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OPTIMAL PROCEDURES FOR STOCHASTICALLY FAILING EQUIPMENT

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PREFACE

When a complex system such as the Apollo vehicle arrives on the Moon, certain parts may no longer function. To restore the system to a working state, these parts must be replaced by new ones; but the vehicle can carry only a limited number of spare parts and must also be able to leave at a specified time after arrival. This Memorandum investigates an abstract (and simplified) version of the problem of the optimal maintenance procedure for such a system. As a bonus of the abstract formulation, the results obtained are applicable to a variety of specific systems.

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SUMMARY

This Memorandum investigates a system that may be classified into one of three possible states: 0, 1, or 2. Intuitively, we think of 0 being a "failed" state, 1 a "turned off" state, and 2 a "working" state.

If the system is in state 2, we may make one of two decisions: "Turn off" or "Let run." With the first decision the system goes instantaneously to state 1, while with the second decision, the system will remain in state 2 with probability β , or go into state 0 with probability $1 - \beta$. If the system is in state 1, we may make the decision "Do nothing," which will leave the system in state 1 at the next time period, or we may make the decision "Turn on." With this latter decision, at the next time period the system will be in state 2 with probability α , or in state 0 with probability $1 - \alpha$. Finally, if the system is in state 0, we may make the decision "Do nothing," which leaves the system in state 0, or we may make the decision "Repair," which will put the system into state 1 after m units of time.

The central problem is, what is the policy which maximizes the probability of having the system in state 2 at time n or $n + k$ when at time 0 we have exactly r spare parts for repairs, and what are these probabilities?

OPTIMAL PROCEDURES FOR STOCHASTICALLY FAILING EQUIPMENT

Suppose we have a system that may be classified into one of three possible states: 0, 1, or 2. Intuitively, we think of 0 being a "failed" state, 1 a "turned off" state, and 2 a "working" state.

If the system is in state 2, we may make one of two decisions: "Turn off" or "Let run." With the first decision the system goes instantaneously to state 1, while with the second decision, the system will remain in state 2 with probability β , or go into state 0 with probability $1 - \beta$. If the system is in state 1, we may make the decision "Do nothing," which will leave the system in state 1 at the next time period, or we may make the decision "Turn on." With this latter decision, at the next time period the system will be in state 2 with probability α , or in state 0 with probability $1 - \alpha$. Finally, if the system is in state 0, we may make the decision "Do nothing," which leaves the system in state 0, or we may make the decision "Repair," which will put the system into state 1 after m units of time.

Given the system described above, let us suppose that "repairs" are made by installing new parts.

The central problem we wish to investigate is what is the policy which maximizes the probability of having the system in state 2 at time n or $n + k$ when at time 0 we have exactly r spare parts for repairs, and what are these

probabilities? For the case when $k = 0$, this problem was solved in [1].

At this point we shall make a few comments on the applicability of the abstract model to concrete situations. It turns out that when the system is initially in state 1,, then we may delay a certain time before attempting to turn it on; but once we do turn it on, we never deliberately turn it off. Thus our results could apply to a system that had no provision for turn-off provided we always start with the system in a "failed" or "just ready to go" state.

There are several specific situations to which our model applies. We mention only two as illustrations.

1. A piece of electrical equipment that is needed to work at set specified times in the future (such as, for example, a radio transmitter). The identification of α and β and the states 0, 1, and 2 are evident.

2. Suppose we wish to use a satellite to record some activity (e.g., solar flares) that we know will occur at specified times in the future. If we have no functioning satellite and none ready to go, we are in state 0. If we have no functioning satellite, but have one on the launching pad ready to fire, we are in state 1. State 2 corresponds to having a satellite in orbit and functioning. One unit of time is the time from firing the missile until the satellite is in orbit, and the "repair" time is the time required to take a missile from storage and get it ready to fire (countdown, etc.). The probability of successful

launch is α . The probability that a functioning satellite continues to function is β .

Another use of the two-time model presented here is to compensate for error in the one-time model given in [1]. That is, we may be interested in having the system in state 2 at some time approximately n . The two-time model is relevant to this problem for the following reason.

If we have a system as described above, except that we wish to be in state 2 at at least one of the times $n_1 < n_2 < \dots < n_j$, then as long as $n_j \leq n_1 + m + 1$, the optimal policy and the associated probabilities are exactly the same as in the two-time case (n_1, n_j) . To see this, simply observe that the dynamic programming equations for the multitime case are the same as those for the two-time case and, moreover, that the two-time case and the multitime case have exactly the same initial conditions and boundary conditions. Hence the probabilities under an optimal policy must be the same.

The above fact enables us then to use the results of the two-time problem to solve the important problem of finding the probabilities under an optimal policy for having the system in state 2 at about time n , that is, of having the system in state 2 in some point in the interval $n \pm \delta$. As long as $2\delta \leq m + 1$, we simply solve the two-time problem for $(n - \delta, n + \delta)$.

Let $Q_i(r; n, n+k)$ denote the probability under an optimal policy that the system is in state 2 at time n or $n+k$, given that at time 0 we have r spares and the initial state is i . Then the following equations govern the Q_i :

$$(1.1) \quad Q_0(0; n, n+k) = 0;$$

$$(1.2) \quad Q_0(r; n, n+k) = \max \{Q_0(r; n-1, n+k-1); \\ Q_1[r-1; (n-m)^+; (n+k-m)^+]\}, \\ r > 0, n > 0;$$

$$(1.3) \quad Q_1(r; n, n+k) = \max \{Q_1(r; n-1, n-1+k); \\ \alpha Q_2(r; n-1, n-1+k) \\ + (1-\alpha) Q_0(r; n-1, n-1+k)\}, \\ r \geq 0, n > 0;$$

$$(1.4) \quad Q_2(r; n, n+k) = \max \{\beta Q_2(r; n-1, n-1+k) \\ + (1-\beta) Q_0(r; n-1, n-1+k)\} \\ r \geq 0, n > 0.$$

