ON INTEGRATION OF HAMILTON-JACOBI
PARTIAL DIFFERENTIAL EQUATION

BY
ABOLGHASSEM GHAFFARI

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SUMMARY

The Hamilton-Jacobi partial differential equation is fundamental in planetary and lunar theories. The solution of many perturbations theories and also numerous problems of the control theory is reduced to the problem of solving the Hamilton-Jacobi equation.

The application of the Hamilton-Jacobi theory to dynamical systems is based on the assumption that the canonical equations can be solved by Jacobi's theorem, which requires essentially the knowledge of any complete integral of Hamilton-Jacobi equation.

The purpose of this paper is to illustrate a method leading to a complete integral of Hamilton-Jacobi equation, which is based on the variational principle and transversality condition. An application of this method to a particular case of Hamilton-Jacobi equation is described, and the classical example of wave fronts or parallel surfaces is considered.

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INTRODUCTION

The equations of motion of a system of n mass-points in terms of generalized coordinates are given [1][1] by Lagrange's equations. These equations are n differential equations of the second order with n unknown functions.

The Lagrangian equations of motion have not been developed with the special reference to the problems of celestial mechanics, and are not yet adapted to the needs of the space mechanics. They have to be transformed into a system of 2n first-order differential equations called Hamiltonian or canonical equations of motion.

In canonical form the 2n variables appear separated into two sets, essentially different in character, namely, into n coordinates of position and n coordinates of momentum. The canonical equations of motion are of the first order and therefore more tractable than the Lagrangian equations. These equations do not facilitate the solution of particular problems and there is no technique known for solving them in general; but, they form the basis for most theoretical discussion since they are amendable to the very considerable body of theory that has been built around first order equations.

The reduction of the order of a differential canonical system can, in theory, be performed by obtaining first integrals of the system.

There are many classical methods such as Hamilton-Jacobi theory, Lindstedt's method, Whittaker's method, Delauny's method, the von Zeipel's method, and others for reducing the order of canonical systems.

The most general technique which has been applied quite often to the problems of celestial and quantum mechanics is the Hamilton-Jacobi method which consists in obtaining a contact-transformation[2] such that the new Hamiltonian is identically zero, and therefore the new variables are all constants.

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[1]Figures in brackets indicate the literature references at the end of this paper.

[2]For a rigorous and general principles of classical dynamics refer to Whittaker's Analytical Dynamics [1] Chapters X-XII.

The integration of the canonical equations of motion can be made to depend on the solution of a first-order partial differential equation called Hamilton-Jacobi (H-J) equation, which does not contain the unknown function (generating function) explicitly.

The integration of canonical equations is thus reduced to the integration of a first-order partial differential equation, which in no way means a simplification of the problem. For the treatment of partial differential equations is usually more complicated than that of ordinary differential equations. But, due to the important applications of H-J equation in dynamical problems and also in the theory of optimal control, the discussion of H-J equation has been very successful in illustrating the intrinsic features of the problem concerned.

The canonical 2n equations can be integrated by Jacobi's theorem which is based on the previous knowledge of any complete integral of H-J equation, a solution containing n arbitrary constants in addition to the additive constant. The integration of canonical equations is, therefore, connected to the general theory of partial differential equations of the first order. One realizes at once that when Hamiltonian function is time-independent, the energy integral constitutes a first integral, and the canonical equations are the differential equations of the characteristics of the H-J equation.

General application of H-J method to planetary and lunar theories is based on the assumption that the canonical equations can be solved by Jacobi's theorem, which requires essentially a complete integral of H-J equation. To obtain, on the other hand, a complete integral of H-J equation one has to set up, formally, its characteristic system of differential equations which is the same as the canonical system of equations. Thus, it seems there is a plausible paradox. Nevertheless, Jacobi's method has been very helpful in many dynamical problems, for one can find out quite often, by some direct procedures and separation of variables, a complete integral of H-J equation without using the canonical equations.

