$ contains every uniformly T-tame subset of R and therefore $\bar{u}$ is an upper bound, and R is the core of R. --

The case when $\alpha > 1$ also yields the fact that $\bar{u}$ is an upper bound in U. The details of the proof are somewhat more complicated and are left out for lack of time. They will be supplied in the next Progress Report. The core of R turns out to be the set $[-A + \frac{\alpha-1}{T}, \quad A - \frac{\alpha-1}{T}]$.

<u>(ii) The System $\ddot{x} = w + u$</u>

Let x_1, x_2 be two real variables and let the vector (x_1,x_2) be denoted by x.

Consider the system

$$\dot{x}_1 = w(t) + u(x) \qquad (3.3)$$
$$\dot{x}_2 = x_1$$

Here $w = w(t)$ is restricted to a certain class W of admissible winds, while $u = u(x)$ is restricted to the class U of admissible controls. The classes W and U are both uniformly bounded in the sense that

$$|w(t)| \leq \alpha \text{ for all } w \in W,$$
$$|u(x)| \leq 1 \text{ for all } u \in U, \qquad (3.4)$$

these inequalities holding true for all t and x for which $w(t)$ and $u(x)$ are defined.

The class U is the class of all real piecewise continuous functions of the real variable x defined in some region R (the constraint set) containing the origin and satisfying (3.4). We are not as specific about the class W except to require that it be reasonable enough to insure that system (3.3) satisfies sufficient conditions for the existence and uniqueness of solutions. We do, however, specifically assume that the class W contains the class of all piecewise constant winds whose range lies in the interval $[-\alpha,\alpha]$. The domain of all winds is $0 \leq t \leq T$, where T is some preassigned positive constant.

It will be shown in the next Progress Report that the function

$$\bar{u}(x) = \begin{cases} -1 & \text{for all } x \text{ such that } x_1 > 0 \\ 0 & \text{for all } x \text{ such that } x_1 = 0 \\ +1 & \text{for all } x \text{ such that } x_1 < 0 \end{cases}$$

turns out to be an upper bound in U. The computation of R_T is also left for the next Report.

4. APPENDIX TO SECTION 1

The striking similarity of the formulas in Theorems 1.1 and 1.2 suggests that these theorems are but special cases of general theorems in set theory. We shall show that this is the case.

Let Φ and Ψ be arbitrary sets and let Λ be a fixed set of certain pairs (f,g) where $f \in \Phi$ and $g \in \Psi$. (Λ does not, however, ordinarily contain all such pairs).

A mapping F of the subsets ψ of Ψ into the subsets φ of Φ is defined as follows: To say that $\varphi = F(\psi)$ means that φ is the largest subset of Φ with the property that $\varphi \times \psi \subset \Lambda$. This means two things:

(1) If $f \in F(\psi)$ and $g \in \psi$, then $(f,g) \in \Lambda$.

(2) If $(f,g) \in \Lambda$ for all $g \in \psi$, then $f \in F(\psi)$.

Theorem Let $\{\psi_\alpha\}$ denote any family of subsets of Ψ. Here α represents an index which ranges over the index set A, which need not be finite or even countable.

Then

$$F[\bigcup_{\alpha\in A} \psi_\alpha] = \bigcap_{\alpha\in A} F(\psi_\alpha)$$

Proof Let $f \in F[\bigcup_\alpha \psi_\alpha]$. By (1), if g is an element of any of the ψ_α, we have $(f,g) \in \Lambda$. Thus, we have $(f,g) \in \Lambda$ for all $g \in \psi_\alpha$, so that by (2) we have $f \in F(\psi_\alpha)$ for all $\alpha \in A$. Hence $f \in \bigcap_\alpha F(\psi_\alpha)$. This establishes the formula of the Theorem when the sign = is replaced by the sign $\subset$.

Next, let $f \in \bigcap_\alpha F(\psi_\alpha)$. Therefore $f \in F(\psi_\alpha)$ for all $\alpha \in A$. If $g \in \psi_\alpha$, we have $(f,g) \in \Lambda$ by (1). That is, $(f,g) \in \Lambda$ for all $g \in \psi_\alpha$. In particular $(f,g) \in \Lambda$ for all $g \in \bigcup_\alpha \psi_\alpha$. Then, by (2), $f \in F[\bigcup_\alpha \psi_\alpha]$. This establishes the formula of the Theorem when the sign $=$ is replaced by the sign $\supset$.

From these two results, the proof of the Theorem is complete.

In the applications to Section 1, the set Φ (or Ψ) is the set R of points and the set Ψ (or Φ) is the set U of controls. The set Λ consists of the pairs (x_o, u), where $x_o \in R$ and $u \in U$, which have the property that $x(t, x_o, w, u) \in R$ for all $t \in [0,T]$ and all $w \in W$. In Section 1, we considered only the case when the index set A contained two elements. A special case in which the index set contains infinitely many elements occurs, however, in the proof of Theorem 3.1.

It is also interesting to observe that the formula of DeMorgan to the effect that

$$C[\bigcup_{\alpha} \psi_{\alpha}] = \bigcap_{\alpha} C(\psi_{\alpha}),$$

where $C[\psi]$ denotes the complement of ψ in Ψ, is also a special case of our theorem. In this case Φ and Ψ are identical and Λ consists of all pairs (f,g) in which $f \in \psi$, $g \in \psi$, and $f \neq g$.

CHAPTER 6

1. An Abstract Theory of Mappings of Subsests into Subsets.

We proceed here to develop an abstract theory of certain mappings from the subsets of a given set Ψ (or Φ) to the subsets of another given set Φ (or Ψ). These mappings are all defined in terms of a subset Ω of the product space $\Phi \times \Psi$. In the application to Control Theory, one can take Φ to be the region R of points x_o in the relevant part of phase space, while Ψ is the set U of allowable controls u, and $(x_o, u) \in \Omega$ if and only if (in the notation of previous progress reports) $x (t, x_o, w, u) \in R$ for all $t \in [0, T]$ and all $w \in W$.

Theorem 1.1 was given in the Appendix to Section 1 of the previous progress report. The Corollary to Theorem 19is essentially an abstract generalization of Proposition 3.1 in the previous progress report, while Theorem 1.11 is an abstract version of Therem 3.1 of the previous progress report. Other relationships between the results of this section and those of previous progress reports will be obvious to the observant reader.

The mappings F and G can be interpreted as the mappings V and Q of the Control Theory as previously developed. But the mappings H and J, even in concrete form, have never previously been considered. Whether H and J will give rise to useful concepts in Control Theory is doubtful. They could indeed be used, but on account of the formulas (1. 5) and (1. 6) they will probably merely lead by another road to results already obtained by way of F and G. Perhaps by using another Ω they may help to obtain new results. In any event, it seems wise to develop such an abstract theory as completely as possible.

Let Φ and Ψ be arbitrary sets and let Ω be a fixed set of certain pairs (f, g) where $f \in \Phi$ and $g \in \Psi$. Ω does not, however, ordinarily contain all such pairs. A mapping F of the subsets ψ of Ψ into the subsets φ of Φ is defined as follows: To say that $\varphi = F (\psi)$ means that φ is the largest subset of Φ with the property that $\varphi \times \psi \subseteq \Omega$.

This means two things:

(I) If $f \in F(\psi)$ and $g \in \psi$, then $(f,g) \in \Omega$.

(II) If $(f,g) \in \Omega$ for all $g \in \psi$, then $f \in F(\psi)$.

Theorem 1.1 Let $\{\psi_\alpha\}$ denote any family of subsets of Ψ. Here α represents an index which ranges over an index set A, which need not be finite or even countable. Then

$$F\left[\bigcup_{\alpha\in A} \psi_\alpha\right] = \bigcap_{\alpha\in A} F(\psi_\alpha)$$

Proof. Let $f \in F[\bigcup \psi_\alpha]$. By (I), if g is an element of any of the ψ_α, we have $(f,g) \in \Omega$. Thus, we have $(f,g) \in \Omega$ for all $g \in \psi_\alpha$, so that by (II), we have $f \in F(\psi_\alpha)$ for all $\alpha \in A$. Hence $f \in \bigcap F(\psi_\alpha)$. This establishes (1.1) when the sign = is replaced by the sign $\subset$.

Next, let $f \in \bigcap F(\psi_\alpha)$. Therefore $f \in F(\psi_\alpha)$ for all $\alpha \in A$. If $g \in \psi_\alpha$, we have $(f,g) \in \Omega$ by (I). That is, $(f,g) \in \Omega$ for all $g \in \psi_\alpha$ and for all $\alpha \in A$, therefore for all $g \in \bigcup \psi_\alpha$. Then, by (II), $f \in F[\bigcup \psi_\alpha]$. This establishes (1.1) when the sign = is replaced by the sign $\supset$.

From these two results, the proof of Theorem 1.1 is complete.

Notice that $F(0) = \Phi$, where 0 is used here to denote the null set. This is true because $\Phi \times 0$ is void. Therefore $\Phi \times 0 \subset \Omega$.

We next introduce a mapping G of the subsets φ of Φ into the subsets ψ of Ψ defined as follows: To say that $\psi = G(\varphi)$ means that ψ is the largest subset of Ψ with the property that $\varphi \times \psi \subset \Omega$. This means two things:

(III) If $g \in G(\varphi)$ and $f \in \varphi$, then $(f,g) \in \Omega$.

(IV) If $(f,g) \in \Omega$ for all $f \in \varphi$, then $g \in G(\varphi)$

Evidently G is a mapping very similar to F, the only difference is that it maps subsets of Φ into subsets of Ψ instead of the other way around. Thus Therem 1.1 can also be stated for the mapping G and we thus have for any family $\{\psi_\alpha\}$

of subsets of Φ, the following formula:

$$G[\bigcup_{\alpha\in A} \varphi_\alpha] = \bigcap_{\alpha\in A} G(\varphi_\alpha) \qquad (1.2)$$

Theorem 1.2 Let $\varphi = F(\psi)$ and $\psi^* = G(\varphi)$. Then $\psi^* \supset \psi$. In other words $G[F(\psi)] \supset \psi$, or, reversing the roles of F and G, $F[G(\varphi)] \supset \varphi$ for any $\varphi \in \Phi$.

Proof. Let $g \in \psi$ and let $f \in \varphi = F(\psi)$. Then $(f,g) \in \Omega$ by (I). Since this is true for all f in φ, we have $g \in G(\varphi) = \psi^*$ by (IV). Thus, every element of ψ is also an element of ψ^*.

Theorem 1.3 If $\psi_1 \subset \psi_2$, then $F(\psi_1) \supset F(\psi_2)$

If $\varphi_1 \subset \varphi_2$, then $G(\varphi_1) \supset G(\varphi_2)$

Proof. Let $\psi_3 = \psi_2 - \psi_1$, so that $\psi_2 = \psi_1 \cup \psi_3$

By Theorem 1.1, $F(\psi_1 \cup \psi_3) = F(\psi_1) \cap F(\psi_3)$

Hence, $F(\psi_2) = F(\psi_1) \cap F(\psi_3) \subset F(\psi_1)$. This establishes the first part of the Theorem. The second part is obtained by using G instead of F and φ instead of ψ.

Theorem 1.4 $$F[\bigcap_{\alpha\in A} \psi_\alpha] \supset \bigcup_{\alpha\in A} F(\psi_\alpha) \qquad (1.3)$$

$$G[\bigcap_{\alpha\in A} \varphi_\alpha] \supset \bigcup_{\alpha\in A} G(\psi_\alpha) \qquad (1.4)$$

We consider only the first of these two formulas, in as much as the second follows from the first merely through a change of notation.

First Proof. Let $\{\psi_\alpha\}$ (indexed by $\alpha \in A$) be arbitrary. Then define $\varphi_\alpha = F(\psi_\alpha)$. By (1.2) $G[\bigcup \varphi_\alpha] = \bigcap G(\varphi_\alpha) = \bigcap G[F(\psi_\alpha)]$

Hence, by Theorem 1.2, $G[\bigcup \varphi_\alpha] \supset \bigcap \psi_\alpha$.

Hence, by Theorems 1.3 and 1.2, $F[\bigcap \psi_\alpha] \supset F\{G[\bigcup \varphi_\alpha]\} \supset \bigcup \varphi_\alpha = \bigcup F(\psi_\alpha)$.

Second Proof. Let $f \in \bigcup F(\psi_\alpha)$. Hence there exists $\alpha \in A$, such that $f \in F(\psi_\alpha)$. Hence, if $g \in \psi_\alpha$, $(f,g) \in \Omega$ by (I). This is true in particular, if $g \in \bigcap_\alpha \psi_\alpha$. In other words $(f,g) \in \Omega$ for all $g \in \bigcap \psi_\alpha$. Hence, by (II) we have

$f \in F[\bigcap \psi_\alpha]$, as we wanted to prove.

The second proof is more fundamental than the first proof. But the first proof also brings out the fact that the sign $\supset$ in the formulas of Theorem 1.4 may be replaced by the sign $=$, whenever the same is true in Theorem 1.2. This is the case in the De Morgan formulas of Boolean algebra. In general, however, this can not be done. As a counter-example consider the following:

Let Ψ denote the set of the four letters a, b, c, d; let Φ denote the set of the four letters α, β, γ, δ; and let Ω consist of the eight pairs αa, βb, γc, δd, αb, βc, γd, δa. Let $\psi_1 = \{c\}$, $\psi_2 = \{d\}$. Then $\psi_1 \cap \psi_2$ is the null set 0. Hence $F[\psi_1 \cap \psi_2] = F[0] = \Phi$ according to a previous remark. On the other hand $F(\psi_1) = F(\{c\}) = \{\beta, \gamma\}$ and $F(\psi_2) = F(\{d\}) = \{\gamma, \delta\}$.

Hence $F(\psi_1) \cup F(\psi_2) = \{\beta, \gamma, \delta\}$ which is a proper subset of $\Phi = \{\alpha, \beta, \gamma, \delta\}$, so that $F(\psi_1) \cup F(\psi_2)$ is a proper subset of $F[\psi_1 \cap \psi_2]$.

Thus we lost the elegant duality of the De Morgan formulas which remain valid when the signs $\cup$ and $\cap$ are interchanged. In order to restore this duality, we may introduce two other mappings H, taking subsets of Ψ into subsets Φ, and J, taking subsets of Φ into subsets of Ψ. The duality is restored in the sense that the formulas remain valid when simultaneously with the interchange of $\cup$ and $\cap$, we interchange F with H, G with J, and $\subset$ with $\supset$.

The needed new definitions are as follows: $\varphi = H(\psi)$ is the smallest subset of Φ such that $(C\psi) \times (C\varphi) \subset \Omega$. Here $C\psi$ means the complement of ψ in Ψ and $C\varphi$ means the complement of φ in Φ. According to the definition of F this is equivalent to saying that $C\varphi = F(C\psi)$. In other words

$$H(\psi) = CF(C\psi) \tag{1.5}$$

It also gives rise to the following characteristic properties

(V) If $f \in CH(\psi)$ and $g \in C\,\psi$, then $(f,g) \in \Omega$

(VI) If $(f,g) \in \Omega$ for all $g \in C\,\psi$, then $f \in C\,H(\psi)$.

Similarly $\psi = J(\varphi)$ is the smallest $\psi \in \Psi$ such that $(C\psi)\,x(C\varphi) \in \Omega$. Equivalently

$$J(\varphi) = C\,G(C\varphi) \tag{1.6}$$

and we have the characteristic properties

(VII) If $g \in C\,J(\varphi)$ and $f \in C\,\varphi$, then $(f,g) \in \Omega$

(VIII) If $(f,g) \in \Omega$ for all $f \in C\,\varphi$, then $g \in C\,J(\varphi)$.

<u>Theorem 1.5</u> $$H\Big(\bigcap_{\alpha\in A} \psi_\alpha\Big) = \bigcup_{\alpha\in A} H(\psi_\alpha) \tag{1.7}$$

$$J\Big(\bigcap_{\alpha\in A} \psi_\alpha\Big) = \bigcup_{\alpha\in A} J(\varphi_\alpha) \tag{1.8}$$

It will obviously be sufficient to consider (1.7), for which we give two proofs, one based on formula (1.5) and Theorem 1.1, while the other is based on the characteristic properties (V) and (VI).

<u>First proof.</u> Let $\bar{\psi}_\alpha = C\,\psi_\alpha$, so that $\psi_\alpha = C\,\bar{\psi}_\alpha$. Then by (1.5)

$H(\bigcap\psi_\alpha) = C\,F\,[C(\bigcap\psi_\alpha)] = C\,F\,[\bigcup(C\,\psi_\alpha)] = C\,F\,[\bigcup\bar{\psi}_\alpha]$.

But this last set, by Theorem 1.1, is the same as

$C[\bigcap F(\bar{\psi}_\alpha)] = \bigcup C\,F\,(\bar{\psi}_\alpha) = \bigcup C\,F\,(C\,\psi_\alpha) = \bigcup H\,(\psi_\alpha)$,

where, of course, the last step is taken in accordance with (1.5).

<u>Second proof.</u> Let $f \in C\,H\,(\bigcap\psi_\alpha)$. Then by (V), if $g \in C[\bigcap\psi_\alpha]$, we must have $(f,g) \in \Omega$. That is, $(f,g) \in \Omega$ for all $g \in C[\bigcap\psi_\alpha] = \bigcup[C\,\psi_\alpha]$. In particular $(f,g) \in \Omega$ for all $g \in C\,\psi_{\bar{\alpha}}$ for each $\bar{\alpha} \in A$. That is, by (VI), $f \in C\,H\,(\psi_{\bar{\alpha}})$, for each $\bar{\alpha} \in A$. Hence, $f \in \bigcap C\,H\,(\psi_\alpha) = C\,[\bigcup H\,(\psi_\alpha)]$. Hence we have proved that

$$C\,H\,(\bigcap\psi_\alpha) \subset C[\bigcup H\,(\psi_\alpha)] \tag{1.9}$$

Next let $f \in C\,[\bigcup H\,(\psi_\alpha)] = \bigcap [C\,H\,(\psi_\alpha)]$. Hence $f \in C\,H\,(\psi_\alpha)$ for each $\alpha \in A$. By (V), if $g \in C\,\psi_\alpha$, then $(f,g) \in \Omega$. That is, $(f,g) \in \Omega$ for all $g \in C\,\psi_\alpha$. Hence $(f,g) \in \Omega$ for all $g \in \bigcup C\,\psi_\alpha = C[\bigcap \psi_\alpha]$. Hence, by (VI) $f \in C\,H\,[\bigcap \psi_\alpha]$. Hence $C\,[\bigcup H(\psi_\alpha)\,] \subset C\,H\,[\bigcap \psi_\alpha]$. Combining this result with (1.9), we have

$$C\,H\,(\bigcap \psi_\alpha) = C\,[\bigcup H\,(\psi_\alpha)]$$

which is equivalent to (1.7)

Theorem 1.6

$$J[\,H\,(\psi)] \subset \psi \qquad (1.10)$$

$$H[\,J\,(\varphi)] \subset \varphi \qquad (1.11)$$

Proof of (1.10) Let $\varphi = H(\psi)$ and $\psi^* = J(\varphi)$. We need to prove that $\psi^* \subset \psi$. From (1.5) we have $C\varphi = F(C\psi)$ and from (1.6) we have $C\,\psi^* = G[C\varphi] = G\{F(C\psi)\}$. Hence, from Theorem 1.2 , we find that $C\,\psi^* \supset C\,\psi$. Hence $\psi^* \subset \psi$ as desired.

The proof of (1.11) is similar.

Theorem 1.7 If $\psi_1 \subset \psi_2$, then $H\,(\psi_1) \supset H\,(\psi_2)$

If $\varphi_1 \subset \varphi_2$, then $J\,(\varphi_1) \supset J\,(\varphi_2)$.

Proof. Since $\psi_1 \subset \psi_2$, we have $\psi_1 \bigcap \psi_2 = \psi_1$. Hence, using Theorem 1.5 we have $H\,(\psi_1) = H\,(\psi_1 \bigcap \psi_2) = H\,(\psi_1) \bigcup H\,(\psi_2) \supset H\,(\psi_2)$. This establishes the first part of Theorem 1.7, and the second part may obviously be proved in the same manner.

Theorem 1.8

$$H[\bigcup_\alpha \psi_\alpha] \subset \bigcap_\alpha H\,(\psi_\alpha) \qquad (1.12)$$

$$J[\bigcup_\alpha \varphi_\alpha] \subset \bigcap_\alpha J\,(\varphi_\alpha) \qquad (1.13)$$

Proof of (1.1 2) By Theorem 1.4,

$$F[\bigcap C\psi_\alpha] \supset \bigcup F\,(C\,\psi_\alpha)$$

Therefore, by making free use of (1.5) , we find that

$$C\,H\,(\bigcup\psi_\alpha) = F\,[\,C\,(\bigcup\psi_\alpha)] = F[\bigcap C\,\psi_\alpha] \supset \bigcup F\,(C\,\psi_\alpha) = \bigcup C\,H\,(\psi_\alpha)$$
$$= C[\bigcap H(\psi_\alpha)]$$

Thus $C\,H\,(\bigcup\psi_\alpha) \supset C[\bigcap H\,(\psi_\alpha)\,]$, which is equivalent to (1.12) .

The proof of (1.13) is entirely similar.

Any one of these functions can be used to introduce a partial ordering among the subsets of Ψ or Φ which, in general, is quite distinct from the natural one afforded directly by the relationship of set inclusion. For instance, we introduce the relationship $\underset{K}{\leq}$ or $\underset{K}{\geq}$ by means of the following

Definitions 1.1. Two subsets ψ_1 and ψ_2 of Ψ satisfy the relationship $\psi_1 \underset{K}{\leq} \psi_2$ or $\psi_2 \underset{K}{\geq} \psi_1$ if and only if $K(\psi_1) \subset K\,(\psi_2)$, where K may stand for either F or H.

Similarly we may also contemplate

Definition 1.2. Two subsets φ_1 and φ_2 of Φ satisfy the relationship $\varphi_1 \underset{K}{\leq} \varphi_2$ or $\varphi_2 \underset{K}{\geq} \varphi_2$ if and only if $K\,(\varphi_1) \subset K\,(\varphi_2)$, where K may stand for either G or J.

It follows from Theorems 1.3 and 1.7 that these relations are transitive and reflexive. These definitions also afford a partial ordering of the individual elements of Ψ or Φ, because an element of Ψ or Φ may always be viewed as a subset of Ψ or Φ containing but a single element. Of course, it is possible in particular instances that the set of order relations between individual elements may turn out to be void.

Theorem 1.9 Let ψ_1 and ψ_2 be subsets of ψ. Then

$$[(f,g) \in \Omega \text{ for all } g \in \psi_1] \Rightarrow [(f,g) \in \Omega \text{ for all } g \in \psi_2] \qquad (1.13)$$

if and only if $\psi_1 \underset{F}{\leq} \psi_2$.

Proof Suppose $\psi_1 \underset{F}{\leq} \psi_2$. Then, by definition, $F(\psi_1) \subset F(\psi_2)$. If now $(f,g) \in \Omega$ for all $g \in \psi_1$, we know from (II) that $f \in F\,(\psi_1) \subset F(\psi_2)$. But from (I) and the fact that $f \in F(\psi_2)$, we conclude that $(f,g) \in \Omega$ as long as $g \in \psi_2$. In other words the assumption $\psi_1 \underset{F}{<} \psi_2$ leads to the implication (1.13)

Next, let us start from (1.13) as our assumption. Let f be an arbitrary element of $F(\psi_1)$. Then, by (I), $(f,g) \in \Omega$ for all $g \in \psi_1$. Then, by our assumption, $(f,g) \in \Omega$ for all $g \in \psi_2$. But, by (II), this means that $f \in F(\psi_2)$. Hence, we have $F(\psi_1) \subset F(\psi_2)$, which duly interpreted, means that $\psi_1 \underset{F}{<} \psi_2$.

Corollary. Let g_1 and g_2 be elements of Ψ. Then $[(f,g_1) \in \Omega] \Longrightarrow$ $[(f,g_2) \in \Omega]$ if and only if $g_1 \underset{F}{<} g_2$.

This follows immediately from Theorem 1.9 by taking $\psi_i = \{g_i\}$, the subset of Ψ consisting of the single element. g_i (i =1 ~~or~~ and 2). We use $g_1 \underset{F}{<} g_2$, of course, as an abbreviation for $\{g_1\} \underset{F}{<} \{g_2\}$,

Definition 1.3 An element $\bar{g} \in \Psi$ is said to be an upper bound in ψ (with respect to F) if and only if $g \underset{F}{<} \bar{g}$ for every $g \in \psi$.

Definition 1.4 A set $\varphi^* \subset \Phi$ is said to be a G- core of Φ iff

(i) $G(\varphi^*)$ is not vacuous

(ii) If φ is any subset of Φ such that $G(\varphi)$ is not vacuous, then $\varphi \supset \varphi^*$.

The set Φ may not have a G- core. However, if it does, then the core is clearly unique. In fact, if φ^* and φ^{**} are cores of Φ , we see from (ii) that $\varphi^* \subset \varphi^{**}$ and $\varphi^{**} \subset \varphi^*$, so that $\varphi^* = \varphi^{**}$.

Theorem 1.10 If $\bar{g}_1$ and $\bar{g}_2$ are upper bounds in Ψ , then $F(\bar{g}_1) = F(\bar{g}_2)$.

Proof. Since $\bar{g}_1 \underset{F}{<} \bar{g}_2$, we have, by Definition 1.1, $F(\bar{g}_1) \subset F(\bar{g}_2)$ and since $\bar{g}_2 \underset{F}{<} \bar{g}_1$ we also have $F(\bar{g}_1) \supset F(\bar{g}_2)$. Hence the theorem.

Theorem 1.11. The G-core of Φ exists if and only if ψ has an upper bound with respect to F. If $\bar{g}$ is an upper bound in Ψ, then $F(\bar{g})$ is the G-core of Φ . If the G-core of Φ, denoted by φ^*, exists, then $G(\varphi^*)$ is the set of all upper bounds with respect to F in Ψ.

Proof. Let $\bar{g}$ be an upper bound in Ψ. Then $F(\bar{g})$ is a set φ^* such that $G(\varphi^*)$ is not vacuous. For $G(\varphi^*) = G[F(\bar{g})]$, in accordance with Theorem 1.2, contains the non-void set $\{\bar{g}\}$. Let φ be an arbitrary subset of Φ such that $G(\varphi) = \psi$ is not vacuous. Since $\psi \subset \Psi$ and $\bar{g}$ was assumed to be an upper bound, we have $g \underset{F}{<} \bar{g}$ for all $g \in \psi$. Hence $F(g) \subset F(\bar{g})$ for all $g \in \psi$. Thus, by Theorem 1.2, $\varphi \subset F[G(\varphi)] = F(\psi)$ (1.14)

By Theorem 1.1, $F(\psi) = F[\bigcup_{g\in\psi} \{g\}] = \bigcap_{g\in\psi} F(\{g\}) = \bigcap_{g\in\psi} F(g)$, which, since $F(g)$ has already been proved to be contained in $F(\bar{g})$ for all $g \in \psi$, leads to the result that $F(\psi) \subset F(\bar{g})$. Hence from (1.14), we have $\varphi \subset F(\bar{g}) = \varphi^*$. Hence φ^* is the G-core of Φ.

Conversely, let φ^* be assumed to be the G-core of Φ and let $\psi^* = G(\varphi^*)$. Let $\bar{g} \in \psi^*$. Then, by Theorems 1.3 and 1.2, we have $F(\{\bar{g}\}) \supset F(\psi^*) = F[G(\varphi^*)] \supset \varphi^*$. Also, since φ^* is the G-core of Φ and since $G[F(\{\bar{g}\})] \supset \{\bar{g}\}$ is not vacuous, we see from (ii) that $F(\{\bar{g}\}) \subset \varphi^*$. Hence $F(\{\bar{g}\}) = \varphi^*$. If g is an arbitrary element of ψ, the set $\varphi = F(\{g\})$ is such that $G[\varphi] = G[F(\{g\})] \supset \{g\}$ is not empty. Hence, by (ii), $\varphi \subset \varphi^*$. Thus $F(\{g\}) \subset F(\{\bar{g}\})$. Hence, by the definition of $\underset{F}{<}$, we must have $g \underset{F}{<} \bar{g}$. Since g was arbitrary, this means that $\bar{g}$ is an upper bound as we desired to prove.