If the initial state is 0 or 1, then the initial condition for $n = 0$ must be that for the optimal probabilities in the case of a one-time system for time k . Let $t_0 > 0$ be such that $\beta^{t_0} \leq \alpha < \beta^{t_0-1}$; then from the results of [1] we have the following:

(i) If $t_0 \leq m + 1$,

$$(1.5) \quad Q_0(r, 0, k) = P_0(r, k) = \begin{cases} 0, & \text{if } k \leq m \text{ or if } r = 0, \\ \alpha, & \text{if } k > m \text{ and if } r > 0; \end{cases}$$

$$(1.6) \quad Q_1(r, 0, k) = P_1(r, k) = \alpha, \quad k > 0;$$

$$(1.7) \quad Q_2(r, 0, k) = 1.$$

(ii) If $t_0 > m + 1$,

$$(1.8) \quad Q_0(r, 0, k) = P_0(r, k) = \begin{cases} 0, & \text{if } k \leq m, \text{ or if } r = 0, \\ \alpha(1+a+\dots+a^{j-1}), & \text{if } j(m+1) - 1 < k \leq (j+1)(m+1) - 1 \\ & \text{for } 0 < j < r, \quad r > 0, \\ \alpha(1+a+\dots+a^{r-1}), & \text{if } r(m+1) \leq k < \infty, \quad r > 0; \end{cases}$$

$$(1.9) \quad Q_1(r, 0, k) = P_1(r, k) = \begin{cases} \alpha(1+a+\dots+a^j), & \text{if } j(m+1) < k \leq (j+1)(m+1) \\ & \text{for } 0 \leq j < r, \quad r > 0, \\ \alpha(1+a+\dots+a^r), & \text{if } r(m+1) < k < \infty, \quad r \geq 0; \end{cases}$$

$$(1.10) \quad Q_2(r, 0, k) = 1, \quad \text{where } a = \beta^{m+1} - \alpha.$$

For the model presented we explicitly find the optimal probabilities and the optimal policy for all values of r, n, k, m , when $\beta^{m+1} \leq \alpha$, and for all r, n, m , when $\beta^{m+1} > \alpha$, provided $k \leq m + 1$. In the remaining case, $\beta^{m+1} > \alpha$ and $k > m + 1$, the problem becomes intractable beyond the value $r = 2$. Experiments on a computer for special values of α, β in this range indicate that the policy tends to become increasingly complex and to depend more and more on the explicit value of α and β .

We state our results as follows:

Theorem 1. Suppose $\beta^{m+1} \leq \alpha$, $0 < k \leq m$, and $t_0 \leq m - k + 1$. Then

(i) if $r = 0$,

$$Q_0 = 0,$$

$$Q_1 = \alpha,$$

$$Q_2(0, n, n+k) = \beta^n, \quad n < t_0,$$

$$= \alpha, \quad n \geq t_0;$$

(ii) if $r \geq 1$,

$$Q_0 = 0, \quad n \leq m - k,$$

$$= \alpha, \quad n > m - k,$$

$$Q_1 = \alpha,$$

$$Q_2 = \beta^n, \quad n < t_0,$$

$$= \alpha, \quad n \geq t_0,$$

The optimal policy is the following. If in state 0 initially, repair (if possible), delay until time $n - 1$ and restart. If in state 1 initially, delay restart until time $n - 1$ and then try. If in state 2, let the system run if $n < t_0$; but if $n \geq t_0$, turn the system off and pursue state 1 policy.

Theorem 2. Suppose $\beta^{m+1} \leq \alpha$, $0 < k \leq m$, and $m - k + 1 < t_0 \leq m + 1$. Then if we set $b = (\beta^{m-k+1} - \alpha)$, we have

(i) if $r = 0$, then Q_0, Q_1, Q_2 are as in Theorem 1(i);

(ii) if $r = 1$,

$$\begin{aligned} Q_0 &= 0, & n \leq m - k, \\ &= \alpha, & n > m - k; \\ Q_1 &= \alpha, & 0 \leq n \leq m - k + 1, \\ &= \alpha + \alpha b, & n > m - k + 1; \\ Q_2 &= \beta^n, & n \leq m - k + 1, \\ &= \beta^{[n-(m-k+1)]} b + \alpha, & m - k + 1 < n \leq m - k + t_0, \\ &= \alpha + \alpha b, & n > m - k + t_0; \end{aligned}$$

(iii) if $r \geq 2$,

$$\begin{aligned} Q_0 &= 0, & n \leq m - k, \\ &= \alpha, & m - k < n \leq 2m - k + 1, \\ &= \alpha + \alpha b, & n > 2m - k + 1; \end{aligned}$$

and Q_1 and Q_2 are the same as in (ii).

The optimal policy for $r = 0$ is the same as in Theorem 1. If $r > 0$ then, if initially in state 0, repair (if possible) and pursue the policy for state 1 over the remaining time period. If in state 1, we delay turn-on until time $n - 1$, and then attempt turn on, if $n \leq m - k + 1$. However, when $n > m - k + 1$, we delay turn-on until time $n - (m - k + 2)$, attempt turn-on, and then pursue the optimal policy for the resulting state over the remaining time period. If initially we are in state 2, we let the system run if $n \leq m - k + t_0$; but if $n > m - k + t_0$, we turn the system off and pursue the policy for state 1.

