DAMPING OF

INTERSTELLAR PLASMA WAVES
BY THE COSMIC RAY GAS

BY

DEREK A. TIDMAN

AUGUST 1965

GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

DAMPING OF INTERSTELLAR PLASMA WAVES BY

THE COSMIC RAY GAS

by

Derek A. Tidman*
Goddard Space Flight Center
Greenbelt, Maryland

(Communicated to the Astrophysical Journal)

(August, 1965)

*Permanent address: Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Maryland.

ABSTRACT

We discuss the propagation of waves in a thermal plasma that co-exists with a tenuous gas of relativistic suprathermal particles. Low frequency modes such as the magnetosonic wave and Alfvèn wave experience a damping due to the suprathermal particles. Waves propagating in such regions as the interstellar medium or the plasma of a supernova remnant would be significantly damped by the cosmic radiation fluxes as a result of this mechanism. Although the decay of disturbances in the interstellar medium (for example produced by stellar winds or old supernova remnants) would be hastened by this process, it is unlikely that it represents a significant energy transfer to the cosmic radiation. For the magnetosonic wave propagating directly across a magnetic field the damping has its origin in a cyclotron resonance between a wave of frequency $\omega \left(\omega^2 \ll \Omega_{0i}^2\right)$ and those relativistic particles for which $\Omega_i = \left(\Omega_{0i}/\gamma\right) \cong \omega$, where $\gamma = \left(1 - v^2/c^2\right)^{-1/2}$ and Ω_{0i} is the nonrelativistic ion cyclotron frequency.

CONTENTS

LIST OF ILLUSTRATIONS

DAMPING OF INTERSTELLAR PLASMA WAVES BY
THE COSMIC RAY GAS

by Derek A. Tidman

1. INTRODUCTION

Many plasmas in nature contain both a thermal and a suprathermal population of particles. Some obvious examples are the cosmic radiation which coexists with the interstellar or interplanetary plasma, or the fluxes of relativistic particles in the Crab nebula or a solar flare, etc. It has been emphasized recently by Parker (1965) that the presence of such energetic particles in a thermal plasma can play an important role in the propagation properties of waves in the medium. This may be the case even though the number density of the energetic particles is much less than that of the thermal plasma since the energy density (pressure) of the suprathermal gas may still exceed that of the thermal plasma.

In this paper we point out that a relativistic suprathermal gas also gives rise to an important damping of waves in a thermal plasma. This damping is automatically excluded if one uses moment or fluid equations as in Parker's treatment (1965). It emerges naturally from the relativistic Vlasov-Maxwell equations in a similar way to the conventional Landau damping (see for example the discussion and references in Chapter 10 of Montgomery and Tidman, 1964). We have calculated this damping for some "hydromagnetic modes," i.e., for waves of frequency much less than the ion cyclotron frequency since such waves are thought to be important for energy transport through the interstellar medium. However it should be born in mind that such damping mechanisms as we are concerned with have their origin in wave-particle resonance effects and are likely to occur in one form or another for astrophysical plasma waves over much wider frequency ranges. They give rise to a damping which is distinct from the viscous damping treated earlier (Parker, 1955; Piddington, 1957).

Consider first the mechanism for cyclotron damping (Stix, 1962) of a wave of frequency $+\omega_r$ and wave number k propagating at an angle θ across a magnetic field B_0. Assume the wave has a nonzero component of its electric vector perpendicular to B_0. Then some of the distribution of particles with velocity v will drift through the wave with just the right velocity along B_0 to experience the wave electric field at a harmonic, n, of their cyclotron frequency. This occurs if (see Figure 1)

$$\left(\frac{\omega_r - \frac{n\Omega_{0i}}{\gamma}}{k \cos \theta} \right) = v_{\parallel} = \frac{v \cdot B_0}{B_0} \, ,$$

and gives rise to a resonant transfer of energy between the wave and these particles and to a net wave damping. We have only considered the ions for simplicity and $\Omega_{0i} = e|B_0|/m_i c$, and $\gamma = \left(1 - v^2/c^2\right)^{-1/2}$. Suppose now one takes the nonrelativistic limit $c \to \infty$, $\gamma \to 1$, and considers the low-frequency hydromagnetic part of the spectrum, $\omega_r^2 \ll \Omega_{0i}^2$. It is then

1

clear since necessarily $|v_{\parallel}| < c$ that only waves propagating in the range $0 < \theta < \theta_M$ $= \cos^{-1}(\Omega_{0i}/kc)$ can experience such cyclotron damping. In this case there is __zero__ damping for waves propagating directly across B_0 $(\theta = \pi/2)$ as is well known.

Suppose next we return to consider a relativistic situation in which the suprathermal tail extends well into the relativistic region. Then some particles in the range $|v_{\parallel}| < c$ can still satisfy the resonance condition if $\left|(\omega_r - n\Omega_{0i}/\gamma)/k\cos\theta\right| < c$. This remains true as $\theta \to \pi/2$ if we choose $\omega_r = n\Omega_{0i}/\gamma$. It gives rise to an essentially __relativistic damping__ of low frequency waves $\left(\omega_r^2 \ll \Omega_{0i}^2\right)$ by highly relativistic particles which experience cyclotron resonance with the wave at their __reduced__ cyclotron frequency $\Omega_i = \Omega_{0i}/\gamma \simeq \omega_r < \Omega_{0i}$. The physical mechanism for this damping is discussed further following its derivation in Equation (32).

In the following sections we shall calculate the damping decrement for the two cases of perpendicular $(\theta = \pi/2)$ and parallel $(\theta = 0)$ propagation, although of course the same effect will occur for arbitrary angles θ. In particular we consider the low frequency magnetosonic wave $(\theta = \pi/2)$ and Alfvén wave $(\theta = 0)$, both of which have phase velocities $\omega_r/k = V_A$ where V_A is the Alfvén speed. If the suprathermal particle energy density W becomes important the magnetosonic wave dispersion relation becomes modified. For the special case of perpendicular propagation it then corresponds to Parker's "suprathermal mode" (Parker, 1965) with a dispersion relation $\omega_r/k \simeq V_A \left(1 + W/n_0 m_i V_A^2\right)^{1/2}$ where m_i and n_0 are the ion mass and thermal number density respectively.

