SUBHARMONIC OSCILLATIONS IN AN ARBITRARY AXISYMMETRIC TANK RESULTING FROM AXIAL EXCITATION

by
Wen-Hwa Chu

Technical Report No. 5
Contract No. NAS8-11045
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SOUTHWEST RESEARCH INSTITUTE
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H. Norman Abramson, Director
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ABSTRACT

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The problem of subharmonic liquid response in a container subject to vertical, axial excitation has been solved to the second approximation* by the method of perturbation, employing characteristic functions. Application of the integral equation method has been made so that numerical construction of the Neumann function and the characteristic functions is possible, at least, for convex axial symmetric tanks. The method can be easily extended to determine the liquid response in an arbitrary tank but its limitation depends on the size of the computer, the accuracy desired, and the cost involved.

Author

* In this report, the example is carried out only to the first approximation.

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SYMBOLS

a	maximum radius of the tank in cylindrical coordinates
$a_{\nu\mu}$	defined in Appendix G
b	maximum radius of F_o for an axial symmetric tank
$C_n^{1/2}, C_n^{3/2}$	binomial coefficients
$E_{s_1 s_2}$	characteristic function of s_1 order and s_2 degree
$E(q_1)$	elliptic integral of the second kind
F_o	mean free surface; $z = 0$ plane
F	free surface
$f_{n,m}^{(i,s)}, f_{n,m}^{(i,c)}$	the Fourier sine and cosine coefficients of the nonhomogeneous term in the free surface condition of the i^{th} approximation [c. f., Equation (26a)]
$\tilde{f}_{l,k}^{(l,s)}, \tilde{f}_{l,k}^{(l,c)}$	see Equations (37a) and (37b)
G	Green's function of the second kind; Neumann function
g_m^*	$\lambda_m^* g^*$
g	downward gravitational acceleration or added regular part of the Neumann function
G_o	singular part of the Neumann function
h	liquid depth
Int. ( )	the integer part of ( ) less or equal to ( )
[I]	the unit matrix

SYMBOLS (Cont'd)

$K(q_1)$	elliptic integral of the first kind
n	outer normal
N	a positive integer
p_o	ullage pressure
p	static pressure
r, σ, θ	spherical coordinates with origin on F_o (Fig. 1)
q, q	magnitude of the velocity vector
t	time
u, v, w	x, y, z component of velocity, respectively
W	surface of the wetted wall
x_o	amplitude of vertical tank oscillation
x, y, z	rectangular coordinates, z positive upward
z_b	vertical tank displacement
$\tilde{\gamma}$	angle between two vectors r and r'
γ	$\theta - \theta'$
δ_{ij}	Kronecker delta, $\delta_{ij} = 0$ for $i \neq j$, $\delta_{ij} = 1$ for $i = j$
ζ	free surface elevation
ρ_L	densify of the fluid
ρ, θ, z	cylindrical coordinates

SYMBOLS (Cont'd)

Φ	velocity potential
Ψ_m	m^{th} characteristic function
ω	frequency of subharmonic oscillation
ω_k	k^{th} liquid sloshing frequency near which the subharmonic oscillations occur

SUPERSCRIPTS AND SUBSCRIPTS

$(\ )^{(\ell)}$	quantities associated with ℓ^{th} approximation
$(\ )_{\ell}^{(i)}$	the ℓ^{th} component of i^{th} approximation
$(\ )_{\ell, m}$	m^{th} component of the ℓ^{th} harmonics of ( )
$(\ )^{(\ell, s)}$	sine component of the ℓ^{th} approximation of ( )
$(\ )^{(\ell, c)}$	cosine component of the ℓ^{th} approximation of ( )
$(\ )_{n, m}^{(\ell, c)}$	n^{th} cosine harmonic of the m^{th} component of the ℓ^{th} approximation
$(\ )_{n, m}^{(\ell, s)}$	n^{th} sine harmonic of the m^{th} component
$\overline{(\ )}$	is not an average value of but related to ( )

INTRODUCTION

The problem of fuel sloshing is an important factor in the dynamic stability of a spacecraft. Relatively extensive theoretical and experimental researches on this subject have been reported. A fuel sloshing monograph (Ref. 1) is in preparation by Abramson, et al., for designers and research workers. Consultation can also be made to References 2 and 3 for literature prior to their publication, respectively.