Theorem 3. Suppose $\beta^{m+1} \leq \alpha$ and $k > m$. Then

(i) if $r = 0$,

$$Q_0 = 0 ;$$

$$Q_1 = \alpha ;$$

$$\begin{aligned} Q_2 &= \beta^n , & n < t_0 , \\ &= \alpha , & n \geq t_0 . \end{aligned}$$

(ii) if $r = 1$,

$$Q_0 = \alpha ;$$

$$\begin{aligned} Q_1 &= \alpha , & n = 0 , \\ &= \alpha + \alpha(1 - \alpha) , & n > 0 ; \end{aligned}$$

$$\begin{aligned} Q_2 &= \beta^n(1 - \alpha) + \alpha , & n < t_0 , \\ &= \alpha + (1 - \alpha)\alpha , & n \geq t_0 ; \end{aligned}$$

(iii) if $r > 1$,

$$\begin{aligned} Q_0 &= \alpha , & n \leq m , \\ &= \alpha + (1 - \alpha) , & n > m ; \end{aligned}$$

and Q_1 and Q_2 are as in case (ii).

The optimal policy is as follows. If initially the state is 0 we "repair" (if possible) and then pursue the policy for state 1 over the remaining time period. If initially we are in state 1 we delay until time $k - 1$ and then turn-on if $n = 0$, but if $n > 0$ we delay until time $n - 1$ and fire and if we miss then we install a new unit if $r > 0$ and delay turn-on until time $k - 1$. If in state 2 we let the system run if $n < t_0$, but if $n \geq t_0$ we turn the system off and pursue the policy for state 1.

Theorem 4. Suppose that $\alpha < \beta^{m+1}$ and $1 \leq k \leq m + 1$. Let $a = \beta^{m+1} - \alpha$ and $b = \beta^{m-k+1} - \alpha$. Then

(i) if $r = 0$,

$$Q_0(0; n, n + k) = 0 , \quad \text{if } n \geq 0 ;$$

$$Q_1(0; n, n + k) = \alpha , \quad \text{if } n \geq 0 ;$$

$$Q_2(0; n, n + k) = \begin{cases} \beta^b, & \text{if } 0 \leq n < t_0 , \\ \alpha , & \text{if } t_0 \leq n ; \end{cases}$$

(ii) if $r \geq 1$,

$$Q_0(r; n, n+k) = \begin{cases} 0, & \text{if } 0 \leq n < m+1-k, \\ \alpha \left[1 + \frac{b(1-a^{j-1})}{1-a} \right], & \text{if } j(m+1) - k \leq n < (j+1)(m+1) - k \\ & \text{and } 1 \leq j < r, \\ \alpha \left[1 + \frac{b(1-a^{r-1})}{1-a} \right], & \text{if } r(m+1) - k \leq n; \end{cases}$$

$$Q_1(r; n, n+k) = \begin{cases} \alpha, & \text{if } 0 \leq n \leq m+1-k, \\ \alpha \left[1 + \frac{b(1-a^j)}{1-a} \right], & \text{if } j(m+1) - k < n \leq (j+1)(m+1) - k \\ & \text{and } 1 \leq j < r, \\ \alpha \left[1 + \frac{b(1-a^r)}{1-a} \right], & \text{if } r(m+1) - k < n; \end{cases}$$

$$Q_2(r; n, n+k) = \begin{cases} \beta^n & \text{if } 0 \leq n \leq m+1-k, \\ \alpha \left[1 + \frac{b(1-a^{j-1})}{1-a} \right] + ba^{j-1}\beta^{n-[j(m+1)-k]}, & \text{if } j(m+1) - k < n \leq (j+1)(m+1) - k \\ & \text{and } 1 \leq j < r, \\ \alpha \left[1 + \frac{b(1-a^{r-1})}{1-a} \right] + ba^{r-1}\beta^{n-[r(m+1)-k]}, & \text{if } r(m+1) - k < n < r(m+1) - k + t_0 \\ \alpha \left[1 + \frac{b(1-a^r)}{1-a} \right], & \text{if } r(m+1) - k + t_0 \leq n. \end{cases}$$

An optimal strategy is:

State 0: If $r = 0$ or $n \leq m - k$, there is no possibility of success. Otherwise, repair and pursue the policy for state 1.

State 1: If $r = 0$, delay until time $n - 1$, then turn on. If $r = 0$ and if $0 \leq n \leq m + 1 - k$, delay until time $n + k - 1$, then turn on. If $r > 0$, $j(m + 1) - k + 1 \leq n \leq (j + 1)(m + 1) - k$, and $1 \leq j < r$, delay until time $n - [j(m + 1) - k + 1]$ and then turn on. If $r > 0$ and if $r(m + 1) - k + 1 \leq n$, delay until time $n - [r(m + 1) - k + 1]$ and then turn on.

State 2: If $r = 0$ and $t_0 \leq n$, or if $r > 0$ and $r(m + 1) - k + t_0 \leq n$, turn off and pursue the policy for state 1. Otherwise, let the system run.

Theorem 5. Suppose $\beta^{m+1} > \alpha$ and $k > r(m + 1)$.

Then

- (i) if $r = 0$,
- $$\begin{aligned} Q_0 &= 0 ; \\ Q_1 &= \alpha ; \\ Q_2 &= \beta^n, & \text{if } n < t_0 , \\ &= \alpha , & \text{if } n \geq t_0 ; \end{aligned}$$

(ii) if $r = 1$,

$$Q_0 = \alpha ;$$

$$Q_1 = \alpha(1 + a) , \quad \text{if } n = 0 ,$$

$$= \alpha(2 - \alpha) , \quad \text{if } n > 0 ;$$

$$Q_2 = \beta^n(1 - \alpha) + \alpha , \quad \text{if } n < t_0 ,$$

$$= \alpha(2 - \alpha) , \quad \text{if } n \geq t_0 ;$$

(iii) if $r = 2$,

$$Q_0 = \alpha(1 + a) , \quad \text{if } n \leq m ,$$

$$= \alpha(2 - \alpha) ; \quad \text{if } n > m ;$$

$$Q_1 = \alpha(1 + a + a^2) , \quad \text{if } n = 0 ,$$

$$= \alpha(1 - \alpha(1+a)) + \alpha(1 + a) , \quad \text{if } 0 < n \leq m + 1 ,$$

$$= \alpha a(1 - \alpha\beta^{m+1}) + \alpha(2 - \alpha) , \quad \text{if } n > m + 1 ;$$

$$Q_2 = \beta^n(1 - \alpha(1+a)) + \alpha(1 + a) , \quad \text{if } 0 \leq n \leq m + 1 ,$$

$$= \beta^{n-(m+1)} a(1 - \alpha\beta^{m+1}) + \alpha(2 - \alpha) , \quad \text{if } m + 1 < n \leq m + t_0 ,$$

$$= \alpha a(1 - \alpha\beta^{m+1}) + \alpha(2 - \alpha) , \quad \text{if } m + t_0 < n .$$

An optimal strategy is:

State 0: If $r = 0$, there is no possibility of success.