Theorem 1.12. Let ψ_1 and ψ_2 be subsets of Ψ. Then, *iff* $\psi_1 \underset{H}{<} \psi_2$,

$[(f,g) \in \Omega$ for all $g \in C\,\psi_1] \Rightarrow [(f,g) \in \Omega$ for all $g \in C\,\psi_2]$ (1.15)

Proof. Suppose $\psi_1 \underset{H}{>} \psi_2$. Then, by definition $H(\psi_1) \supset H(\psi_2)$. Hence, taking complements $CH(\psi_1) \subset CH(\psi_2)$. If now $(f,g) \in \Omega$ for all $g \in C\,\psi_1$, we know from (VI) that $f \in CH(\psi_1) \subset CH(\psi_2)$. But from (V) we conclude (from the fact that $f \in C\,H\,(\psi_2)$) that $(f,g) \in \Omega$ as long as $g \in C\,\psi_2$. In other words the assumption $\psi_1 \underset{H}{>} \psi_2$ leads to the implication (1.15).

Next let us start from (1.15) as our assumption. Let f be an arbitrary element of $C\,H(\psi_1)$. Then, by (V), $(f,g) \in \Omega$ for all $g \in C\,\psi_1$. Then, by assumption, we have $(f,g) \in \Omega$ for all $g \in C\,\psi_2$. But, by (VI), this means that $f \in C\,H(\psi_2)$. Hence, we have $C\,H(\psi_1) \subset CH(\psi_2)$. Therefore $H(\psi_1) \supset H(\psi_2)$, which duly interpreted means that $\psi_1 \underset{H}{>} \psi_2$.

2. ON THE EXISTENCE OF UPPER BOUNDS . SOME ELEMENTARY EXAMPLES.

(i) The System $\dot{x}=w+u$.

Let x be a real variable and consider the system

$$\dot{x} = w(t) + u(x) \tag{2.1}$$

Here $w = w(t)$ is restricted to a certain class W of admissible winds while $u = u(x)$ is restricted to the class U of admissible controls. The class U is the class of all real piecewise continuous functions of the real variable x which map some given interval R containing the origin (the constraint set) into the interval $[-1,1]$. The class W is a given collection of admissible winds whose domain is $[0,T]$ and whose range is $[-\alpha,\alpha]$. It is assumed that W contains all piecewise constant winds (whose domain and range are restricted as above) and that it is, moreover, reasonable enough to insure that system (2.1) satisfies sufficient conditions for the existence and uniqueness of solutions.

Let R be the interval $-A \leq x \leq A$. We shall show that the core of R exists. This will be done by showing that U has an upper bound. The core of R will also be computed.

Let

$$\bar{u}(x) = \begin{cases} -1 & \text{for all } 0 < x \leq 1 \\ 0 & \text{for all } x = 0 \\ +1 & \text{for all } -1 \leq x < 0. \end{cases}$$

We shall show that $\bar{u}$ is an upper bound in U.

Suppose first that $\alpha \leq 1$. Consider the system $\dot{x} = w(t) + \bar{u}(x)$. Let $x_o > 0$, $x_o \in R$. Then $w(t) + \bar{u}(x(t,x_o,w,\bar{u})) \leq 0$ for any $w \in W$ so long as

$x(t,x_o, w, \bar{u}) > 0$. Moreover, the trajectory $x(t,x_o,w,\bar{u})$ cannot cross over into the negative half of R, since if it were to do so the direction of the vector field would be reversed. Hence

$$0 \leq x(t) \leq x_o \quad \text{for all} \quad 0 \leq t \leq T \quad \text{and every} \quad w \in W.$$

Therefore, $(0,A] \subset Q(\bar{u})$. Similarly $[-A,0) \subset Q(\bar{u})$. We also have $x(t,0,w,\bar{u}) = 0$ for all $0 \leq t \leq T$ and all $w \in W$. Hence, $0 \in Q(\bar{u})$. Thus $R \subset Q(\bar{u})$ whence $R = Q(\bar{u})$. It follows that $Q(\bar{u})$ contains every uniformly T-tame subset of R and therefore $\bar{u}$ is an upper bound, and R is the core of R.

Consider next the case when $\alpha > 1$. There are two possibilities: either $(\alpha-1)\,T > A$ or $(\alpha-1)\,T \leq A$. If the first inequality holds it is not hard to show that $Q(u) = \emptyset$ for every u in U. In fact, let $x_o \geq 0$ and let u be arbitrary. Consider the system

$$\dot{x} = w_1(t) + u(x) \quad , \quad x(0) = x_o \tag{2.2}$$

where $w_1(t) \equiv \alpha$. We have

$$\dot{x} = \alpha + u \geq \alpha - 1,$$

whence

$$x(T) - x(0) = x(T) - x_o \geq (\alpha-1)\,T$$

and therefore

$$x(T) \geq x_o + (\alpha-1)\,T > (\alpha-1)\,T > A.$$

It follows that $x_o \notin Q(u)$. We have thus shown that $[0,A] \cap Q(u) = \emptyset$ for every $u \in U$. Now taking $x_o \leq 0$ and $w_2(t) \equiv -\alpha$ we obtain, in a completely analogous fashion, that $[-A,0] \cap Q(u) = \emptyset$ for every $u \in U$. Combining these two facts we get that $R \cap Q(u) = \emptyset$ for every $u \in U$. In other words, the given system admits no uniformly T-tame set. Such a system plays no role within the context of the minimax problem.

We may therefore assume, without loss of generality, that

$(\alpha-1)\, T \leq A$.

Proposition 2.1 Let I be the interval $[-A + (\alpha-1)\, T,\ A - (\alpha-1)\, T]$. Then $Q(u) \subset I$ for every $u \in U$

Proof. The argument of the preceding paragraph shows that $(R-I) \cap Q(u) = \emptyset$ for every $u \in U$. The result follows.

Proposition 2.2 $I = Q(\bar{u})$.

Proof. Let $x_o \in I$, let w be an arbitrary member of W and let $x(t, x_o, w, \bar{u})$ be denoted by $x(t)$. Let

$$T^+ = \{ t \in [0, T] \mid x(t) > 0 \}$$
$$T^- = \{ t \in [0, T] \mid x(t) < 0 \}$$

Then for all $t \in T^+$ one has

$$w(t) + \bar{u}(x(t)) = w(t) - 1 \leq \alpha - 1,$$

whereas for all $t \in T^-$ one has

$$w(t) + \bar{u}(x(t)) = w(t) + 1 \geq -\alpha + 1.$$

Let

$$t^+ = \underset{t \,\in\, T^+}{\text{g.l.b.}}\ t \qquad , \qquad t^- = \underset{t \,\in\, T^-}{\text{g.l.b.}}\ t$$

Then for every $t \in T^+$ one has

$$0 < x(t) = x(t^+) + \int_{T^+ \cap [0,t]} [w(\tau) + \bar{u}(x(\tau))]\, d\tau$$

$$\leq x(t^+) + \int_{T^+} (\alpha-1)\, d\tau$$

$$\leq x(t^+) + (\alpha-1)T \leq A.$$

Similarly, for every $t \in T^-$ one has

$$0 > x(t) = x(t^-) + \int_{T^- \cap [0,t]} [w(\tau) + \bar{u}(x(\tau))] \, d\tau$$

$$\geq x(t^-) + \int_{T^-} (-\alpha + 1) \, d\tau$$

$$\geq x(t^-) + (-\alpha + 1) T \geq -A.$$

Hence $x_o \in Q(\bar{u})$. This completes the proof.

Proposition 2.3 Let $\alpha > 1$. Then $\bar{u}$ is an upper bound in U and I is the core of R.

Proof. Propositions 2.1 and 2.2.

(ii) The System $\mathring{x}^{\bullet} = w + u$

Let x_1, x_2 be two real variables and let the vector (x_1, x_2) be denoted by x.

Consider the system

$$\mathring{x}_1 = w(t) + u(x) \tag{2.3}$$

$$\mathring{x}_2 = x_1$$

Here $w = w(t)$ is restricted to a certain class W of admissible winds, while $u = u(x)$ is restricted to the class U of admissible controls. The classes W and U are both uniformly bounded in the sense that

$$|w(t)| \leq \alpha \quad \text{for all} \quad w \in W,$$

$$|u(x)| \leq 1 \quad \text{for all} \quad u \in U, \tag{2.4}$$

these inequalities holding true for all t and x for which $w(t)$ and $u(x)$ are defined.

The class U is the class of all real piecewise continuous functions of the real variable x defined in some given rectangular region R (the constraint set) containing the origin and satisfying (2.4). We are not as specific about the class W except to require that it be reasonable enough to insure that system (2.3) satisfies sufficient conditions for the existence and uniqueness of solutions. We do, however, specifically assume that the class W contains the class of all piecewise constant winds whose range lies in the interval $[-\alpha, \alpha]$. The domain of all winds is $0 \leq t \leq T$, where T is some preassigned positive constant. The region R is defined by means of inequalities such as $|x_1| \leq A$, $|x_2| \leq B$, where A and B are given constants.

For the time being we shall restrict our attention to the case when $\alpha < 1$.

Let

$$\bar{u}(x) = \begin{cases} -1 & \text{for all } x \text{ such that } x_1 > 0 \\ 0 & \text{for all } x \text{ such that } x_1 = 0 \\ +1 & \text{for all } x \text{ such that } x_1 < 0. \end{cases}$$

We shall show that $\bar{u}(x)$ is an upper bound in U.

Let w be an arbitrary (but fixed) wind in W and consider the system

$$\begin{aligned} \dot{x}_1 &= w(t) + \bar{u}(x) \\ \dot{x}_2 &= x_1 \end{aligned} \tag{2.5}$$

Let $-B \leq x_2^o \leq B$. Let J be the interval

$$J = \left\{ (0,x_2) \mid -B \leq x_2 \leq B \right\}.$$

The vector field defined by system (2.5) is discontinuous at every point of J. In fact,

$$\lim_{\substack{x_1 \to 0+ \\ x_2 \to x_2^o}} \dot{x}_1 < 0 \quad , \qquad \lim_{\substack{x_1 \to 0- \\ x_2 \to x_2^o}} \dot{x}_1 > 0.$$

In other words, the vector field of (2.5) reverses its direction across J. Hence a trajectory of (2.5) which passes through $(0,x_2^o)$ at time t_J will be "stopped" there for all subsequent time. This fact will be employed in the sequel.

Let $u \in U$ and let $(\xi, \eta) \in Q(u)$. Suppose first that $\xi > 0$. Let $w_o(t) \in W$ and let

$$x_1(t) = x_1(t,\xi,\eta,w_o,u) \qquad \bar{x}_1(t) = x_1(t,\xi,\eta,w_o,\bar{u})$$

$$x_2(t) = x_2(t,\xi,\eta,w_o,u) \qquad \bar{x}_2(t) = x_2(t,\xi,\eta,w_o,\bar{u}).$$

As long as both $x_1(t)$ and $\bar{x}_1(t)$ are positive, we have

$$d\, x_1(t)/dt = w(t) + u(x) \geq w(t) - 1 = d\, \bar{x}_1(t)/dt.$$

Let $[0,t_1)$ be the largest subinterval in $[0,T]$ throughout which $x_1(t)>0$; let $[0,t_2)$ be the largest subinterval in $[0,T]$ throughout which $\bar{x}_1(t) > 0$; let $t_3 = \min(t_1,t_2)$. Then, for every $t \in [0,t_3]$ we have

$$0 \leq \bar{x}_1(t) = \xi + \int_0^t [w(\tau)-1]d\,\tau$$
$$\leq \xi + \int_0^t [w(\tau)+u(x\,(\tau))]d\,\tau = x_1(t) \leq A.$$

It follows, in particular, that $t_3 = t_2$. Hence

$$0 \leq \bar{x}_1(t) \leq x_1(t) \leq A \quad \text{for all} \quad t \in [0,t_2] \tag{2.6}$$

and therefore, for all $t \in [0,t_2]$,

$$\bar{x}_2(t) = \eta + \int_0^t \bar{x}_1(\tau)d\,\tau \leq \eta + \int_0^t x_1(\tau)d\,\tau = x_2(t) \leq B.$$

Moreover, $d\,\bar{x}_2(t)/d\,t > 0$ throughout $[0,t_2)$, and since $\eta \geq -B$ we conclude that $\bar{x}_2(t) > -B$ for all $t \in [0,t_2)$. Therefore

$$-B \leq \bar{x}_2(t) \leq B \quad \text{for all} \quad t \in [0,t_2]. \tag{2.7}$$

If $t_2 = T$, one now has

$$(\bar{x}_1(t),\, \bar{x}_2(t)) \in R \quad \text{for all} \quad t \in [0,T]. \tag{2.8}$$

If $t_2 < T$ then $\bar{x}_1(t) = 0$ and $\bar{x}_2(t) = \bar{x}_2(t_2)$ for all $t \geq t_2$. This together with (2.6) and (2.7), implies that (2.8) still holds. Since these considerations apply to any wind $w_o \in W$ it follows that $(\xi,\eta) \in Q(\bar{u})$.

Proposition 2.4 $Q(u) \subset Q(\bar{u})$ for every $u \in U$.

Proof. It has been shown that if $(\xi,\eta) \in Q(u)$ and $\xi > 0$ then $(\xi,\eta) \in Q(\bar{u})$.

The case when $\xi < 0$ is analogous. Moreover, $J \subset Q(\bar{u})$. This completes the proof.

Corollary 2.1 $\bar{u}$ is an upper bound in U.

Proposition 2.5 Let $w^*(t) \equiv \alpha$; let $\xi > 0$; let $x_i^*(t) = x_i(t,\xi,\eta,w^*,\bar{u})$, $i=1,2$. If $(x_1^*(t), x_2^*(t)) \in R$ for all $t \in [0,T]$ then $(\xi,\eta) \in Q(\bar{u})$.

Proof. Let $w \in W$; let $x_i(t) = x_i(t,\xi,\eta,w,\bar{u})$, $i= 1,2$. If $(x_1(t), x_2(t))$ intersects J at some time t_J it remains "stopped" there for all subsequent times. We therefore need concern ourselves only with the interval $[0,t_J)$. For all t in this interval one has

$$d\, x_1(t)/dt = w(t) - 1 \leq \alpha - 1 = d\, x_1^*(t)/dt.$$

Hence, for all $t \in [0,t_J)$,

$$0 < x_1(t) \leq x_1^*(t) \leq A,$$

whence

$$x_2(t) \leq x_2^*(t) \leq B.$$

Morever,

$$- B \leq \eta \leq x_2(t),$$

whence finally

$$(\xi,\eta) \in Q\,(\bar{u}).$$

This completes the proof.

Proposition 2.6 Let $w_*(t) \equiv -\alpha$; let $\xi < 0$; let $x_i'(t) = x_i(t,\xi,\eta,w_*,\bar{u})$, $i = 1, 2$. If $(x_1'(t), x_2'(t)) \in R$ for all $t \in [0,T]$ then $(\xi,\eta) \in Q(\bar{u})$.

Proof. The proof is analogous to the proof of Proposition 2. 5.

Let R^+ be the set $\{ (x_1, x_2) \mid 0 < x_1 \leq A, -B \leq x_2 \leq B \}$
We shall use Proposition 2. 5 to determine $Q(\bar{u}) \cap R^+$.

Consider the system

$$\dot{x}_1 = \alpha - 1$$
$$\dot{x}_2 = x_1. \tag{2.9}$$

Clearly $\dot{x}_1 < 0$ and $\dot{x}_2 > 0$ throughout R^+. If $(\xi,\eta) \in R^+$ then the solution $(x_1(t,\xi,\eta), x_2(t,\xi,\eta))$ of (2.9), when continued in the direction of increasing t, can only reach the boundary of R^+ on the set J or on the set

$$K = \{ (x_1, B) \mid 0 < x_1 \leq A \}.$$

One has

$$x_1(t,\xi,\eta) = \xi + (\alpha - 1)t$$
$$x_2(t,\xi,\eta) = \eta + \xi t + (1/2)(\alpha - 1)t^2.$$

The solution of (2.9) passing through the point (0,B) lies on the parabola

$$x_2 = B + \frac{x_1^2}{2(\alpha - 1)} .$$

Therefore, if $\eta \leq B + \xi^2/2(\alpha - 1)$ then $(x_1(t,\xi,\eta), x_2(t,\xi,\eta))$, when continued in the direction of increasing t, will reach the set J. All such points are clearly in $Q(\bar{u})$, To get the remainder of $Q(\bar{u}) \cap R^+$, continue the solutions of (2.9), starting on K, for a time interval of length T, in the direction of decreasing time. One has

$$x_1(-T,\xi,B) = \xi - (\alpha - 1)T$$
$$x_2(-T,\xi,B) = B - \xi T + (1/2)(\alpha - 1)T^2 \tag{2.11}$$

Elimination of ξ yields

$$x_2(-T,\xi,B) = [B+(1/2)(1-\alpha)T^2] - T x_1(-T,\xi,B).$$

Let $B+(1/2)(1-\alpha)T^2 = \beta$. Let L be the line $x_2 = -T x_1 + \beta$. On K we have $0 < x_1 \leq A$. Hence in (2.11), $0 < \xi \leq A$, and $(1-\alpha)T < x_1(-T,\xi,B) \leq A + (1-\alpha)T$. However, since we are restricted to R, we must limit our attention to those points of L which satisfy

$$(1-\alpha)T < x_1 \leq A.$$

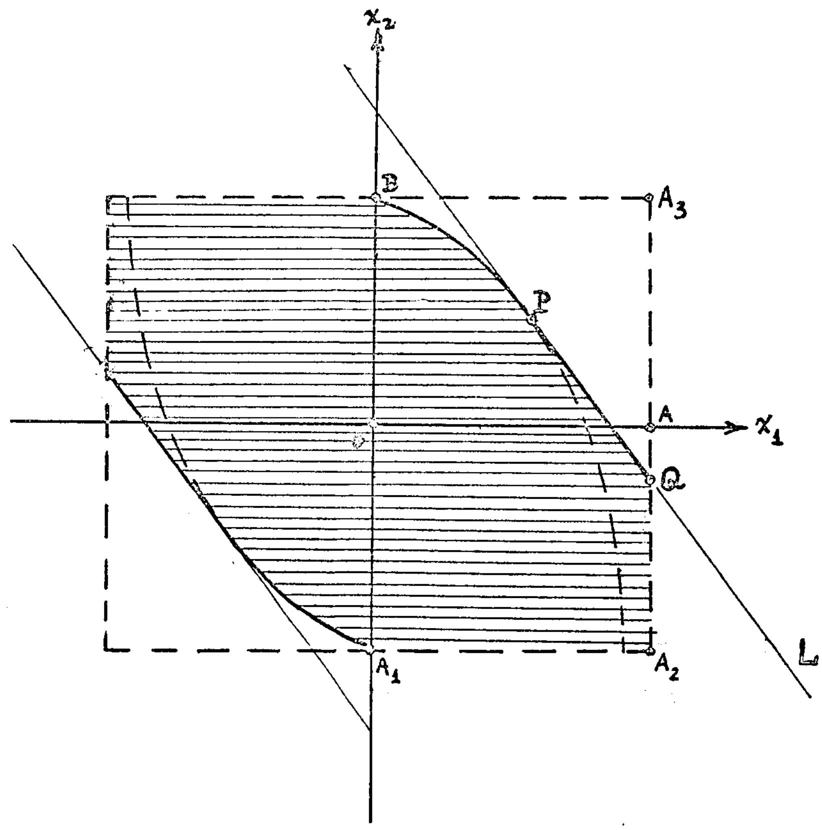
The point P on L whose abscissa is $(1-\alpha)T$ has ordinate $B+(1/2)(\alpha-1)T^2$. This point also lies on the parabola (2.10); in fact, this is the point obtained by solving (2.9) starting at (0,B), in the direction of decreasing time to the point where $t = -T$. If $P \notin R^+$ then

$$S^+ = Q(\bar{u}) \cap R^+ = \left\{ (\xi,\eta) \in R^+ \mid \eta \leq B + \xi^2/2(\alpha-1) \right\}$$

If $P \in R^+$, let Q be the point at which the line L intersects the set $\overline{A_1A_2} \cup \overline{A_2A_3}$ (Figure 1). (The point Q may be on $\overline{A_1A_2}$ or on $\overline{A_2A_3}$). In this case S^+ is the set obtained by deleting J from the closed set bounded by the curve BOA_1QPB (Figure 1). Let S^- be the set obtained by reflecting S^+ through the origin. Let $S = S^- \cup J \cup S^+$.

Proposition 2.7 S is the core of R.

Proof: By definition, $S^+ = Q(\bar{u}) \cap R^+$. The fact that $S^- = Q(\bar{u}) \cap R^-$ follows in complete analogy from proposition 2.6. This completes the proof.



The shaded region is the core of R

Figure 1

CHAPTER 7

1. COMPARABILITY OF THE SOLUTIONS OF TWO DIFFERENTIAL EQUATIONS OF THE SAME ORDER

We have previously indicated a method of partially ordering the controls $u \in \mathcal{U}$ and we have shown that an important part of our problem is solved if we can identify the upper bounds of these controls. To implement this procedure, we need to compare the trajectories of the system under different controls. This amounts to comparing the solutions of pairs of systems of differential equations with respect to various criteria. In this and the next section we present several theorems which we hope will be useful in this connection.

In this section the system is represented by means of a single differential equation of possibly high order. It is always possible, at least in the linear case according to a known theorem, to reduce the system to this form, if there is but a single actuator and if the system is controllable at all.

Theorem 1.1 Suppose $x(t)$ and $\xi(t)$ are both of class C^{n+1} on $[0,T]$ and satisfy the initial conditions,

$$\xi^{(k)}(0) \geq x^{(k)}(0),\ k=0,1,\cdots,n \tag{1.1}$$

and the differential equations,

$$x^{(n+1)}(t) = f[x^{(n)}(t),\cdots,x^{(h)}(t),\cdots,t],\ h=0,\cdots,n-1 \tag{1.2}$$

$$\xi^{(n+1)}(t) = \varphi[\xi^{(n)}(t),\cdots\quad \xi^{(h)}(t),\cdots t] \tag{1.3}$$

Suppose, furthermore, that

$$\varphi[x^{(n)},\cdots,\xi^{(h)},\cdots,t] > f[x^{(n)},\cdots x^{(h)},\cdots t] \tag{1.4}$$

as long as

$$\xi^{(h)} \geqq x^{(h)}, \quad \text{for} \quad h = 0, \cdot \cdot \cdot , n - 1 \tag{1.5}$$

Then

$$\xi^{(k)}(t) > x^{(k)}(t) \quad \text{for} \quad 0 < t \leq T \quad \text{and } k = 0, 1, \cdot \cdot \cdot , n.$$

<u>Proof.</u> By Taylor's theorem with the remainder

$$\xi^{(k)}(t)-x^{(k)}(t)=\sum_{\ell=0}^{n-k} \left\{\frac{\xi^{(k+\ell)}(0)-x^{(k+\ell)}(0)}{\ell!}\right\} t^{\ell}+ \left\{\frac{\xi^{(n+1)}(t^*)-x^{(n+1)}(t^*)}{(n-k+1)!}\right\} t^{n-k+1}$$

where t^* is a suitably chosen positive number not greater than t. According to (1.1), the first $(n-k+1)$ terms in the above sum are non-negative. If they are actually positive, then the whole sum must be positive for sufficiently small $t > 0$, since the last term is an infinitesimal of higher order than any of the other non-vanishing terms. If, however, the first $(n-k+1)$ terms of the above sum happen to vanish, we have

$$\xi^{(k+\ell)}(0) = x^{(k+\ell)}(0), \ \ell = 0, 1, \cdot \cdot \cdot , n - k$$

and in particular (by taking $\ell = n - k$),

$$\xi^{(n)}(0) = x^{(n)}(0) \tag{1.6}$$

We also observe that the last term in the above sum differs from

$$\left\{\frac{\xi^{(n+1)}(0) - x^{(n+1)}(0)}{(n-k+1)!}\right\} t^{n-k+1}$$

by an infinitesimal of order higher than t^{n-k+1}. This is because of the fact that $\xi(t)$ and $x(t)$ are of class C^{n+1}. Hence, if t is sufficiently small, $\xi^{(k)}(t) - x^{(k)}(t)$ must have the same sign as

$$\xi^{(n+1)}(0) - x^{(n+1)}(0) = \varphi[\xi^{(n)}(0), \cdot \cdot \cdot , \xi^{(h)}(0), \cdot \cdot \cdot, 0] - f[x^{(n)}(0), \cdot \cdot \cdot, x^{(h)}(0), \cdot \cdot \cdot 0],$$

which, by (1.6), is the same as

$$\varphi\,[x^{(n)}(0),\cdots,\ \xi^{(h)}(0),\cdots,0] - f[x^{(n)}(0),\cdots,\ x^{(h)}(0),\cdots,\ 0]$$

But the sign of this latter quantity is positive because of (1.1) , (1.4) and (1.5). Hence, we have proved that in all cases there exists a positive number δ such that

$$\xi^{(k)}(t) > x^{(k)}(t) \quad \text{for} \quad 0 < t \leq \delta \tag{1.7}$$

and $k = 0, 1, 2, \cdots, n$.

If $\delta \geq T$, there would be nothing more to be proved. We therefore assume in the sequel that $\delta < T$. For the sake of brevity we let $z(t)=\xi(t)-x(t)$. Then (1.7) means that $z(t)$ and its first n derivatives are positive for $0 < t \leq \delta$. We wish to prove that they are also positive for $\delta < t \leq T$. If this were not true, one at least of the functions $z^{(k)}(t)$ would have to vanish on the interval $(\delta,T]$ at some point t_k $(k=0,1,\cdots,n)$. But, if $k < n$, we can then say that $z^{(k+1)}(t)$ would have to vanish on the interval $(\delta,T]$ at some point $t_{k+1} < t_k$. For $z^{(k)}(\delta)$ and $z^{(k+1)}(\delta)$ are both known to be positive while $z^{(k)}(t_k) = 0$. Hence $z^{(k)}(t)$ would have to have a maximum at some point t_{k+1} interior to the interval (δ,t_k). At such a maximum, we, of course have

$$z^{(k+1)}(t_{k+1}) = 0.$$

It follows that, if the theorem were false, there would exist a point t_n on $(\delta,T]$ such that

$$z^{(n)}(t_n) = 0 \tag{1.8}$$

$$z^{(n)}(t) > 0 \quad \text{for } \delta \leq t < t_n \tag{1.9}$$

$$z^{(h)}(t_n) = \xi^{(h)}(t_n) - x^{(h)}(t_n) > 0 \tag{1.10}$$

$h = 0, 1, 2, \cdots, n-1$

From (1.8) and (1.9) we have $z^{(n+1)}(t_n) \leq 0$. Hence from (1.1) and (1.2) we have

$$z^{(n+1)}(t_n) = \xi^{(n+1)}(t_n) - x^{(n+1)}(t_n)$$

$$= \varphi[\xi^{(n)}(t_n), \cdots, \xi^{(h)}(t_n), \cdots, t_n] - f[x^{(n)}(t_n), \cdots, x^{(h)}(t_n), \cdots, t_n] \leq 0.$$

From (1.8), we have $\xi^{(n)}(t_n) = x^{(n)}(t_n)$. Hence, we have shown that

$$\varphi[x^{(n)}(t_n), \cdots, \xi^{(h)}(t_n), \cdots, t_n] \leq f[x^{(n)}(t_n), \cdots, x^{(h)}(t_n), \cdots, t_n].$$

But, in virtue of (1.10), this is a contradiction of (1.4) and (1.5). The theorem follows at once from this contradiction.

The hypothesis, that the inequality (1.4) should hold as long as the inequalities (1.5) hold, cannot be replaced by the less restrictive hypothesis that merely

$$\varphi[x^{(n)}, \cdots, x^{(h)}, \cdots, t] > f[x^{(n)}, \cdots, x^{(h)}, \cdots, t] \qquad (1.11)$$

To show this we consider an elementary example in which $n=1$. (The case $n=0$ is also covered by Theorem 1.1, but in this case the set of the variables $x^{(h)}$ for $h=0, \cdots, n-1$ becomes vacuous, so that the weaker hypothesis (1.11) coincides with the hypothesis of the theorem embodied by (1.4) and (1.5)). Our example with $n = 1$ is as follows:

Let $\varphi(\dot{\xi}, \xi, t) = -(\alpha + \beta)\dot{\xi} - \alpha\beta\xi + \epsilon$

and $f(\dot{x}, x, t) = -(\alpha+\beta)\dot{x} - \alpha\beta x$,

where α, β, and ϵ are all real non-zero constants and ϵ is positive. We are thus concerned with the equations $\ddot{\xi} + (\alpha + \beta)\dot{\xi} + \alpha\beta\xi = \epsilon$ and $\ddot{x} + (\alpha + \beta)\dot{x} + \alpha\beta x = 0$.