For the case of hydromagnetic waves propagating through the interstellar medium the damping which is produced by the cosmic ray gas depends sensitively on the direction of propagation θ. The damping lengths are also frequency dependent and range from a value much less than a light year up to indefinitely high values as $\omega_r \to 0$. The phenomenon thus provides a mechanism by which energy is continuously transferred from waves to

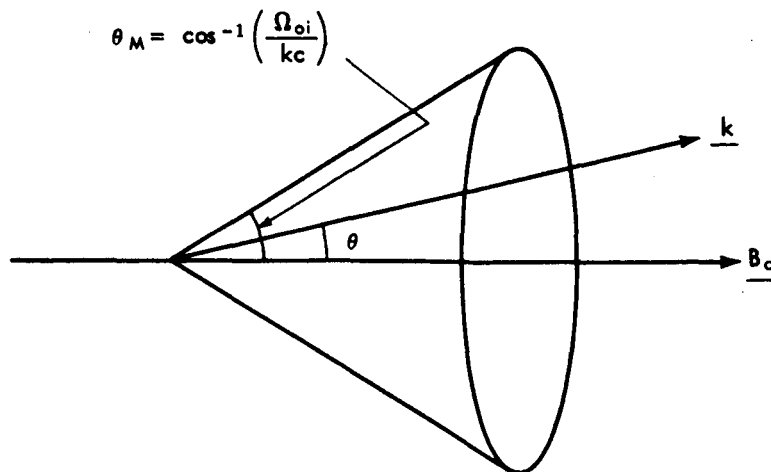$$\theta_M = \cos^{-1}\left(\frac{\Omega_{0i}}{kc}\right)$$

Figure 1—Cone of wave vectors for which cyclotron wave damping can occur for a nonrelativistic plasma.

2

suprathermal particles in such regions. In this very wide sense it is thus related to the Fermi mechanism - just as the attendant plasma wave damping can be regarded as a relativistic form of cyclotron Landau damping. However in the classical Fermi mechanism particles are visualized as being reflected in collisions with nonlinear wavefronts of magnetic field and travel freely between such collisions. In our case the suprathermal particles are a part of the plasma and are trapped along a magnetic field B_0 together with the thermal plasma. A wave then propagates past these particles through the medium continuously depositing energy preferentially into the suprathermal particles through the cyclotron mechanism. The waves involved are natural modes of the plasma and one does not have to construct models for them.

It should be pointed out of course that the Fermi mechanism is a more appropriate model for the stochastic acceleration of particles in a region of violent nonlinear plasma turbulence such as a supernova. Our calculations are appropriate to the dissipation of small-amplitude waves in the cosmic ray gas such as might be excited by the old decaying remnants of a supernova or by stellar winds, etc. Such wave damping however represents only a small energy input into the cosmic ray gas compared to that available for particle acceleration in the early phase of violent turbulence in a supernova.

2. BASIC EQUATIONS

The relativistic Vlasov-Maxwell equations are,

$$\frac{\partial f_a}{\partial t} + v \cdot \frac{\partial f_a}{\partial x} + e_a \left(E + \frac{1}{c} v \times B \right) \cdot \frac{\partial f_a}{\partial p} = 0 , \tag{1}$$

$$\frac{\partial}{\partial x} \times E = -\frac{1}{c} \frac{\partial B}{\partial t} , \tag{2}$$

$$\frac{\partial}{\partial x} \times B = \frac{1}{c} \frac{\partial E}{\partial t} + \frac{4\pi}{c} \sum n_0 e_a \int f_a v \, dp , \tag{3}$$

where $f_a(p)$ is the momentum distribution function for the ath species and is normalized so that the probable number density $n_a(x, t)$ of the ath species is

$$n_a = n_0 \int f_a \, dp ,$$

where n_0 is the average number density which is taken to be the same for both species (we shall only consider an electron-ion plasma for which $a = e$ or i). The relativistic

3

particle momentum is given by,

$$p = m_\alpha v\gamma , \qquad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \equiv \left(1 + \frac{p^2}{m_\alpha^2 c^2}\right)^{1/2} . \qquad (4)$$

Note that the distribution function $f_\alpha(p)$ contains <u>both</u> the "thermal" and any relativistic "suprathermal" populations of particles which may be present.

Now in the usual way we consider the behaviour of a small amplitude disturbance in a homogeneous plasma in which there is a spatially uniform magnetic field B_0 and for which the unperturbed distributions $f_{0\alpha}(|p|)$ are isotropic functions of p. Thus we write $f_\alpha = f_{0\alpha} + f_\alpha^{(1)}$, $E = 0 + E^{(1)}$, $B = B_0 + B^{(1)}$, and linearize in the perturbations. The linearized Equations (1) - (3) can then be solved for an initial value problem by the method of Laplace-Fourier transforms. Defining

$$\mathcal{E}(k, \omega) = \int_0^\infty dt \int_{-\infty}^\infty \frac{dx}{(2\pi)^3} e^{i(\omega t - k \cdot x)} E^{(1)}(x, t) , \qquad (5)$$

with $\mathrm{Im}(\omega) > 0$, one finds a relation of the form $R \cdot \mathcal{E} = I$ for the transform of the electric field where I represents initial value terms which we will not write out - they are listed in Montgomery and Tidman (1964).

3. PROPAGATION ACROSS THE MAGNETIC FIELD

We choose a co-ordinate system as shown in Figure 2 with B_0 along the z-axis and consider a disturbance for which all the k vectors are along the x-axis. The matrix relation for $\mathcal{E}$ then simplifies to,

$$\begin{bmatrix} R_{xx} & R_{xy} & 0 \\ R_{yx} & R_{yy} & 0 \\ 0 & 0 & R_{zz} \end{bmatrix} \begin{bmatrix} \mathcal{E}_x \\ \mathcal{E}_y \\ \mathcal{E}_z \end{bmatrix} = \begin{bmatrix} I_x \\ I_y \\ I_z \end{bmatrix} \qquad (6)$$