If $r > 0$, repair and then pursue the strategy for state 1.

State 1: If $r = 0$ or 1, do nothing until time $n - 1$

and then turn on. If $r = 2$ and $n \geq m + 2$,

do nothing until time $n - (m + 2)$ then

turn on. If $r = 2$ and $n < m + 2$, do nothing

until time $n - 1$ then turn on.

State 2: If $r = 0$ or 1 and $n \geq t_0$ or if $r = 2$ and $n \geq t_0 + m + 1$, turn off and pursue the strategy for state 1. Otherwise let the system run.

PROOFS

Proof of Theorem 1. (i) The result for Q_0 is obvious. $Q_1(0, 0, k) = \alpha$ and $Q_2(0, 0, k) = 1$. Assume that we have established the result for all $n \leq n_0$, $n_0 + 1 < t_0$. Then for $n_0 + 1$ we have by Eqs. (1.3) and (1.4) that

$$Q_1(0, n_0 + 1, n_0 + 1 + k) = \max \{ \alpha; \alpha \beta^{n_0} \} = \alpha ;$$

$$Q_2(0, n_0 + 1, n_0 + 1 + k) = \max \{ \beta^{n_0+1}; \alpha \} = \beta^{n_0+1} .$$

Hence (i) holds for all $n < t_0$. If $n = t_0$, we have

$$Q_1(0, t_0, t_0 + k) = \max \{ \alpha; \alpha \beta^{t_0-1} \} = \alpha ;$$

$$Q_2(0, t_0, t_0 + k) = \max \{ \beta^{t_0}; \alpha \} = \alpha .$$

Induction on n now completes the proof of (i).

(ii) Clearly $Q_0 = 0$ if $n \leq m - k$. Hence no matter what r is, we must have that Q_1 and Q_2 are the same as the $r = 0$ case for all $n \leq m - k$. Now

$$Q_1(r; m - k + 1, m + 1) = \max \{ \alpha; \alpha^2 \} = \alpha ;$$

$$Q_2(r; m - k + 1, m + 1) = \max \{ \alpha; \alpha \} = \alpha ;$$

and induction then shows that $Q_1 = \alpha$, $Q_2 = \alpha$, all $n \geq m - k + 1$. Since $Q_0(r, n, n + k) = Q_1(r - 1, n - m, n - m + k)$ for all $n \geq m$, we have by induction on r that $Q_0(r, n, n + k) = \alpha$, all $n > m - k$. This completes the proof of Theorem 1.

Proof of Theorem 2. (i) The same argument used to establish Theorem 1(i) shows that (i) is valid. (ii) If $n \leq m - k$, then Q_0 must be 0. Induction on n shows that our formulas for Q_1 and Q_2 are certainly valid for $n \leq m - k$:

$$Q_0(1; m - k + 1, m + 1) = Q_1(0, 0, 1) = \alpha ;$$

$$Q_1(1; m - k + 1, m + 1) = \max \{ \alpha_1; \alpha \beta^{m-k} \} = \alpha ;$$

$$Q_2(1; m - k + 1, m + 1) = \max \{ \beta^{m-k+1}; \alpha \} = \beta^{m-k+1} ;$$

which establishes the formula for $n = m - k + 1$. For $n \geq m - k + 1$ we have

$$Q_0(1; n, n + k) = Q_1[0; (n - m)^+; n + k - m] = \alpha ;$$

$$\begin{aligned} Q_1(1, m - k + 2; m + 2) &= \max \{ \alpha; \alpha \beta^{m-k+1} + (1 - \alpha) \} \\ &= \alpha + \alpha(\beta^{m-k+1} - \alpha) ; \end{aligned}$$

$$Q_2(1, m - k + 2) = \max \{ \beta^{m-k+2} + (1 - \beta)\alpha; \alpha + \alpha\beta \} = \alpha + \beta\alpha.$$

Suppose we have established the formula for $n, m - k + 1 < n < m - k + t_0$.

Then

$$\begin{aligned} Q_1(1, n+1, n+1+k) &= \max \{ \alpha + \alpha b; \alpha(\alpha + \beta^{n-(m-k+1)}) + (1-\alpha)\alpha \} \\ &= \alpha + \alpha b ; \\ Q_2(1, n+1, n+1+k) &= \max \{ \beta^{n+1-(m-k+1)} + (1-\beta)\alpha; \alpha + \alpha b \} \\ &= \alpha + \beta^{n+1-(m-k+1)} b . \end{aligned}$$

So the formulas are valid for all n , $m-k+1 < n \leq m-k+t_0$. For $n = m-k+t_0+1$, we readily compute that $Q_1 = \alpha + \alpha b$, while Q_2 becomes $\alpha + \alpha b$. Induction on n now completes the proof. (iii) $Q_0(r; n, n+k) = Q_1[r-1, (n-m)^+, (n+k-m)^+]$, and thus for $r = 2$, we have that the formula for Q_0 is valid. Hence for all $n \leq 2m+1-k$, we have that the expression for Q_1 and Q_2 are the same as in the $r = 1$ case. But for $n \geq 2m-k+2$, we have that $Q_2 = Q_1 = \alpha + \alpha b$, and so we must have that $Q_1 = Q_2 = \alpha + \alpha b$ for all $n > 2m+k+1$. This establishes the formulas for $r = 2$. Assuming that we have established them for $r_0 \geq 2$, we then have, by the same argument, that they are valid for $r_0 + 1$.