Most of the literature known concerns linear fuel sloshing due to the lateral motion of the containers Near resonance, rotational and nonplanar motion were observed (Ref. 2). The stability boundary of a circular cylindrical tank has been predicted by Hutton (Ref. 4) through a simplified third order theory. The same theory had been applied to non-linear force responses in compartmented tanks (Ref. 5) in fair agreement with experiments. Nonlinear shell responses of breathing modes had been observed and reported in Reference 6. These phenomena include softening* of zero force response, jump in amplitude and instability to beat-type shell response with nonsinusoidal periodic low frequency liquid motion, (Refs. 6, 10). Quantitative nonlinear analysis for a breathing tank would be laborious as evidenced by the prediction of the natural frequencies (Refs. 11, 12)

*For standing gravity waves in a rigid rectangular tank, there is a transition depth below which the liquid amplitude increases with the frequency (hardening), Refs. 7, 8, 9.

which may require as much as ten terms or more in shell modes; a qualitative theoretical study, which is in progress, will not be presented herein.

The problem of vertical (axial) excitation was probably first investigated by Yarymovych (Ref. 13) for a rectangular tank, while the large amplitude surface waves were studied earlier in References 11, 12. Taylor (Ref. 14) obtained 1/2-subharmonic motion due to lateral excitation when trying to obtain large amplitude wave profile near the lowest resonant frequency. Yarymovych analyzed and measured 1/2-subharmonic vertical excitation. The measured surface response curve in Reference 13 is in good agreement with the third order theory (Ref. 4), although the error of prediction increases with the maximum surface displacement as expected. Capillary waves, spray, and effect of damping were also discussed in Reference 13. Subsequently, Dodge, Kana and Abramson (Ref. 15) studied the subharmonic motions in a circular tank under vertical (axial) excitation. The stability of harmonic response was shown to be governed by the Mathieu stability diagram which was converted to stability regions in an amplitude versus frequency plot. The 1/2-subharmonic motion was observed and compared with a third order theory. The theory yielded fair amplitude responses in comparison with measured data.

In Reference 16, the finite amplitude standing wave in an arbitrary tank was studied by Moiseev. It also indicated how the analysis can be extended to the Kravtchenko problem, a tank with a partially moving wall.

Furthermore, Moiseev considered only superharmonic response to the excitation force and did not indicate how the characteristic function could be constructed.

The present paper follows Moiseev's approach, employing a perturbation method and characteristic functions for the subharmonic response to an oscillatory axial excitation. The integral equation method for the numerical construction of characteristic functions analogous to that in Reference 17 is devised, so that actual prediction or correlation with experiments can be made.

FORMULATION OF THE PROBLEM

In the vertically oscillating tank-fixed coordinates (Fig. 1), the equations of motion* are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_L}\frac{\partial p}{\partial x} \qquad (1a)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho_L}\frac{\partial p}{\partial y}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_L}\frac{\partial p}{\partial z} - g + N^2\omega^2 x_o \cos(N\omega t) \qquad (1c)$$

where the last term is the inertial acceleration due to the vertical oscillation

$$z_b = x_o \cos(N\omega t) \qquad (2)$$

Assuming that the flow is irrotational, there exists a velocity potential such that

$$\nabla\Phi = \underset{\rightarrow}{q} \qquad (3)$$

where $\underset{\rightarrow}{q}$ is the velocity vector in the oscillating frame. Integration of Equations (1a-c) yields the unsteady Bernoulli's equation in the oscillating frame

$$\frac{\partial\Phi}{\partial t} + [g - N^2\omega^2 x_o \cos(N\omega t)]\, z + \frac{p}{\rho_L} + \frac{1}{2} q^2 = f(t) + \frac{p_o}{\rho_L} \qquad (4)$$

where p_o is the constant ullage pressure and $f(t)$ can be absorbed by the velocity potential Φ. The continuity equation and the irrotationality condition yield

$$\nabla^2\Phi = 0 \qquad (5)$$

*All equations will be given in the tank-fixed coordinates unless specified otherwise.