where

$$R_{xx} = -\omega^2$$

$$- 2\pi\omega \sum_\alpha \frac{\omega_\alpha^2}{\Omega_{0\alpha}} \sum_{n=-\infty}^\infty \int_{-\infty}^\infty dp_\parallel \int_0^\infty \frac{p_\perp^2 dp_\perp \, n^2 \Omega_\alpha^3 J_n^2 \left(\frac{\partial f_{0\alpha}}{\partial p_\perp}\right)}{k^2 v_\perp^2 \left(\omega - n\Omega_\alpha\right)} \qquad (7)$$
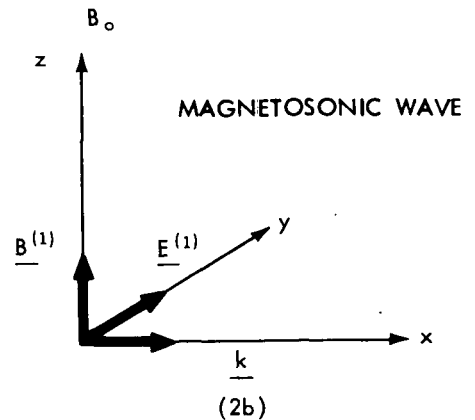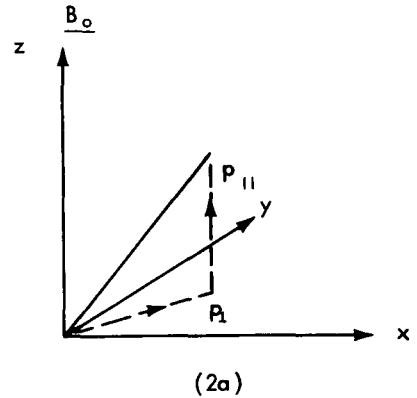

(2a)

MAGNETOSONIC WAVE

(2b)

Figure 2—Co-ordinate system for propagation of waves across a uniform magnetic field.

4

$$R_{xy} = - R_{yx} = - 2i\pi\omega \sum_{\alpha}' \frac{\omega_\alpha^2}{\Omega_{0\alpha}} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dp_{\parallel} \int_0^{\infty} \frac{p_{\perp}^2 \, dp_{\perp} \, n\Omega_\alpha^2 \, J_n J_n' \left(\frac{\partial f_{0\alpha}}{\partial p_{\perp}}\right)}{k \, v_{\perp} \left(\omega - n\Omega_\alpha\right)} , \tag{8}$$

$$R_{yy} = - \omega^2 + c^2 k^2 - 2\pi\omega \sum_{\alpha}' \frac{\omega_\alpha^2}{\Omega_{0\alpha}} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dp_{\parallel} \int_0^{\infty} \frac{p_{\perp}^2 \, dp_{\perp} \, \Omega_\alpha \, (J_n')^2 \left(\frac{\partial f_{0\alpha}}{\partial p_{\perp}}\right)}{\left(\omega - n\Omega_\alpha\right)} , \tag{9}$$

and

$$\omega_\alpha^2 = \frac{4\pi n_0 e_\alpha^2}{m_\alpha} \tag{10}$$

where there are two cyclotron frequencies,

$$\Omega_{0\alpha} = \frac{e_\alpha |B_0|}{m_\alpha c} , \qquad \Omega_\alpha = \frac{\Omega_{0\alpha}}{\gamma} , \tag{11}$$

and the arguments of the Bessel functions J_n are $kv_{\perp}/\Omega_\alpha = kp_{\perp}/m_\alpha \Omega_{0\alpha}$. It is also important to notice that the integrals may go through a singularity of $\left(\omega - n\Omega_\alpha\right)^{-1}$, and they are defined as written for $Im(\omega) > 0$. They can be continued to $Im(\omega) < 0$ in the usual way.

Now the process of solving for $E^{(1)}(t)$ involves inverting equation (6) for $\mathcal{E}$ and then the transforms (5). The field $E^{(1)}$ then has components proportional to $e^{-i\omega_n t}$ from the zeros of $|R| = 0$ and also some contributions from branch cuts and possible singularities of I. We shall assume that the distributions are such that asymptotically for large t the disturbance in the plasma resolves itself into modes with frequencies given by the "dispersion relation" $|R| = 0$ and that other contributions decay more rapidly in time. For more discussion of this point see Montgomery and Tidman, 1964.

We have not listed R_{zz} in (7) - (9) since it corresponds to a pure electromagnetic transverse wave propagating across B_0 with its electric field $E^{(1)}$ along B_0, and is not of interest to us here. The modes of present interest are those for which $E_x^{(1)}$ and $E_y^{(1)}$ are nonzero and which are described by the dispersion relation,

$$R_{xx} R_{yy} - R_{xy} R_{yx} = 0 \tag{11}$$

Among these "mixed" modes (i.e., they are generally neither purely longitudinal or transverse) is one which becomes the magnetosonic wave in the low frequency limit.
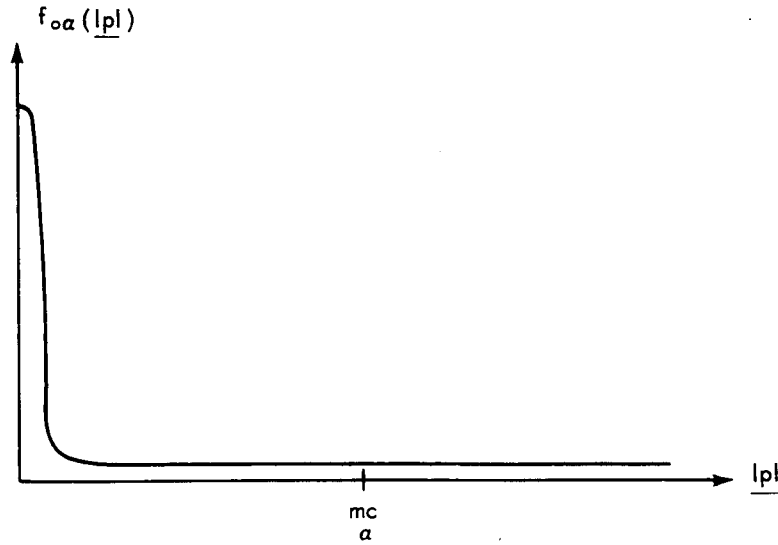
$f_{o\alpha}(|\underline{p}|)$

$\frac{mc}{\alpha}$

$|\underline{p}|$

Figure 3—Schematic plot of a momentum distribution function
with a relativistic suprathermal tail.

By inspection of (7) – (9) we see that in the nonrelativistic limit $c \to \infty$, $\Omega_\alpha = \Omega_{0\alpha}$ and the factor $(\omega - \Omega_{\alpha 0})$ comes outside the momentum integrations. In this limit there is no Landau damping of these waves propagating across B_0. The damping has its origin in a contribution from the singularity of $(\omega - n\Omega_\alpha)^{-1}$ occurring inside the range of momentum integration which can only occur in the relativistic case.