It is seen that the hypothesis (1.4) - (1.5) is fulfilled if and only if α and β have opposite signs, whereas the weaker hypothesis (1.11) is fulfilled in any case. We consider initial conditions $\dot{x}(0) = \dot{\xi}(0)$ and $x(0) < \xi(0)$. For brevity, we let $Z(t) = \xi(t) - x(t)$. Then $Z(t)$ satisfies the differential equation $\ddot{Z} + (\alpha+\beta)\dot{Z} + \alpha\beta Z = \epsilon$ and the initial conditions $Z(0) > 0$, $\dot{Z}(0)=0$. Evidently $\ddot{Z}(0) = -\alpha\beta Z(0) + \epsilon$. Hence $\ddot{Z}(0) < 0$ if α and β have the same sign and $Z(0) > \epsilon / (\alpha\beta)$. Hence, in such a case $\ddot{Z}(t) = \ddot{\xi}(t)-\ddot{x}(t)$ is negative for all sufficiently small positive values of t. Since $\dot{Z}(0) = 0$, the same is true for $\dot{Z}(t) = \dot{\xi}(t) - \dot{x}(t)$, contrary to the conclusion of the theorem. On the other hand, if α and β have opposite sign, it is obvious that both $\dot{\xi}(t)-\dot{x}(t) = \dot{Z}(t)$ and $\xi(t) - x(t) = Z(t)$ are positive for sufficiently small positive t in accordance with the theorem. The theorem also asserts (in case $\alpha\beta < 0$) that these differences are positive for all $t > 0$; but this fact is not quite so obvious if $\alpha + \beta > 0$, even though the equations are, of course, easily integrated.

Theorem 1.2 Same hypotheses as in Theorem 1.1, except that (1.4) is modified to read

$$\varphi[x^{(n)}, \cdots, \xi^{(h)}, \cdots, t] \geq f[x^{(n)}, \cdots, x^{(h)}, \cdots, t]$$

as long as (1.5) holds.

Then $\xi^{(k)}(t) \geq x^{(k)}(t)$ for $0 < t \leq T$ and $k = 0, 1, \cdots, n$.

Proof. Let $\eta(t,\epsilon)$ satisfy the differential equation $\eta^{(n+1)}(t,\epsilon) = \varphi[\eta^{(n)}(t,\epsilon), \cdots, \eta^{(h)}(t,\epsilon), \cdots, t] + \epsilon$ and the initial conditions $\eta^{(k)}(0,\epsilon) = \xi^{(k)}(0) \geq x^{(k)}(0)$ for $k = 0, 1, \cdots, n$.

Then, by Theorem 1.1, $\eta^{(k)}(t,\epsilon) > x^{(k)}(t)$, for $k = 0, 1, \cdots, n$, and for $t \in (0,T]$, if $\epsilon > 0$. It is also known from theorems on the solutions of differential equations containing parameters that

$$\lim_{\epsilon \to 0} \eta^{(k)}(t,\epsilon) = \xi^{(k)}(t).$$

Hence, passing to the limit in the above inequality, we get $\xi^{(k)}(t) \geq x^{(k)}(t)$, as desired.

2. COMPARABILITY OF THE SOLUTIONS OF TWO SYSTEMS OF DIFFERENTIAL EQUATIONS

In this section we prove some theorems similar to those of the preceding section. But they apply directly to a pair of systems of differential equations rather than to a pair of single differential equations. We write the systems in the form

$$\dot{\xi} = \varphi(t,\xi) \tag{2.1}$$

and

$$\dot{x} = f(t,x) \tag{2.2}$$

where ξ, x, φ, and f are n-vectors, while φ and f are continuous and Lipschitzian with respect to ξ and x respectively. We consider in connection with these systems a scalar function $H(x)$ of class C^1 in the components of x. Starting with solutions of (2.1) and (2.2) satisfying the same initial conditions,

$$x(0) = \xi(0) = x_0 \tag{2.3}$$

we wish to compare $H[\xi(t)]$ and $H[x(t)]$ for $t > 0$. If we let $Q_\varphi(x,t) = H_x(x)\,\varphi(t,x)$ and $Q_f(x,t) = H_x(x)\,f(t,x)$, it is clear that $Q_\varphi[\xi(t),t]$ is the rate of increase of $H[\xi(t)]$, while $Q_f[x(t),t]$ is the rate of increase of $H[x(t)]$. Hence it might be conjectured that $H[x(t)] \leq H[\xi(t)]$, if, for all x and t, we had

$$Q_\varphi(x,t) > Q_f(x,t) \tag{2.4}$$

or, in other words, if $Q_\varphi(x,t) - Q_f(x,t)$ were positive definite.

This turns out not always to be true (as we will show immediately below by means of a counter-example.). This is because the two trajectories $\xi(t)$ and $x(t)$ may be quite distinct. Hence we will need to replace the hypothesis (2.4) by a more stringent one. Before stating and proving the main theorem, we first present our counter-example in order to establish the necessity for such a more stringent hypothesis.

Let us consider the two systems

$$\dot{\xi} = \alpha\,\xi, \quad \dot{\eta} = \beta\,\xi + \gamma\,\eta \tag{2.5}$$

and

$$\dot{x} = a\,x + by, \; \dot{y} = cy \tag{2.6}$$

together with the initial conditions

$$\xi(0) = x(0) = x_o, \; \eta(0) = y(0) = y_o \tag{2.7}$$

We are here temporarily using (ξ,η) and (x,y) to represent the vectors previously denoted by ξ and x respectively, n being equal to 2. If we now take $H(x,y) = (1/2)(x^2 + y^2)$, we have $Q_\varphi(\xi,\eta) = \alpha\,\xi^2 + \beta\,\xi\,\eta + \gamma\,\eta^2$

and $Q_f(x,y) = a\,x^2 + b\,x\,y + c\,y^2$. Then

$Q_\varphi(x,y) - Q_f(x,y) = (\alpha - a)\,x^2 + (\beta - b)\,x\,y + (\gamma - c)\,y^2$

is certainly positive definite if $\alpha > a$, $\gamma > c$ and $|\beta - b|$ is sufficiently small. These conditions are fulfilled if we assume that

$$\gamma > c > \alpha > a, \quad \beta = b \neq 0. \tag{2.8}$$

Both systems are easily integrated. The solution of (2.5) is

$$\xi = x_o\,e^{\alpha t}, \quad \eta = [y_o - \frac{\beta\,x_o}{\alpha-\gamma}]\,e^{\gamma t} + \frac{\beta\,x_o}{\alpha-\gamma}\,e^{\alpha t} \tag{2.9}$$

while the solution of (2.6) is

$$x = [x_o - \frac{b\,y_o}{c-a}]\,e^{at} + \frac{b\,y_o}{c-a}\,e^{ct}, \quad y = y_o\,e^{ct}. \tag{2.10}$$

From these explicit solutions and from (2.8) we see that

$H[\xi(t),\eta(t)] = O(e^{2\gamma t})$, except when $y_o - \frac{\beta\,x_o}{\alpha-\gamma} = 0$. But

$$H[\xi(t),\eta(t)] = O(e^{2\alpha t}), \text{ if } y_o - \frac{\beta\,x_o}{\alpha-\gamma} = 0 \text{ and } x_o \neq 0.$$

(Here we use $f(t) = O(g(t))$ to mean that $f(t)$ and $g(t)$ are defined for t sufficiently large, that $g(t) \neq 0$, and that $\lim_{t\to\infty} f(t)/g(t)$ exists and is not zero. This convention is more strict than the one sometimes used, which is merely to the effect that $f(t)/g(t)$ is bounded). In the case cited above, where $y_o - \frac{\beta\,x_o}{\alpha-\gamma} = 0$ and $x_o \neq 0$, we also have $y_o \neq 0$, since $\beta \neq 0$. Hence, we see from (2.10) that $H[x(t),y(t)] = O(e^{2ct})$. Since $c > \alpha$, it is evident that for solutions starting at any point of the straight line $(\alpha-\gamma)y-\beta x = 0$, except the origin, we must eventually have $H[x(t),y(t)] > H[\xi(t),\eta(t)]$. This is contrary to the conjectured theorem.

Theorem 2.1. Suppose that

$$Q_\varphi(\xi,t) > Q_f(x,t) \tag{2.11}$$

whenever

$$H(\xi) \geq H(x). \tag{2.12}$$

Then the vector functions $\xi(t)$ and $x(t)$, satisfying (2.1), (2.2), and (2.3) for $0 \leq t \leq T$, have the property that

$$H[\xi(t)] > H[x(t)] \tag{2.13}$$

for all $t \in (0,T]$

Proof. Let $w(t) = H[\xi(t)] - H[x(t)]$. Then according to (2.11) and (2.12), $\dot{w}(0) = Q_\varphi(x_o,0) - Q_f(x_o,0) > 0$. Therefore, since $w(0) = 0$, there exists a positive number δ such that $w(t) > 0$ for $0 < t \leq \delta$. If $\delta \geqq T$, there is nothing further to be proved. We therefore assume from now on that $0 < \delta < T$; and, of course, $w(\delta) > 0$.

If the theorem were false, there would exist a number $t^* \epsilon (\delta,T]$ such that $w(t^*) = 0$ and such that

$$w(t) > 0 \quad \text{for} \quad t \in (\delta,t^*) \tag{2.14}$$

Since $w(\delta) > 0$, there would have to exist numbers $\bar{t}$ between δ and t^* where $w(t)$ is not increasing; for otherwise $w(t^*) > w(\delta) > 0$ contrary to the definition of t^*. For such a $\bar{t}$ we would therefore have simultaneously $\dot{w}'(\bar{t}) = Q_\varphi[\xi(\bar{t}),\bar{t}] - Q_f[x(\bar{t}),\bar{t}] \leq 0$ and $w(\bar{t}) = H[\xi(\bar{t})] - H[x(\bar{t})] > 0$, the latter following from (2.14). We thus have a contradiction of the hypothesis expressed by (2.11) and (2.12). From this contradiction, the theorem is established.

Theorem 2.2. Suppose that

$$Q_\varphi(\xi,t) \geq Q_f(x,t) \tag{2.15}$$

whenever

$$H(\xi) \geq H(x). \tag{2.16}$$

Suppose also that the two vector functions $\xi(t)$ and $x(t)$ satisfying (2.1), (2.2), and (2.3) for $0 \leq t \leq T$ have the property of not simultaneously passing through a point where the vector $H_x(x)$ vanishes. Then

$$H[\xi(t)] \geq H[x(t)] \tag{2.17}$$

for all $t \in [0,T]$.

Proof. Let ϵ be a positive parameter and consider the two systems

$$\dot{\xi} = \varphi_\epsilon(t,\xi) = \varphi(t,\xi) + \epsilon H_x(\xi) \tag{2.18}$$

and

$$\dot{x} = f_\epsilon(t,x) = f(t,x) - \epsilon H_x(x) \tag{2.19}$$

and the vectors $\xi(t,\epsilon)$ and $x(t,\epsilon)$ satisfying (2.18) and (2.19) respectively and the initial conditions,

$$\xi(0,\epsilon) = x(0,\epsilon) = x_o \tag{2.20}$$

If σ is a preassigned positive number, one knows that these vectors will be defined for $0 \leq t \leq T - \sigma$ (at least, if ϵ is sufficiently small) and that

$$\lim_{\epsilon \to 0} \xi(t,\epsilon) = \xi(t), \quad \lim_{\epsilon \to 0} x(t,\epsilon) = x(t), \tag{2.21}$$

uniformly for $0 \leq t \leq T - \sigma$. By hypothesis the two vectors $H_x[\xi(t)]$ and $H_x[x(t)]$ never vanish simultaneously (i.e. for the same value of t). Hence, if ϵ is sufficiently small $H_x[\xi(t,\epsilon)]$ and $H_x[x(t,\epsilon)]$ can never vanish simultaneously for $0 \leq t \leq T - \sigma$. To substantiate this last statement, we observe that otherwise there would exist sequences, $\epsilon_1, \epsilon_2, \ldots$ and $t_1, t_2 \ldots$ such that $\epsilon_n \to 0$, $0 \leq t_n \leq T - \sigma$ and such that

$$H_x[\xi(t_n,\epsilon_n)] = H_x[x(t_n, \epsilon_n)] = 0. \tag{2.22}$$

From the compactness of the interval $[0,T-\sigma]$ we may also assume, by confining attention to suitable subsequences if necessary, that $t_n \to t^*$, where $0 \leq t^* \leq T - \sigma$. We now omit numerous details regarding the passage to the limit in (2.22). Suffice it to say that the uniformity in (2.21) and the uniform continuity of H_x in a suitably chosen region lead to the result that

$$H_x[\xi(t^*)] = H_x[x(t^*)] = 0,$$

which contradicts the hypothesis. Hence, $H_x[\xi(t,\epsilon)]$ and $H_x[x(t,\epsilon)]$ can never vanish simultaneously, as already stated, at least if ϵ is sufficiently small.

It follows that $\left\{H_x[\xi(t,\epsilon)] \cdot H_x[\xi(t,\epsilon)]\right\}$ and

$\epsilon\left\{H_x[x(t,\epsilon)] \quad H_x[x(t,\epsilon)]\right\}$ can never both fail simultaneously to be positive.

We also have

$$Q_{\varphi\epsilon}[\xi(t,\epsilon),t] = Q_\varphi[\xi(t,\epsilon),t] + \epsilon\left\{H_x[\xi(t,\epsilon)] \cdot H_x[\xi(t,\epsilon)]\right\}$$

and

$$Q_{f\epsilon}[x(t,\epsilon),t] = Q_f[x(t,\epsilon),t] - \epsilon\left\{H_x[x(t,\epsilon)] \cdot H_x[x(t,\epsilon)]\right\}$$

We thus see from (2.15) and (2.16) that we can confine attention to a region in which

$$Q_{\varphi\epsilon}[\xi,t] > Q_{f\epsilon}[x,t]$$

as long as $H[\xi] \geq H[x]$. Hence applying Theorem 2.1, we have

$$H[\xi(t,\epsilon)] > H[x(t,\epsilon)] \quad \text{for} \quad 0 < t \leqq T - \sigma.$$

Passing to the limit as $\epsilon \to 0$, we thus find that

$$H[\xi(t)] \geqq H[x(t)] \quad \text{for} \quad 0 < t \leqq T - \sigma$$

Since equality holds for $t = 0$ because of (2.3) and since σ is an arbitrary positive number, we have

$$H[\xi(t)] \geqq H[x(t)] \quad \text{for } 0 \leqq t < T.$$

Finally, by continuity at $t= T$, we see that the theorem is true as stated.

3. ON THE EXISTENCE OF UPPER BOUNDS FOR THE SYSTEM

$$\ddot{x} = h(\dot{x},x,t) + w + u.$$

In this section we wish to indicate the manner in which Theorem 1.2 may be used to generalize the work of the previous Progress Report on the system $\ddot{x} = u + w$. We hereby consider a system of the form

$$\ddot{x} = h(\dot{x},x,t) + u + w \tag{3.1}$$

in which $h(\dot{x},x,t)$ is monotone non-decreasing in x for each fixed $\dot{x}$ and t. The system can also be written in the form

$$\dot{x}_1 = h(x_1,x_2,t) + u + w \tag{3.2}$$

$$\dot{x}_2 = x_1,$$

if we set $x_2 = x$ and $x_1 = \dot{x}$. In the sequel we use (x_1,x_2) and $(\dot{x},x)$ interchangeably without further comment according to convenience.

We wish the motion to be confined in the rectangle R : $|x_1| \leq A$ and $|x_2| \leq B$ for $t \in [0,T]$. The " control " $u = u(x_1,x_2) \in U$ is subject to various restrictions which may be used in the definition of U. The only one of these conditions essential for the moment is the condition $|u| \leq 1$. Similarly the " wind " $w = w(t) \in W$ is assumed to satisfy the condition $|w| \leq \alpha$, where $\alpha < 1$.

We suppose we have a non-negative function $\lambda(x_2)$ defined for $|x_2| \leq B$ and such that when the point (x_1,x_2) is initially within the set J defined by $|x_1| \leq \lambda(x_2)$ it remains trapped within this set for all time during the motion defined by (3.2) with $u(x_1,x_2)$ replaced by $\tilde{u}(x_1,x_2)$ where $\tilde{u}(x_1,x_2) = -\operatorname{sgn} x_1$ for (x_1,x_2) <u>outside</u> the set J. This supposition is supposed to hold for any $w \in W$ and no matter how $\tilde{u}$ is defined inside the set J, so long as it is such that $\bar{u} \in U$.

In the special case $h(x_1, x_2) \equiv 0$ treated in the previous Progress Report the set J could be the interior and boundary of the ellipse

$$\frac{x_2^2}{B^2} + \frac{x_1^2}{C^2} = 1$$

where C is sufficiently small. We omit the proof of this fact. In the previous Progress Report only the limiting case $C \to 0$ was considered, so that the set J was the line segment $x_1 = 0$, $|x_2| \leq B$. This necessitated the assumption that the class U contained discontinuous functions. But for $C > 0$, we can consider the class U as containing only functions that are continuous in the interior of R and $\bar{u}$ might be defined throughout the interior of R as follows:

$$\bar{u} = -\operatorname{sgn} x_1 \quad \text{if} \quad \lambda(x_2) < |x_1| \leq A$$

$$\bar{u} = -x_1 / \lambda(x_2) \quad \text{if} \quad |x_1| \leq \lambda(x_2).$$

In the special case where J is the interior and boundary of the ellipse mentioned above $\lambda = \frac{C}{B}\sqrt{B^2 - x_2^2}$. The above definition then makes $\bar{u}$ continuous on the boundary of R as well as in the interior except at the points $(0, \pm B)$, where it is left undefined.

Let u be an arbitrary control $\in U$. By $Q(u)$ we mean (as in previous Progress Reports) the maximal subset of R that is controllable by u. We shall now prove that $Q(u) \subset Q(\bar{u})$, or, in other words, $u \lesssim_Q \bar{u}$. Let $(\dot{x}_0, x_0)$ be an arbitrary point of $Q(u)$. We then consider the solution $x = \xi(t)$ of the system

$$\ddot{\xi} = \varphi(\dot{\xi}, \xi, t) = h(\dot{\xi}, \xi, t) + u(\dot{\xi}, \xi) + w(t)$$

such that $\xi(0) = x_0$ and $\dot{\xi}(0) = \dot{x}_0$. Since the initial point is in the controllable region, we have for all $t \in [0,T]$ and all $w \in W$

$$|\xi(t)| \leq B, \quad |\dot{\xi}(t)| \leq A \tag{3.3}$$

We also consider the solution $x = x(t)$ of the system

$$\ddot{x} = f(\dot{x},x,t) = h(\dot{x},x,t) + \bar{u}(\dot{x},x) + w(t) \tag{3.4}$$

such that $x(0) = x_o$ and $\dot{x}(0) = \dot{x}_o$. If the initial point $(\dot{x}_o, x_o)$ is in J, the complete motion is trapped in J under the control $\bar{u}$ by definition of J. Hence $(\dot{x}_o, x_o) \in Q(\bar{u})$ and there is nothing further to be proved in this case.

Suppose therefore that the initial point is not in J. Then $|x_1| = |\dot{x}| > \lambda(x_2)$ and initially $\bar{u}(\dot{x},x) = -\operatorname{sgn} x_1$. We consider the case $x_1 > 0$ so that $\bar{u} = -1$. If the solution $[\dot{x}(t),x(t)]$ ever enters J for any $t < T$, it will be trapped there, nor can x_1 ever become negative without the solution first passing into J. Hence, it is only necessary to consider the system (3.4) with $\bar{u} = -1$. Since $|u(\dot{x},x)| \leq 1$, we see at once that

$\varphi(\dot{x},x,t) = h(\dot{x},x,t) + u(\dot{x},x) + w(t) \geq h(\dot{x},x,t) - 1 + w(t) = f(\dot{x},x,t)$

and moreover $f(\dot{x},x,t)$ is monotone non-decreasing in x. Hence, the hypotheses of Theorem 1.2 are fulfilled and we find from this theorem and (3.3) that

$x(t) \leq \xi(t) \leq B$ and $\dot{x}(t) \leq \dot{\xi}(t) \leq A$ for $0 \leq t \leq T$.

We have already noted that in the case now under consideration $\dot{x}(t) > 0 > -A$, and, since $\dot{x}(t) > 0$, we have also $x(t) \geq x(0) = x_o \geq -B$.

Hence $[\dot{x}(t), x(t)] \in R$ for $0 \leq t \leq T$. The case in which x_1 is initially negative can be similarly treated. Hence, in all cases in which $(\dot{x}_o, x_o) \in Q(u)$ we have also $(\dot{x}_o, x_o) \in Q(\bar{u})$. Hence $Q(u) \subset Q(\bar{u})$, as stated. $\bar{u}$ is an upper bound.

Note: Strictly speaking we have not used Theorem 1.2 in exactly the form there stated. For the hypothesis there used was to the effect that $\varphi(\dot{x},\xi,t) \geq t(\dot{x},x,t)$ whenever $\xi \geq x$. However, it is obvious that this hypothesis holds if $\varphi(\dot{x},x,t) \geq f(\dot{x},x,t)$ for all $\dot{x},x$, and t and if, in addition, either φ or f is monotonic non-decreasing in its second argument. Observe that it is not necessary that both φ and f should have this property of monotonicity.

The theorems of Section 1 also enable us to prove that, if $(\dot{x}_o,x_o) \in R$ and $\dot{x}_o > \lambda(x_o)$ and if the solution $\dot{\xi}(t)$ of the system,

$$\ddot{\xi} = h(\dot{\xi},\xi,t) - 1 + \alpha = \varphi(\dot{\xi},\xi,t), \tag{3.5}$$

with initial conditions $\xi(0) = x_o$, $\dot{\xi}(0) = \dot{x}_o$, remains within R for $t \in [0,T]$, then $(\dot{x}_o,x_o) \in Q(\bar{u})$. In other words $(\dot{x}_o, x_o)$ is in the core of R.

To prove this we consider the solution of the system

$$\ddot{x} = h(\dot{x},x,t) + \bar{u} + w \tag{3.6}$$

such that $x(0) = x_o$, $\dot{x}(0) = \dot{x}_o$. Here w is an arbitrary element of W, and and $\bar{u} = \bar{u}(\dot{x},x) = -1$ when $\dot{x} > \lambda(x)$. Thus $\bar{u}$ is initially -1. If the solution were to enter J, it would be trapped there and there would be nothing more to prove. Hence in the only case of interest $\dot{x} > 0$ and $\bar{u} = -1$ as long as the solution stays in R. Thus we are led to compare the system

$$\ddot{x} = f(\dot{x},x,t) = h(\dot{x},x,t) - 1 + w \tag{3.7}$$

with the system (3.5). Since $w \leq \alpha$, we see that $\varphi(\dot{x},x,t) \geq f(\dot{x},x,t)$, and the monotonicity requirement is also satisfied. Since, by hypothesis,

$|\dot{\xi}(t)| \leq A$ and $|\xi(t)| \leq B$ for $0 \leq t \leq T$, we therefore have by Theorem 1.2, $\dot{x}(t) \leq \dot{\xi}(t) \leq A$ and $x(t) \leq \xi(t) \leq B$ for $0 \leq t \leq T$. Moreover, since in the only case of interest $\dot{x}(t) > 0$, we have both $x(t) \geq x(0) \geq -B$ and $\dot{x}(t) > -A$. Hence the solution of (3.6) under the given initial conditions remains within R for the given time interval and this completes the proof.

A similar result holds if $\dot{x}_o < -\lambda(x_o)$. Thus the way remains wide open for describing the complete core of R as in Section 2 (ii), of the previous Progress Report, although the description will necessarily not be quite so explicit because of the greater generality of the system here considered.

A further generalization which will probably work is not to define J as a set in which the motion would be permanently trapped but merely as a suitably chosen set from which the motion (no matter what the wind $w \in W$) could not reach the boundary of R within the time T. This would free us from supposing necessarily that $\alpha < 1$.

CHAPTER 8

1. ON UPPER BOUNDS FOR THREE DIMENSIONAL SYSTEMS.

We have indicated in our previous Progress Reports how to partially order the admissible controls and we have shown that an important part of the minimax problem is solved if we can identify the upper bounds of these controls. These ideas were extensively illustrated by means of two dimensional examples. However, the usefulness of the notion of upper bounds as well as the corollary notion of a core would be rather limited if it were not possible to identify upper bounds for systems whose dimension is higher than 2.

We have therefore expended considerable effort in the analysis of various three dimensional systems, notably the familiar one with three zero eigenvalues. Most of our attempts led to a dead end. In fact, not only were we unsuccessful in finding an upper bound, but we even came to consider the very existence of such an upper bound as questionable.

Yet during the past week we conceived of a control function which appears to have the properties we were searching for. A quick preliminary study encourages us to believe that we have indeed come upon an upper bound for the controllable system of dimension 3 with 3 zero eigenvalues. However, the proof is still heuristic in nature and we prefer to postpone the presentation of this "result" until an exact proof is available. At the present time we would like to restrict ourselves to the remark that our conjecture involves the definition of an upper bound for the 3 (i.e. n) dimensional system by means of the time-optimal solution of an appropriate

2 (i.e. n-1) dimensional system. If our conjecture turns out to be correct, it will be discussed in detail in our next Progress Report.

2. REMARKS ON THE COMPARABILITY OF SOLUTIONS OF DIFFERENTIAL EQUATIONS.

In Section 1. of the eighth Progress Report were proved some theorems on camparing the solutions of two differential equations of order, say n + 1. The equations were written in the form

$$x^{(n+1)}(t) = f\,[x^{(n)}(t), \ldots, x^{(h)}(t), \ldots, t] \tag{1}$$

and

$$\xi^{(n+1)}(t) = \varphi[\xi^{(n)}(t), \ldots, \xi^{(h)}(t), \ldots, t] \tag{2}$$

where $h = 0, 1, \ldots, n-1$. Both f and φ are supposed to satisfy hypotheses sufficient to insure existence and uniqueness of solutions in appropriate regions starting with given initial conditions. In Theorem 1.2 the fundamental additional hypothesis was made that

$$\varphi[x^{(n)}, \ldots, \xi^{(h)}, \ldots, t] \geq f[x^{(n)}, \ldots; x^{(h)}, \ldots; t]$$

as long as $\xi^{(h)} \geq x^{(h)}$, for $h = 0, 1, \ldots, n-1$. We shall refer to this as hypothesis H_1. It is clear that this hypothesis will be automatically satisfied if

$$\varphi[x^{(n)}, \ldots, x^{(h)}, \ldots, t] \geq f[x^{(n)}, \ldots, x^{(h)}, \ldots, t] \tag{3}$$

and if either φ or f (But not necessarily both) is monotonic non-decreasing in $x^{(o)}, x^{(1)}, \ldots, x^{(n-1)}$ for each fixed $x^{(n)}$ and t. We shall refer to this latter hypothesis, formulated in terms of monotonicity, as hypothesis H_2. It was found in Section 3 of the previous Progress Report that hypothesis H_2 was more convenient to use than hypothesis H_1.