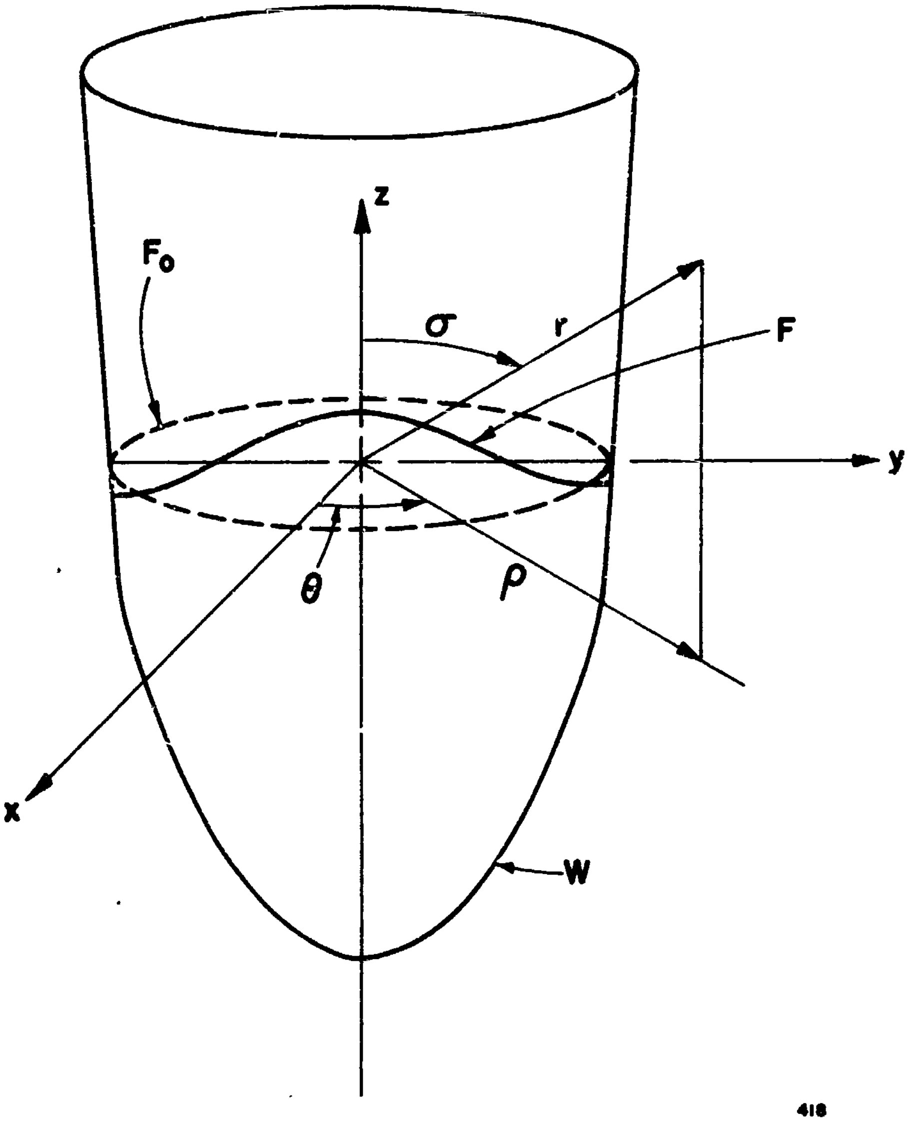

FIGURE 1. COORDINATES AND BOUNDING SURFACES

Assuming that the tank is rigid, then in the tank-fixed frame the normal velocity to the wetted wall, W, must vanish, i. e.,

$$\frac{\partial \Phi}{\partial n} = 0 \qquad \text{on W} \tag{6a}$$

On the free surface F, assume that the free surface curvatures are small so that the surface tension can be neglected. Further, assume that the gravitational acceleration is not small so that the surface elevation will not be too large for applying the free surface condition on a flat plane $z = 0$ through Taylor's expansion. The dynamic condition is $p = p_o$ on F and

$$\left[\frac{\partial \Phi}{\partial t} + [g - N^2 \omega x_o \cos(N\omega t)]\ z + \frac{1}{2} q^2\right]_{z = p} = 0 \qquad \text{on F} \tag{6b}$$