Proof of Theorem 3. (i) A simple induction argument on n establishes this result. (ii) $Q_0(1, 0, k) = \alpha$, $Q_1(1, 0, k) = \alpha$, and $Q_2(1, 0, k) = 1$. We have that $Q_0(1, n, n+k) = \alpha$ for all n .

$$\begin{aligned} Q_1(1, 1, k+1) &= \max \{ \alpha; \alpha + (1-\alpha)\alpha \} = \alpha + 1-\alpha \alpha ; \\ Q_2(1, 1, k+1) &= \max \{ \beta + (1-\beta)\alpha; \alpha \} = \beta(1-\alpha) + \alpha . \end{aligned}$$

Induction on n now easily establishes the formula. (iii)
 It is clear that $Q_0(r; n, n+k) = Q_0(1, n, n+k)$ if $n \leq m$, and thus Q_1 and Q_2 are the same as in the $r = 1$ case, at least for all $n \leq m+1$. Since $Q_0(r; n, n+k) = Q_1(r-1, n-m, n+k-m)$ for $n > m$, we see that if we have established the formulas for r , then we will have that they hold for $r+1$, since Q_1 and $Q_2 = \alpha + (1-\alpha)\alpha$ for all $n > m$.

Proof of Theorem 5. Proof of Theorem 5 will precede that of Theorem 4. First let $r = 0$. $Q_0 = 0$ by (1.1). $Q_1(0; 0, k) = \alpha$, and $Q_2(0; 0, k) = 1$ by (1.9) and (1.10). Suppose $n \geq 0$ and the formulas for $Q_i(0; n, n+k)$, $i = 1$ or 2 , are correct. Then

$$\begin{aligned} Q_1(0; n+1, n+1+k) \\ = \max \{ \alpha, \alpha Q_2(0; n, n+k) + (1-\alpha) \cdot 0 \} = \alpha . \end{aligned}$$

If $n < t_0$,

$$\begin{aligned} Q_2(0; n+1, n+1+k) \\ = \max \{ \alpha, \beta^{n+1} + (1-\beta) \cdot 0 \} = \beta^{n+1} , \quad \text{if } n+1 < t_0 , \\ = \alpha , \quad \text{if } n+1 \geq t_0 . \end{aligned}$$

If $n \geq t_0$,

$$Q_2(0; n+1, n+1+k) = \max \{ \alpha, \beta \alpha + (1-\beta) \cdot 0 \} = \alpha .$$

Now suppose $r = 1$. The formulas for $Q_i(1; 0, k)$ follow from (1.8) - (1.10), since $k > m+1$. Suppose $n \geq 0$

and $Q_0(1; n, n+k) = \alpha$.

Then

$$\begin{aligned} Q_0(1; n+1, n+1+k) \\ = \max(\alpha, Q_1(0; (n+1-m)^+, (n+1-m+k)^+)) = \alpha \end{aligned}$$

by (1.9) or the case $r = 0$.

We have

$$Q_1(1; 1, 1+k) = \max(\alpha(1+a), \alpha + (1-\alpha)\alpha) = \alpha(2-\alpha),$$

since $a = \beta^{m+1} - \alpha < 1 - \alpha$. Suppose $n \geq 1$ and

$Q_1(1; n, n+k) = \alpha(2-\alpha)$. Then

$$\begin{aligned} Q_1(1; n+1, n+1+k) \\ = \max(\alpha(2-\alpha), \alpha Q_2(1; n, n+k) + (1-\alpha)\alpha) \\ = \alpha(2-\alpha), \end{aligned}$$

since $Q_2(1; n, n+k) < 1$ for $n > 0$.

Suppose $n \geq 0$ and the formula for $Q_2(1; n, n+k)$ is correct. If $n = 0$,

$$\begin{aligned} Q_2(1; 1, 1+k) &= \max(\alpha(1+a), \beta + (1-\beta)\alpha) \\ &= \max(\alpha + \alpha a, \alpha + \beta(1-\alpha)) = \alpha + \beta(1-\alpha), \end{aligned}$$

since $\beta \geq \beta^{m+1} > \alpha$ and $1-\alpha > \beta^{m+1} - \alpha = a$.

If $0 < n < t_0$,

$$\begin{aligned} Q_2(1; n+1, n+1+k) \\ = \max(\alpha(2-\alpha), \beta(\beta^n(1-\alpha) + \alpha) + (1-\beta)\alpha) \\ = \max(\alpha(1-\alpha) + \alpha, \beta^{n+1}(1-\alpha) + \alpha) \\ = \beta^{n+1}(1-\alpha) + \alpha, \quad \text{if } n+1 < t_0, \\ = \alpha(1-\alpha) + \alpha, \quad \text{if } n+1 = t_0. \end{aligned}$$

If $t_0 \leq n$,

$$\begin{aligned} Q_2(1; n+1, n+1+k) \\ &= \max (\alpha(2-\alpha), \beta(\alpha(2-\alpha)) + (1-\beta)\alpha) \\ &= \max (\alpha(2-\alpha), \alpha(2-\alpha) - (1-\beta)(\alpha(1-\alpha))) \\ &= \alpha(2-\alpha). \end{aligned}$$

Finally let $r = 2$. The values for $Q_i(2; 0, k)$ follow from (1.8)–(1.10) since $k > 2(m+1)$. Suppose $0 \leq n$ and the formula for $Q_0(2; n, n+k)$ is correct. Then if $n < m$,

$$\begin{aligned} Q_0(2; n+1, n+1+k) \\ &= \max (\alpha(1+a), Q_1(1; 0, n+1+k-m)) \\ &= \max (\alpha(1+a), \alpha(1+a)) = \alpha(1+a). \end{aligned}$$

If $n \geq m$,

$$\begin{aligned} Q_0(2; n+1, n+1+k) \\ &= \max (Q_0(2; n, n+k), Q_1(1; n-m+1, n-m+1+k)) \\ &= \max (Q_0(2; n, n+k), \alpha(2-\alpha)) = \alpha(2-\alpha), \end{aligned}$$

since $\alpha(1+a) < \alpha(2-\alpha)$.