Unfortunately hypothesis H_2 is not likely to be satisfied in problems of significance. For example, a problem in the control of a Saturn Launch vehicle leads to a differential equation such as

$$\dddot{\varphi} = - .022\ddot{\varphi} + .200\dot{\varphi} - .006\varphi + \psi \tag{4}$$

where $\psi = \psi(\varphi, \dot{\varphi}, \ddot{\varphi}, t)$ will depend on the control law adopted as well as the wind. This equation was obtained by eliminating all dependent variables except φ from the third order system of equations obtained from the Marshall Space Flight Center at Huntsville and using the numerical values for certain constants obtained from the same source. One might then wish to compare the solutions of (4) with the solutions of some such equation as the following:

$$\dddot{y} = - .002\ddot{y} + .200\dot{y} - .006y + \alpha(t). \tag{5}$$

where $\alpha(t)$ is a bound for ψ; say, a lower bound, so that

$$\psi(\varphi, \dot{\varphi}, \ddot{\varphi}, t) \geqq \alpha(t) \tag{6}$$

Then assuming $y(0) = \varphi(0)$, $\dot{y}(0) = \dot{\varphi}(0)$, $\ddot{y}(0) = \ddot{\varphi}(0)$, one might wish to conclude that $\varphi(t) \geqq y(t)$, $\dot{\varphi}(t) \geqq y(t)$, $\ddot{\varphi}(t) \geqq \ddot{y}(t)$ for $t > 0$. This conclusion would, however, be unjustified on the basis of Theorem 1.2; for although the condition (3) is satisfied the right hand member of (5) is not monotonic non-decreasing in both $\dot{y}$ and y (but only in $\dot{y}$), while nothing at all is known about the monotonicity of the right member of (4) in Φ and φ (without further knowledge of ψ). Thus hypothesis H_2 is not satisfied. It is found, however, that a suitably chosen transformation on the dependent variables reduces the equations to a form in which Theorem 1.2 may be used to obtain a modified result. Thus, if we set

$$\xi = e^{-(1/5)t}\varphi \quad \text{and} \quad x = e^{-(1/5)t}y \tag{7}$$

we find that equation (4) appears in the form

$$\dddot{\xi} = -.622\,\ddot{\xi} + .0712\,\dot{\xi} + .02512\,\xi + e^{-(1/5)t}\psi \tag{8}$$

and that equation (5) appears in the form

$$\dddot{x} = -.622\,\ddot{x} + .0712\,\dot{x} + .02512\,x + e^{-(1/5)t}\alpha(t) \tag{9}$$

Since $\alpha(t) \leqq \psi[e^{(1/5)t}\xi,\ e^{(1/5)t}(\dot{\xi}+(1/5)\xi),\ e^{(1/5)t}(\ddot{\xi}+(2/5)\dot{\xi}+(1/25)\xi),t]$

we also have

$$e^{-(1/5)}\alpha(t) \leqq e^{-(1/5)t}\psi[e^{(1/5)t}\xi, e^{(1/5)t}(\dot{\xi}+(1/5)\xi), e^{(1/5)t}(\ddot{\xi}+(2/5)\dot{\xi}+(1/25)\xi),t].$$

Morever, the right hand member of (9) is monotonic increasing in both $\dot{x}$ and x. In other words the hypothesis H_2 (and hence H_1) is fulfilled. Hence Theorem 1.2 is immediately applicable to equations (8) and (9). We conclude that, if $\xi(0)=x(0)$, $\dot{\xi}(0)=\dot{x}(0)$, and $\ddot{\xi}(0)=\ddot{x}(0)$, then $x(t)\leqq\xi(t)$, $\dot{x}(t)\leqq \dot{\xi}(t)$, $\ddot{x}(t)\leqq\ddot{\xi}(t)$ for $t > 0$. Interpreting these results in terms of (4) and (5) with the help of (7) the following may be stated:

If $y(0) = \varphi(0), \dot{y}(0)=\Phi(0), \ddot{y}(0)=\ddot{\varphi}(0)$, then, for $t > 0$, we must have

$$y(t) \leqq \varphi(t)$$

$$\dot{y}(t) - (1/5)y(t) \leqq \dot{\varphi}(t) - (1/5)\varphi(t)$$

$$\ddot{y}(t) - (2/5)\dot{y}(t) + (1/25)y(t) \leqq \ddot{\varphi}(t) - (2/5)\dot{\varphi}(t) + (1/25)\varphi(t).$$

It is not clear what application can be made of these results.

CHAPTER 9

1. THE CORE OF A REGION RELATIVE TO A SUBSET THEREOF.

The program outlined in the Sixth Progress Report (for initiating an approach to the minimax problem by searching for the upper bounds in a certain partial ordering of the control functions) has run into difficulties whenever we have attempted to apply the idea to systems of order greater than two. This has certainly been the case for systems with only one actuator and with a finite allowable region R. It just so happens that in such cases there is no reason to suppose that an upper bound need exist. We now wish to discuss and illustrate a simple modification which will permit the ideas of the Sixth Progress Report to be retained for some systems of higher order.

The modification consists, roughly speaking, in the preliminary abandonment of any hope of controlling all points initially in R. More precisely we make a preliminary specification of a suitably chosen region $R^* \subset R$ and define, for each $V \subset \mathcal{U}$, the set $Q(V)$ to be the largest set of points in R^* (rather than in R, as previously) which are uniformly T-tame with respect to R under V. With this modification, we define $Q(u) = Q(\{u\})$ and say that $u_1 \underset{Q}{<} u_2$ if and only if $Q(u_1) \subset Q(u_2)$, just as we did previously. Again, if $\bar{u}$ is an upper bound under the order relation $\underset{Q}{<}$, we can prove, as previously, that $Q(\bar{u})$ is independent of the particular upper bound $\bar{u}$ considered; and we may thus call $Q(\bar{u})$ the core of R(relative to R^*). Evidently the core is now always a subset of R^*.

We wish to illustrate the foregoing by considering a system of the form

$$\dddot{x} = h(\ddot{x},\dot{x},x,t) + u + w \qquad (1.1)$$

in which $h(\ddot{x},\dot{x},x,t)$ is monotone non-decreasing in x and $\dot{x}$. The system can also be written in the form

$$\begin{aligned} \dot{x}_1 &= h(x_1,x_2,x_3,t) + u + w \\ \dot{x}_2 &= x_1 \\ \dot{x}_3 &= x_2 \end{aligned} \qquad (1.2)$$

if we set $x_3 = x$, $x_2 = \dot{x}$, and $x_1 = \ddot{x}$. In the sequel we use (x_1, x_2, x_3) and $(\ddot{x},\dot{x},x)$ interchangeably without further comment according to convenience.

We wish the motion to be confined in the rectangular parallelepiped R: $|x_1| \leq A$, $|x_2| \leq B$, $|x_3| \leq C+BT$ for $t \epsilon [0,T]$. The region $R^* \subset R$ of initial states is to be the rectangular parallelepiped, $|x_1| \leq A, |x_2| \leq B, |x_3| \leq C$. The "control" $u = u(x_1,x_2,x_3,t) \epsilon \mathcal{U}$ is subject to various restrictions which may be used in the definition of $\mathcal{U}$. The only one of these conditions essential for the moment is the condition $|u(x_1,x_2,x_3,t)| \leq \beta(t)$, where $\beta(t)$ is a given positive function defined for $0 \leq t \leq T$. Similarly the "wind" $w = w(t) \epsilon W$ is assumed to satisfy the condition $|w| \leq \alpha(t) < \beta(t)$.

Parenthetically we note that our conception of the classes $\mathcal{U}$ and W is somewhat more general than that displayed in previous progress reports. We introduce this generalization in order to take care of certain transformed equations such as equations (8) or (9) of the Ninth Progress Report. For a bi-product of the transformation there discussed is a modification of the classes $\mathcal{U}$ and W.

We suppose we have a non-negative function $\lambda(x_2, x_3) < A$ defined for $|x_2| \leq B$ and $|x_3| \leq C + BT$ and such that when the point (x_1, x_2, x_3) is in the set J defined by $|x_1| \leq \lambda(x_2, x_3)$ at time t_0, (assuming that it started at time 0 from a point of R^*) it remains trapped within this set for $t_0 \leq t \leq T$ during the motion defined by (1.2) with $u(x_1, x_2, x_3, t)$ replaced by $\bar{u}(x_1, x_2, x_3, t)$, where $\bar{u}(x_1, x_2, x_3, t) = -\beta(t) \operatorname{sgn} x_1$ for (x_1, x_2, x_3) <u>outside</u> the set J (but within R). This supposition is supposed to hold for any $w \in W$ and no matter how $\bar{u}$ is defined inside the set J, so long as it is such that $\bar{u} \in \mathcal{U}$.

Let u be an arbitrary element of $\mathcal{U}$. By $Q(u)$ we mean the maximal subset of R^* that is controllable by u. We shall now prove that $Q(u) \subset Q(\bar{u})$, or, in other words, $u \underset{Q}{<} \bar{u}$.

Let $(\ddot{x}_0, \dot{x}_0, x_0)$ be an arbitrary point of $Q(u)$. We then consider the solution $x = \xi(t)$ of the equation,

$$\dddot{\xi} = \varphi(\ddot{\xi}, \dot{\xi}, \xi, t) = h(\ddot{\xi}, \dot{\xi}, \xi, t) + u(\ddot{\xi}, \dot{\xi}, \xi, t) + w(t),$$

such that $\ddot{\xi}(0) = \ddot{x}_0$, $\dot{\xi}(0) = \dot{x}_0$, $\xi(0) = x_0$. Since the initial point is in the controllable region, we have for all $t \in [0,T]$ and all $w \in W$

$$|\ddot{\xi}(t)| \leq A, \quad |\dot{\xi}(t)| \leq B, \quad |\xi(t)| \leq C + BT \tag{1.3}$$

We also consider the solution $x = x(t)$ of the equation

$$\dddot{x} = f(\ddot{x},\dot{x},x,t) = h(\ddot{x},\dot{x},x,t) + \bar{u}(\ddot{x},\dot{x},x,t) + w(t) \qquad (1.4)$$

such that $\ddot{x}(0) = \ddot{x}_o$, $\dot{x}(0) = \dot{x}_o$, $x(0) = x_o$. If the initial point $(\ddot{x}_o,\dot{x}_o,x_o)$ is in J, the complete motion is trapped in J under the control $\bar{u}$ by definition of J. Hence $(\ddot{x}_o,\dot{x}_o,x_o) \in Q(\bar{u})$ and there is nothing further to be proved in this case.

Suppose therefore that the initial point is not in J. Then $|x_1| = |\ddot{x}| > \lambda(x_2,x_3)$ and initially $\bar{u}(x,x,x,t) = -\beta(t) \operatorname{sgn} x_1$. We first consider the case $x_1 > 0$, so that $\bar{u} = -\beta(t)$. If the solution $[\ddot{x}(t),\dot{x}(t),x(t)]$ ever enters J for any $t < T$, it will be trapped there, nor can x_1 ever become negative without the solution first passing into J. Hence, in the case now under consideration, it is only necessary to consider the equation (1.4) with $\bar{u} = -\beta(t)$. Since $|u(\ddot{x}, \dot{x}, x, t)| \leq \beta(t)$, we see at once that $\varphi(\ddot{x}, \dot{x}, x, t) = h(\ddot{x}, \dot{x}, x, t) + u(\ddot{x}, \dot{x}, x, t) + w(t) \geq h(\ddot{x}, \dot{x}, x, t) - \beta(t) + w(t) = f(\ddot{x}, \dot{x}, x, t)$, and moreover $f(\ddot{x}, \dot{x}, x, t)$ is monotone non-decreasing in $\dot{x}$ and x. Hence, the hypotheses of Theorem 1.2 of the Eighth Progress Report are fulfilled and we find from this Theorem and (1.3) that, for $t \in [0, T]$,

$$\ddot{x}(t) \leq \ddot{\xi}(t) \leq A$$

$$\dot{x}(t) \leq \dot{\xi}(t) \leq B$$

$$x(t) \leq \xi(t) \leq C + BT$$

Moreover $x_1(t) = \ddot{x}(t) > 0 > -A$ in the case under present consideration so that $|\ddot{x}| \leq A$. And, since $\ddot{x} > 0$, $\dot{x} \geq \dot{x}(0) \geq -B$, so that $|\dot{x}| \leq B$. Finally, since $\dot{x} \geq -B$, $x \geq x_o - BT \geq -C-BT$, so that $|x| \leq C + BT$. Hence, in this case, in which x_1 is initially positive, $[\ddot{x}(t), \dot{x}(t), x(t)] \in R$ for $0 \leq t \leq T$ and so $[\ddot{x}_o, \dot{x}_o, x] \in Q(\bar{u})$.

The case in which x_1 is initially negative is treated as follows: Here, we have initially $\bar{u} = +\beta(t)$. If the solution $[\ddot{x}(t), \dot{x}(t), x(t)]$ ever enters J for any $t < T$, it will be trapped there, nor can x_1 ever become positive without the solution first passing into J. Hence, in the case now under consideration, it is only necessary to consider the equation (1.4) with $\bar{u} = + \beta(t)$. Since $|u(\ddot{x}, \dot{x}, x, t)| \leq \beta(t)$, we see at once that

$$\varphi(\ddot{x}, \dot{x}, x, t) = h(\ddot{x}, \dot{x}, x, t) + u(\ddot{x}, \dot{x}, x, t) + w(t)$$
$$\leq h(\ddot{x}, \dot{x}, x, t) + \beta(t) + w(t) = f(\ddot{x}, \dot{x}, x, t),$$

and moreover $f(\ddot{x}, \dot{x}, x, t)$ is monotone non-decreasing in $\dot{x}$ and x. Hence, Theorem 1.2 of the Eighth Progress Report, together with (1.3), shows that, for $t \epsilon [0, T]$,

$$\ddot{x}(t) \geq \ddot{\xi}(t) \geq -A$$
$$\dot{x}(t) \geq \dot{\xi}(t) \geq -B$$
$$x(t) \geq \xi(t) \geq -C-BT$$

Moreover $x_1(t) = \ddot{x}(t) < 0 < +A$ in the case now being considered, so that $|\ddot{x}| \leq A$. And, since $\ddot{x} < 0$, $\dot{x} < \dot{x}_0 < +B$, so that $|\dot{x}| \leq B$. Finally, since $\dot{x} \leq B$, $x \leq x_0 + BT \leq C + BT$, so that $|x| \leq C + BT$. Hence, in this case also, in which x_1 is initially negative $[\ddot{x}(t), \dot{x}(t), x(t)] \epsilon R$ for $0 \leq t \leq T$ and so $[\ddot{x}_0, \dot{x}_0, x] \epsilon Q(\bar{u})$.

We have thus proved completely that $Q(u) \subset Q(\bar{u})$, as we desired to do. $\bar{u}$ is an upper bound. $Q(\bar{u})$ is the core of R relative to R^*.

We now pass to a more explicit representation of the core of R relative to R^*. Namely it is possible to show that this core is the union of three sets, say S_1, S_2, and S_3 defined as follows:

$$S_1 = J \cap R^*$$

S_2 is the set of all points $(\ddot{x}, \dot{x}_0, x_0)$ in R^* with $\ddot{x} > \lambda(\dot{x}_0, x_0)$ such that the solution $\xi(t)$ of the equation

$$\dddot{\xi} = h(\ddot{\xi}, \dot{\xi}, \xi, t) - \beta(t) + \alpha(t) = \varphi(\ddot{\xi}, \dot{\xi}, \xi, t) \qquad (1.5)$$

with initial conditions $\ddot{\xi}(0) = \ddot{x}_0$, $\dot{\xi}(0) = \dot{x}_0$, $\xi(0) = x_0$, does not leave R for $0 \leq t \leq T$, unless it previously enters J.

S_3 is the set of all points $(\ddot{x}_0, \dot{x}_0, x_0)$ in R^* with $\ddot{x}_0 < -\lambda(\dot{x}_0, x_0)$ such that the solution $\xi(t)$ of the equation $\dddot{\xi} = h(\ddot{\xi}, \dot{\xi}, \xi, t) + \beta(t) - \alpha(t) = \varphi(\ddot{\xi}, \dot{\xi}, \xi, t)$ with the initial conditions $\ddot{\xi}(0) = \ddot{x}_0$, $\dot{\xi}(0) = \dot{x}_0$, $\xi(0) = x_0$, does not leave R for $0 \leq t \leq T$, unless it previously enters J.

By definition of J, it is obvious that S_1 is part of the core.

We next prove that S_2 is part of the core, namely that $S_2 \subset Q(\bar{u})$. To prove this we consider the equation

$$\dddot{x} = h(\ddot{x}, \dot{x}, x, t) + \bar{u} + w \qquad (1.6)$$

and its solution such that $\ddot{x}(0) = \ddot{x}_0$, $\dot{x}(0) = \dot{x}_0$, $x(0) = x_0$. Here w is an arbitrary element of W, and $\bar{u}\,(\ddot{x}, \dot{x}, x, t) = -\beta(t)$ when $\ddot{x} > \lambda(\dot{x}, x)$. Thus $\bar{u}$ is initially $-\beta(t)$. If the solution were to enter J, it would be trapped there, and there would be nothing more to prove. Hence, in the only case of interest $\ddot{x} > 0$ and $\bar{u} = -\beta(t)$ as long as the solution stays in R. Thus we are led to compare the system

$$\dddot{x} = f(\ddot{x}, \dot{x}, x, t) = h(\ddot{x}, \dot{x}, x, t) - \beta(t) + w \qquad (1.7)$$

with the system (1.5). Since $w \leq \alpha$, we see that $\varphi(\ddot{x}, \dot{x}, x, t) \geq f(\ddot{x}, \dot{x}, x, t)$, and the monotonicity requirement is satisfied. Since, by hypothesis, $|\ddot{\xi}(t)| \leq A$, $|\dot{\xi}(t)| \leq B$, and $|\xi(t)| \leq C + BT$ for $0 \leq t \leq T$ (at least as long as J is not entered), we therefore have by Theorem 1.β of the Eighth Progress Report

$$\ddot{x}(t) \leq \ddot{\xi}(t) \leq A$$

$$\dot{x}(t) \leq \dot{\xi}(t) \leq B$$

$$x(t) \leq \xi(t) \leq C + BT$$

for $0 \leq t \leq T$. Moreover. since $\ddot{x}(t) > 0$, we have $\dot{x}(t) \geq 0 \geq -B$ and $\ddot{x}(t) > -A$. Hence $|\ddot{x}(t)| \leq A$ and $|\dot{x}(t)| \leq B$. Finally, from the fact that $\dot{x}(t) \geq -B$, we find that $x(t) \geq x_0 - BT \geq -C - BT$, so that $|x(t)| \leq C + BT$. Hence, the solution of (1.6) under the given initial conditions remains in R for the given time interval. This completes the proof that $S_2 \subset Q(\bar{u})$.

The fact that those points of $Q(\bar{u})$ for which $x_1 > \lambda(x_2, x_3)$ must belong to S_2 is obvious from the fact that such points are controlled by $\bar{u}$ no matter what the wind is and hence, in particular, if $w = \alpha(t)$, while $\bar{u} = -\beta(t)$ until J is entered, if it ever is entered.

Similar statements can be made about S_3 and so it is easy to see that the core of R_1 relative to R^*, is

$$Q(\bar{u}) = S_1 \cup S_2 \cup S_3.$$

2. A GENERALIZATION OF THE MINIMAX PROBLEM.

Our concept of the minimax problem has undergone a number of changes in the course of this study. The latest formulation occurred in the Fifth Progress Report. Although this formulation seems to meet the main requirements, there are two respects in which it now seems wise to introduce a slight generalization. In the first place, in Section 1, we have already indicated a reason for allowing the control function to depend explicitly upon t as well as implicitly through the mediation of x. In the second place, the quantity F to be minimaxed, should be allowed to depend not only on the state variables x but also upon both the control function u, and the wind w. For instance, from the physical point of view, F might represent the bending moment of a launch vehicle, a quantity which could depend very much on both the wind and the control function as well as on the state variables. Hence we would wish to replace the continuous function $F(x)$ of Section 1 of the Fifth Progress Report by a continuous function $F(x,w^*,u^*)$, where w^* is an element in the union of the ranges of all the functions $w \in W$ and where u^* is an element in the union of the ranges of all the functions $u \in U$, and x has the same meaning as before. For the sake of clarity we give the complete present formulation as follows

We consider the system

$$S: \quad \dot{x} = f(x, w, u)$$

where x is an n-vector representing the system's state, the dot represents differentiation with respect to time t, w and u are vectors (of any dimensionality, not necessarily n), and where f is an n-vector function of x, w, u, defined, say, for $x \epsilon X$, $w \epsilon W^*$, and $u \epsilon U^*$.

We consider a class W of functions w (referred to as "winds") which map the time interval $[0, T]$ into the set W^* and a class U of functions u (referred to as "controls") which map the cartesian product of X by $[0, T]$ into U^*.

Under suitable conditions on f and the class W and U, the system S, which is now thought of as taking the form

$$\dot{x} = f[x, w(t), u(x, t)],$$

admits, through each initial point x_0, a unique solution, $x = x(t, x_0, w, u)$ such that $x(o, x_0, w, u) \equiv x_0$.

Moreover, the system is supposed to disintegrate whenever the solution emerges from some given region $R \subset X$. We find it therefore useful to say that the system S is uniformly T-tame with respect to R, if there exists a subregion $R_T (\subset R)$ and a non-vacuous sub-class $V (\subset U)$ of controls, such that $x(t, x_0, w, u) \epsilon R$ as long as $x_0 \epsilon R_T$, $t \epsilon [0, T]$, $w \epsilon W$ and $u \epsilon V$. We henceforth assume that S is uniformly T-tame in the sense of this definition.

Suppose next that it is desirable that a given continuous function $F(x, w, u)$ defined for $x \epsilon R$, $w \epsilon W^*$, and $u \epsilon U^*$ be kept as small as possible during the motion. More precisely, we are interested in minimizing by proper choice of the control u the maximum value of

$$F[x(t, x_o, w, u), w(t), u(x(t, x_o, w, u), t)]$$

for $x_o \epsilon R_T$, $t \epsilon [0, T]$ and for $w \epsilon W$. This is a peculiar expression because of the way the three u's enter between the square brackets. The first and third u represent an element of U, which is, of course, a function, whereas the second u represents the value of this function at a certain point and is thus an element of U^*. Nevertheless it is intuitively evident that this F depends continusouly on x_o, t, w, and u; and in the sequel we give a formal proof of this fact under suitable hypothesis regarding U and W. Thus, letting

$$g(w, u) = \max_{x_o \epsilon R_T} \left\{ \max_{0 \leq t \leq T} F[x(t,x_o,w,u), w(t), u(x(t,x_o,w,u),t)] \right\},$$

we pose the question as to the existence of a "bad" wind $\bar{w} \epsilon W$ and a "good" control $\bar{u} \epsilon V$ such that

$$\underset{u \epsilon V}{\text{g.l.b.}} \left[\underset{w \epsilon W}{\text{l.u.b.}} \; g(w, u) \right] = g(\bar{w}, \bar{u}).$$

The answer to this question is in the affirmative, at least if $f[x, w(t), u(x,t)]$ satisfies the Lipschitz condition demanded in formula (2.2) of the Fifth Progress Report, with $\varphi = \{w, u\}$, and if R_T, W, and V are compact. The proof of this existence theorem is the same as that previously given for the problem formulated in the Fifth Progress Report, the proof being given in Section 3 thereof.

The only thing which needs to be added is a formal proof of the continuity of the F introduced above, considered as a function of t, x_o, w, and u. We turn our attention immediately to this proof after first making the explanation that the symbol $\|\ldots\|$ is used to denote the norm of a finite dimensional vector, while $|\ldots|$ is used to denote a norm of a vector function, with a finite number of scalar functions as components. Specifically, if h is a vector function with components $h_1, \cdots, h_k$ with domain Y, we take $|h| = \underset{y \in Y}{\text{l.u.b.}} \|h(y)\|$.

We seek to prove continuity of F at an arbitrary fixed point $\bar{t}$, $\bar{x}_o$, $\bar{w}$, $\bar{u}$, where $\bar{t} \in [0, T]$, $\bar{x}_o \in R_T$, $\bar{w} \in W$, $\bar{u} \in V$.

In our proof we have the following four facts to work with

I $F(x, w^*, u^*)$ is continuous in x, w^*, u^*, by hypothesis.

II $x(t, x_o, w, u)$ is continuous in t, x_o, w, u, as proved in the Fifth Progress Report.

III If $u \in U$, $u(x, t)$ is continuous in x and t, by hypothesis.

IV If $w \in W$, $w(t)$ is continuous in t, by hypothesis.

We let $\bar{w}^* = \bar{w}(\bar{t})$, $\bar{u}^* = \bar{u}(x(\bar{t}, \bar{x}_o, \bar{w}, \bar{u}), \bar{t})$ and $\bar{x} = x(\bar{t}, \bar{x}_o, \bar{w}, \bar{u})$. If $\epsilon > 0$ is preassigned we choose δ_1, in accordance with I, so that

$$|F(x, w^*, u^*) - F(\bar{x}, \bar{w}^*, \bar{u}^*)| < \epsilon \tag{2.1}$$

as long as $\|x - \bar{x}\|$, $\|w^* - \bar{w}^*\|$, $\|u - \bar{u}^*\| < \delta_1$

By definition of $|u|$ as $\underset{x}{\max} \underset{t}{\max} \|u(x, t)\|$, it is evident that $\|u(x, t) - \bar{u}(x, t)\| < (1/2)\delta_1$ if $|u - \bar{u}| < (1/2)\delta_1$. Using III choose $\delta_2 < (1/2)\delta_1 < \delta_1$ so that $\|\bar{u}(x, t) - \bar{u}(\bar{x}, \bar{t})\| < (1/2)\delta_1$ as long as $\|x - \bar{x}\|$, $|t - \bar{t}| < \delta_2$. Hence by the triangle inequality we see that

$$\|u(x, t) - \bar{u}(\bar{x}, \bar{t})\| < \delta_1 \tag{2.2}$$

as long as $\|x - \bar{x}\|$, $|t - \bar{t}|$, $|u - \bar{u}| < \delta_2$.

By definition of $\|w\|$ as $\max_t \|w(t)\|$, it is clear that $\|w(t) - \bar{w}(t)\| < (1/2)\delta_1$ if $\|w - \bar{w}\| < (1/2)\delta_1$. Using IV choose $\delta_3 < \delta_2 < (1/2)\delta_1 < \delta_1$ so that $\|\bar{w}(t) - \bar{w}(\bar{t})\| < (1/2)\delta_1$ as long as $|t - \bar{t}| < \delta_3$.

Hence by the triangle inequality we see that

$$\|w(t) - \bar{w}(\bar{t})\| < \delta_1 \tag{2.3}$$

as long as $|t - \bar{t}|$, $\|w - \bar{w}\| < \delta_3$.

Using III choose $\delta_4 < \delta_3 < \delta_2 < (1/2)\delta_1 < \delta_1$ so that

$$\|x(t,x_o,w,u) - x(\bar{t},\bar{x}_o,\bar{w},\bar{u})\| < \delta_2 < \delta_1 \tag{2.4}$$

as long as $|t - \bar{t}|$, $\|x_o - \bar{x}_o\|$, $\|w - \bar{w}\|$, $\|u - \bar{u}\| < \delta_4$.

From (2.4) and (2.2), it is seen that

$$\|u^* - \bar{u}^*\| < \delta_1 \tag{2.5}$$

as long as $|t - \bar{t}|$, $\|x_o - \bar{x}_o\|$, $\|w - \bar{w}\|$, $\|u - \bar{u}\| < \delta_4$, where, of course, we have set $u^* = u(x(t, x_o, w, u), t)$. From (2.3) we also have

$$\|w^* - \bar{w}^*\| < \delta_1 \tag{2.6}$$

as long as $|t - \bar{t}|$, $\|w - \bar{w}\| < \delta_4 < \delta_3$, where, of course, we have set $w^* = w(t)$. Finally, from (2.4) we have

$$\|x - \bar{x}\| < \delta_1, \tag{2.7}$$

as long as $|t - \bar{t}|$, $\|x_o - \bar{x}_o\|$, $\|w - \bar{w}\|$, $\|u - \bar{u}\| < \delta_4$, where, of course, we have written x as an abbreviation for $x(t, x_o, w, u)$. Hence from (2.1), (2.5), (2.6) and (2.7), we find that

$$|F[x(t,x_o,w,u), w(t), u(x(t,x_o,w,u), t)] - F[x(\bar{t},\bar{x}_o,\bar{w},\bar{u}), \bar{w}(t), \bar{u}(x(\bar{t},\bar{x}_o,\bar{w},\bar{u}),\bar{t})]| < \epsilon$$

as long as $|t - \bar{t}|$, $\|x_o - \bar{x}_o\|$, $\|w - \bar{w}\|$, $\|u - \bar{u}\| < \delta_4$, which completes the proof of the desired result.

CHAPTER 10

ON THE EXISTENCE OF UPPER BOUNDS: A THREE DIMENSIONAL EXAMPLE.