Finally let the geometric equation of the free surface be

$$z - \zeta\,(\rho, \theta, 0, t) = 0 \tag{7}$$

Then the kinematic condition is

$$w = \frac{\partial \Phi}{\partial z} = \frac{\partial \zeta}{\partial t} + \frac{\partial \Phi}{\partial \rho}\frac{\partial \xi}{\partial \rho} + \frac{\partial \Phi}{\rho \partial \theta}\frac{\partial \xi}{\rho \partial \theta} \qquad \text{on F} \tag{8}$$

It seems convenient to use the cylindrical coordinates in which the unperturbed frozen free surface is $z = 0$.

In this report, the initial conditions will not be imposed, as part of the general solution, which depends on the initial conditions, will be damped out in a real system with damping. The part annihilating the secular term must exist* in order to render definitely a nonlinear periodic solution.

*There is some argument which indicates that secular terms can appear in a finite term approximation (Ref. 18).

METHOD OF PERTURBATION

In this report, the analytic problem will be solved by the method of perturbation, employing characteristic functions.

Let Ψ_m be the m^{th} characteristic function satisfying the Laplace equation and the conditions of

$$\frac{\partial \Psi_m}{\partial n} = 0 \qquad \text{on the wetted wall} \tag{9}$$

and

$$\left.\frac{\partial \Psi_m}{\partial z}\right|_{z=0} = \lambda_m \Psi_m(x, y, 0) = \lambda_m \Psi_m(x, y) \tag{10}$$

Assume that the tank is axially symmetric. For simplicity, the characteristic function, Ψ_m, will be said to be proportional to $\cos(m\theta)$. In reality, there could be more than one mode for the same $\cos(m\theta)$ variation. For finite term approximations, one may consider

$$\psi_m = E_{s_1 s_2} \tag{11}$$

where, letting S_{mx} and M_{mx} be the number of columns and the number of total elements of $E_{s_1 s_2}$, respectively,

$$m = S_{mx}(s_1 - 1) + s_2 \tag{11a}$$

$$s_1 = \text{Int.}\left(\frac{m-1}{S_{mx}}\right) + 1 \tag{11b}$$

$$s_2 = m - s_1 S_{mx}$$

$$0 \le s_2 \le S_{mx}, \quad 0 < s_1 \le \frac{M_{mx}}{S_{mx}}, \quad 0 \lesssim m \le M_{mx} \tag{11c}$$

and the m^{th} characteristic function is proportional to $\cos(s_1 \theta)$.

For the m^{th} vibrational mode of infinitesimally small amplitude without motions of the tank, the free surface elevation and velocity potential are, respectively,

$$\zeta_m^{(\ell r)} = A \psi_m(x, y) \sin(\omega_m t)\, \omega_m \tag{12}$$

$$\phi_m^{(\ell r)} = A g \Psi_m(x, y, z) \cos(\omega_m t) \tag{13}$$

where

$$\Psi_m(x, y, z) = \lambda_m \int_F G(x, y, z; x'y', 0)\, \psi_m(x', y')\, ds \tag{14}$$

It is understood that

$$\Psi_m(x, y, 0) = \psi_m(x, y) \tag{14a}$$

After the characteristic function is defined, one may expand the velocity potential, Φ, and elevation, ζ, into a power series of a small parameter

$$\Phi = \epsilon \sum_{n=0}^{\infty} \phi_n \epsilon^n \tag{15a}$$

$$\zeta = \epsilon \sum_{n=0}^{\infty} \zeta_n \epsilon^n \tag{15b}$$

Moiseev in Reference 17 defined a nondimensional time

$$\tau = \frac{t\, \omega_k}{1 + \sum_{n=0}^{\infty} h_n \epsilon^n} \tag{16a}$$

In the present problem, one must further assume

$$\omega = \frac{\omega_k}{1 + \sum_{n=0}^{\infty} h_n \epsilon^n} \tag{16b}$$

so that cos $(N\omega t)$ in the time dependent coefficient will be cos $(N\tau)$, a function of τ. Let the nondimensional tank amplitude be