Hamilton-Jacobi equation can be integrated, in general, by the application of Charpit's method which provides an infinite number of complete integrals. Nevertheless, the method of the separation of variables, subject to Levi-Civita's conditions, can also be applied.

The purpose of this paper is to illustrate a different method leading to a complete integral of H-J equation, which is based on the variational principle and transversality condition. An application of this method to a particular case of H-J equation is described, and the example of parallel surfaces or wave fronts is considered. The application of this method to some problems of space mechanics and optimal control theory will be treated later on.

METHOD OF INTEGRATION

Extermals with Variable End-Points on Fixed Curves

Let us consider the functional

$$I[E_{AB}] = \int_{A}^{B} f(x,\ y,\ z,\ \dot{y},\ \dot{z})\,dx \tag{1}$$

taken along a variable admissible arc E_{AB} (Figure 1) whose end-points A (a, c) and B (b, d) describe the two fixed and given curves C and D.

Dot denotes the derivatives with respect to x. The coordinates x, y, and z along the arc E_{AB} may be taken as functions of the parameter time t.
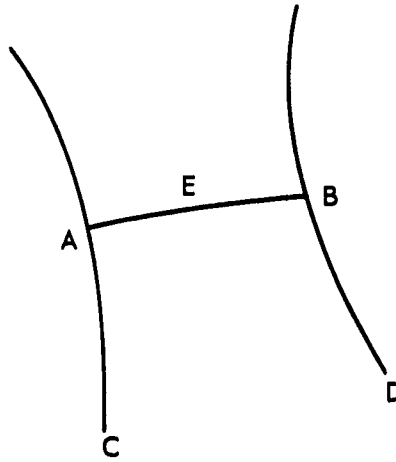

Figure 1

The necessary conditions [2][3] that the admissible arc E_{AB} extremizes the functional $I[E_{AB}]$, i.e.

$$\delta I[E_{AB}] = 0,$$

where δ indicates the first variation are that:

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[3]For more details, see Bliss [2], Chapter VI.

1. The admissible arc E_{AB} represents a non-singular extremal of the functional I, i.e., y(x) and z(x) must have continuous first and second order derivatives on (a,b), and E_{AB} satisfies the two Euler-Lagrange differential equations:

$$\left. \begin{array}{r} \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial \dot{y}}\right) - \dfrac{\partial f}{\partial y} = 0 \\[4mm] \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial \dot{z}}\right) - \dfrac{\partial f}{\partial z} = 0 \end{array} \right\} \tag{2}$$

where the function f is supposed to be twice continuously differentiable with respect to each arguments x, y, z, $\dot{y}$ and $\dot{z}$.

2. The admissible arc E_{AB} satisfies the transversality condition

$$\left(f - \dot{y}\,\frac{\partial f}{\partial \dot{y}} - \dot{z}\,\frac{\partial f}{\partial \dot{z}}\right)dx + \frac{\partial f}{\partial \dot{y}}\,dy + \frac{\partial f}{\partial \dot{z}}\,dz = 0 \tag{3}$$

or

$$f = \left(\dot{y} - \frac{dy}{dx}\right)\frac{\partial f}{\partial \dot{y}} + \left(\dot{z} - \frac{dz}{dx}\right)\frac{\partial f}{\partial \dot{z}}$$

at its intersections point A and B with the fixed curves C and D, i.e., the non-singular extremal arc E_{AB} must cut the curves C and D transversally at the endpoint A and B respectively.

Needless to say that the arguments x, y, z, $\dot{y}$, $\dot{z}$ belong to the extremal E_{AB} and dx, dy and dz represent a direction tangent to the curves C and D at the points A and B respectively.

The general solution of Euler's equations (2) form a four-parameter family of extremals. In order to determine these four parameters one has to verify that:

1. The general solution satisfies the two unknown abscissas a, b of the varying end-points A and B.

2. The transversality condition is satisfied at both end-points A and B.

Elimination of a and b between the six conditions obtained leads to four relations for the determination of the four parameters.