(i) Introduction

Consider the third order linear controllable system with three zero eigenvalues, namely

$$S_3: \quad \begin{aligned} \dot{x}_1 &= w(t) + u(x_1, x_2, x_3) \\ \dot{x}_2 &= x_1 \\ \dot{x}_3 &= x_2 \end{aligned}$$

We are interested in solutions to this system subject to the constraints

$$C_1: \quad |x_i(t)| \leq A_i \text{ , } i = 1, 2, 3 \text{ , for all } t \geq 0.$$

$$C_2: \quad |w(t)| \leq \alpha < 1 \text{ for all } t \geq 0 \text{ .}$$

$$C_3: \quad |u(x_1, x_2, x_3)| \leq 1.$$

It is assumed that u and w belong, respectively, to classes of functions $\mathcal{U}$ and W which assure the existence and uniqueness of solutions for the system S_3. We shall show that if A_1, A_2 and A_3 satisfy certain inequalities, the given system admits an upper bound within the parallelopiped R_3 defined by C_1.

(ii) The System S_2

It will prove useful to refer to the system

$$S_2 : \quad \begin{aligned} \dot{x}_1 &= w(t) + u(x_1, x_2) \\ \dot{x}_2 &= x_1 \end{aligned}$$

together with the constraints

$$\begin{aligned} C_1' &: \\ C_2' &: \\ C_3' &: \end{aligned} \quad \begin{cases} |x_i(t)| \leq A_i, \; i = 1, 2, & \text{for all } t \geq 0 \\ |w(t)| \leq \alpha < 1 & \text{for all } t \geq 0 \\ |u(x_1, x_2)| \leq 1. \end{cases}$$

We recall that the system S_2 was shown in our previous progress reports to admit an upper bound within the rectangle R_2 defined by C_1'. Indeed, it was shown that the function

$$\bar{u}(x_1, x_2) = \begin{cases} -1 & \text{whenever } x_1 > 0 \\ 0 & \text{whenever } x_1 = 0 \\ +1 & \text{whenever } x_1 < 0 \end{cases}$$

was such an upper bound. The core K_2 of S_2 consists of that region of R_2 which lies between the two parabolic arcs

$$\pi_1 : \; x_2 = A_2 + x_1^2 /2 (\alpha-1) \; , \; x_1 \geq 0 \; , \qquad (1)$$

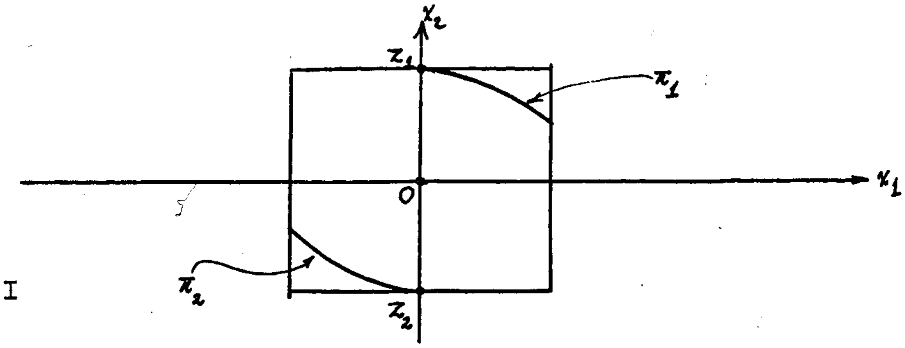
and

$$\pi_2 : \; x_2 = -A_2 + x_1^2 /2 (1-\alpha) \; , \; x_1 \leq 0 \; , \qquad (2)$$

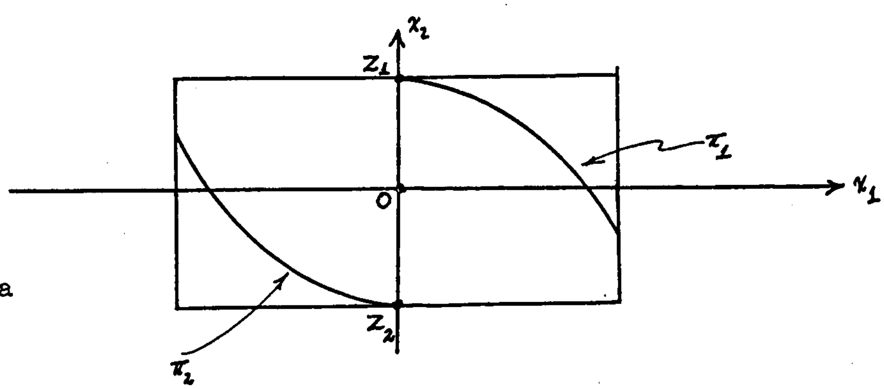
(Figure 1). However, upper bounds in S_2 are not unique. In fact, any member $u'(x_1, x_2)$ of $\mathcal{U}$ satisfying

$$u'(x_1,x_2) = \begin{cases} -1 & \text{on the line } x_1 = A_1 , \\ -1 & \text{in the region } x_1 > 0,\ x_2 \geq A_2 + x_1^2/2(\alpha-1) , \\ +1 & \text{on the line } x_1 = -A_1 , \\ +1 & \text{in the region } x_1 < 0,\ x_2 \leq -A_2 + x_1^2/2(1-\alpha) , \end{cases}$$

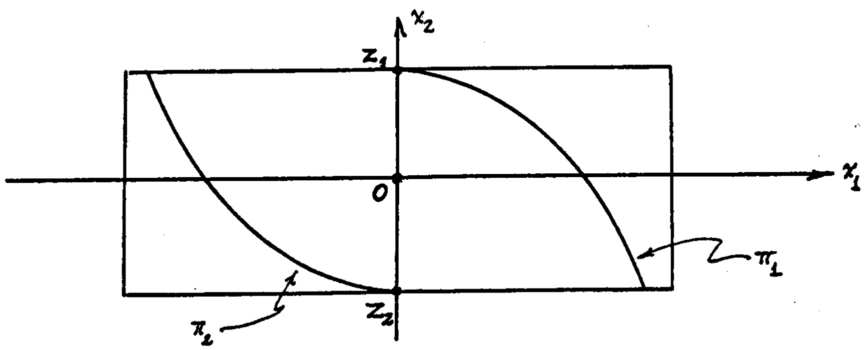
is an upper bound. The reader should have no difficulty in establishing this fact by showing that $Q(u') = K_2$. For the sake of clarity we note that the difference between the core K_2 of the present example and the core established for S_2 in our previous work results from the fact that here we take T to be $+\infty$. The case of a finite T also yields an upper bound; however, we prefer to deal first with the case $T=+\infty$ in order to simplify our proofs.



Case I



Case Ia



Case II

Figure 1

Let

$$G_1 = \left\{(x_1, x_2) \mid 0 < x_1 \leq A_1 , A_2 + x_1^2/2(\alpha-1) < x_2 \leq A_2 \right\}$$

$$G_2 = \left\{(x_1, x_2) \mid -A_1 \leq x_1 < 0 , -A_2 \leq x_2 < -A_2 + x_1^2/2 (1-\alpha) \right\}.$$

Proposition 1. Let u be an upper bound for the system S_2 with respect to the rectangle R_2, the class W and the time $T = +\infty$. Then u controls the core K_2 within itself with respect to any wind $w \in W$.

Proof. Let $p \in K_2$ and let $\Gamma(t, p, u, w,)$ be the (unique) trajectory with wind w and control u such that $\Gamma(o, p, u, w) = p$. Suppose $q = \Gamma(t_1, p, u, w_1) \notin K_2$ for some $t_1 > 0$ and some $w_1 \in W$. Then $q \in G_1$ or $q \in G_2$. Suppose $q \in G_1$. Let

$$w_2 = \begin{cases} w_1 & \text{for all } 0 \leq t \leq t_1 \\ \alpha & \text{for all } t_1 < t. \end{cases}$$

(It is hereby assumed that W is large enough so that $w_2 \in W$). The trajectory $\Gamma(t, p, u, w_2)$, when continued beyond q in the positive direction, must leave the rectangle R_2. But then $p \notin Q(u) = K_2$, contrary to assumption. If $q \in G_2$, modify the definition of w_2, replacing α by $-\alpha$. This completes the proof.

(iii) <u>The Function $u^*(x_1, x_2, x_3)$.</u>

Let

$$R_2 = \left\{ (x_1, x_2, x_3) \mid |x_i| \leq A_i , i = 1, 2; x_3 = 0 \right\} ,$$

$$X_3^+ = \left\{ (x_1, x_2, x_3) \mid 0 < x_3 \leq A_3 \right\} ,$$

$$X_3^- = \left\{ (x_1, x_2, x_3) \mid 0 > x_3 \geq - A_3 \right\} , \quad P_2 = \left\{ x \mid x_1 = -A_1 \right\} \cap R_3$$

$$P_1 = \left\{ (x_1, x_2, x_3) \mid x_1 = A_1 \right\} \cap R_3 ,$$

$$H_i = (\bar{G}_i \cap R_2) \cup P_i , i = 1, 2.$$

$$L_i = R_2 - H_i , i = 1, 2.$$

For any set $A \subset R_2$ we now define

$$A^+ = A \times X_3^+ , \quad A^- = A \times X_3^-$$

Let

$$u^*(x_1,x_2,x_3) = \begin{cases} -1 & \text{in } L_2^+ \cup H_1^- \\ +1 & \text{in } L_1^- \cup H_2^+ \\ \bar{u}(x_1, x_2) & \text{in } R_2. \end{cases}$$

(Figure 2). We shall show that if A_3 is large enough, the function u^* is an upper bound in R_3.

<u>Remark:</u> Throughout the remainder of this proof we shall refer only to Case II (Figure 1). Cases I and Ia require slight modifications of little significance. The inclusion of these cases does not change the nature of the results; it only produces a slight refinement of the inequality constraint affecting A_3.

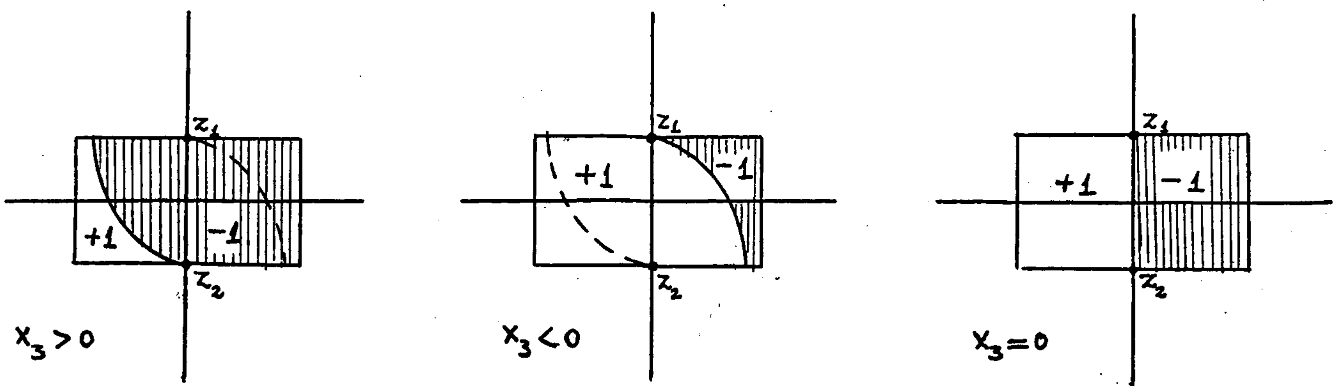


Figure 2

(iv) <u>Some Properties Of $u^*(x_1, x_2, x_3)$.</u>

For each fixed x_3, u^* satisfies (3) and therefore, if P is any point in the set $K_2^+ \cup K_2 \cup K_2^-$ and if $\Gamma(t, p, w, u^*)$ is the trajectory through p with wind w and control u^* satisfying $\Gamma(0, p, w, u^*) = p$, then $\Gamma(t, p, w, u^*)$ cannot leave R_3 through any of the four walls $x_i = \pm A_i$, $i = 1, 2$. It follows that $\Gamma(t, p, w, u^*)$ can leave R_3 only through the wall $x_3 = A_3$ or the wall $x_3 = - A_3$.

We shall denote the winds $w(t) \equiv \alpha$ and $w(t) \equiv - \alpha$ by α and $- \alpha$, respectively.

<u>Lemma 1.</u> Let $A_3 \geq \frac{4}{3} \sqrt{2} \, A_2^{3/2} \, (1-\alpha)^{-1/2}$. Let $p \in K_2^+ \cup K_2 \cup K_2^-$. If $\Gamma(t, p, \alpha, u^*)$ does not leave R_3 through the wall $x_3 = A_3$, then $\Gamma(t, p, w, u^*)$ does not leave R_3 through $x_3 = A_3$ for any $w \in W$.

<u>Proof.</u> Let $p = (x_1^0, x_2^0, x_3^0)$. Suppose $x_3^0 < 0$. Let $\Gamma(t) = \Gamma(t, p, w, u^*)$. In order for $\Gamma(t)$ to reach the plane $x_3 = A_3$ it must first cross the plane $x_3 = 0$ and we may as well choose this point as our point of departure. We may thus assume, without loss of generality, that $x_3^0 \geq 0$.

Suppose next that $x_2^o < 0$. Then at time $t = 0$ the derivative $\dot{x}_3$ along $\Gamma(t)$ is negative and the trajectory $\Gamma(t)$ begins its motion by descending below plane $x_3 = x_3^o$. In order for $\Gamma(t)$ to leave R_3 through the wall $x_3 = A_3$ it would have to return to this plane and cross it. We may therefore assume, without loss of generality that $x_2^o \geq 0$. (This last assumption is particularly desirable when $x_3^o = 0$).

Denote the plane $x_3 = 0$ by π. Let $x_i(t)$, $x_i^1(t)$, $i = 1, 2, 3$, denote the x_i cordinates along $\Gamma(t, p, \alpha, u^*)$ and $\Gamma(t, p, w, u^*)$, respectively. We note first that along π_2 the vector field of the system S_2 is discontinuous. Therefore any trajectory whose projection on π reaches π_2 will proceed in such a manner that its projection remain on π_2 until it (the projection) reaches Z_2. Moreover, except for the point Z_2 itself, throughout the duration of this part of the motion we have

$$\dot{x}_2 = \sqrt{2(1-\alpha)\ (A_2 + x_2)} \quad \text{along } \pi_2$$

independently of the wind.

If $p \in \pi_2$ then $x_2(t) = x_2^1(t)$ until the respective projections reach the point Z_2 (simultaneously) at time t^*.

For $t \geqslant t^*$ we have $x_1(t) = x_1^1(t) = 0$; $x_2(t) = x_2^1(t) = - A_2$ (and therefore $x_3^1(t) = x_3(t) < A_3$) as long as $x_3 > 0$. Hence if $p \in \pi_2$ the proof of the Lemma is complete.

In the set $K_2 - \pi_2$ we have

$$w + u^* = w(t) - 1 \leq \alpha - 1 < 0$$

Therefore, if $p \in K_2 - \pi_2$, we get

$$x_1'(t) \leq x_1(t)$$

so long as $(x_1(t), x_2(t))$ and $(x_1^1(t), x_2^1(t))$ remain in $K_2 - \pi_2$.
Throughout this part of the motion we therefore have

$$x_2^1(t) \leq x_2(t)$$

$$x_3^1(t) \leq x_3(t) \leq A_3.$$

The slope of π_2 is negative throughout. Hence $(x_1^1(t), x_2^1(t))$ reaches π_2 before $(x_1(t), x_2(t))$ does.

(Figure 3).

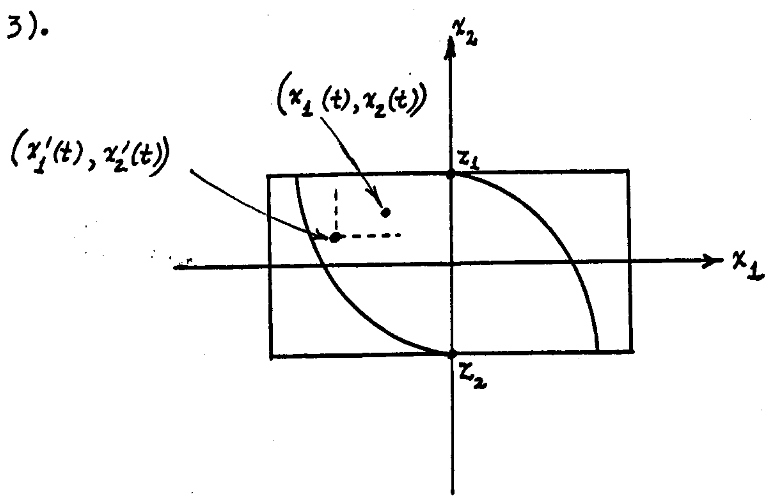


Figure 3

Let the time of impact be t_o. Then $x_1'(t_o) \leq x_1(t_o)$. So long as $x_1'(t) \leq x_1(t)$, $t \geq t_o$, we still have $x_2'(t) \leq x_2(t)$. Let $t_1 > t_o$ be the first (if there are any) time such that $x_1'(t_1) = x_1(t_1)$. Then clearly $x_2'(t_1) \leq x_2(t_1)$, whence $(x_1(t_1), x_2(t_1))$ lies either on π_2 or above it. In the first case we have $(x_1(t_1), x_2(t_1)) = (x_1'(t_1), x_2'(t_1))$, whence $(x_1(t_1), x_2(t_1)) = (x_1'(t_1), x_2'(t_1))$ for all $t \geq t_1$ for which both trajectories are still in the half space $x_3 > 0$. In the latter case we note that $\dot{x}_1(t_1) = \alpha-1 < 0$ and therefore the projection $(x_1(t), x_2(t))$ moves to the left of the line $x_1 = x_1'(t_1)$. At the same time $(x_1'(t), x_2'(t))$ moves to the right of this line. Hence $(x_1(t), x_2(t))$ will intersect π_2 (if at all) at some time $t_2 > t_1$ at a point " above " $(x_1'(t_2), x_2'(t_2))$. therefore, again, so long as both trajectories are still in the half space $x_3 > 0$, we have

$$x_2'(t) \leq x_2(t).$$

Hence, so long as $x_3(t) > 0$ and $x_3'(t) > 0$ we have

$$x_3'(t) \leq x_3(t) \leq A_3. \qquad (4)$$

As long as $x_3'(t) > 0$, the projection (x_1', x_2') proceeds in K_2 towards π_2 and then along π_2 towards Z_2. If the plane π has not yet been reached, the projection is " stopped " at Z_2 and $x_3'(t)$ decreases monotonically at a fixed rate $\dot{x}_3' = - A_3$ towards 0. Eventually, $\Gamma(t, p, w, u^*)$ descends into the lower half space $x_3 < 0$. From that point on the inequalities (4) can no longer be applied for all subsequent time. Surely, however, there is no risk of $\Gamma(t, p, w, u^*)$ emerging from R_3 through the wall $x_3 = A_3$ so long as $x_3'(t) < 0$. Suppose then that $\Gamma(t, p, w, u^*)$ returns to π at some subsequent point P_1. The proof of Lemma 1 would be complete if we knew that $\Gamma(t,\ p_1,\ \alpha,\ u^*)$ does not leave R_3 through the wall $x_3 = A_3$ for any point $p_1 \in K_2$; for then we could simply apply the results obtained above to the point p_1. The fact that $\Gamma(t,\ p_1,\ \alpha,\ u^*)$ does not leave R_3 through the wall $x_3 = A_3$ will be proved in the following Lemma.

Lemma 2. Let $A_3 \geq \frac{4}{3} \sqrt{2}\, A^{3/2}\, (1-\alpha)^{-1/2}$ and let $p \in K_2$. Then $\Gamma(t,\ p, \alpha,\ u^*)$ does not leave R_3 through the wall $x_3 = A_3$.

Proof. Let $p = (x_1^0,\ x_2^0,\ 0)$. As in the proof of Lemma 1 we may assume, without loss of generality, that $x_2^0 \geq 0$.

Suppose first that $p = M_1 = (\sqrt{2(1-\alpha)\, A_2},\ 0\ ,\ 0\ )$.

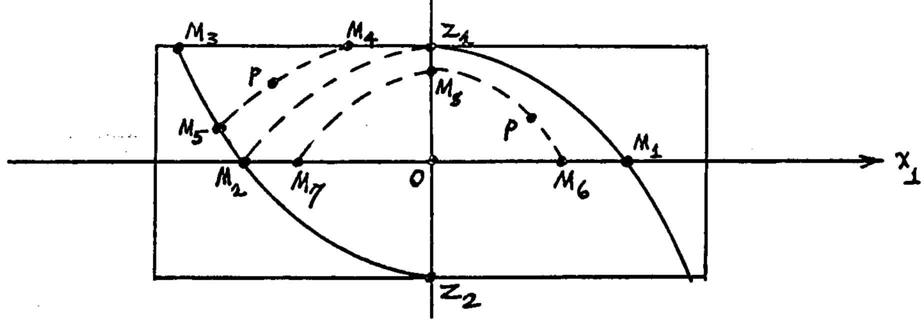


Figure 4

The solution $\Gamma(t,\ M_1,\ \alpha,\ u^*)$ is given by

$$x_1(t) = \sqrt{2(1-\alpha)\, A_2} + (\alpha-1)\, t$$

$$x_2(t) = (\sqrt{2(1-\alpha)\, A_2})t + \frac{(\alpha-1)\, t^2}{2}$$

$$x_3(t) = (\sqrt{2(1-\alpha)\, A_2})\ \frac{t^2}{2} + \frac{(\alpha-1)}{6}\ t^3$$

Starting at M_1 at time $t = 0$, the projection of this trajectory reaches M_2 at time $t_1 = 2\sqrt{\frac{2 A_2}{1-\alpha}}$. The value of x_3 at that time is found to be

$$x_3(t_1) = \frac{4}{3} \sqrt{2} \; A_2^{3/2} \; (1-\alpha)^{-1/2} = \beta$$

Since $A_3 \geq \beta$, and since $x_3(t)$ is monotonically increasing for all $0 \leq t \leq t_1$, we conclude that for all t in this interval $x_3(t) \leq A_3$.

After the projection of $\Gamma(t, M_1, \alpha, u^*)$ reaches M_2 it proceeds along π_2 towards Z_2 and continues to do so as long as $x_3(t) > 0$. If it reaches Z_2 it is " stopped " there until $x_3(t) = 0$, that is, until $\Gamma(t, M_1, \alpha, u^*)$ returns to the plane π at time, say t_2. At that point $x_2(t_2) < 0$ and $\Gamma(t, M_1, w, u^*)$ descends into the half space $x_3 < 0$. It reemerges on the plane π at some later time and some other point $q = (x_1', x_2', 0)$, with $x_2' \geq 0$.

This phenomenon is general: starting with any point $q = (x_1', x_2', 0) \in K_2$, with $x_2' \geq 0$, the trajectory $\Gamma(t, g, \alpha, u^*)$ rises monotonically into the half space $x_3 > 0$ and continues to do so as long as $x_2(t, g, \alpha, u^*) > 0$. Eventually $x_2(t, g, \alpha, u^*)$ becomes < 0, the trajectory descends to π, enters the lower half space whence it returns to π, enters the lower half space whence it returns to π at a later time and at another point.

Clearly then, $\Gamma(t, M_1, \alpha, u^*)$ or for that matter any $\Gamma(t, p, \alpha, u^*)$, can cross the plane $x_3 = A_3$ only during one of these intervals of monotonic increase in x_3. We shall show that the maximum rise in the value of x_3 attainable during one of these intervals, starting from any point in the upper half of K_2, is β. Once this is shown the proof of Lemma 2 would be complete.

Suppose first that p lies in the region $Z_1 M_2 M_5 M_3 Z_1$ (Figure 4). The projection of the trajectory through p would proceed along a parabolic arc until it reaches π_2, then descend to M_2. This is the interval of rise in the value of $x_3(t)$. It is clear that if a given point p is replaced by the point on (the line segment) $M_3 Z_1$ which precedes it on the trajectory through p, the rise in x_3 would be increased (See M_4, Figure 4). Moreover, $\dot{x}_2$, viewed as a function of x_1 is a monotonically increasing function. It is therefore easy to see that

$$\int_{M_3}^{M_2} x_2(t)\,dt \leq \int_{M_4}^{M_5} x_2(t)dt + \int_{M_5}^{M_2} x_2(t)\,dt \leq \int_{Z_1}^{M_2} x_2(t)\,dt.$$

But

$$\int_{Z_1}^{M_2} x_2(t)\,dt < \int_{M_1}^{Z_1} x_2(t)\,dt + \int_{Z_1}^{M_2} x_2(t)\,dt = \beta$$

Hence our assertion is proved when $p \in Z_1 M_2 M_5 M_3 Z_1$.

If $p \in M_1 Z_1 M_2 0 M_1$, one has

$$\int_{P}^{M_8} x_2(t)dt + \int_{M_8}^{M_7} x_2(t)dt \leq \int_{M_6}^{M_8} x_2(t)\,dt + \int_{M_8}^{M_7} x_2(t)dt = 2\int_{M_6}^{M_8} x_2(t)dt.$$

Let $M_6 = (a,0,0)$. The trajectory through M_6 (whose projection on π is given by $M_6 M_8 M_7$) is given by

$$x_1(t) = a + (\alpha-1)\, t$$

$$x_2(t) = a\, t + (\alpha-1)\, \frac{t^2}{2}$$

$$x_3(t) = \frac{at^2}{2} + (\alpha-1)\, \frac{t^3}{6}\, .$$

The value of $x_3(t)$ is a monotonically increasing functions of a.

Hence

$$2\int_{M_6}^{M_8} x_2(t)dt \leq 2\int_{M_1}^{Z_1} x_2(t)dt = \beta.$$

It follows that the maximum rise attainable (in the direction of x_3) by any trajectory originating in K_2^- is β. This completes the proof of Lemma 2.

Lemma 3. Let $A_3 \geq \beta$. Let $u \in \mathcal{U}$, $p \in R_3$. If $p \in Q(u)$ then $\Gamma(t, p, \alpha, u^*)$ cannot leave R_3 through the wall $x_3 = A_3$.

Proof. Let $p = (x_1^o, x_2^o, x_3^o)$. Since $p \in Q(u)$ it follows that $(x_1^o, x_2^o) \in K_2$. Hence $(x,(t,p,\alpha,u^*), x_2(t,p,\alpha,u^*)) \in K_2$ for all $t \geq 0$. Hence if $\Gamma(t, p, \alpha, u^*)$ ever reaches the plane π, it could not (by Lemma 2) subsequently cross the plane $x_3 = A_3$. Thus if $x_3^o \leq 0$ the proof is complete. We therefore restrict our attention to the case when $x_3^o > 0$.

Denote $x_i(t, p, \alpha, u^*)$ and $x_i(t, p, \alpha, u)$, $i = 1, 2, 3$, by $x_i(t)$ and $y_i(t)$, respectively.

Suppose first that $(x_1^o, x_2^o) \in K_2 - \pi_2$. Then as long as both $(x,(t), x_2(t))$ and $(y_1(t), y_2(t))$ belong to $K_2 - \pi_2$ and both $x_3(t)$ and

$y_3(t)$ are positive, we have

$$\alpha + u > \alpha + u^* = \alpha - 1,$$

whence

$$x_1(t) \leq y_1(t)\ ,\ x_2(t) \leq y_2(t),\ x_3\ (t) \leq y_3\ (t) \leq\ A_3 \qquad (5)$$

If $x_3(t)$ vanishes at time t^* before $(x_1(t),\ x_2(t))$ reaches π_2, the proof is complete on account of the fact that $\Gamma(t,\ p,\ \alpha,\ u^*)$ reached π. Otherwise $(x_1(t),\ x_2(t))$ reaches π_2 at some time t_o, at which time $x_3(t_o)$ is still > 0.

The situation is now somewhat analogous to that which obtained in the proof of Lemma 1. We first note that due to the negative slope of π_2 and the inequalities (5) it is necessary that $(x_1(t),\ x_2\ (t))$ reach π_2 before $(y_1\ (t),\ y_2(t))$ does. Moreover

$$x_1(t_o) \leq\ y_1(t_o) \quad , \quad x_2(t_o) \leq y_2(t_o).$$

So long as $x_1(t) \leq y_1(t),\ t \geq t_o$, we still have $x_2(t) \leq y_2(t)$. In order for $y_2(t)$ to "catch up" and become $\leq\ x_2(t)$ the projection $(y_1(t),\ y_2(t))$ must proceed to left of the (moving) line $x_1 = x_1(t)$. However, since $p \in Q\ (u)$ this projection must remain to the right of π_2. This "catching up" process can therefore proceed only through the wedge indicated by diagonal shading in Figure 5. But $(x_1(t),\ x_2(t))$ has the least possible value throughout this wedge. Hence

$$x_2(t) \leq\ y_2(t)$$

as long as $x_3(t)\ > 0$. Hence

$$x_3(t) \leq y_3(t) \leq A_3$$

as long as $x_3(t) > 0$. This completes the proof in the case when $x_3^o > 0$ and $(x_1^o, x_2^o) \in K_2 - \pi_2$.