Magnetosonic Waves

In this section we calculate the damping decrement for magnetosonic waves propagating through a plasma containing relativistic suprathermal particles. We represent the plasma by the distributions,

$$f_{0\alpha} = \kappa \frac{\delta(p_\perp)}{\pi p_\perp} \delta(p_\parallel) + (1 - \kappa) g_\alpha(|p|) \qquad (12)$$

where the constant $\kappa \cong 1$ and $0 < (1 - \kappa) << 1$. The δ-function represents a zero-temperature "thermal" plasma containing most of the particles, and the function g_α represents the tenuous relativistic suprathermal tail (see Figure 3). We have defined

$$\int_0^\infty \delta(p_\perp) \, dp_\perp = \frac{1}{2} \, .$$

The constant κ has been chosen to be the same for electrons and protons although generally $g_e \neq g_i$. This restriction is not important. Further all distributions are normalized to unity, i.e.,

$$\int dp \, f_{0\alpha} = \int g_\alpha \, dp = 1 \, .$$

6

Making use of (12) in (7) - (9) the matrix elements become,

$$R_{xx} = -\omega^2 - \kappa \sum_{\alpha} \frac{\omega_\alpha^2 \omega^2}{\left(\Omega_{0\alpha}^2 - \omega^2\right)} - 2\pi\omega\,(1-\kappa) \sum_{\alpha} \frac{\omega_\alpha^2}{\Omega_{0\alpha}} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dp_{\parallel} \int_0^{\infty} \frac{p_{\perp}^2\, d p_{\perp}\, n^2 \Omega_\alpha^3\, J_n^2\, \frac{\partial g_\alpha}{\partial p_{\perp}}}{k^2 v_{\perp}^2 \left(\omega - n\Omega_\alpha\right)}, \quad (13)$$

$$R_{xy} = -R_{yx} = -i\kappa \sum_{\alpha} \frac{\Omega_{\alpha 0}\, \omega_\alpha^2\, \omega}{\left(\Omega_{\alpha 0}^2 - \omega^2\right)} - 2\omega i\pi\,(1-\kappa) \sum_{\alpha} \frac{\omega_\alpha^2}{\Omega_{0\alpha}} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dp_{\parallel} \int_0^{\infty} \frac{p_{\perp}^2\, dp_{\perp}\, n\Omega_\alpha^2\, J_n J_n'\, \frac{\partial g_\alpha}{\partial p_{\perp}}}{k v_{\perp} \left(\omega - n\Omega_\alpha\right)}, \quad (14)$$

$$R_{yy} = -\omega^2 + c^2 k^2 - \kappa \sum_{\alpha} \frac{\omega_\alpha^2 \omega^2}{\left(\Omega_{\alpha 0}^2 - \omega^2\right)} - 2\pi\omega(1-\kappa) \sum_{\alpha} \frac{\omega_\alpha^2}{\Omega_{0\alpha}} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dp_{\parallel} \int_0^{\infty} \frac{p_{\perp}^2\, dp_{\perp}\, \Omega_\alpha \left(J_n'\right)^2\, \frac{\partial g_\alpha}{\partial p_{\perp}}}{\left(\omega - n\Omega_\alpha\right)}. \quad (15)$$

Now we seek roots $\omega = \omega_r(k) + i\omega_i(k)$ of the dispersion relation $R_{xx} R_{yy} - R_{xy} R_{yx} = 0$. In order to make this practical we shall make use of three small parameters:

$$\left|\frac{\omega_i}{\omega_r}\right| \ll 1 \quad \text{(damping decrement is assumed small)} \qquad (16)$$

$$\left|\frac{\omega_r^2}{\Omega_{0i}^2}\right| \ll 1 \quad \text{(hydromagnetic wave limit)} \qquad (17)$$

$$(1-\kappa) \ll 1 \quad \text{(suprathermal gas density is small)} \qquad (18)$$

Thus making use of (17) and noting that $\sum \omega_\alpha^2/\Omega_{\alpha 0} = 0$ for an electron-ion plasma, it follows that R_{xy} has two terms of order $\omega^3/\Omega_{\alpha 0}^3$ and $(1-\kappa)$ respectively. Thus the term $R_{xy} R_{yx}$ in the dispersion relation contains only terms $O\left[\omega^6/\Omega_{\alpha 0}^6, (1-\kappa)^2, (1-\kappa)\,\omega^3/\Omega_{\alpha 0}^3\right]$ all of which we shall neglect compared to the leading terms of $R_{xx} R_{yy}$. One can verify that the roots are thus given approximately by $R_{xx} = R_{yy} = 0$. It should be noted that in applying condition (18) we shall also neglect the energy density of the suprathermal particle gas compared with $B_0^2/8\pi$ although we are aware that the cosmic ray energy density is of the same order as the field energy density for the interstellar medium. We are principally concerned with a calculation of the damping decrements and allowing for the suprathermal energy density within the framework of the Vlasov equations generates dispersion relations considerably more complicated than those obtained by Parker (1965) without much alteration of the damping results for the modes we consider.

7

The magnetosonic mode corresponds to the dispersion relation

$$R_{yy} \cong 0$$

and has a wave electric vector along the y axis, and as one easily verifies from Maxwell's equations, a wave magnetic vector along the z axis (see Figure 1b). The magnetic lines of force thus crowd together without bending at the wave crests perpendicular to the direction of propagation.

Now the dispersion relation $R_{yy} = 0$ has the form (noting (17))

$$-\omega^2 \left(1 + \kappa \sum \frac{\omega_\alpha^2}{\Omega_{0\alpha}^2}\right) + c^2 k^2 + \omega(1 - \kappa) F(\omega) = 0 \qquad (19)$$

where

$$F(\omega) = -2\pi \sum_\alpha \frac{\omega_\alpha^2}{\Omega_{0\alpha}} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{\infty} dp_\parallel \int_0^\infty \frac{p_\perp^2 \, dp_\perp \, \Omega_\alpha \, (J_n')^2 \, \frac{\partial g_\alpha}{\partial p_\perp}}{(\omega - n\Omega_\alpha)}$$

$$\equiv F_r + i F_i \qquad (20)$$

Thus setting $\omega = \omega_r + i \omega_i$ in (19) and (20) and equating the real and imaginary parts of (19) to zero and dropping terms of $O(\omega_i^2)$ or $O[\omega_i (1 - \kappa)]$ we readily find,

$$\frac{\omega_r}{k} = \pm \frac{c}{\left[1 + \sum \frac{\omega_\alpha^2}{\Omega_{0\alpha}^2}\right]^{1/2}} \cong \pm V_A = \pm \frac{B_0}{\sqrt{4\pi n_0 \left(m_e + m_i\right)}}, \qquad (21)$$

and for the damping decrement,

$$\omega_i \cong \frac{(1 - \kappa) F_i \left(\omega_r + i0\right)}{2\left[1 + \sum \frac{\omega_\alpha^2}{\Omega_{0\alpha}^2}\right]} \cong \frac{(1 - \kappa) V_A^2}{2c^2} F_i \left(\omega_r + i0\right). \qquad (22)$$

The approximations in the last terms of (21) and (22) follow if the Alfven speed V_A satisfies $V_A^2 \ll c^2$, which it usually does.