Suppose $n \geq 0$ and the formulas for $Q_i(2; n, n+k)$, $i = 1$ or 2 , are correct.

Case 1. $n = 0$:

$$\begin{aligned} Q_1(2; 1, 1+k) &= \max (\alpha(1+a+a^2), \alpha + (1-\alpha)\alpha(1+a)) \\ &= \max (\alpha(1+a(1+a)), \alpha(1+(1-\alpha)(1+a))) \\ &= \alpha(1+(1-\alpha)(1+a)) = \alpha(1-\alpha(1+a)) + \alpha(1+a). \end{aligned}$$

$$Q_2(2; 1, 1 + k)$$

$$\begin{aligned} &= \max (\alpha(1 - \alpha(1 + a)) + \alpha(1 + a), \beta + (1 - \beta)\alpha(1 + a)) \\ &= \max (\alpha(1 - \alpha(1 + a)) + \alpha(1 + a), \beta(1 - \alpha(1 + a)) + \alpha(1 + a)) \\ &= \beta(1 - \alpha(1 + a)) + \alpha(1 + a). \end{aligned}$$

Case 2. $1 \leq n \leq m$:

$$Q_1(2; n + 1, n + 1 + k)$$

$$\begin{aligned} &= \max (\alpha(1 - \alpha(1 + a)) + \alpha(1 + a), \alpha\beta^n(1 - \alpha(1 + a)) + \alpha(1 + a)) \\ &= \alpha(1 - \alpha(1 + a)) + \alpha(1 + a); \end{aligned}$$

$$Q_2(2; n + 1, n + 1 + k)$$

$$\begin{aligned} &= \max(\alpha(1 - \alpha(1 + a)) + \alpha(1 + a), \beta^{n+1}(1 - \alpha(1 + a)) + \alpha(1 + a)) \\ &= \beta^{n+1}(1 - \alpha(1 + a)) + \alpha(1 + a). \end{aligned}$$

Case 3. $n = m + 1$:

$$Q_1(2; n + 1, n + 1 + k)$$

$$\begin{aligned} &= \max (\alpha(1 - \alpha(1 + a)) + \alpha(1 + a), \alpha\beta^{m+1}(1 - \alpha(1 + a)) + \alpha^2(1 + a) + (1 - \alpha)\alpha(2 - \alpha)) \\ &= \max (\alpha a(1 - \alpha) + \alpha(2 - \alpha), \alpha a(1 - \alpha\beta^{m+1}) + \alpha(2 - \alpha)) \\ &= \alpha a(1 - \alpha\beta^{m+1}) + \alpha(2 - \alpha); \end{aligned}$$

$$Q_2(2; n + 1, n + 1 + k)$$

$$\begin{aligned} &= \max (\alpha a(1 - \alpha\beta^{m+1}) + \alpha(2 - \alpha), \beta^{m+2}(1 - \alpha(1 + a)) + \beta\alpha(1 + a) + (1 - \beta)\alpha(2 - \alpha)) \\ &= \max (\alpha a(1 - \alpha\beta^{m+1}) + \alpha(2 - \alpha), \beta a(1 - \alpha\beta^{m+1}) + \alpha(2 - \alpha)) \\ &= \beta a(1 - \alpha\beta^{m+1}) + \alpha(2 - \alpha). \end{aligned}$$

Case 4. $m + 1 < n \leq m + t_0$:

$$Q_1(2; n + 1, n + 1 + k)$$

$$\begin{aligned} &= \max (\alpha a(1 - \alpha\beta^{m+1}) + \alpha(2 - \alpha), \alpha\beta^{n-(m+1)} a(1 - \alpha\beta^{m+1}) + \alpha(2 - \alpha)) \\ &= \alpha a(1 - \alpha\beta^{m+1}) + \alpha(2 - \alpha); \end{aligned}$$

$$\begin{aligned}
 Q_2(2; n+1, n+1+k) &= \max (\alpha a(1-\alpha\beta^{m+1})+\alpha(2-\alpha), \beta^{n+1-(m+1)} a(1-\alpha\beta^{m+1})+\alpha(2-\alpha)) \\
 &= \beta^{n+1-(m+1)} a(1-\alpha\beta^{m+1}) + \alpha(2-\alpha), \quad \text{if } n+1 \leq m+t_0, \\
 &= \alpha a(1-\alpha\beta^{m+1}) + \alpha(2-\alpha), \quad \text{if } n+1 > m+t_0.
 \end{aligned}$$

Case 5. $m+t_0 < n$:

$$\begin{aligned}
 Q_1(2; n+1, n+1+k) &= \max (\alpha a(1-\alpha\beta^{m+1})+\alpha(2-\alpha), \alpha^2 a(1-\alpha\beta^{m+1})+\alpha(2-\alpha)) \\
 &= \alpha a(1-\alpha\beta^{m+1}) + \alpha(2-\alpha);
 \end{aligned}$$

$$\begin{aligned}
 Q_2(2; n+1, n+1+k) &= \max (\alpha a(1-\alpha\beta^{m+1}) + \alpha(2-\alpha), \beta\alpha a(1-\alpha\beta^{m+1}) + \alpha(2-\alpha)) \\
 &= \alpha a(1-\alpha\beta^{m+1}) + \alpha(2-\alpha).
 \end{aligned}$$

The proof is now complete by induction on n .