There remains the case when $x_3^o > 0$ and $(x_1^o, x_2^o) \in \pi_2$. This case is disposed of by using the "wedge argument" given above: In order to surpass $(x_1(t), x_2(t))$ the point $(y_1(t), y_2(t))$ must proceed through the wedge. If at any time t' the latter point coincides with the former it can either continue to move on π_2 (in which case it would coincide with $(x_1(t), x_2(t))$ since the vertical component $\dot{x}_2$ of the vector field is forced on π_2) or move to the right of π_2. In neither case can $y_2(t)$ become $\leq x_2(t)$. This completes the proof of Lemma 3.

The following Lemmas are proved in complete analogy with Lemmas 1,2 and 3.

<u>Lemma 4</u>. let $A_3 \geq \beta$. Let $p \in K_2^+ \cup K_2 \cup K_2^-$. If $\Gamma(t, p, -\alpha, u^*)$ does not leave R_3 through the wall $x_3 = -A_3$, then $\Gamma(t, p, w, u^*)$ does not leave R_3 through $x_3 = -A_3$ for any $w \in W$.

<u>Lemma 5.</u> Let $A_3 \geq \beta$ and let $p \in K_2$. Then $\Gamma(t, p, -\alpha, u^*)$ does not leave R_3 through the wall $x_3 = -A_3$.

<u>Lemma 6.</u> Let $A_3 \geq \beta$. Let $u \in U$, $p \in R_3$. If $p \in Q(u)$ then $\Gamma(t, p, -\alpha, u^*)$ cannot leave R_3 through the wall $x_3 = -A_3$.

We are now in a position to state our main result.

Theorem. Let $\beta = \frac{4}{3}\sqrt{2}\ A_2^{3/2}(1-\alpha)^{-1/2}$. If $A_3 \geq \beta$ then $u^*(x_1, x_2, x_3)$ is an upper bound for S_3 in R_3. Moreover, $K_2 \subset Q(u^*)$.

Proof. Let $u \in U$ and let $p \in Q(u)$. If $p = (x_1^0, x_2^0, x_3^0)$ then $(x_1^0, x_2^0) \in K_2$. It now follows from Lemmas 3, 1, 6 and 4 that $p \in Q(u^*)$. Hence $Q(u) \subset Q(u^*)$ for all $u \in U$.

The fact that $K_2 \subset Q(u^*)$ follows from Lemmas 2, 1, 5 and 4. This completes the proof.

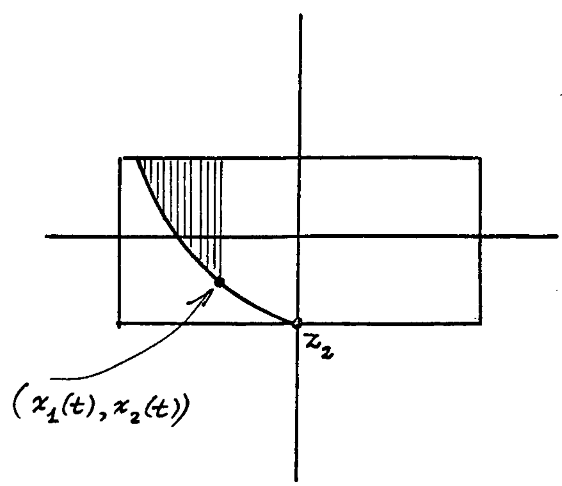


Figure 5

CHAPTER 11

1. THE CORE FOR THE SYSTEM $\ddot{x} = h(\dot{x},x,t) + w + u$.

Eventually we shall restrict h to be independent of t, but We can begin without making this restriction.

We refer to Section 3 of the Eighth Progress Report, in which the system

$$\ddot{x} = h(\dot{x}, x, t) + w + u \tag{1.1}$$

was considered subject to certain constraints. We suppose $h(\dot{x}, x, t)$ to be monotone non-decreasing in x for each fixed $\dot{x}$ and t. The system can also be written in the form

$$\begin{aligned} \dot{x}_1 &= h(x_1,x_2,t) + u(x_1,x_2) + w(t) \\ \dot{x}_2 &= x_1 \end{aligned} \tag{1.2}$$

The constraints to be considered are

$$\begin{aligned} C_1' &: \quad |x_i(t)| \leq A_i \ , \ i = 1, 2, \ \text{for al} \ \ t \in [0,T) \\ C_2' &: \quad |w(t)| \leq \alpha < 1, \quad w \in W, \\ C_3' &: \quad |u(x_1, x_2)| \leq 1, \quad u \in \mathcal{U}. \end{aligned} \tag{1.3}$$

We hereby use A_1 instead of A and A_2 instead of B as in the Eighth Progress Report.

We suppose we have a non-negative continuous function $\lambda(x_2)$ defined for $|x_2| \leq A_2$ and such that when the point (x_1, x_2) is initially within the set J defined by $|x_1| \leq \lambda(x_2)$ it remains trapped within this set for all time during the motion defined by (1.2) with $u(x_1,x_2)$ replaced by $\bar{u}(x_1,x_2)$, where $\bar{u}(x_1,x_2) = -\operatorname{sgn} x_1$ for (x_1,x_2) outside the set J. This supposition is supposed to hold for any $w \in W$ and no matter how $\bar{u}$ is defined inside the set J, so long as it is such that $\bar{u} \in \mathcal{U}$.

The first main result of Section 3 of the Eighth Progress Report was to the effect that such a function $\bar{u} \in \mathcal{U}$ was an upper bound for the rectangle R_2 defined by C_1' and for any time interval $[0,T]$.

The second main result of this section of the Eighth Progress Report was concerned with the systems

$$\sum\nolimits_1 : \quad \begin{aligned} \dot{x}_1 &= h(x_1, x_2, t) - 1 + \alpha \\ \dot{x}_2 &= x_1 \end{aligned}$$

(cf. equation (3.5) of the Eighth Progress Report where the system was written as a single equation of the second order $\ddot{\xi} = h(\xi, \dot{\xi}, t) - 1 + \alpha$) and

$$\sum\nolimits_2 : \quad \begin{aligned} \dot{x}_1 &= h(x_1, x_2, t) + 1 - \alpha \\ \dot{x}_2 &= x_1 . \end{aligned}$$

The following result was implicitly developed although it was not explicitly stated in the Eighth Progress Report:

Theorem 1.1 The core of the system (1.2), that is $Q(\bar{u})$, is $J \cup L_1 \cup L_2$, where L_i consists of those points $(\dot{x}_o, x_o)$ with

$$\lambda(x_o) < (-1)^{i-1} \dot{x}_o \leq A_1 \text{ and } |x_o| \leq A_2$$

such that the trajectory of $\sum_i$, with initial point $(\dot{x}_o, x_o)$, either enters the set J (where it would, of course, be trapped) at some time $t \leqq T$ without previously having left the rectangle R_2, or else it leaves the rectangle R_2 only at some time $t \geqq T$.

If $T = \infty$, the second alternative in the definition of L_i may be omitted, and the picture becomes correspondingly simpler. As in the Twelfth Progress Report, we restrict attention to this case.

A further simplifying assumption will also be made to the effect that throughout R_2, $h(x_1, x_2, t) - 1 + \alpha$ is negative while $h(x_1, x_2, t) + 1 - \alpha$ is positive. This is, of course, true in the particular case $h(x_1, x_2, t) \equiv 0$ studied in the Twelfth Progress Report and is true also in the slightly more general case in which h (being continuous) is independent of t and vanishes at $x_1 = x_2 = 0$, and in which A_1 and A_2 are sufficiently small. Physically it means that a sufficiently powerful control is always available to prevail against the combined effects of the wind and the natural characteristics of the system embodied in the function h.

As a result of these assumptions, it is seen that the trajectories of $\sum_1$ have positive slopes when $x_1 < 0$ and negative slopes when $x_1 > 0$. The opposite is true of the trajectories of $\sum_2$. The trajectories of both $\sum_1$ and $\sum_2$ cut the x_2 - axis at right angles. Thus trajectories of $\sum_1$ and $\sum_2$ follow curves similar to parabolas, symmetric in the x_2 - axis, which played such an important role in the Twelfth Progress Report. We therefore call these curves pseudo-parabolas and arcs of these curves are called pseudo-parabolic arcs. Although these pseudo-parabolas have " pseudo-vertices " on the x_2 - axis, where they intersect this axis at right angles, they are not necessarily symmetric in this axis, unless $h(-x_1, x_2, -t) \equiv h(x_1, x_2, t)$, an identity which we do not assume.

If $T = \infty$ and if h does not depend explicitly on t, as we henceforth assume, it is easily seen from Theorem 1.1 that the core K_2, illustrated in Figure 1, of the system (1.2) is bounded by the following:

A pseudo-parabolic arc π_1, associated with the system $\sum_1$, reaching from the point $Z_1 = (0, A_2)$ to a point N_1 on the boundary of R_2 with positive abscissa x_1. (N_1 may be on $x_1 = A_1$ or on $x_2 = - A_2$).

The part of the right-hand boundary of R_2 extending from the last mentioned point N_1 to the point $Z_2 = (0, - A_2)$.

A pseudo-parabolic arc π_2, associated with the system $\sum_2$, reaching from the point $Z_2 = (0, - A_2)$ to a point N_2 on the boundary of R_2 with negative abscissa x_1. (N_2 may be on $x_1 = - A_1$ or on $x_2 = + A_2$-.

The part of the left - hand boundary of R_2 extending from the last mentioned point N_2 to the point $Z_1 = (0, A_2)$.

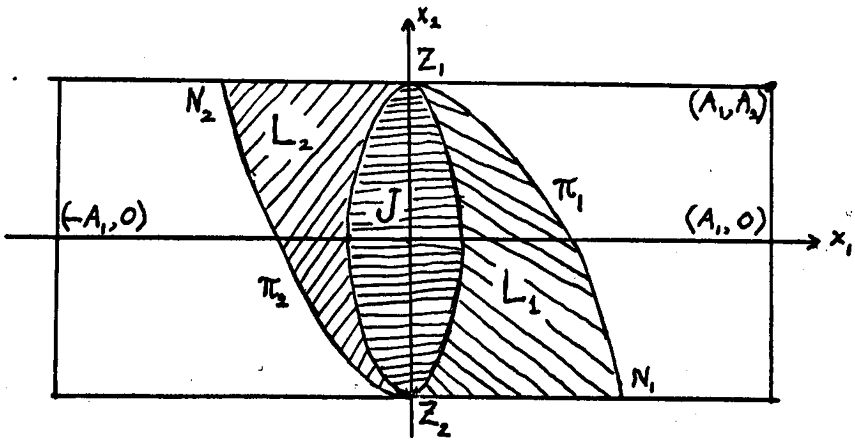


Figure 1

In following π_1 from the point N_1 we first reach the boundary of the set J either at the point Z_1, or at an earlier point P. But since π_1 is part of the boundary of K_2, it is easily seen that it can not enter into the interior of J. And from the characteristic property that J has of trapping within its interior, when $u = \bar{u}$, anything that ever gets there including some of the points near P, it is not hard to prove that π_1 must coincide with the boundary of J from P until it reaches its terminal at Z_1.

We have already restricted ourselves to taking $T = \infty$. The class W of functions defined on $[0, \infty)$ is, as previously, assumed to be bounded in absolute value by the positive number $\alpha < 1$. We now suppose furthermore that W is wide enough so that, for any $t_1 \in [0, \infty)$, whenever $w_1 \in W$, and $w_2 \in W$, the function w^* and w, defined as follows also belong to W.

$$\begin{aligned} w^*(t) &= w_1(t_1 + t) \quad \text{for} \quad t \geq 0 \\ w(t) &= w_1(t) \quad \text{for} \quad 0 \leq t < t_1 \\ w(t) &= w_2(t) \quad \text{for} \quad t \geq t_1 \end{aligned} \tag{1.4}$$

The following theorem which generalizes Proposition 1 of the Twelfth Progress Report gives a further indication of the advantage of taking $T = \infty$.

Theorem 1.2 Let u be an upper bound for the system (1.2), assuming that $h(x_1, x_2, t) = h(x_1, x_2)$ is independent of t. The upper bound u is assumed with respect to the rectangle R_2 and the class W with the properties specified above. Then u controls the core K_2 within itself with respect to any wind $w \in W$.

Proof. If $p \in R_2$, let $\Gamma(t, p, u, w)$ be the unique trajectory with the wind w and control u such that $\Gamma(0, p, u, w) = p$. Suppose now that $p \in K_2$ but that $q = \Gamma(t_1, p, u, w_1) \notin K_2$ for some $t_1 > 0$ and some $w_1 \in W$. Then q is still in R_2, since $p \in Q(u) = K_2$. But since $q \notin K_2$, there will exist a $w_2 \in W$, such that the trajectory $\Gamma(t, q, u, w_2)$ will leave the rectangle. We now define $w(t)$ in terms of $w_1(t)$ and $w_2(t)$ in accordance with (1.4). Then the trajectory $\Gamma(t,p,u,w)$ follows the same path as $\Gamma(t, q, u, w_2)$ for $t > t_1$ and will hence leave the rectangle R_2. But this is absurd since $w \in W$ and u controls the system within R_2 relative to the whole class W. Hence there can exist no point $q = \Gamma(t, p, u, w,) \notin K_2$ (if $p \in K_2$), as we desired to prove.

2. ON THE EXISTENCE OF UPPER BOUNDS FOR THE SYSTEM

$\dddot{x} = h(\ddot{x}, \dot{x}) + w(t) + u(\ddot{x}, \dot{x}, x)$

By setting $\ddot{x} = x_1$, $\dot{x} = x_2$, $x = x_3$ we write the system in the form

$$\begin{aligned} \dot{x}_1 &= h(x_1, x_2) + u(x_1, x_2, x_3) + w(t) \\ \dot{x}_2 &= x_1 \\ \dot{x}_3 &= x_2 \end{aligned} \qquad (2.1)$$

Our immediate purpose is to define a function $u^*(x_1, x_2, x_3)$ which will later be proved to be an upper bound. The definition is analogous to the one that should have been given on p. 6 of the Twelfth Progress Report. It will be observed, however, that this definition (though its intention is fairly clear) contains a couple of errors. The definition is also slightly changed in some other unessential respects. Thus the following is a modification, generalization, and correction of the one which appeared in the Twelfth Progress Report.

Let

$$R_3 = \{ (x_1, x_2, x_3) \mid |x_i| \leq A_i, \ i = 1, 2, 3 \}$$

$$R_2 = \{ (x_1, x_2, 0) \mid |x_i| \leq A_i, \ i = 1, 2 \}$$

$$X_3^+ = \{ (0, 0, x_3) \mid 0 < x_3 \leq A_3 \}$$

$$X_3^- = \{ (0, 0, x_3) \mid 0 > x_3 \geq -A_3 \}$$

$$P_i = \{ (-(-1)^i A_1, x_2, 0) \} \cap R_3, \ i = 1, 2$$

$$G_i = \{ (x_1, x_2, 0) \mid -(-1)^i g_i(x_2) < -(-1)^i x_1 \} \cap R_2$$

where $x_1 = g_i(x_2)$ is an equation for the half of the pseudo-parabola belonging to $\sum_i$ and passing through the point Z_i for which $-(-1)^i x_1 > 0$. See the previous section for the meaning of $\sum_i$ and Z_i. Let

$$H_i = \bar{G}_i \cup P_i, \qquad i = 1, 2.$$

$$L_i = R_2 - H_i, \qquad i = 1, 2.$$

For any set $A \subset R_2$ we now define

$$A^+ = A \times X_3^+, \quad A^- = A \times X_3^-$$

Let

$$u^*(x_1, x_2, x_3) = \begin{cases} -1 & \text{in } L_2^+ \cup H_1^- \\ +1 & \text{in } L_2^- \cup H_2^+ \\ -1 & \text{in } R_2 \text{ if } x_1 > 0 \\ +1 & \text{in } R_2 \text{ if } x_1 < 0 \end{cases}$$

See Figure 2 of the Twelfth Progress Report. We shall show that, if A_3 is large enough, the function u^* is an upper bound in R_3.

For each fixed x_3, $u^*(x_1, x_2, x_3)$ is seen to be an upper bound for the system

$$\dot{x}_1 = h(x_1, x_2) + u + w \tag{2.2}$$

$$\dot{x}_2 = x_1.$$

Therefore, if p is any point in the set $K_2^+ \cup K_2 \cup K_2^-$ and if $\Gamma(t, p, w, u^*)$ is the trajectory through p with wind w and control U^* satisfying $\Gamma(0, p, w, u^*) = p$, then $\Gamma(t, p, w, u^*)$ cannot leave R_3 through any of the four walls $x_i = \pm A_i$, $i = 2$. This is a consequence of Theorem 1.2.

It follows that $\Gamma(t, p, w, u^*)$ can leave R_3 only through the wall $x_3 = A_3$ or the wall $x_3 = - A_3$.

We shall denote the winds $w(t) \equiv \alpha$ and $w(t) \equiv - \alpha$ by α and $-\alpha$, respectively.

Let $x_k^{(i)}(t)$ be the solution of the system $\sum_1$ $(i = 1, 2; k = 1, 2)$ satisfying the initial conditions $x_k^{(i)}(0) = \delta_2 k\,(-1)^{i+1} A_2$ and let it intersect the x_1 - axis at $t = \tau_1^{(i)} < 0$ and at $t = \tau_2^{(i)} > 0$.

Then we set $\beta = \max_i \left(\left| \int_{\tau_1^{(i)}}^{\tau_2^{(i)}} x_2^{(i)}(t)\, dt \right| \right)$

In the special case considered in the Twelfth Progress Report, the solutions $x_k^{(1)}(t)$ and $x_k^{(2)}(t)$ of systems $\sum_1$ and $\sum_2$ respectively intersect each other at two points M_1 and M_2 on the x_1 axis. See Figure 4 of that report. This need not be the case for the present more general system. If, however, A_1 is large enough relative to a fixed A_2, it is geometrically obvious that these two pseudo-parabolas intersect each other at two points within R_2, one on each side of the x_2 - axis. We restrict ourselves to this case. The other cases are not conceptually more difficult but they do present more annoying details. Let the point of intersection on the right side of the x_2 - axis have the ordinate $x_2 = a_1$ and the one on the left side have the ordinate $x_2 = a_2$. These points lie on the pseudo-parabolic arcs π_1 and π_2 respectively. We again refer to the equation for the pseudo-parabolic arc π_i, given in the form $x_1 = g_i(x_2)$ for $|x_2| \leq A_2$.

If $(-1)^i a_i \leq 0$, define $\gamma_i = 0$. If, however, $(-1)^i a_i > 0_1$, we let

$$\gamma_i = \int_{a_i}^{0} \frac{dx}{g_i(x)}$$

In this latter case, the significance of γ_i is that it gives the time it takes for a point (x_1, x_2) to pass from the intersection point $(g_i(a_i), a_i)$ on

π_i to the x_1 - axis whenever the control is such that it compels the point to move along the arc π_i. The result is, of course, obtained by integrating $\dot{x}_2 = x_1 = g_i\,(x_2)$.

Moreover, if we let $x_2(t)$ be the solution of the differential equation $\dot{x}_2 = g_i(x_2)$ and the initial condition $x_2(0) = a_i$, then

$$\lambda_i = \int_0^{\gamma} |\; x_2(t)\;|\; dt$$

is the increase in $(-1)^i\, x_3$ (under the assumption $\dot{x}_3 = x_2$) as the projection of the point (x_1, x_2, x_3) on the plane $x_3 = 0$ moves along π_i from $(g_i, (a_i), a_i)$ to the x_2 - axis as previously described, in the case that $(-1)^i\, a_i > 0$. Otherwise, of course, $\lambda_i = 0$.

We now take $\lambda = \max\,(\lambda_1, \lambda_2)$ and state our main result to the effect that u^* is an upper bound for the system (2.1) with respect to R_3 provided that $A_3 \geq \beta + \lambda$.

We have not had time to write up the complete details of the proof of this result. We will do this for the next Progress Report. Actually the proof for the special case treated in the Twelfth Progress Report needs only to be modified so as to avoid the explicit integration of the systems $\sum_1$ and $\sum_2$. It turns out that this can be done by appealing to Theorem 1.2 of the Eighth Progress Report and the following simple corollary thereof:

Lemma 2.1 Let $f(x_1, x_2)$ be a monotonic non-decreasing function of x_2 for each fixed x_1. Let $f(x_1, x_2) \leq -\mu$, where μ is positive. Let $x_1(t,a)$, $x_2(t,a)$, $x_3(t,a)$ satisfy the system of differential equations $\dot{x}_1 = f(x_1, x_2)$, $\dot{x}_2 = x_1$, $\dot{x}_3 = x_2$ and the initial conditions, $x_1(0,a) = a$, $x_2(0,a) = x_3(0,a) = 0$, $a > 0$. Then there exists a positive number $T = T\,(a)$

such that $x_2(t,a) > 0$ for $0 < t < T$ and such that $x_2(T) = 0$. Moreover $T(a)$ and $x_3(T(a), a)$ are both monotonic increasing functions of a.

<u>Proof</u> Since $x_2(0) = 0$, and $\dot{x}_2(0) = x_1(0) = a > 0$, $x_2(t) > 0$ for sufficiently small positive values of t. Now evidently $x_1(t) \leq a - \mu t$. Hence $x_2(t) \leq at - \frac{1}{2}\mu t^2$, which is negative for sufficiently large values of t. Hence we let T be the (first) point where $x_2(t)$ ceases to be positive. This proves the first statement of the Lemma.

Let $a < a^1$, $x_i(t) = x_i(t,a)$ and $x_i^1(t) = x_i(t,a^1)$, $i = 1, 2, 3$. Then $\ddot{x}_2 = f[\dot{x}_2(t), x_2(t)]$ and $\ddot{x}_2^1 = f[\dot{x}_2^1(t), x_2^1(t)]$ and we also have the initial condidtions, $x_2(0) = x_2^1(0) = 0$.

$$\dot{x}_2(0) = x_2(0) = a < a^1 = x_1^1 = \dot{x}_2^1(0).$$

It follows from Theorem 1.2 of the Eighth Progress Report that

$$x_2(t) \leq x_2^1(t) \text{ and that } x_1(t) = \dot{x}_2(t) \leq \dot{x}_2^1(t) = x_1^1(t)$$

for $t > 0$. Since $x_2(t) > 0$ on the interval $(0,T(a))$, it follows that $x_2^1(t)$ is also positive on this interval. Hence $T(a) \leq T(a^1)$. Finally

$$x_3[T(a)] = \int_0^{T(a)} x_2(t)\,dt \leq \int_0^{T(a)} x_2^1(t)\,dt \leq \int_0^{T(a^1)} x_2^1(t)\,dt = x_3^1[T(a^1)].$$

This proves the last statement of the Lemma.

3. ON THE EXISTENCE OF UPPER BOUNDS: MODIFICATION OF A PREVIOUS EXAMPLE.

(i) Introduction.

As in our Twelfth Progress Report (T P R) we consider the third order linear controllable system with three zero eigenvalues, namely

$$S_3 : \begin{cases} \dot{x}_1 = w(t) + u\,(x_1, x_2, x_3) \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_2 \,. \end{cases}$$

We are interested in solutions to this system subject to the constraints C_1, C_2, C_3 of T P R. It is assumed that u and w belong, respectively, to classes of function $\mathcal{U}$ and W which assure the existence and uniqueness of solutions for the system S_3. In T P R we showed that S_3 admits an upper bound u^* within the parallelopiped R_3 defined by C_1. In the present section we shall exhibit the existence of still another, more sophisticated, upper bound v^* which, apart from being an upper bound in R_3, has certain additional desirable properties within the core of R_3.

Throughout this section we shall refer freely to the Figures, definitions and notations of T P R.

(ii) The Function $v^*(x_1, x_2, x_3)$

Let R_2, X_3^+, X_3^-, P_1, P_2 be as in Section 2 of the present report. Let π_3, π_4 be the two parabolic arcs

$$\pi_3: \quad x_2 = x_1^2/2\,(\alpha - 1), \; x_1 \geq 0$$

$$\pi_4: \quad x_2 = x_1^2/2\,(1 - \alpha), \; x_1 \leq 0,$$

(Figure 2). Let M_1 and M_2 be the regions shaded, respectively, in Figure 2 by horizontal and vertical lines. Let G_1, G_2

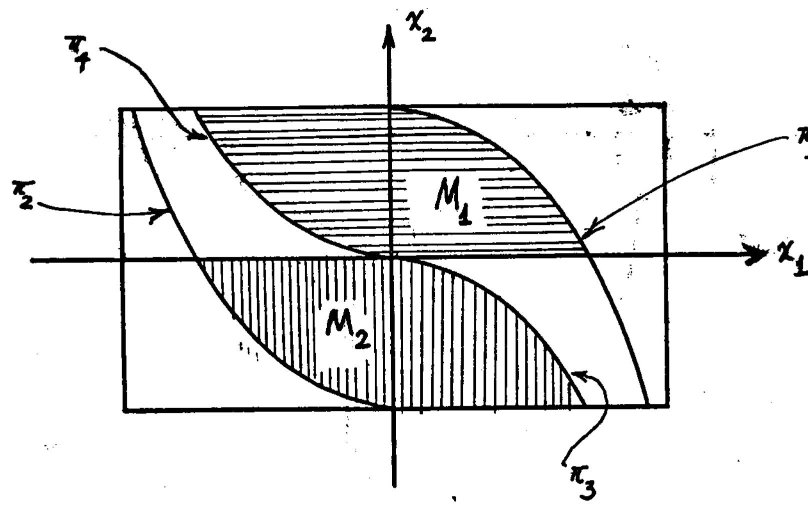


Figure 2

be as in T P R. (These are NOT the same as the sets G_i of Section 2 of the present report). Let

$$J_i = (\bar{G}_i \cap R_2) \cup P_i \cup \bar{M}_i \ , \qquad i = 1, 2.$$

$$L_i = R_2 - J_i \ , \quad i = 1, 2.$$

For any set $A \subset R_2$ we now define

$$A^+ = A \times X_3^+ \ , \quad A^- = A \times X_3^-$$

Let

$$v^*(x_1,x_2,x_3) = \begin{cases} -1 \text{ in } L_2^+ \cup J_1^- \\ +1 \text{ in } L_1^- \cup J_2^+ \\ u^*(x_1, x_2, x_3) \text{ in } R_2, \end{cases}$$

where u^* is as defined in T P R (Figure 3).