In order to further simplify the expression for ω_i we need the function $F_i \left(\omega_r + i0\right)$. Consider the quantity $(\omega - n\Omega_\alpha)^{-1}$ in (20). This contributes a pair of branch points at

$p_{||} = \pm i(m_a^2 c^2 + p_\perp^2)^{1/2}$ which can be connected by a cut in the $p_{||}$ plane passing through the point at infinity, and a pair of simple poles at the zeros of $(\omega - n\Omega_a)$. Regarding $p_\perp$ as fixed for the time being suppose $p_{||}^R(p_\perp, n, a)$ is a value of $p_{||}$ for which $\omega_r - n\Omega_a = 0$. Then making a Taylor expansion for $p_{||} \cong p_{||}^R$ we find

$$(\omega_r - n\Omega_a) \cong (p_{||} - p_{||}^R) \frac{n\Omega_{0a} \, p_{||}^R}{\gamma_R^3 \, m_a^2 \, c^2} \tag{23}$$

where $\gamma_R = \gamma(p_\perp, p_{||}^R) = n\Omega_{0a}/\omega_r$. Thus the Plemelj formulas become in this case,

$$\lim_{\omega_i \to +0} \frac{1}{(\omega_r - n\Omega_a + i\omega_i)} = \frac{P}{(\omega_r - n\Omega_a)} - \pi i \sum_R \delta(p_{||} - p_{||}^R) \frac{\gamma_R^3 \, m_a^2 \, c^2}{|n\Omega_{0a} \, p_{||}^R|} \, I(ne_a) \, J(p_\perp) \tag{24}$$

where for the sake of definiteness we have assumed $\omega_r > 0$, and $I(ne_a) = 0$ if $ne_a \leq 0$ or $= 1$ if $ne_a > 0$. The summation $\sum_R$ goes over the two roots $p_{||}^R$ of $\omega_r - n\Omega_a = 0$, which are of equal magnitude and opposite sign, and P is the principal value operation. It should also be noted that a zero of $\omega_r - n\Omega_a$ only exists for a finite range $0 \leq p_\perp \leq p_0$ which is accounted for in (24) by defining $J = 1$ in this range and $J = 0$ for $p_\perp > p_0$ where $\gamma_R = (1 + p_0^2/m_a^2 c^2)^{1/2}$.

Now the complex part of $F_r + iF_i$ derives from the second term of (24). If we note that at $p_{||} = p_{||}^R$, $\gamma_R = n\Omega_{0a}/\omega_r$ it follows from (22), (20), and (24) that

$$\omega_i = \frac{(1 - \kappa)\pi^2 V_A^2}{\omega_r^2} \sum_a m_a^2 \omega_a^2 |\Omega_{0a}| \sum_n |n| \, I(ne_a) \sum_R \int_{-\infty}^{\infty} dp_{||} \cdot \int_0^{p_0} \frac{p_\perp^2 \, dp_\perp}{|p_{||}^R|} (J_n')^2 \frac{\partial g_a}{\partial p_\perp} \delta(p_{||} - p_{||}^R) . \tag{25}$$

Next, since $p_0 = (p_\perp^2 + (p_{||}^R)^2)^{1/2}$ we have

$$\gamma_R = \frac{n\Omega_{0a}}{\omega_r} = \left(1 + \frac{p_0^2}{m_a^2 c^2}\right)^{1/2} , \tag{26}$$

so that

$$p_0 = \left(\frac{n^2 \Omega_{0a}^2}{\omega_r^2} - 1\right)^{1/2} m_a c , \tag{27}$$

9

and the quantity $\partial g_a/\partial p_\perp$ in (25) can be written as $(p_\perp/p_0)\left(\partial g_a(p_0)/\partial p_0\right)$ because of the δ-function in the integrand. Thus (25) reduces to

$$\omega_i = \frac{2(1-\kappa)\pi^2 V_A^2}{\omega_r^2} \sum_a{}' m_a^2 \omega_a^2 |\Omega_{0a}| \sum_{n=-\infty}^{+\infty} |n| \, I(ne_a) \, \frac{g_a'(p_0)}{p_0} \int_0^{p_0} \frac{p_\perp^3 \, dp_\perp}{\sqrt{p_0^2 - p_\perp^2}} \left[J_n'\left(\frac{kp_\perp}{m_a \Omega_{0a}}\right) \right]^2 \quad (28)$$

Finally we note that $m_e^2 \omega_e^2 |\Omega_{0e}| = m_i^2 \omega_i^2 |\Omega_{0i}|$, $J_{-n}(-z) = J_n(z)$, and $p_0 \cong nc \, |\Omega_{0a} m_a|/\omega_r$ is independent of the species a. Thus,

$$\omega_i \cong 2\pi^2 (1-\kappa) \left(\frac{V_A}{c}\right)^2 \left(\frac{\omega_e}{\Omega_{0e}}\right)^2 |\Omega_{0e}| \, \psi \quad (29)$$

where the dimensionless number ψ depends on the details of the relativistic suprathermal particle distributions and is given by,

$$\psi = \sum_{n=1}^{\infty} \frac{1}{|n|} \left[p_0^4 \, g_e'(p_0) + p_0^4 \, g_i'(p_0) \right] \int_0^1 \frac{x^3 \, dx}{\sqrt{1-x^2}} \left[J_n'\left(\frac{nc}{V_A} x\right) \right]^2 \quad (30)$$

with $p_0 \cong nc \, |\Omega_{0e} m_e|/\omega_r$.