$$\frac{x_o}{a} = x_o{}^* \epsilon \tag{17a}$$

This is a simplifying assumption as x_o/a may be also expanded into a power series of ϵ as t, but there will be more than one unknown constant for a given x_o. If ϵ is defined as the amplitude of the tank motion, $x_o^* = 1$, then as amplitude approaches zero, $\omega = \omega_k/h_o$ which is not amplitude dependent as shown in experiments for a circular cylindrical tank (Ref. 15). If ϵ is taken as a parameter defined by the frequency ω, one may assume

$$\omega = \frac{\omega_k}{h_o + h_1 \epsilon + h_2 \epsilon^2} \tag{17b}$$

where $h_1 \neq 0$ will be shown for 1/2 subharmonics, N = 2. For N = 2, $h_2 \neq 0$ will also be assumed later. With the exception of x_o^* in the equation, the superscript star will be used to indicate a nondimensionalized quantity; length being nondimensionalized by the maximum radius, a, of the tank and frequency by the k^{th} natural frequency, ω_k. Then one has, for instance,

$$\lambda_m^* g^* = \omega_m^{*2} = \omega_m^2/\omega_k^2 \tag{18a}$$

$$\phi_\ell^* = \sum_{m=0}^{\infty} \phi_m^{(\ell)} \Psi_m(x, y, z); \; \phi^* = \sum_{\ell=0}^{\infty} \epsilon^\ell \phi_\ell^* \tag{18b}$$

$$\phi_\ell^*(x, y, 0) = \sum_{m=0}^{\infty} \phi_m^{(\ell)} \psi_m(x, y) \tag{18c}$$

$$\left[\frac{\partial \phi_\ell}{\partial z^*}\right]_{z=0} = \sum_{m=0}^{\infty} \phi_m \lambda_m^* \psi_m(x, y) \tag{18d}$$

$$\zeta_\ell^* = \sum_{n=0}^{\infty} \zeta_m^{(\ell)} \psi_m ; \; \zeta^* = \sum_{\ell=0}^{\infty} \zeta_\ell^* \epsilon^\ell \tag{18e}$$

The second superscripts s and c will be used to designate quantities associated with sine and cosine harmonics, respectively; e. g.,

$$\zeta_m^{(\ell)} = \sum_{n=0}^{\infty} \zeta_{n,m}^{(\ell,c)} \cos(n\tau) + \sum_{n=0}^{\infty} \zeta_{n,m}^{(\ell,s)} \sin(n\tau) \tag{19}$$

The boundary conditions on z = 0 are derived in Appendix A. By equating terms of equal powers of ϵ, solutions by the method of perturbation are found. Approximations up to the second order are given in the present study.

Zeroth Approximation

In the zeroth approximation, components of the velocity potential are governed by

$$\frac{\partial^2 \phi_{01}^*}{\partial \tau^2} + g^* h_0^2 \frac{\partial \phi_0^*}{\partial z^*} = 0 \tag{20a}$$

$$\zeta_0^* = - \frac{1}{g^* h_o} \frac{\partial \phi_0^*}{\partial \tau} \tag{20b}$$

It can be shown by Green's theorem that the characteristic functions $\psi_m(x, y)$ are orthogonal on the plane, z = 0. The mth component of Equation (20) is

$$\frac{\partial^2 \phi_m^{(0)}}{\partial \tau^2} + g^* h_o^2 \frac{\partial \phi_m^{(0)}}{\partial z^*} = 0 \tag{21}$$

i.e.,

$$\frac{\partial^2 \phi_m^{(0)}}{\partial \tau^2} + h_o^2 \omega_m^{*2} \phi_m^{(0)} = 0 \tag{21a}$$

for $m = k$, $\omega_m^{*2} = 1$

$$\frac{\partial^2 \phi_k^{(0)}}{\partial \tau^2} + h_o^2 \phi_k = 0 \tag{21b}$$

The oscillation of the period, 2π, in the nondimensional time τ is sought in the present study.