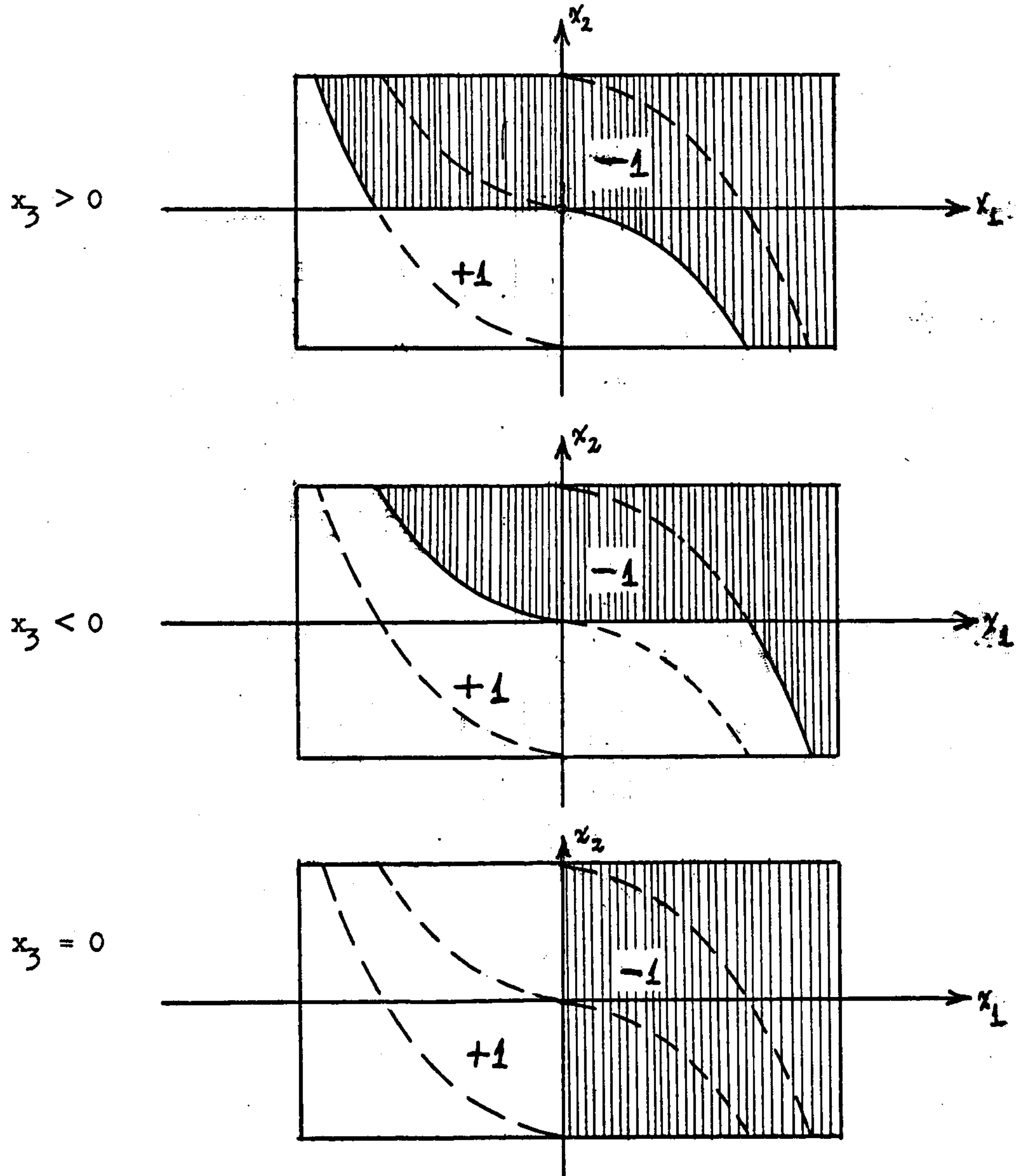


Figure 3

Theorem 3.1. Let $\beta = \frac{4}{3}\sqrt{2}\ A_2^{3/2}\ (1-\alpha)^{-1/2}$. If $A_3 \geq \beta$ then $v^*(x_1, x_2, x_3)$ is an upper bound for S_3 in R_3. Moreover, $K_2 \subset Q\ (v^*)$, where K_2 is as in T P R.

The proof proceeds along lines which are fairly analogous to the proof in T P R. The reader may therefore wish to review the latter proof before proceeding with the present one.

(iii) Proof of Theorem 3.1

Lemma 3.1 Let $A_3 \geq \beta$ and let $p \in K_2$. Then $\Gamma(t, p, \alpha, v^*)$ does not leave R_3 through the wall $x_3 = A_3$.

Proof. Let $p = (x_1^o, x_2^o, 0)$. As in the proof of Lemma 2, T P R, we may assume, without loss of generality, that $x_2^o \geq 0$. Starting with any point $p = (x_1^o, x_2^o, 0) \in K_2$, with $x_2^o \geq 0$, the trajectory $\Gamma(t, p, \alpha, v^*)$ rises monotonically into the half space $x_3 > 0$ and continues to do so as long as $x_2\ (t, p, \alpha, v^*) > 0$. Eventually, $x_2\ (t, p, \alpha, v^*)$ becomes < 0, at which time $x_3\ (t, p, \alpha, v^*)$ begins to decrease monotonically. Moreover, once $x_2\ (t, p, \alpha, v^*)$ becomes < 0 it remains ≤ 0 as long as $x_3(t, p, \alpha, v^*)$ is still ≥ 0. Thus, once $x_2\ (t, p, \alpha, v^*)$ becomes < 0, the trajectory $\Gamma\ (t, p, \alpha, v^*)$ begins a monotonic descent and it cannot start a new ascent before first returning to the plane π(see T P R). Finally, if a subsequent ascent of $\Gamma(t, p, \alpha, v^*)$ towards the wall $x_3 = A_3$ does take place, then it can only caome after $\Gamma(t, p, \alpha, v^*)$ first passed through some point $q = (x_1^1, x_2^1, 0)$ with $x_2^1 \geq 0$. It is then sufficient to study $\Gamma(t, q, \alpha, v^*)$, $t \geq 0$, directly, without reference to its "past" history. In other words, for the purpose of this Lemma, it is sufficient to take $p = (x_1^o, x_2^o, 0) \in K_2$ with $x_2^o \geq 0$ and consider the arc of $\Gamma(t, p, \alpha, v^*)$ corresponding to the first maximal interval, say $0 \leq t \leq t_1$, during which $\Gamma(t, p, \alpha, v^*)$ is ascending. However, for all $0 \leq t \leq t_1$ we have $x_i(t, p, \alpha, v^*) \geq 0$, $i = 2, 3$, and therefore

$$v^*(x_1(t, p, \alpha, v^*), \ldots, x_3(t, p, \alpha, v^*)) = u^*(x_1,(t, p, \alpha, u^*), \ldots, x_3(t, p, \alpha, u^*))$$

whence

$$\Gamma(t, p, \alpha, v^*) = \Gamma(t, p, \alpha, u^*)$$

for all $0 \leq t \leq t_1$. This, together with Lemma 2 of T P R, completes the proof of Lemma 3.1.

Lemma 3.2 Let $A_3 \geq \beta$. Let $p \in K_2^+ \cup K_2 \cup K_2^-$. If $\Gamma(t, p, \alpha, v^*)$ does not leave R_3 through the wall $x_3 = A_3$, then $\Gamma(t, p, w, v^*)$ does not leave R_3 through $x_3 = A_3$ for any $w \in W$.

Proof. Let $p = (x_1^o, x_2^o, x_3^o)$. As in Lemma 1 of T P R we may assume, without loss of generality, that $x_3^o \geq 0$, $x_2^o \geq 0$. Let $x_i(t)$, $x_i^1(t)$, $i = 1, 2, 3$, denote, respectively, the x_i coordinates of $\Gamma(t, p, \alpha, v^*)$ and $\Gamma(t, p, w, v^*)$. Let $[0, t_1]$ be the maximal interval throughout which both $x_2(t) \geq 0$ and $x_2^1(t) \geq 0$. Then for all $t \in [0, t_1]$ we have $x_i(t) \geq 0$ and $x_i^1(t) \geq 0$, $i = 2, 3$,

whence

$$\Gamma(t, p, \alpha, v^*) = \Gamma(t, p, \alpha, u^*)$$

$$\Gamma(t, p, w, v^*) = \Gamma(t, p, w, u^*)$$

where u^* is as in T P R. Hence it follows from the proof of Lemma 1 in T P R that

$$x_3^1(t) \leq x_3(t) \leq A_3$$

for all $0 \leq t \leq t_1$. Thus $\Gamma(t, p, w, v^*)$ cannot reach the wall $x_3 = A_3$ during the interval $[0, t_1]$. A subsequent ascent towards this wall would have to start from some point $q = (x_1^1, x_2^1, 0) \in K_2$ with $x_2^1 \geq 0$. But then we may choose q as our starting point and argue as above.
This completes the proof.

<u>Lemma 3.3.</u> Let $A_3 \geq \beta$. Let $u \in U$, $p \in R_3$. If $p \in Q(u)$ then $\Gamma(t, p, \alpha, v^*)$ cannot leave R_3 through the wall $x_3 = A_3$.

<u>Proof.</u> Let $p = (x_1^o, x_2^o, x_3^o)$. Since $p \in Q(u)$ it follows that $(x_1^o, x_2^o) \in K_2$. Hence $(x_1(t, p, \alpha, v^*), x_2(t, p, \alpha, v^*)) \in K_2$ for all $t \geq 0$. Hence if $\Gamma(t, p, \alpha, v^*)$ ever reaches the plane π, it could not (by Lemma 3.1) subsequently cross the plane $x_3 = A_3$. Thus if $x_3^o \leq 0$ the proof is complete. We therefore restrict our attention to the case when $x_3^o > 0$.

Denote $x_i(t, p, \alpha, v^*)$ and $x_i(t, p, \alpha, u)$, $i = 1, 2, 3$ by $x_i(t)$ and $y_i(t)$, respectively.

If $(x_1^o, x_2^o) \in \bar{M}_2$ then $x_2(t, p, \alpha, v^*)$ remains ≤ 0 as long as $x_3(t) \geq 0$. Hence $\Gamma(t, p, \alpha, v^*)$ cannot start an ascent towards the plane $x_3 = A_3$ until and unless it first reaches the plane π. This completes the proof in the case when $(x_1^o, x_2^o) \in \bar{M}_2$. We may therefore assume that $x_3^o \geq 0$ and $(x_1^o, x_2^o) \in K_2 - \bar{M}_2$. However, in $K_2 - \bar{M}_2$ the function v^* is identical with the function u^* of T P R and the proof of Lemma 3 in T P R applies. Therefore, as long as $(x_1(t), x_2(t))$

and $(y_1(t), y_2(t))$ are both in $K_2 - \bar{M}_2$ we have

$$x_2(t) \leq y_2(t)$$

and

$$x_3(t) \leq y_3(t) \leq A_3.$$

It follows that $(x_1(t), x_2(t))$ reaches the set $\bar{M}_2$ before $y_2(t)$ becomes negative. But once $(x_1(t), x_2(t))$ reaches the set $\bar{M}_2$ it cannot leave it until and unless $\Gamma(t, p, \alpha, v^*)$ reaches the plane π. This completes the proof of Lemma 3.3.

The following Lemmas are proved in complete analogy with Lemmas 3.1, 3.2 and 3.3.

Lemma 3.4 Let $A_3 \geq \beta$ and let $p \in K_2$. Then $\Gamma(t, p, -\alpha, v^*)$ does not leave R_3 through the wall $x_3 = - A_3$.

Lemma 3.5. Let $A_3 \geq \beta$. Let $p \in K_2^+ \cup K_2 \cup K_2^-$. If $\Gamma(t, p, -\alpha, v^*)$ does not leave R_3 through the wall $x_3 = - A_3$, then $\Gamma(t, p, w, v^*)$ does not leave R_3. through $x_3 = - A_3$ for any $w \in W$.

Lemma 3.6. Let $A_3 \geq \beta$. Let $u \in U$, $p \in R_3$. If $p \in Q(u)$ then $\Gamma(t, p, -\alpha, v^*)$ cannot leave R_3 through the wall $x_3 = - A_3$.

The reader will easily convince himself that the proof of Theorem 3.1 is contained in Lemmas 3.1 - 3.6.

The function v^* is clearly superior to the function u^* of T P R inasmuch as it eliminates the worst oscillatory features of a trajectory moving under the influence of u^*. A more detailed discussion of the exact nature of v^* does not seem warranted at this time.

CHAPTER 12

1. PROOF OF THE EXISTENCE OF UPPER BOUNDS FOR THE SYSTEM.

$$\dddot{x} = h(\ddot{x}, \dot{x}) + w(t) + u(\ddot{x}, \dot{x}, x)$$

The conditions under which the system referred to in the section title has an upper bound were stated in Section 2 of the Thirteenth Progress Report. In the present section we present the details of the proof that an upper bound actually exists under the stated conditions. The proof depends upon six lemmas, which are discussed in the following pages, after which the main theorem becomes almost self evident.

The notation of the section is intended to be the same as the notation of Sections 1 and 2 of the Thirteenth Progress Report. It may be noted, however, that the λ of this section is the same as the λ of Section 2 of the Thirteenth Progress Report, but has nothing to do with λ of Section 1 of the Thirteenth Progress Report.

Lemma 1. Let $A_3 \geq \beta + \lambda$ and let $p \in K_2$. Then $\Gamma(t, p, \alpha, u^*)$ does not leave R_3 through the wall $x_3 = A_3$.

Proof. Let $p = (x_1^0, x_2^0, 0)$. If $x_2^0 < 0$, we see, from the fact that $\dot{x}_3 = x_2$, that the system begins its motion by descending below the plane $x_3 = 0$. It can never reach the plane $x_3 = + A_3$ until it first returns to the plane $x_3 = 0$, and this must happen (if it happens at all) at a point where $x_2 \geq 0$. Hence, without loss of generality we may assume in the first place that $x_2^0 \geq 0$.

Suppose first that $p = M_1$, the point where the pseudo-parabolic arc π_1 intersects the x_1 - axis.

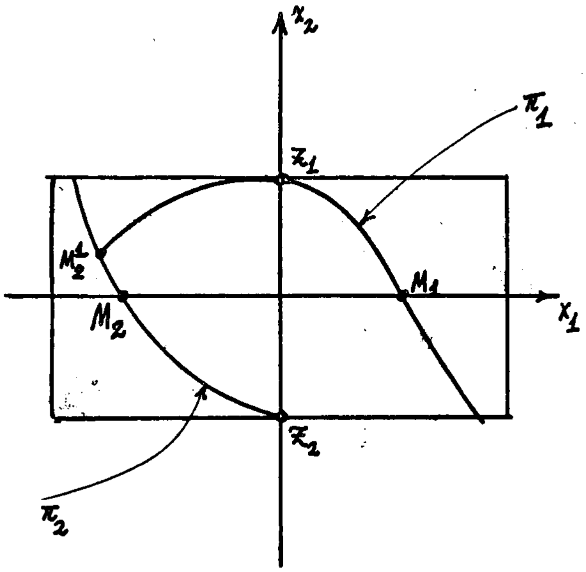


Figure 1a

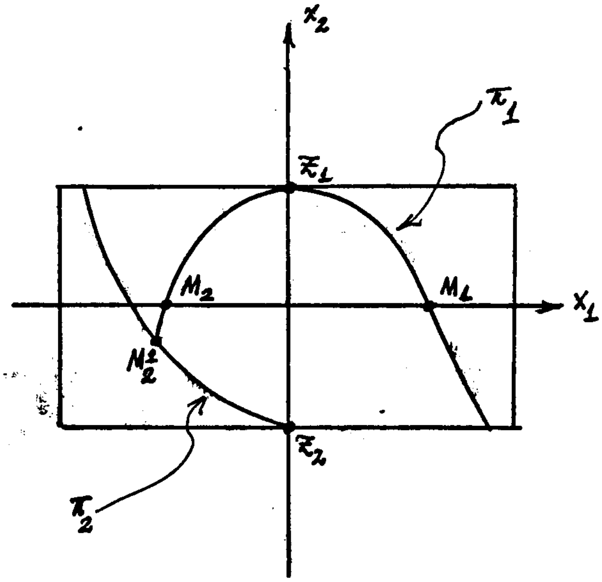


Figure 1b

The solution $\Gamma\,(t, M_1, \alpha, u^*)$ in terms of the $x_k^{(i)}\,(t)$ introduced in the Thirteenth Progress Report, bottom of p. 8, may be written as follows

$$x_1(t) = x_1^{(1)}\,(t + \tau_1^{(1)})$$

$$x_2(t) = x_2^{(1)}\,(t + \tau_1^{(1)})$$

$$x_3(t) = \int_0^t x_2\,(s)ds = \int_0^t x_2^{(1)}(s+\tau_1^{(1)})ds = \int_{\tau_1^{(1)}}^{\tau_1^{(1)}+t} x_2^{(1)}\,(\sigma)\,d\sigma.$$

Starting at M_1 the projection of this trajectory reaches Z_1 at time $-\tau_1^{(1)}$, by definition of $\tau_1^{(1)}$. The projection of the trajectory may then return to the x_1 - axis at a point M_2 before it meets the pseudo-parabolic arc π_2 at M_2^1 as shown in Figure 1a or it may meet the arc π_2 at M_2^1 and then follow along π_2 until it reaches the x_1- axis at the point M_2 where π_2 crosses the x_1- axis, as shown in Figure 1b.

In the case illustrated by Figure 1a, the projection of the trajectory proceeds from Z_1 to M_2 after the lapse of a time $\tau_2^{(1)}$. Hence it passes from M_1 to M_2 after a total time $t_1 = \tau_2^{(1)} - \tau_1^{(1)}$. The value of x_3 at that time is found to be

$$x_3(t_1) = \int_{\tau_1^{(1)}}^{\tau_1^{(1)} + t_1} x_2^{(1)}(s)ds = \int_{\tau_1^{(1)}}^{\tau_2^{(1)}} x_2^{(1)}(s)ds \leq \beta \leq \beta + \lambda$$

Since $A_3 \geq \beta + \lambda$, and since $x_3(t)$ is monotonically increasing for all $t \in (0, t_1)$, we conclude that for all t in this interval $x_3(t) \leq A_3$. After the projection of $\Gamma(t, M_1, \alpha, u^*)$ reaches M_2, $x_3(t)$ starts to decrease while the projection goes toward M_2^1 and, if it actually reaches M_2^1, it then continues along π_2 toward Z_2 as long as $x_3(t)$ is still positive. When once $x_3(t)$ becomes ≤ 0, there is no possibility for the trajectory to pass through the wall $x_3 = + A_3$ until it emerges again at some point $(x_1, x_2, 0) \in K_2$ for which $x_2 \geq 0$. Also, if the projection reaches Z_2 before $x_3(t) \leq 0$, the projection will be stopped at Z_2 until $x_3(t) \leq 0$, and the previous sentence is again applicable.

In the case illustrated by Figure 1b, the projection of the trajectory proceeds from Z_1 to M_2^1 after the lapse of a time $\tau^* < \tau_2^{(1)}$. Hence it passes from M_1 to M_2^1 after a total time $t_1^* = \tau^* - \tau_1^{(1)}$. The value of x_3 at that time is found to be

$$x_3(t_1^*) = \int_{\tau_1^{(1)}}^{\tau_1^{(1)} + t_1^*} x_2^{(1)}(s)ds \leq \int_{\tau_1^{(1)}}^{\tau_2^{(1)}} x_2^{(1)}(s)ds \leq \beta$$

After reaching M_2^1 the projection proceeds along π_2 until M_2 is reached at time t_1. In terms of previous notation, Thirteenth Progress Report pp. 9 and 10,

$$x_3(t_1) - x_3(t_1^*) = \int_0^{\gamma_2} x_2\,(s)ds = \lambda_2 \leq \lambda$$

Hence, again, we find that $x_3(t_1) \leq \beta + \lambda \leq A_3$, and thus, as before, we see that, for all $t \in (0, t_1)$, $x_3(t) \leq A_3$.

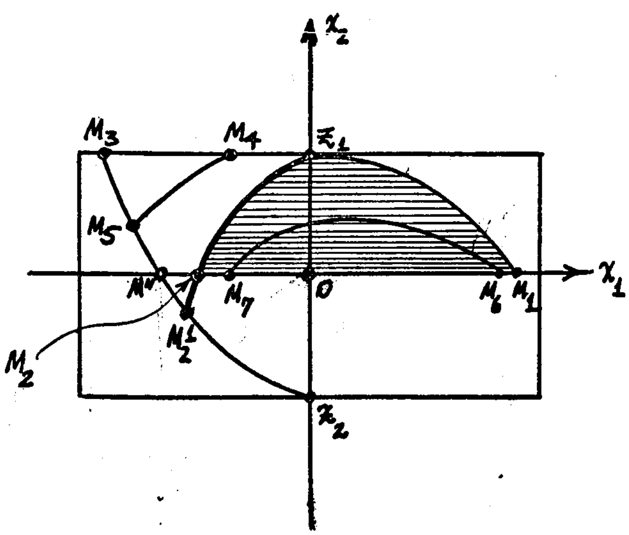


Figure 2a

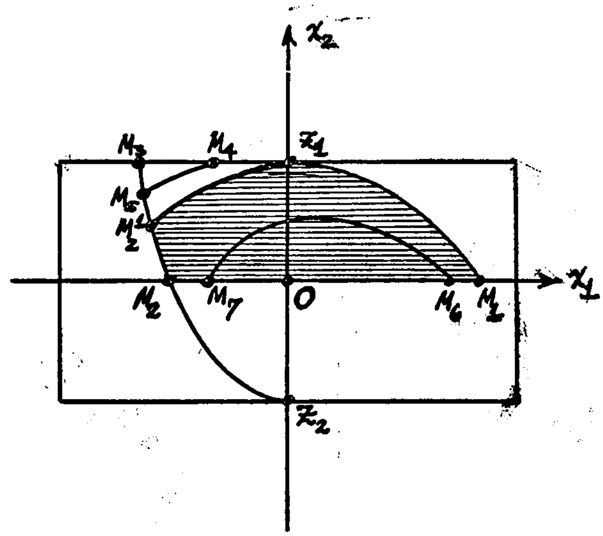


Figure 2b

Suppose first that p lies in the region bounded by $Z_1 M_2^1 M_3 Z_1$ in the case of Figure 2b. The projection of the trajectory through p would proceed along a pseudo-parabolic arc until it reaches π_2 and descend to M_2^1 and on to M_2. We already know that the amount that x_3 increases during the passage from M_2^1 to M_2 is $\leq \gamma$. Hence we examine the amount that x_3 increases during the passage from p to M_2^1. It is clear that, if a given point p is replaced by the point M_4 on the line segment $M_3 Z_1$ which precedes it on the projection of the trajectory through p, the increase in x_3 would be made greater. Even this greater increase in x_3 will be shown to be $< \beta$.

Let $[x_1(t), x_2(t)]$ denote the motion of the projection along the path $M_4 M_5 M_2^1$ starting at M_4 when $t = 0$ and ending at M_2^1 when $t = T_x$, say. Let $[\xi_1(t), \xi_2(t)]$ denote a corresponding motion along $Z_1 M_2^1$ starting at Z_1 when $t = 0$ and ending at M_2^1 when $t = T_\xi$. Since $x_1(0) < \xi_1(0)$, $x_2(0) = \xi_2(0)$, $\dot{x}_2(t) = x_1(t)$, and $\dot{\xi}_2(t) = \xi_1(t)$, it is clear that $x_2(t) < \xi_2(t)$ for positive t at least as long as $x_1(t) \leq \xi_1(t)$. But, if the point (x_1, x_2) is below (ξ_1, ξ_2), it is clear from Figure 2b, that it is impossible for it to be to the right of (ξ_1, ξ_2). This is because the slope of the pseudo-parabolic arc $M_2^1 Z_1$ is everywhere positive and the point (x_1, x_2) is at all times in the closure of the region bounded by $Z_1 M_2^1 M_3 Z_1$. Thus it is impossible for $x_1(t)$ ever to exceed $\xi_1(t)$ during the motion toward M_2^1. It is therefore clear that $x_2(t)$ remains $< \xi_2(t)$ during this motion. Hence the point M_2^1 is reached more quickly along $M_4 M_5 M_2^1$ than along $Z_1 M_2^1$.

In other words $T_x < T_\xi$. Hence

$$\int_0^{T_x} x_2(t)dt < \int_0^{T_x} \xi_2(t)dt < \int_0^{T_\xi} \xi_2(t)dt. \qquad (1.1)$$

Since $\dot{x}_3 = x_2$, it is therefore seen that the rise in x_3 during the passage from M_4 to M_2^1 is less than the rise in x_3 during the passage from Z_1 to M_2^1. But this latter is less than the rise in x_3 during the passage from M_1 to M_2^1 which is already known to be $< \beta$. We have therefore disposed with the case illustrated in Figure 2b.

Suppose next that p lies in the region bounded by $Z_1M_2M_2''M_3Z_1$ in the case of Figure 2 a. The projection of the trajectory through p would proceed along a pseudo-parabolic arc until it reaches π_2 or the line segment $M_2'' M_2$. In the former case it proceeds along π_2 until it reaches M_2'' on the x_1 - axis. As in the case of Figure 2b we can, without loss of generality, take p at M_4. Again we consider the motion of the point $[x_1 (t), x_2(t)]$ beginning at M_4 when $t = 0$ and ending on the x_1- axis when $t = T_x$ with $\dot{x}_2(t) = x_1(t)$. We also consider the motion of $[\xi_1(t), \xi_2(t)]$ along $Z_1 M_2$ beginning at Z_1 when $t = 0$ and ending at M_2 when $t = T_\xi$. Exactly as in the case of Figure 2b, we prove that $T_x < T_\xi$ and that $x_2(t) < \xi_2(t)$ for $0 < t \leq T_x$. Hence the formula (1.1) is again valid. This disposes of the case illustrated in Figure 2a.

The only remaining cases to be considered occur when the initial point is in the shaded area illustrated in either Figure 2 a or Figure 2 b. Without loss of generality we may clearly suppose that the initial point is at M_6 on the positive x_1 - axis. In the case of Figure 2a, it then follows at once from Lemma 2.1 of the Thirteenth Progress Report that the increase of x_3 during the passage from M_6 to M_7 along the indicated pseudo-parabola is less than the rise of x_3 during the passage along $M_1 Z_1 M_2$, which is known to be $\leq \beta$. The case of Figure 2 b is slightly more difficult but the details may still be left to the reader. Before applying the Lemma one should extend the pseudo-parabolic arc $Z_1 M_2^1$ until it intersects the negative x_1- axis. The point M_7 may be on the arc π_2 instead of on the negative x_1- axis as shown in Figure 2 a. If this is the case one should also extend the arc $M_6 M_7$ until it crosses the negative x_1- axis. One thus finds that the rise in x_3 during the passage from M_6 to M_7 is $< \beta$. If M_7 is on π_2, there is a further rise in x_3 which, however, is obviously less than λ.

Lemma 2. Let $A_3 \geq \beta + \lambda$. Let $p \in K_2^+ \cup K_2 \cup K_2^-$. If $\Gamma(t, p, \alpha, u^*)$ does not leave R_3 through the wall $x_3 = A_3$, then $\Gamma(t, p, w, u^*)$ does not leave R_3 through the wall $x_3 = A_3$ for any $w \in W$.

Proof. Let $p = (x_1^0, x_2^0, x_3^0)$. Suppose $x_3^0 < 0$. Let $\Gamma(t) = \Gamma(t, p, w, u^*)$. In order for $\Gamma(t)$ to reach the plane $x_3 = A_3$ it must first cross the plane $x_3 = 0$ and we may as well choose this point as our point of departure. We may thus assume, without loss of generality, that $x_3^0 \geq 0$.

Suppose next that $x_2^0 < 0$. Then at time $t = 0$ the derivative $\dot{x}_3$ along $\Gamma(t)$ is negative and the trajectory $\Gamma(t)$ begins its motion by descending below the plane $x_3 = x_3^0$. In order for $\Gamma(t)$ to leave R_3 through the wall $x_3 = A_3$ it would have to return to this plane and cross it. We may therefore assume, without loss of generality, that $x_2^0 \geq 0$.

Let $x_i(t), \bar{x}_i(t)$, i=1,2,3, denote the x_i coordinate along $\Gamma(t, p, \alpha, u^*)$ and $P(t, p, w, u^*)$, respectively. Our hypothesis is to the effect that $x_3(t) \leq A_3$ for all $t \geq 0$. We wish to prove that $\bar{x}_3(t) \leq A_3$.

We note first that along π_2 the vector fields of the systems governing the motions of the projections (on the plane $x_3 = 0$) of both $\Gamma(t, p, \alpha, u^*)$ and $\Gamma(t, p, w, u^*)$ are discontinuous as long as x_3(or $\bar{x}_3$) > 0. Morever, any trajectory whose projection on the plane $x_3 = 0$ reaches π_2 will proceed in such a manner that its projection remains on π_2 until the projection reaches Z_2 or until the trajectory has an x_3-coordinate which is no longer positive. Except for the point Z_2 itself, throughout this part of the motion we have

$$\dot{x}_2 = g_2(x_2)$$

where $x_1 = g_2(x_2)$ is the equation of π_2. This is true independently of the wind. If the projection of $p \in \pi_2$, then $x_2(t) = \bar{x}_2(t)$ and $x_1(t) = \bar{x}_1(t)$ until the respective projections reach the point Z_2 simultaneously or until the trajectories simulteneously meet the plane $x_3 = 0$. In the former case, after Z_2 is reached, we have $x_1(t) = \bar{x}_1(t) = 0$, $x_2(t) = \bar{x}_2(t) = -A_2$. Hence, in either case, $x_1(t) = \bar{x}_1(t)$ and $x_2(t) = \bar{x}_2(t)$ as long as $\bar{x}_3(t) = x_3(t) > 0$. Since $x_3(t) \leq A_3$ by hypothesis, we have $\bar{x}_3(t) \leq A_3$ on the time interval during which x_3 is initially positive. We next wish to obtain a similar result when the projection of $p \in K_2 - \pi_2$.