Approximate Expressions and Physical Mechanism for the Damping of Magnetosonic Waves

Consider the last integral in (30). If we are only interested in an approximate result we can split the range of integration up as follows,

$$\int_0^1 dx \rightarrow \int_0^{nV_A/c} dx + \int_{nV_A/c}^1 dx \quad (31)$$

and use the asymptotic forms of the Bessel function in each range, i.e.,

$$J_n(z) \cong \sqrt{\frac{2}{\pi z}} \cos\left[z - \left(\frac{2n+1}{4}\right)\pi \right] \quad \text{for} \quad |z| \gg 1, \quad |z| \gg n \, .$$

and

$$J_n(z) \cong \frac{1}{n!} \left(\frac{z}{2}\right)^n \quad \text{for} \quad |z| \ll 1, \quad n > 0 \, .$$

10

The contribution from the first range is then negligible, and that from the second part of (31) gives

$$\int_0^1 \frac{x^3 \, dx}{\sqrt{1-x^2}} \left[J_n' \left(\frac{nc}{V_A} x \right) \right]^2 \cong \frac{V_A}{4nc}$$

If further, g_e' and g_i' are monotonically decreasing functions of p_0, then due to the factor n^{-2} in (30) we can approximately neglect all terms except $n = 1$. Thus ω_i reduces to

$$\omega_i \equiv \omega_i^1 \cong \frac{\pi}{2} (1 - \kappa) \left(\frac{V_A}{c} \right)^3 \left(\frac{\omega_e}{\Omega_{0e}} \right)^2 |\Omega_{0e}| \, P_0^4 \left[g_e'(P_0) + g_i'(P_0) \right] \qquad (32)$$

where

$$P_0 = \left(\frac{\Omega_{0i}}{\omega_r} \right) m_i c \qquad (33)$$

The physical mechanism giving rise to this damping decrement can be understood as follows. Consider a wave $E \sin \omega_r t$ of frequency ω_r propagating through the plasma and consider only its interaction with the ions for the present. An ion of momentum P_0 is in exact cyclotron resonance with the frequency $\omega_r = \Omega_{0i}/\gamma_0$, and those particles which are nearly in resonance (see Figure 4) have frequencies,

$$\frac{\Omega_{0i}}{\gamma} \equiv \omega_r \pm \Delta \omega_r$$

Those particles of slightly less momentum $P_0 - \Delta P$ have a slightly higher cyclotron frequency $\omega_r + \Delta \omega_r$. If their phase in the electric field $E \sin \omega_r t$ is such that they lose energy initially they further increase their frequency Ω_{0i}/γ and get rapidly further from exact resonance. On the other hand particles with momentum $P_0 - \Delta P$ and initial phase such that they gain energy undergo a decrease in $\Omega_{0i}/\gamma = \omega_r + \Delta \omega_r$ which brings them into <u>closer</u> resonance for further acceleration. Thus the net result for particles in the range $P_0 - \Delta P < p < P_0$ is a gain of energy from the wave. Similarly the net result for particles in the range $P_0 < p < P_0 + \Delta P$ is a loss of energy to the wave. There are more particles in the first range than the second which gives therefore a net damping of the wave with a damping decrement ω_i proportional to the derivative of the distribution function f_{0i}.
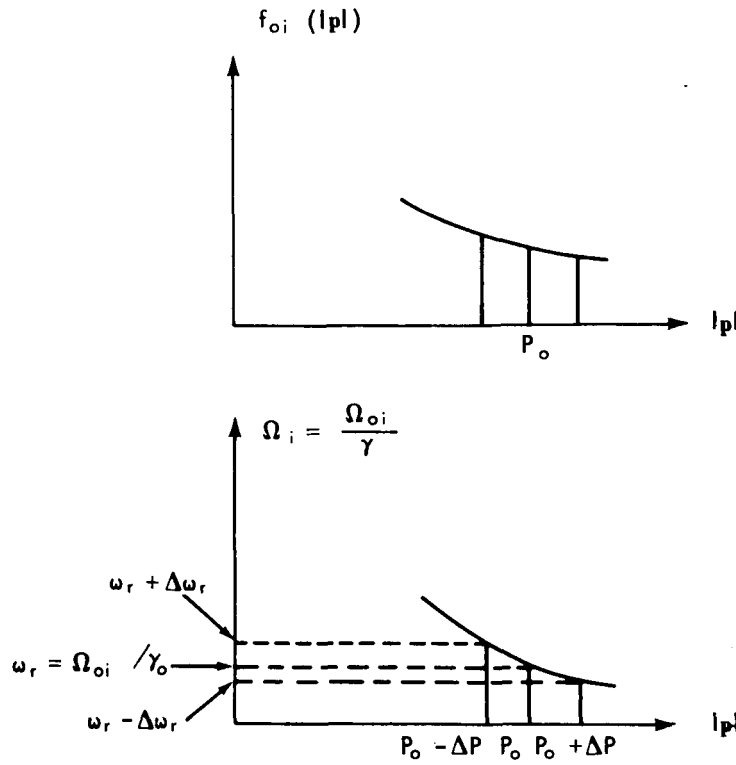
11

Figure 4—Schematic plots of the variation of the ion momentum distribution and cyclotron frequency.

4. PROPAGATION PARALLEL TO THE MAGNETIC FIELD - ALFVÈN WAVES

The dispersion relation for circularly polarized cyclotron waves propagating through a relativistic Vlasov plasma is well known (see for example Eq. (10.75), Montgomery and Tidman, 1964),

$$\omega^2 - c^2 k^2 + \pi\omega \sum_\alpha \omega_\alpha^2 \int_{-\infty}^\infty dp_\parallel \int_0^\infty \frac{p_\perp^2 dp_\perp \dfrac{\partial f_{0\alpha}(|\mathbf{p}|)}{\partial p_\perp}}{\gamma(\omega - kv_\parallel \pm \Omega_\alpha)} = 0 \tag{34}$$

The ± signs refer to right and left polarizations respectively. In the limit $\omega^2 \ll \Omega_{0i}^2$ this reduces to the case of Alfvèn waves propagating along the field B_0.