The Euler equations (2) are of the second order and are, therefore, equivalent to the following system of the first order:

$$\frac{dy}{dx} = \dot{y} \ , \quad \frac{dz}{dx} = \dot{z}$$

$$\left. \begin{array}{l} \frac{\partial^2 f}{\partial \dot{y} \partial x} + \frac{\partial^2 f}{\partial \dot{y} \partial y} \dot{y} + \frac{\partial^2 f}{\partial \dot{y} \partial z} \dot{z} + \frac{\partial^2 f}{\partial \dot{y} \partial \dot{y}} \frac{d\dot{y}}{dx} + \frac{\partial^2 f}{\partial \dot{y} \partial \dot{z}} \frac{d\dot{z}}{dx} = \frac{\partial f}{\partial y} \\[4mm] \frac{\partial^2 f}{\partial \dot{z} \partial x} + \frac{\partial^2 f}{\partial \dot{z} \partial y} \dot{y} + \frac{\partial^2 f}{\partial \dot{z} \partial z} \dot{z} + \frac{\partial^2 f}{\partial \dot{z} \partial \dot{y}} \frac{d\dot{y}}{dx} + \frac{\partial^2 f}{\partial \dot{z} \partial \dot{z}} \frac{d\dot{z}}{dx} = \frac{\partial f}{\partial z} \end{array} \right\} \quad (4)$$

where $\dot{y}$ and $\dot{z}$ are considered as new unknown functions.

In system (4) the arguments y, z, $\dot{y}$, $\dot{z}$ figure in a nonsymmetric manner and the equations are not solved for the derivative of the all unknown functions.

The application of Legendre's transformation [3][4]

$$\left. \begin{array}{l} p = \frac{\partial f}{\partial \dot{y}} \ , \quad q = \frac{\partial f}{\partial \dot{z}} \\[6mm] h(x, \ y, \ z, \ p, \ q) = \dot{y} \, \frac{\partial f}{\partial \dot{y}} + \dot{z} \, \frac{\partial f}{\partial \dot{z}} - f \end{array} \right\} \quad (5)$$

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[4]See [3] Chapter IV, and [4] Chapter II.

leads to a more satisfactory first-order system:

$$\frac{dy}{dx} = \frac{\partial h}{\partial p} , \frac{dz}{dx} = \frac{\partial h}{\partial q}$$

$$\frac{dp}{dx} = - \frac{\partial h}{\partial y} , \frac{dq}{dx} = - \frac{\partial h}{\partial z}$$

(6)

called the canonical differential equations of the variational problem, provided the Hessian of $f \neq 0$, i.e.

$$D = \left\| \frac{\partial^2 f}{\partial \dot{y} \partial \dot{z}} \right\| \neq 0. \tag{7}$$

Taking into account of Legendre's transformation (5) the differential of the functional I, taken along a variable non-singular extremal arc E_{AB} whose end-points A and B describe two fixed curves C and D, can be written [2]

$$dI = [-h(x, y, z, p, q)dx + p \, dy + q \, dz]_A^B . \tag{8}$$

If we limit ourselves to the case where the extremal E_{AB} is transversal only at curve C at A, then

$$dI = [-h \, dx + p \, dy + q \, dz]_B . \tag{9}$$

Suppose now by each point B of curve D passes one and only one extremal, then the value of I as well as the values of p and q depend on the coordinates x, y, z of variable end-point B.

Equation (9) can be written:

$$\frac{\partial I}{\partial x} = - h(x, y, z, p, q) , \frac{\partial I}{\partial y} = p, \frac{\partial I}{\partial z} = q$$

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or

$$\frac{\partial I}{\partial x} + h\left(x, \ y, \ z, \ \frac{\partial I}{\partial y}, \ \frac{\partial I}{\partial z}\right) = 0 \qquad (10)$$

Therefore the functional I satisfies a first-order partial differential equation solved for $\partial I/\partial x$, which does not contain the unknown functional I explicitly. This equation is called Hamilton-Jacobi (H-J) partial differential equation.