Assume that ω_m^* is not an integer except when $m = k$, for which $\omega_m^* = 1$. Thus, in Equation (22b)

$$h_o = 1 \tag{22}$$

$$\phi_m^{(0)} = \delta_{m,k}[\phi_{1,k}^{(0,c)} \cos \tau + \phi_{1,k}^{(0,s)} \sin \tau] \tag{23}$$

$$\phi_{1,k}^{(0,c)} = A^* g^* \quad \text{or} \quad \phi_{1,k}^{(0,s)} = B^* g^* \tag{23a, b}$$

From Equations (20b), (22) and (23)

$$\zeta_m^{(0)} = \delta_{mk} \psi_k [\zeta_{1,k}^{(0,s)} \sin \tau + \zeta_{1,k}^{(0,c)} \cos \tau] \tag{24}$$

$$\zeta_{1,k}^{(0,s)} = A^*, \quad \zeta_{1,k}^{(0,c)} = -B^* \tag{24a, b}$$

In the first approximation, it will be shown that either $A^* = 0$ or $B^* = 0$. In order to have a linear velocity potential inphase or out of phase with the velocity of excitation, $A^* \neq 0$, $B^* = 0$ will be selected. Otherwise, the choice might be determined by a stability analysis.

In the case where $B^* = 0$, the velocity potential and surface elevation for the zero[th] approximation can be written as

$$\phi_0^* = \sum_{m=1}^{\infty} \sum_{n=0}^{N} \phi_{nm}^{(0,c)} \psi_m \cos(n\tau) + \sum_{m=1}^{\infty} \sum_{n=1}^{N} \phi_{n,m}^{(0,s)} \psi_m \sin(n\tau)$$

$$= \phi_{1,k}^{(0,c)} \psi_m \cos \tau \tag{25a}$$

$$\zeta_0^* = \sum_{m=1}^{\infty} \sum_{n=0}^{N} \zeta_{nm}^{(0,c)} \psi_m \cos(n\tau) + \sum_{m=1}^{\infty} \sum_{n=1}^{N} \zeta_{n,m}^{(0,s)} \psi_m \sin(n\tau)$$

$$= \zeta_{1,k}^{(0,s)} \psi_k \sin\tau \qquad (25b)$$

where

$$\phi_{nm}^{(0,c)} = \tilde{\phi}_{nm}^{(0,c)} + \bar{\phi}_{nm}^{(0,c)} ; \; \phi_{nm}^{(0,s)} = \tilde{\phi}_{nm}^{(0,s)} + \bar{\phi}_{nm}^{(0,s)} \qquad (25aa)$$

with

$$\tilde{\phi}_{nm}^{(0,c)} = 0 \; ; \; \tilde{\phi}_{nm}^{(0,s)} = 0 \qquad (25ab)$$

$$\bar{\phi}_{nm}^{(0,c)} = \delta_{mk} \delta_{n1} \phi_{1,k}^{(0,c)} \; ; \; \bar{\phi}_{nm}^{(0,s)} = 0 \qquad (25ac)$$

and

$$\zeta_{nm}^{(0,s)} = \tilde{\zeta}_{nm}^{(0,s)} + \bar{\zeta}_{nm}^{(0,s)} \; , \; \zeta_{nm}^{(0,c)} = \tilde{\zeta}_{nm}^{(0,c)} + \bar{\zeta}_{nm}^{(0,c)} \qquad (25ba)$$

$$\tilde{\zeta}^{(0,s)} = 0 \; ; \; \tilde{\zeta}^{(0,c)} = 0 \qquad (25bb)$$

$$\bar{\zeta}_{nm}^{(0,s)} = \delta_{mk} \delta_{n1} \zeta_{1,k}^{(0,s)} \; ; \; \bar{\zeta}_{nm}^{(0,c)} = 0 \qquad (25bc)$$

First Order Approximation

In the first approximation, one collects the second order terms to get

$$\frac{\partial^2 \phi_m^{(1)}}{\partial \tau^2} + g_m^* \frac{\partial \phi_m^{(1)}}{\partial z^*} = \sum_{n=1}^{N+1} f_{n,m}^{(1,s)} \sin(n\tau) + \sum_{n=0}^{N+1} f_{nm}^{(1,c)} \cos(n\tau) = R_{1m} \qquad (26a)$$