Substituting the distribution functions (12) equation (34) becomes,

$$\omega^2 - c^2 k^2 - \kappa \sum_\alpha \frac{\omega \omega_\alpha^2}{(\omega \pm \Omega_{0\alpha})} + \pi\omega(1 - \kappa) \sum_\alpha \omega_\alpha^2 \int_{-\infty}^\infty dp_\parallel \int_0^\infty \frac{p_\perp^2 dp_\perp \dfrac{\partial g_\alpha}{\partial p_\perp}}{\gamma(\omega - kv_\parallel \pm \Omega_\alpha)} = 0 \tag{35}$$

12

We shall now make the same approximations as those in (16) - (18) in order to calculate the small damping decrement for low-frequency Alfvèn waves by a tenuous suprathermal particle flux. We also assume that the zeros of $\left(\omega - kv_{\parallel} \pm \Omega_a\right)$ in (35) occur at values of $v_{\parallel}$ such that $\left(v_{\parallel}^R\right)^2 \gg \omega_r^2/k^2 \cong V_A^2$, and therefore neglect the real part ω_r of ω in this factor. Writing $\omega = \omega_r + i\omega_i$ and dropping small terms (35) then reads,

$$\left(\omega_r^2 + 2i\omega_i\,\omega_r\right)\left(1 + \sum \frac{\omega_a^2}{\Omega_{0a}^2}\right) - c^2 k^2 - \pi\omega_r\,(1-\kappa)\sum \frac{m_a\,\omega_a^2}{k}\int_{-\infty}^{\infty} dp_{\parallel}\int_0^{\infty}\frac{p_{\perp}^2\,dp_{\perp}\,\frac{\partial g_a}{\partial p_{\perp}}}{\left(p_{\parallel} \mp \frac{m_a\,\Omega_{0a}}{k} - i0\right)} \cong 0 \ .$$

Making use of the Plemelj formulas and the fact that g_a is isotropic and $V_A^2 \ll c^2$ leads directly to,

$$\frac{\omega_r}{k} \cong + V_A \ , \tag{36}$$

and

$$\omega_i \equiv \omega_i^{\parallel} \cong -\pi^2(1-\kappa)\left(\frac{V_A}{c}\right)^2\left(\frac{\omega_e}{\Omega_{0e}}\right)^2 |\Omega_{0e}|\,\Phi \ , \tag{37}$$

where the dimensionless number Φ is given by

$$\Phi = \int_{p_m}^{\infty} p_m\,p\,dp\,\left[g_e(p) + g_i(p)\right] \ , \tag{38}$$

with

$$p_m = \left|\frac{m_i\,\Omega_{0i}}{k}\right| \equiv m_i\,V_A\left(\frac{\Omega_{0i}}{\omega_r}\right) \ . \tag{39}$$

5. DISCUSSION

Equations (32) and (37) represent the damping decrements for hydromagnetic waves propagating in directions perpendicular and parallel to the magnetic field B_0 respectively. It is clear by inspection of these expressions that $\omega_i^{\parallel} \gg \omega_i^{\perp}$. This is because for parallel propagation waves interact with a lower energy part of the suprathermal particle spectrum than for perpendicular propagation. The lower energy particles both gain energy more rapidly and are likely to be more numerous.

In the following discussion we estimate some characteristic damping lengths for waves propagating through the interstellar medium. For this purpose let us consider a general integral power law momentum spectrum for the flux of suprathermal protons,

$$N(>p) = C \left(\frac{m_i \, c}{p}\right)^{\Gamma} \text{cm}^{-2} \text{ sec}^{-1} \text{ sterad}^{-1} \quad . \tag{40}$$

We shall assume this is valid for $p > p_{min}$, and neglect electron fluxes for the present. Then equating dN/dp to $n_0 \, v(1 - \kappa) \, g_i \, (p) \, p^2$ where $v = p/m_i \, \gamma$ is the speed of a proton of momentum p, we have for the momentum distribution function as defined in (12),

$$(1 - \kappa) \, g_i \, (p) = \left(\frac{C}{n_0 \, v}\right) \frac{\Gamma(m_i \, c)^{\Gamma}}{p^{\Gamma+3}} \quad . \tag{41}$$

If we define characteristic damping lengths $L_\perp$ and $L_{\parallel}$ for hydromagnetic waves propagating perpendicular and parallel to the magnetic field respectively, it readily follows from (32), (37), and (41), that

$$L_\perp \equiv \frac{V_A}{|\omega_i^{\perp}|} = \frac{2}{\pi} \frac{c}{|\Omega_{0e}|} \left(\frac{c}{V_A}\right)^2 \left(\frac{\Omega_{0e}}{\omega_e}\right)^2 \left(\frac{n_0 \, v_0}{C}\right) \frac{1}{\Gamma(3+\Gamma)} \left(\frac{\Omega_{0i}}{\omega_r}\right)^{\Gamma} \tag{42}$$

provided

$$P_0 = \left(\frac{\Omega_{0i}}{\omega_r}\right) m_i \, c > p_{min} \quad , \tag{43}$$

and

$$L_{\parallel} \equiv \frac{V_A}{|\omega_i^{\parallel}|} = \frac{1}{\pi^2} \frac{c}{|\Omega_{0e}|} \left(\frac{\Omega_{0e}}{\omega_e}\right)^2 \left(\frac{n_0 \, v_m}{C}\right) \left(\frac{2+\Gamma}{\Gamma}\right) \left(\frac{\Omega_{0i}}{\omega_r}\right)^{\Gamma} \left(\frac{V_A}{c}\right)^{\Gamma-1} \quad , \tag{44}$$

provided

$$P_m = \left(\frac{V_A}{c}\right)\left(\frac{\Omega_{0i}}{\omega_r}\right) m_i \, c > p_{min} \quad . \tag{45}$$

The momentum conditions (43) and (45) are simply that those particles responsible for the damping should lie in the range $p > p_{min}$ for which (40) is valid. The velocity v_0 is

14

that for an ion of momentum P_0, i.e., $v_0 = P_0/m_i\gamma_0$, and v_m for an ion of momentum P_m assuming however that $P_m \cong m_i v_m$. In equation (44) if P_m is in the relativistic range i.e., $P_m > m_i c$, we can replace v_m by c to obtain an approximate result for $L_\parallel$.

Corresponding propagation lengths for waves propagating at arbitrary angles to the magnetic field will probably lie between these two values. We see from the conditions (43) and (45) that the perpendicular propagation waves are damped only by highly relativistic particles, whereas the parallel wave interacts with the low energy part of the suprathermal spectrum. We shall next consider as an example the interstellar medium, although it should be noted that in regions of stronger magnetic field and more intense suprathermal fluxes such as the Crab nebula, or a solar flare, these effects could be more important. There are also many forms of such damping for the wider frequency range of plasma waves.