Conversely, this result leads to a general method of integration of first-order partial differential equations not containing the unknown function. In fact, in solving such an equation for one of its derivative, say $\partial W/\partial x$, it takes the form:

$$\frac{\partial W}{\partial x} + H\left(x, \ y, \ z, \ \frac{\partial W}{\partial y}, \ \frac{\partial W}{\partial z}\right) = 0. \qquad (11)$$

The corresponding canonical equations are

$$\left.\begin{array}{c} \dfrac{dy}{dx} = \dfrac{\partial H}{\partial p}, \ \dfrac{dz}{dx} = \dfrac{\partial H}{\partial q} \\[4mm] \\[4mm] \dfrac{dp}{dx} = -\dfrac{\partial H}{\partial y}, \ \dfrac{dq}{dx} = -\dfrac{\partial H}{\partial z} \end{array}\right\} \qquad (12)$$

where $H = H(x, y, z, p, q)$.

The integration of (12) gives y, z, p and q in terms of x and four arbitrary constants.

Now we consider the functional

$$I = \int_A^B f \, dx$$

where f is the function of x, y, z, p and q defined by Legendre's transformation (5) and the canonical equations (6). Therefore

$$f = p \frac{\partial H}{\partial p} + q \frac{\partial H}{\partial q} - H . \tag{13}$$

Substituting in f for y, z, p and q their expressions (solution of 6) in terms of x, we will get the functional I in terms of the coordinates of B, and it satisfies the H-J equation (11). Thus we obtain, by this method, a solution of (6) which depends on the arbitrary constants or functions, the coordinates of A or the functions determining the curve C. It is supposed, of course, that the equation to be solved may be considered as a consequence of a variational problem, that is to say, in applying Legendre's transformation to Euler's equation one can get canonical equations and vice versa. Therefore the resolution of the system

$$\frac{dy}{dx} = \frac{\partial H}{\partial p} , \frac{dz}{dx} = \frac{\partial H}{\partial q}$$

for p and q is possible if the Hessian of H $\neq$ 0, i.e.

$$\left(\frac{\partial^2 H}{\partial p \partial p} \right) \left(\frac{\partial^2 H}{\partial q \partial q} \right) - \left(\frac{\partial^2 H}{\partial p \partial q} \right)^2 \not\lt 0. \tag{14}$$

This implies that the function H (Hamiltonian function) must be different from an homogeneous function of the first degree in p and q, which excludes the case f = 0.

As a result we can deduct that this method provides us the integrals of the given equation. In order to obtain a general solution of H-j equation one can take an extremal transversal at curve C at the end-point A. If the end-point A is taken as a fixed point, one obtains for the functional I an integral whose integrand depends on x and four arbitrary constants, the same number of arbitrary constants which arises from the integration of the canonical system. These arbitrary constants are the initial values of y, z, p and q for x = a. The initial values of x, y and z represent the coordinates of the end-point A. It remains two more arbitrary constants which will be used for describing that the extremal passes also through the end-point B.

Therefore, in varying the end-point A on curve C, one obtains an integral depending of these two arbitrary constants which are the coordinates of A and one additive constant γ, since W + γ is also a solution.

PARTICULAR CASE OF HAMILTON-JACOBI EQUATION

The application of the preceding method to the following special case of Hamilton-Jacobi equation:

$$\frac{\partial W}{\partial t} + H \left(\frac{\partial W}{\partial x} , \frac{\partial W}{\partial y} \right) = 0 \tag{15}$$

gives easily the most general expression for its complete integral.

In fact, setting

$$\frac{\partial W}{\partial x} = p , \quad \frac{\partial W}{\partial y} = q$$

we get

$$f = p \frac{\partial H}{\partial p} + q \frac{\partial H}{\partial q} - H = R(p, q) \tag{16}$$

where R is a known function of p and q.