In the set $K_2 - \pi_2$, we have

$h(x_1, x_2) + w + u^* = h(x_1, x_2) + w(t) - 1 \leq h(x_1, x_2) + \alpha - 1 < 0$, where, by hypothesis, $h(x_1, x_2)$ is monotonic non decreasing in x_2. By Theorem 1.2 of the Eighth Progress Report, we therefore have

$$\bar{x}_1(t) \leq x_1(t),\ \bar{x}_2(t) \leq x_2(t) \quad \text{and} \quad \bar{x}_3(t) \leq x_3(t),$$

so long as $(x_1(t), x_2(t))$ and $(\bar{x}_1(t), \bar{x}_2(t))$ remain in $K_2 - \pi_2$ and as long as $\bar{x}_3(t) > 0$.

Throughout this part of the motion we therefore have $\bar{x}_3(t) \leq A_3$ since, by hypothesis $x_3(t) \leq A_3$.

The slope of π_2 is negative. Hence $(\bar{x}_1(t), \bar{x}_2(t))$ reaches π_2 before $(x_1(t), x_2(t))$ does. Let the time of impact be t_o. Then $\bar{x}_1(t_o) \leq x_1(t_o)$. So long as $\bar{x}_1(t) \leq x_1(t)$, we still have $\bar{x}_2(t) \leq x_2(t)$. Let $t_1 > t_o$ be the first (if there is any) time such that $\bar{x}_1(t_1) = x_1(t_1)$. Then clearly $\bar{x}_2(t_1) \leq x_2(t_1)$, whence $(x_1(t_1), x_2(t_1))$ lies either on π_2 or above it. In the first case we have $(x_1(t_1), x_2(t_1))=(\bar{x}_1(t_1), \bar{x}_2(t_1))$ whence $(x_1(t), x_2(t)) = (\bar{x}_1(t), \bar{x}_2(t))$ for all $t \geq t_1$ for which both trajectories are still in the half space $x_3 > 0$. In the latter case we note that $\dot{x}_1(t_1) = h(x_1(t), x_2(t)) + \alpha - 1 < 0$ and therefore the projection $(x_1(t), x_2(t))$ moves to the left of the line $x_1 = \bar{x}_1(t_1)$. At the same time $(\bar{x}_1(t), \bar{x}_2(t))$ moves to the right of this line. Hence $(x_1(t), x_2(t))$ will intersect π_2 (if at all) at some time $t_2 > t_1$ at a point above $(\bar{x}_1(t_2), \bar{x}_2(t_2))$. Therefore, again, so long as both trajectories are still in the half space $x_3 > 0$, we have $\bar{x}_2(t) \leq x_2(t)$. Hence so long as $x_3(t) > 0$ and $\bar{x}_3(t) > 0$ we have $\bar{x}_3(t) \leq x_3(t) \leq A_3$.

As long as $\bar{x}_3(t) > 0$, the projection $(\bar{x}_1, \bar{x}_2)$ proceeds in K_2 toward π_2 and then along π_2 toward Z_2. If the plane $x_3 = 0$ has not yet been reached, the projection is "stopped" at Z_2 and $\bar{x}_3(t)$ decreases monotonically at a fixed rate $\dot{\bar{x}}_3 = - A_3$ toward 0. Eventually,

$\Gamma(t, p, w, u^*)$ descends into the lower half space $x_3 < 0$. From that point on the inequality $\bar{x}_3(t) \leq x_3(t)$ can no longer be applied for all subsequent time. But surely $\Gamma(t, p, w, u^*)$ can not escape from R_3 through the wall $x_3 = A_3$ as long as $\bar{x}_3(t) < 0$. Suppose then that $\Gamma(t, p, w, u^*)$ returns to the plane $x_3 = 0$ at some subsequent point p_1. This can only happen if $p_1 \epsilon K_2$. But it is already known from Lemma 1, that $\Gamma(t, p_1, \alpha, u^*)$ does not leave R_3 through the wall $x_3 = A_3$. (The p of that Lemma is the present p_1). Hence we can apply the results obtained above for the point p to the point p_1 to show that $\Gamma(t, p_1, w, u^*)$ can not leave R_3 through the wall $x_3 = A_3$ unless it first descends below the plane $x_3 = 0$ and then emerges there again at a point $p_2 \in K_2$ and so forth. The complete proof of Lemma 2 by induction is thus clearly indicated.

<u>Lemma 3</u>. Let $A_3 \geq \beta + \lambda$. Let $u \in U$, $p \in R_3$. If $p \in Q(u)$, then $\Gamma(t, p, a, u^*)$ cannot leave R_3 through the wall $x_3 = A_3$.

<u>Proof</u>. Let $p = (x_1^o, x_2^o, x_x^o)$. Since $p \in Q(u)$ it follows that $(x_1^o, x_2^o) \epsilon K_2$. Hence $(x_1(t, p, \alpha, u^*), x_2(t, p, \alpha, u^*)) \in K_2$ for all $t \geq 0$. Hence if $\Gamma(t, p, \alpha, u^*)$ ever reaches the plane $x_3 = 0$, it could not (by Lemma 1) subsequently cross the plane $x_3 = A_3$. Thus, if $x_3^o \leq 0$ the proof is complete. We therefore restrict attention to the case when $x_3^o > 0$.

Denote $x_i(t, p, \alpha, u^*)$ and $x_i(t, p, \alpha, u)$, $i = 1, 2, 3$, by $x_i(t)$ and $y_i(t)$, respectively.

Suppose first that $(x_1^o, x_2^o) \in K_2 - \pi_2$. Then as long as both $(x_1(t), x_2(t))$ and $(y_1(t), y_2(t))$ below to $K_2 - \pi_2$ and both $x_3(t)$ and $y_3(t)$ are positive, we have

$$h(x_1, x_2) + \alpha + u \geq h(x_1, x_2) + a + u^* = h(x_1, x_2) + \alpha - 1.$$

Since h is monotonic non-decreasing in x_2, we have from Theorem 1.2 of the Eighth Progress Report

$$x_1(t) \leq y_1(t), \quad x_2(t) \leq y_2(t), \quad x_3(t) \leq y_3(t) \leq A_3 \qquad (1.2)$$

If $x_3(t)$ vanishes at time t^* before $(x_1(t), x_2(t))$ reaches π_2, the proof is complete on account of the fact that $\Gamma(t, p, \alpha, u^*)$ reached the plane $x_3 = 0$ so that Lemma 1 can be invoked. Otherwise $(x_1(t), x_2(t))$ reaches π_2 at some time t_o, at which time, $x_3(t_o)$ is still > 0.

Since π_2 has a negative slope, we see from (1.2) that $(x_1(t), x_2(t))$ reaches π_2 before $(y_1(t), y_2(t))$ does. Moreover

$$x_1(t_o) \leq y_1(t_o), \quad x_2(t_o) \leq y_2(t_o).$$

As long as $x_1(t) \leq y_1(t)$, $t \geq t_o$, we still have $x_2(t) \leq y_2(t)$. In order for $y_2(t)$ to "catch up" and become $\leq x_2(t)$ the projection $(y_1(t), y_2(t))$ must proceed to the left of the moving line $x_1 = x_1(t)$. However, since $p \in Q(u)$, this projection must remain to the right of π_2. This " catching up" process can therefore proceed only through the "wedge" indicated by shading in Figure 3. But for points (y_1, y_2) in this

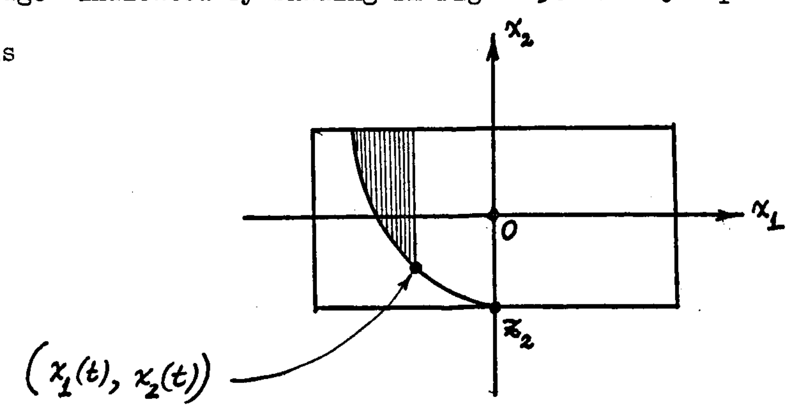


Figure 3

wedge it is geometrically obvious that $y_2 \geq x_2(t)$. Hence

$$x_2(t) \leq y_2(t)$$

as long as $x_3(t) > 0$. Hence $x_3(t) \leq y_3(t) \leq A_3$ as long as $x_3(t) > 0$. This completes the proof in the case when $x_3^o > 0$ and $(x_1^o, x_2^o) \in K_2 - \pi_2$.

There remains the case $x_3^o > 0$ and $(x_1^o, x_2^o) \in \pi_2$. Initially the two points $(x_1(t), x_2(t))$ and $(y_1(t), y_2(t))$ coincide. The former point must, of course, move along π_2; and, if the latter point also moves along π_2 it must coincide with the former point, since motion along π_2 is always governed by $\dot{x}_2 = g_2(x_2)$. If at any time t^1 the point $(y_1(t), y_2(t))$ leaves π_2 it can only do so by moving to the right of π_2, since $(x_1^o, x_2^o, x_3^o) \in Q(u)$.

Since the point (x_1, x_2) and (y_1, y_2) part company at time $t = t^1$, we see that the non-negative function

$$p(t) = [y_1(t) - x_1(t)]^2 + [y_2(t) - x_2(t)]^2$$

is not identically zero on any interval $t^1 < t < t^1 + \delta$ no matter how small the positive number δ is. It might, however, vanish infinitely often on such an interval. Let t'' be any point on the interval $(t^1, t^1 + \delta)$ such that $p(t'') > 0$. Since $p(t)$ is continuous, there is an open interval (α, β) containing t'' such that $p(\alpha) = 0$ and such that $p(t) > 0$ for $t \in (\alpha, \beta) \subset (t^1, t^1 + \delta)$. We wish to show that it is not possible to have $x_2(t_1) > y_2(t_1)$ at any point $t_1 \in (\alpha, \beta)$. Choose $\alpha^* \geq \alpha$ so that $x_2(\alpha^*) - y_2(\alpha^*) = 0$ and so that $x_2(t) - y_2(t) > 0$ for $t \in (\alpha^*, t_1)$. Then we can find $t^* \in (\alpha^*, t_1)$ such that $\dot{x}_2(t^*) - \dot{y}_2(t^*) > 0$. But, since $\dot{x}_2 = x_1$ and $\dot{y}_2 = y_1$, this would mean that $x_1(t^*) > y_1(t^*)$, while simultaneously (since $t^* \in (\alpha^*, t_1)$) we also have $x_2(t^*) > y_2(t^*)$.

Hence the point $(y_1(t^*), y_2(t^*))$ would lie to the left of π_2, which is impossible. Hence we must have $x_2(t_1) \leq y_2(t_1)$ for any $t_1 \in (\alpha, \beta)$. Hence, in particular, $x_2(t'') \leq y_2(t'')$.

Since t'' was any point on $(t^1, t^1 + \delta)$ for which $p(t'') > 0$, we see that $x_2(t) \leq y_2(t)$ on any sufficiently small interval $(t^1, t^1 + \delta)$. Moreover $x_2(t)$ is not identically $y_2(t)$ on any such interval, because then we would have $x_1(t) \equiv \dot{x}_2(t) \equiv \dot{y}_2(t) \equiv y_1(t)$, which contradicts the assumption that (x_1, x_2) and (y_1, y_2) actually parted company at t^1.

When once we know that $x_2(t) < y_2(t)$ for some $t > t^1$ (as we know now), we may apply the argument about the wedge which was previously discussed, at least if $x_1(t) < y_1(t)$. If, however, $x_1(t) \geq y_1(t)$, the point (y_1, y_2) is already in the wedge. In either case the argument shows that $y_2(t)$ can never become $< x_2(t)$. This completes the proof of Lemma 3.

The following Lemmas are proved in complete analogy with Lemmas 1, 2, 3.

Lemma 4. Let $A_3 \geq \beta + \lambda$ and let $p \epsilon K_2$. Then $\Gamma(t, p, -\alpha, u^*)$ does not leave R_3 through the wall $x_3 = -A_3$.

Lemma 5. Let $A_3 \geq \beta + \lambda$. Let $p \epsilon K_2^+ \cup K_2 \cup K_2^-$. If $\Gamma(t, p, -\alpha, u^*)$ does not leave R_3 through the wall $x_3 = -A_3$, then $\Gamma(t, p, w, u^*)$ does not leave R_3 through the wall $x_3 = -A_3$.

Lemma 6. Let $A_3 \geq \beta + \lambda$. Let $u \epsilon U$, $p \epsilon R_3$. If $p \epsilon Q(u)$, then $\Gamma(t, p, -\alpha, u^*)$ cannot leave R_3 through the wall $x_3 = A_3$.

Our main result is an almost obvious consequence of these six lemmas. It may be stated as follows:

<u>Theorem</u> Let β be defined as at the top of p. 9 of the Thirteenth Progress Report and let λ be defined as on p. 10 of the same report. If $A_3 \geq \beta + \lambda$, then $u^*(x_1, x_2, x_3)$ is an upper bound for the system

$$\dot{x}_1 = h(x_1, x_2) + u(x_1, x_2, x_3) + w(t)$$

$$\dot{x}_2 = x_1$$

$$\dot{x}_3 = x_2$$

in R_3. Moreover $K_2 \subset Q(u^*)$.

<u>Proof.</u> Let $u \epsilon U$ and let $p \epsilon Q(u)$. Then the projection of p on the plane $x_3 = 0$ lies in K_2. It now follows from Lemmas 3, 2, 6, 5 that $p \epsilon Q(u^*)$. Hence $Q(u) \subset Q(u^*)$ for all $u \epsilon U$. In other words u^* is an upper bound. The fact that $K_2 \subset Q(u^*)$ follows from Lemmas 1, 2, 4, and 5.

CHAPTER 13

During the month of March 1965 attempts were made to describe the cores of the third order systems considered in the Thirteenth and Fourteenth Progress Report. Since it was proved in these reports that these systems admitted upper bounds, it is known that the cores exist, but we have not yet succeeded in describing them in a simple explicit manner. On the contrary, the present status of this work is both tentative and complicated, and we can not report in detail upon it here.

We have also attempted to generalize the results of the last two progress reports by omitting the requirement on $h(x_1, x_2)$ that it should be monotonic increasing in x_2 for each fixed x_1, or at least by replacing this requirement by a weaker one. In particular it was thought for a while that the Lemma 2.1* of the Thirteenth Progress Report could be proved without the monotonicity requirement, since it is not hard to see that it does indeed hold in many special cases when the monotonicity requirement is violated. Although we have failed to find a satisfactory substitute for this Lemma 2.1, the following considerations, besides indicating the nature of part of last month's work, many yield some insight into this problem.

<u>Theorem 1.</u> In the half-plane $x_2 \geq 0$, suppose $f(x_1, x_2) \leq -\mu$ where μ is positive. Suppose also that $f \epsilon C'$. Let $x_1(t, a), x_2(t, a)$ satisfy the system of differential equations

$$\dot{x}_1 = f(x_1, x_2) \tag{1}$$

$$\dot{x}_2 = x_1 \tag{2}$$

and the initial conditions

$$x_1(0, a) = a > 0 \tag{3}$$

$$x_2(0, a) = 0 \tag{4}$$

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*There were some errors in the typing of the proof of this Lemma 2.1. Line 11 on p. 11 should read "$\dot{x}_2(0) = x_1(0) = a < a^1 = x_1^1(0) = \dot{x}_2^1(0)$" and line 13 of the same page should read "$x_2(t) \leq x_2^1(t)$ and that $x_1(t) = \dot{x}_2(t) \leq \dot{x}_2^1(t) = x_1^1(t)$."

Then there exists a positive number $T = T(a)$ such that $x_2(t, a) > 0$ for $0<t<T$ and such that $x_2(T,a) = 0$. Moreover, T, considered as a function of a has a derivative, $T'(a)$, given by the formula

$$\left\{ \frac{\partial x_2(t, a)}{\partial t} \bigg|_{t = T(a)} \right\} T'(a) + \left\{ \frac{\partial x_2(t,a)}{\partial a} \bigg|_{t = T(a)} \right\} = 0$$

where

$$\frac{\partial x_2(t,a)}{\partial t} \bigg|_{t = T(a)} < 0. \qquad (5)$$

Proof. Since $x_2(0, a) = 0$ and $\dot{x}_2(0, a) = x_1(0, a) = a > 0$, $x_2(t, a)$ must be positive for sufficiently small positive values of t. Since $\dot{x}_1 = f(x_1, x_2) \leq -\mu$, we evidently have $x_1(t, a) \leq a - \mu t$, at least as long as $[x_1(t,a), x_2(t,a)]$ stays in the half-plane $x_2 \geq 0$. Hence, from (2), we find that $x_2(t,a) \leq at - (1/2)\mu t^2$ which is negative for sufficiently large values of t. We let T be the first point where $x_2(t,a)$ ceases to be positive. This proves the first statement of the theorem.

As noted above, $x_2(t, a) > 0$ for sufficiently small positive values of t. Because of (2), $x_2(t, a)$ continues to increase as long as $x_1(t, a) > 0$. Hence $x_2(t, a) > 0$ as long as $0 \leq x_1(t, a) < a$. In this connection it should be remembered that $x_1(t, a)$ is monotonic decreasing, according to (1) and the fact that $f \leq -\mu$, at least as long as $[x_1, x_2]$ stays in the half plane $x_2 \geq 0$. But $x_2(T, a) = 0$ by definition of T. Hence $x_1(T,a)$ can not lie on the interval $[0, a)$. Since $T > 0$ and $x_1(t, a)$ is monotonic decreasing in t, we also know from (3) that $x_1(T,a)$ can not be greater than or even equal to a. The only alternative is that $x_1(T, a) < 0$. From (2), we now see that (5) must hold.

Now $T(a)$, of course, satisfies the equation

$$x_2[T(a), a] = 0 \qquad (6)$$

Since (5) has been shown to hold, the equation (6) suffices for a local unique determination of $T(a)$ in accordance with the implicit function theorem. The implicit function theorem yields the further information that $T'(a)$ exists and satisfies the equation written above in the statement of Theorem 1. This completes the proof of Theorem 1.

It is clear from Theorem 1 that the sign of $T'(a)$ is the same as the sign of

$$\left. \frac{\partial x_2(t,a)}{\partial a} \right|_{t = T(a)}$$

and the sign of this latter may be referred to the variational equations of the system (1), (2). But no useful criterion has been discovered in this way.

Another way of proceeding is first to eliminate t from the autonomous system (1), (2), thus obtaining

$$\frac{dx_2}{dx_1} = \frac{x_1}{f(x_1,x_2)} \qquad (7)$$

where the denominator on the right is not zero, for $x_2 \geq 0$, since

$$f(x_1, x_2) \leq - \mu < 0. \qquad (8)$$

Equation (7) is used to define a solution $x_2 = y(x_1, a)$, such that $y(a, a) = 0$. This solution is defined for

$$-b \leq x_1 \leq a \qquad (9)$$

where $b = b(a) = -x_1[T(a), a] > 0$ in accordance with Theorem 1, and we know also that $y[-b, a] = 0$. From (1) we obviously get an explicit formula for $T(a)$ in terms of $y(x, a)$, namely

$$T(a) = -\int_{-b(a)}^{a} \frac{dx}{f[x, y(x,a)]} \tag{10}$$

The graphs of the two functions $y(x,a)$ and $y(x,a')$ can not intersect if $a \neq a'$; for, if they did, the uniqueness theorem for the solutions of (7) would be violated. It follows then that

$$b'(a) \geq 0, \qquad \frac{\partial y(x, a)}{\partial a} \geq 0 \tag{11}$$

Differentiating the integral in (10) we obtain

$$T'(a) = -\frac{1}{f(a, 0)} + \frac{-b'(a)}{f(-b(a), 0)} + \int_{-b(a)}^{a} \frac{f_y[x, y(x,a)]}{f[x,y(x,a)]^2} \cdot \frac{\partial y(x, a)}{\partial a} dx \tag{12}$$

Because of (11), the last written integral is ≥ 0 if $f[x_1, x_2]$ is monotonic increasing in x_2 for each fixed x_1, while the other two terms on the right hand side of (12) are ≥ 0 in any case because of (8) and (11). We may state this result in the form of a theorem.

Theorem 2. $T(a)$ is a monotonic increasing function of a if the right hand side of (12) is always non-negative. This will automatically be the case if $f(x_1, x_2)$ is monotonic increasing in x_2 for each fixed x_1.

CHAPTER 14

1. Some Final Remarks.

During the last part of the present contract a substantial amount of work was invested in the following two directions:

(a) A complete analysis of the cores of the third order systems considered in the thirteenth and fourteenth Progress Reports was completed. The geometrical configuration of these cores was carefully analyzed in detail in order to determine whether any pathological behavior of the systems had been overlooked. This analysis was both tedious and complicated and led to the conclusion that no pathological behavior occurs. Apart from this fact, which does have some significance, the exact geometrical configurations obtained by us are not of sufficient importance to warrant the additional investment of time and effort which would be required in order to present them here in detail. They are therefore omitted.

(b) Investigations designed to explore various approaches to the solution of the minimax problem within the core were initiated. The two main approaches attempted fell in the categories of functional analysis and differential games, respectively. The first of these led to negative results. The second was terminated at an early and inconclusive stage.

During the same period the preparation and editing of the final report was undertaken and completed.

2. Recommendations for Future Research on the Minimax Problem

I. It appears to us at the present time that further research on the minimax problem should proceed mainly along lines suggested by differential games.

II. Further research along lines suggested by our own approach during the past year should also yield additional results of interest. The following objectives of such research can be specifically formulated:

A. Research Relating to Upper Bounds

1. On the existence and nature of upper bounds for linear systems.

2. On the existence and nature of upper bounds with respect to a subset of R.

3. (i) Relation between the existence of upper bounds and the set R.

   (ii) Relation between the existence of upper bounds and the given system.

4. (i) The existence and nature of upper bounds for the controllable linear system with three zero roots and suitable R.

   (ii) The existence and nature of upper bounds for the controllable linear system with four zero roots and suitable R.

   (iii) The existence and nature of upper bounds for the linear system with roots 0, λ, $-\lambda$ and suitable R.

5. The existence and nature of upper bounds for other (particular systems) of interest.

B. The Minimax Problem Inside the Core of R.

1. Re-examination of the minimax problem inside the core of R.

2. Possible new criteria for choice of control inside the core.

3. On the qualitative nature of solutions to the general minimax problem inside the core.

4. On the quantitative nature of solutions to the general minimax problem.

5. Qualitative and quantitative investigation of the minimax problem (inside the core) for the particular systems listed in Part A.

Methods suggested by the theory of differential games should prove useful in pursuing items 3, 4, 5 above.

III. It appears to us at the present time that additional research on the minimax problem would yield various results of a mathematical nature which would provide deeper insight into the problem. Such insight may have useful engineering and/or computational implications. However, the probability of obtaining closed form solutions to the minimax problem for systems which are of practical relevance to the Saturn Launch Vehicle is at best difficult to evaluate at the present time and at worst somewhat pessimistic.

APPENDIX

APPENDIX

NOTES AND ERRATA

Chapter 1, Section 1:

The statement of the problem given here, in spite of its vagueness, is sufficiently precise to have started the investigation on a somewhat wrong track. Namely the class U was supposed to have been a class of functions of t rather than functions of x, or at least rather than functions of x as well as of t. It was not until the Fourth Progress Report (i.e. Chapter 3) that this point of view was abandoned. The following sections were written under this assumption (that the class U was composed of functions of t alone): Sections 5 and 6 of Chapter 1, pp. 1-21 to 1-25. Sections 1 and 2 of Chapter 2, pp. 2-1 to 2-15. The other sections in Chapters 1 and 2, even though they were written before the faulty assumption was abandoned, contain material of such generality as to be applicable to cases where the class U contains functions of x.

Chapter 1, Section 2:

The first comma of l. 11, p. 1-10, should be omitted.

Chapter 1, Section 3:

The K in formula (3.2) should be A

On p. 1-13, l. 2, the word "say" should be "any".

On p. 1-17, last line, F should be G.

Chapter 1 Section 4:

On p. 1-19, second line, $x \to Y$ should be $X \to Y$.

On p. 1-19, line 9, "be" should be "by".

Chapter 1, Section 5:

On p. 1-21, last line, $[0, t]$ should be $[0, T]$.

The title of this Section is misleading. The problem considered is essentially a "maximin" problem rather than a "minimax" problem. We were led to it because of the faulty assumption that U contained functions of t only.

On p. 1-22, line 2, $u \in U$ should be $u^* \in U$.

Chapter 1, Section 6:

This section is defective in several respects and is superceded by Chapter 2, Section 2.

Chapter 2, Section 1:

This section is still based on the assumption that the class U of admissible controls consists of functions of time. It sets forth a maximin problem in WxU and shows how this may be transformed into a minimax problem in a new cross product set WxH.

p. 2-5, line 4, WxH should be WxU.

Chapter 2, Section 2:

On p. 2-9, l. 6. "number" should be "member".

This section, although it satisfactorily corrects the errors of Chapter 1, Section 6, is itself superceded by the more general Sections 2 and 3 of Chapter 4, pp. 4-4 to 4-11.

Chapter 3:

Here is found the exact formulation of the minimax problem using control functions which depend on the state-variables alone.

Chapter 5, Section 3.2:

On p. 5-17, Definition 3.1 is correct albeit awkwardly stated. It may be simplified as follows:

Definition 3.1 Let $u_1, u_2 \in U$. We say that u_2 contains u_1, written $u_2 \supset u_1$, iff $Q(u_2) \supset Q(u_1)$.

Proposition 3.1 then becomes true by definition.

p. 5-21, line 12, $\left[-A + \frac{\alpha-1}{T}, A - \frac{\alpha-1}{T}\right]$ should be $\left[-A + (\alpha-1) T, A - (\alpha-1) T\right]$.

Chapter 5, Appendix to Section 1:

p. 5-23. Part of the proof of the Theorem on this page is incorrect, but this material is superceded by Chapter 6, Section 1.

Chapter 6, Section 1:

p. 6-3, l. 7. The formula in Theorem 1.1 should be numbered (1.1).

Chapter 9, Section 2:

p. 9-11, l. 7. $|h|$ should be $\|h\|$.

Chapter 10:

p. 10-6. The definitions of sets X_3^+ and X_3^- are defective in that the conditions $x_1 = x_2 = 0$ are missing. The definitions of P_1 and P_2 are defective in that the condition $x_3 = 0$ is missing.

p. 10-8, line 3, "below plane" should read "below the plane".

p. 10-12, ll. 14, 15, 16, the g in formulas $\Gamma(t, g, \alpha, u^*)$ and $x_2(t, g, \alpha, u^*)$ should be replaced by q.

p. 10-12, line 17. The first nine words and π should be omitted.

p. 10-14, last line, $(x, (t), x_2(t))$ should read $(x_1(t), x_2(t))$.

p. 10-15, ll. 19 and 20. The sentence, "But $(x_1(t), x_2(t))$ has the least possible value throughout this wedge", should read as follows: "But for points (y_1, y_2) in this wedge it is geometrically obvious that $y_2 \geqslant x_2$".

Chapter 11, Section 1:

p. 11-5, l. 16. $w_2(t)$ should be replaced by $w_2(t-t_1)$.

Chapter 11, Section 2:

p. 11.9, l. 17. O_1 should be O.

p. 11.11, l. 11. $x_2(o)$ should be $x_1(o)$.

Chapter 11, Section 3:

p. 11-15, line 4 from bottom, "caome" should read "come".

p. 11-16, line 5, x_1, (t, p, α, u^*) should read $x_1(t, p, \alpha, u^*)$.