Damping of Waves in the Interstellar Medium by the Cosmic Ray Gas

For highly relativistic protons the total energy $E \cong pc$ and equation (40) can be written approximately $N \cong C(m_i c^2/E)^\Gamma$. The values of Γ and C appropriate to various ranges are given by (Waddington, 1960),

$$\Gamma \cong 1.4 , \quad C \cong .5 \text{ for } 4 < pc < 10^4 \text{ BeV}$$

$$\Gamma \cong 1.5 , \quad C \cong 1 \text{ for } 10^4 < pc < 10^6 \text{ BeV}$$

$$\Gamma \cong 2 , \quad C \cong 1,400 \text{ for } 10^6 < pc < 10^9 \text{ BeV} \qquad (46)$$

Assuming the following parameters for the interstellar medium: n_0 = number density for the ionized component $\cong .1 \text{ cm}^{-3}$, and $B_0 \cong 10^{-5}$ gauss, i.e., $|\Omega_{0e}| \cong 175$, $\omega_e \cong 1.8 \ 10^4$, $V_A/c \cong 2.3 \ 10^{-4}$ then leads to

$$L_\perp \cong 200 \left(\frac{\Omega_{0i}}{\omega_r}\right)^{1.4} \text{ light years}$$

for $\Omega_{0i} > \omega_r > 10^{-4} \Omega_{0i}$. Note that if the suprathermal spectrum parameters Γ and C remain constant then the numerical factor in this expression for $L_\perp$ is proportional to (n_0/B_0). It thus decreases in regions of smaller n_0 or larger B_0.

Next consider the case of parallel propagation. Here we have the difficulty that we do not know what the very low energy part of the cosmic ray spectrum is in the interstellar medium. Thus assuming we know the values of C and Γ for equation (40) only down to P_{min}, it follows from (45) that we can only calculate the damping decrement for

$$\omega_r < \left(\frac{V_A}{c}\right)\left(\frac{m_i\,c}{p_{min}}\right)\Omega_{0i} \quad .$$

(47)

Frequencies higher than this would be damped by unknown fluxes of lower energy particles with $p < p_{min}$.

Suppose for example we assume that the values in the top line of (46), i.e., $C \simeq .5$, $\Gamma \simeq 1.4$, are valid down to some value p_{min}. Then it follows from (44) and (45) that Alfvèn waves in the frequency range given by (47) are damped in the cosmic ray gas in a length scale,

$$L_{\parallel} \simeq 10^{-6} \left(\frac{\Omega_{0i}}{\omega_r}\right)^{1.4} \left(\frac{v_m}{c}\right) \text{ light years} \quad .$$

Now we know that the spectrum (40) is a reasonable valid one down to $p_{min}\,c \simeq 1\,\text{BeV}$. From this it follows that frequencies in the range $\omega_r < 2.10^{-4}\,\Omega_{0i}$ will be damped in a length scale which has a minimum value for $\omega_r = \left(V_A/c\right)\left(m_i\,c/p_{min}\right)\Omega_{0i} = 2.10^{-4}\,\Omega_{0i}$ of $L_{\parallel} \simeq 10^{-1}$ light years.

Further, suppose we were to assume the spectrum (40) remained valid down to $p_{min}\,c = 100\,\text{MeV}$. This would lead to a damped frequency range $\omega_r < 2.10^{-3}\,\Omega_{0i}$ with a minimum damping length at $\omega_r = 2.10^{-3}\,\Omega_{0i}$ of $L_{\parallel} \simeq 10^{-3}$ light years.

It is thus clear that if there exist any suprathermal particle fluxes in the range $m_i\,c^2 > pc >$ a few times the thermal energy, then these would heavily damp a wide frequency range $\left(\omega_r < \Omega_{0i}\right)$ of Alfvèn waves. On the other hand the damping length $L_{\perp}$ for perpendicular propagation involves the highly relativistic particles and is considerably longer than $L_{\parallel}$, although for the range $10^{-2}\,\Omega_{0i} < \omega_r < \Omega_{0i}$ it is still less than a galactic radius. It should be noted of course that the interstellar magnetic field B_0 is nonuniform with some characteristic length scale ℓ. Thus strictly speaking our results only apply if $L_{\parallel}$ or $L_{\perp}$ are much less than ℓ. However even if the reverse situation holds, i.e., $\ell < L_{\parallel}$ or $L_{\perp}$ for some frequencies, we would still expect that the same basic process of cyclotron damping should occur for waves of wavelength $\lambda \ll \ell$ propagating through such regions. These waves would suffer refraction and propagate at varying angles across B_0.

Since interstellar plasma waves are damped by the cosmic ray gas it is natural to consider whether this contributes a significant energy imput to the cosmic radiation. We first note however that the waves discussed here are of small enough amplitude that the cosmic ray particles are able to freely traverse the waves with only a minor perturbation in their trajectories. This is distinct from the reflection of energetic particles by a

nonlinear wavefront which is the scattering process basic to the statistical Fermi acceleration of cosmic ray particles. Suppose further we accept the theory that the bulk of the cosmic rays in our galaxy have their origin in the supernova explosions* (Ginzburg and Syrovatskii, 1964). It is then clear that there is much more energy available for energizing cosmic rays in the early violently nonlinear turbulence described by the Fermi mechanism than is available in the final decaying phase when the expanding supernova remnant has degenerated to a region of small-amplitude turbulence. We conclude that this wave damping is not likely to be an important source of energy for the cosmic ray gas. However it could play an important role in the dissipation of disturbances in the interstellar medium caused by old supernova remnants, or stellar winds, etc.

ACKNOWLEDGMENT

REFERENCES

Ginzburg, V. L. and Syrovatskii, S. I. 1964, The Origin of Cosmic Rays, Pergamon Press, New York.

Montgomery, D. C. and Tidman, D. A. 1964, Plasma Kinetic Theory, McGraw-Hill, New York.

Parker, E. N. 1965, University of Chicago Tech. Notes EFINS-65-19, EFINS-65-27.

Parker, E. N. 1955, Phys. Rev. 99, 241.

Piddington, J. H. 1957, Austr. J. Phys. 10, 515.

Stix, T. H. 1962, Theory of Plasma Waves, McGraw-Hill, New York.

Waddington, C. J. 1960, Progr. in Nuclear Phys. 8, 1.

---

*Ginzburg and Syrovatskii (see pages 326-329 of "The Origin of Cosmic Rays") point out that with plausible assumptions for the acceleration and escape of suprathermal particles from bounded turbulent plasmas such as supernova shells, one can obtain a "universal" integral power law spectrum $E^{-1.5}$ for the energetic particles escaping from such plasmas into the interstellar medium.