Proof of Theorem 4: First we need a lemma.

Lemma. Let $r > 0$ and $1 \leq k \leq m+1$. Then

$$\begin{aligned}
 Q_0(r; n, n+k) &= 0, & \text{if } 0 \leq n < m+1-k, \\
 &= \alpha, & \text{if } m+1-k \leq n < m, \\
 &= Q_1(r-1; n-m, n-m+k) & \text{if } m \leq n.
 \end{aligned}$$

Proof. First suppose $0 \leq n < m+1-k$. Then $k < m+1$,

so $Q_0(r; 0, k) = 0$ by (1.5) and (1.8). If $n > 0$, then

$$\begin{aligned}
 Q_0(r; n, n+k) &= \max (Q_0(r; n-1, n-1+k), Q_1(r-1; 0, 0)) \\
 &= Q_0(r; n-1, n-1+k) = 0
 \end{aligned}$$

by induction. Now suppose $n = m+1-k$. If $m+1-k = 0$,

then $Q_0(r; n, n+k) = Q_0(r; 0, m+1) = \alpha$ by (1.5) and (1.8).

If $m + 1 - k > 0$, then

$$\begin{aligned} Q_0(r; n, n+k) &= \max (Q_0(r; m-k, m), Q_1(r-1; 0, 1)) \\ &= \max (0, \alpha) = \alpha. \end{aligned}$$

If $m + 1 - k < n \leq m$, then

$$\begin{aligned} Q_0(r; n, n+k) &= \max (Q_0(r; n-1, n-1+k), Q_1(r-1; 0, n-m+k)) \\ &= \max (Q_0(r; n-1, n-1+k), \alpha) \text{ by (1.6) and (1.9)} \\ &= \max (\alpha, \alpha) = \alpha \end{aligned}$$

by induction on n . Since

$$Q_0(r; m, m+k) = \alpha = Q_1(r-1; 0, k),$$

we have now established the lemma for $0 \leq n \leq m$.

Finally, suppose $n > m$ and the lemma is true for $n-1$.

Then

$$\begin{aligned} Q_0(r; n, n+k) &= \max (Q_1(r-1; n-1-m, n-1-m+k), Q_1(r-1; n-m, n-m+k)) \\ &= Q_1(r-1; n-m, n-m+k) \end{aligned}$$

by (1.3). This completes the proof of the lemma.

The theorem for $r = 0$ follows from Theorem 5.

Suppose $r > 0$ and the formulas for $Q_i(r-1; n, n+k)$, $i = 0, 1, 2$, are correct for all $n \geq 0$. Then the formula for $Q_0(r; n, n+k)$ for all $n \geq 0$ follows from this inductive assumption and the lemma. The formulas for $Q_i(r; n, n+k)$, $i = 1$ or 2 , follow from (1.9) and (1.10) for $n = 0$.

Suppose that they are correct for some $n \geq 0$, and we will prove them for $n + 1$.

Case 1. $0 \leq n < m + 1 - k$:

$$\begin{aligned} Q_1(r; n + 1, n + 1 + k) &= \max \{ \alpha, \alpha\beta^n + (1 - \alpha) \cdot 0 \} = \alpha ; \\ Q_2(r; n + 1, n + 1 + k) &= \max \{ \alpha, \beta^{n+1} + (1 - \beta) \cdot 0 \} = \beta^{n+1} . \end{aligned}$$

Case 2. $n = m + 1 - k$:

$$\begin{aligned} Q_1(r; n + 1, n + 1 + k) &= \max \{ \alpha, \alpha\beta^{m+1-k} + (1 - \alpha)\alpha \} \\ &= \max \{ \alpha, \alpha(1 + b) \} \\ &= \alpha(1 + b) = \alpha \left[1 + \frac{b(1 - a)}{1 - a} \right] ; \\ Q_2(r; n + 1, n + 1 + k) &= \max \{ \alpha(1 + b), \beta^{m+2-k} + (1 - \beta)\alpha \} \\ &= \max \{ \alpha + \alpha b, \alpha + \beta b \} \\ &= \alpha + \beta b \\ &= \alpha \left[1 + \frac{b(1 - a^0)}{1 - a} \right] + ba^0 \beta^{n+1-(m+1-k)} . \end{aligned}$$

Case 3. $j(m + 1) - k < n < (j + 1)(m + 1) - k$ and $1 \leq j \leq r$:

Set $i = n - (j(m + 1) - k)$. Then

$$\begin{aligned} Q_1(r; n + 1, n + 1 + k) &= \max \left\{ \alpha \left[1 + \frac{b(1 - a^j)}{1 - a} \right], \alpha^2 \left[1 + \frac{b(1 - a^{j-1})}{1 - a} \right] + \alpha ba^{j-1} \beta^i \right. \\ &\quad \left. + (1 - \alpha) \alpha \left[1 + \frac{b(1 - a^{j-1})}{1 - a} \right] \right\} \\ &= \max \left\{ \alpha \left[1 + \frac{b(1 - a^j)}{1 - a} \right], \alpha \left[1 + \frac{b(1 - a^j)}{1 - a} \right] - \alpha(1 - \beta^i) ba^{j-1} \right\} \\ &= \alpha \left[1 + \frac{b(1 - a^j)}{1 - a} \right] ; \end{aligned}$$

$$\begin{aligned} Q_2(r; n + 1, n + 1 + k) &= \max \left\{ \alpha \left[1 + \frac{b(1 - a^j)}{1 - a} \right], \beta \alpha \left[1 + \frac{b(1 - a^{j-1})}{1 - a} \right] + ba^{j-1} \beta^{i+1} \right. \\ &\quad \left. + (1 - \beta) \alpha \left[1 + \frac{b(1 - a^{j-1})}{1 - a} \right] \right\} \end{aligned}$$

$$\begin{aligned}
 &= \max \left\{ \alpha \left[1 + \frac{b(1 - a^{j-1})}{1 - a} \right] + \alpha b a^{j-1}, \right. \\
 &\quad \left. \alpha \left[1 + \frac{b(1 - a^{j-1})}{1 - a} \right] + \beta^{i+1} b a^{j-1} \right\} \\
 &= \alpha \left[1 + \frac{b(1 - a^{j-1})}{1 - a} \right] + \beta^{i+1} b a^{j-1},
 \end{aligned}$$

since $\beta^{i+1} \geq \beta^{m+1} > \alpha$.