The Hamiltonian function is only a function of generalized momenta p, q and the corresponding canonical equations have the two first integrals

$$p = cont. \ \Box \ a, \quad q = cont. = b \tag{17}$$

The momenta are, therefore, preserved under a contact-transformation, and the cyclic coordinates x, y are expressible as linear functions of time:

$$x - \alpha = \frac{\partial H(a, b)}{\partial a} (t - t_0), \quad y - \beta = \frac{\partial H(a, b)}{\partial b} (t - t_0) \tag{18}$$

where t_0 and t correspond to the fixed end-point A (α, β) and variable end-point B (x, y) respectively.

The trajectories (18) represent the characteristics of (15) and also the extremals of the functional

$$I\,[E_{AB}] \;=\; \int_A^B f\,dt \;=\; \int_A^B R(a,\,b)\,dt \;=\; (t - t_0)\,R(a,\,b)$$

If the Hessian of $H \neq 0$, i.e.,

$$\left(\frac{\partial^2 H}{\partial a\,\partial a}\right)\left(\frac{\partial^2 H}{\partial b\,\partial b}\right) - \left(\frac{\partial^2 H}{\partial a\,\partial b}\right) \neq 0, \tag{19}$$

a and b can be obtained from the system (18).

In fact, setting $\xi = \partial H/\partial a$, $\eta = \partial H/\partial b$ we get, according to the properties of Legendre's transformation

$$a = \frac{\partial f(\xi,\,\eta)}{\partial \xi}\;,\quad b = \frac{\partial f(\xi,\,\eta)}{\partial \eta}$$

$$R(a,\,b) = f(\xi,\,\eta)$$

Therefore

$$I = (t - t_0)R(a,\,b) = (t - t_0)\,f(\xi,\,\eta)$$

$$= (t - t_0)\,f\left[\frac{x - \alpha}{t - t_0},\,\frac{y - \beta}{t - t_0}\right] = F(t - t_0,\,x - \alpha,\,y - \beta),$$

which gives the most general expression of the complete integral in the form

$$W = \gamma + F(t - t_0,\,x - \alpha,\,y - \beta) \tag{20}$$

where F is a certain homogeneous function of the first degree in $t - t_0$, $x - \alpha$, $y - \beta$, and γ is an additive constant to W, since the equation still stands if we replace W by $W + \gamma$.

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The function F is obtained, through the known Hamiltonial H, by

$$F(t - t_0, \ x - \alpha, \ y - \beta) = (t - t_0) \ f\left[\frac{x - \alpha}{t - t_0}, \ \frac{y - \beta}{t - t_0}\right]$$

and the relation (16).

As the motion of the free particle governed by equation (15) is conservative its solution (complete integral) can be obtained by applying the method of separation of the variables. This method gives the complete integral

$$W = -ht + \alpha x + \beta y + \gamma \tag{21}$$

which is a special form of (20), where α, β, γ are arbitrary constants and h is another constant called energy integral.

EXAMPLE

Because of the analogy between the motion of particles and the propagation of light rays one can also apply the method discussed in this paper to the special form of eikonal equation.

$$(\nabla W)^2 = \left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 + \left(\frac{\partial W}{\partial z}\right)^2 = 1 \tag{22}$$

which is fundamental in geometrical optics. Each solution of (22) describes a definite beam of light rays, and the wave surfaces are the surfaces of constant eikonal, i.e. the parallel surfaces of the form

$$W(x, y, z) = \text{cont.}$$

Equation (22) has the complete integral

$$W = \alpha x + \beta y + (1 - \alpha^2 - \beta^2)^{1/2} \ z + \gamma \tag{23}$$

or more specifically

$$W = [(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2]^{1/2},$$ (24)

which represents a family of parallel surfaces with α, β, γ and w as arbitrary constants.

From the eikonal equation (22) is easily deduced that we are dealing with the ordinary Euclidean metric

$$ds = (dx^2 + dy^2 + dz^2)^{1/2},$$

the geodesics of which are of course the straight lines.

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