Case 4. $n = (j + 1)(m + 1) - k$ and $1 \leq j < r$:

$Q_1(r; n + 1, n + 1 + k)$

$$\begin{aligned}
 &= \max \left\{ \alpha \left[1 + \frac{b(1 - a^j)}{1 - a} \right], \alpha^2 \left[1 + \frac{b(1 - a^{j-1})}{1 - a} \right] + \alpha b a^{j-1} \beta^{m+1} \right. \\
 &\quad \left. + \alpha(1 - \alpha) \left[1 + \frac{b(1 - a^j)}{1 - a} \right] \right\} \\
 &= \max \left\{ \alpha \left[1 + \frac{b(1 - a^j)}{1 - a} \right], \alpha \left[1 + \frac{b(1 - a^{j+1})}{1 - a} \right] \right\} \\
 &= \alpha \left[1 + \frac{b(1 - a^{j+1})}{1 - a} \right];
 \end{aligned}$$

$Q_2(r; n + 1, n + 1 + k)$

$$\begin{aligned}
 &= \max \left\{ \alpha \left[1 + \frac{b(1 - a^{j+1})}{1 - a} \right], \alpha \beta \left[1 + \frac{b(1 - a^{j-1})}{1 - a} \right] + b a^{j-1} \beta^{m+2} \right. \\
 &\quad \left. + (1 - \beta) \alpha \left[1 + \frac{b(1 - a^j)}{1 - a} \right] \right\} \\
 &= \max \left\{ \alpha \left[1 + \frac{b(1 - a^j)}{1 - a} \right] + \alpha b a^j, \alpha \left[1 + \frac{b(1 - a^j)}{1 - a} \right] + \beta b a^j \right\} \\
 &= \alpha \left[1 + \frac{b(1 - a^j)}{1 - a} \right] + \beta b a^j.
 \end{aligned}$$

Case 5. $r(m + 1) - k < n < r(m + 1) - k + t_0$:

Let $i = n - [r(m + 1) - k]$. Then

$$Q_1(r; n+1, n+1+k)$$

$$\begin{aligned} &= \max \left\{ \alpha \left[1 + \frac{b(1-a^r)}{1-a} \right], \alpha^2 \left[1 + \frac{b(1-a^{r-1})}{1-a} \right] \right. \\ &\quad \left. + \alpha b a^{r-1} \beta^i + (1-\alpha) \alpha \left[1 + \frac{b(1-a^{r-1})}{1-a} \right] \right\} \\ &= \max \left\{ \alpha \left[1 + \frac{b(1-a^r)}{1-a} \right], \alpha \left[1 + \frac{b(1-a^r)}{1-a} \right] \right. \\ &\quad \left. - \alpha(1-\beta^i) b a^{r-1} \right\} \\ &= \alpha \left[1 + \frac{b(1-a^r)}{1-a} \right] ; \end{aligned}$$

$$Q_2(r; n+1, n+1+k)$$

$$\begin{aligned} &= \max \left\{ \alpha \left[1 + \frac{b(1-a^r)}{1-a} \right], \beta \alpha \left[1 + \frac{b(1-a^{r-1})}{1-a} \right] \right. \\ &\quad \left. + b a^{r-1} \beta^{i+1} + (1-\beta) \alpha \left[1 + \frac{b(1-a^{r-1})}{1-a} \right] \right\} \\ &= \max \left\{ \alpha \left[1 + \frac{b(1-a^{r-1})}{1-a} \right] + \alpha b a^{r-1}, \alpha \left[1 + \frac{b(1-a^{r-1})}{1-a} \right] \right. \\ &\quad \left. + \beta^{i+1} b a^{r-1} \right\} \\ &= \alpha \left[1 + \frac{b(1-a^{r-1})}{1-a} \right] + \beta^{i+1} b a^{r-1}, \quad \text{if } i < t_0 - 1, \\ &= \alpha \left[1 + \frac{b(1-a^r)}{1-a} \right], \quad \text{if } i = t_0 - 1, \end{aligned}$$

since for $i < t_0 - 1$, $\beta^{i+1} \geq \beta^{t_0-1} > \alpha$, while for $i = t_0 - 1$, $\beta^{i+1} = \beta^{t_0} \leq \alpha$.

Case 6. $n \geq r(m+1) - k + t_0$:

$$\text{Let } A = \alpha \left[1 + \frac{b(1-a^{r-1})}{1-a} \right] \quad \text{and} \quad B = \alpha \left[1 + \frac{b(1-a^r)}{1-a} \right] .$$

Then $A < B$, so

$$\begin{aligned} Q_1(r; n+1, n+1+k) &= \max \{B, \alpha B + (1-\alpha)A\} \\ &= \max \{B, B - (1-\alpha)(B-A)\} = B, \end{aligned}$$

and

$$\begin{aligned} Q_2(r; n+1, n+1+k) &= \max \{B, \beta B + (1-\beta)A\} \\ &= \max \{B, B - (1-\beta)(B-A)\} = B. \end{aligned}$$

By induction on n , the theorem is proved for r and any $n \geq 0$. By induction on r , the theorem is established.

REFERENCE

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