The dynamic condition yields

$$\zeta_1^* = -\frac{1}{g^* h_o} \left\{ h_o \frac{\partial \phi_1^*}{\partial \tau} + h_1 \left(\frac{\partial \phi_0}{\partial \tau} + 2 g_o^* \zeta_o^* h_o \right) - N^2 \zeta_o^* x_o^* \cos(N\tau) + \frac{1}{2} \sum_{i=1}^{2} h_o^2 \left(\frac{\partial \phi_o}{\partial x^*} \right) + \frac{1}{2} h_o^2 \left(\frac{\partial \phi_o}{\partial z} \right)^2 \right\} \text{ on } F \qquad (26b)$$

In general, on F_o, by expansion into Taylor's series and recollection of equal power terms, the dynamic and kinematic conditions are:

$$\zeta_n^* + \frac{1}{h_o g^*} \frac{\partial \phi_n}{\partial \tau} = R_{d_n} \tag{27a}$$

$$\frac{\partial \zeta_n^*}{\partial \tau} - h_o \frac{\partial \phi_n}{\partial z^*} = R_{k_n} \tag{27b}$$

where R_{d_n} and R_{k_n} are given in Appendix B and C, respectively, for $n = 0, 1, 2$ and 3. As a result of Equations (27a, b),

$$\frac{\partial^2 \phi_n}{\partial \tau^2} + h_o^2 g^* \frac{\partial^2 \phi_n}{\partial z^2} = h_o g^* \frac{\partial R_{d_n}}{\partial \tau} - h_o g^* R_{k_n} = R_n \tag{27c}$$

For convenience, the cylindrical coordinates will b employed in which dx_i $(i = 1, 2, 3)$ are

$$dx_1 = dr, \quad dx_2 = r d\theta, \quad dx_3 = dz \tag{28}$$

It is found that

$$f_{n,m}^{(1,s)} = \frac{1}{2} x_o^* B^* (N^3 - N^2) \delta_{m,k} \delta_{n,N+1} + \frac{B^*}{2} x_o^* (N^3 + N^2) \delta_{m,k} \delta_{n,N-1}$$
$$- 2g^* h_1 \delta_{m,k} \delta_{n,1} B^* + \frac{1}{2}\left[\sum_{i=1}^{2} \frac{1}{a_m^2} \int_F \left(\frac{\partial \psi_k}{\partial x_i^*}\right)^2 \psi_m ds \cdot 2\right.$$
$$\left. + \lambda_k^2 \frac{1}{a_m^2} \int_F \psi_k^2 \psi_m ds\right] (A^{*2} g^{*2} - B^{*2}) \delta_{n,2} + f_{mna}^{(1,s)} \tag{29a}$$

$$f_{nm}^{(1,c)} = \frac{1}{2} x_o^* B^* (N^3 - N^2) \delta_{mk} \delta_{n,N+1} + \frac{1}{2} x_o^* B^* (-N^3 + N^2) \delta_{mk} \delta_{n,N-1}$$
$$- 2g^* h_1 \delta_{m,k} \delta_{n,1} A^* - A^* B^* g^{*2} \delta_{n,2} \sum_{i=1}^{2} \left[2 \cdot \frac{1}{a_m^2} \int_F \left(\frac{\partial \psi_k}{\partial x_i^*}\right)^2 \psi_m ds\right.$$
$$\left. + \lambda_k^{*2} \frac{1}{a_m^2} \int_F \psi_k^2 \psi_m ds\right] + f_{nma}^{(1,c)} \tag{29b}$$

where

$$f^{(1,s)}_{nma} = \delta_{m,k} \quad \frac{1}{2}(A^{*2} - B^{*2})\,\delta_{n,2}\, h_o^2 g^{*2} \frac{1}{a_m^2} \int_F \psi_k \frac{\partial^2 \psi_k}{\partial z^{*2}} \psi_m ds$$

$$+ (A^{*2} - B^{*2})\,\delta_{n,2}\,\delta_{mk} \frac{1}{a_m^2} \int_F \psi_k \frac{\partial \psi_k}{\partial z^*} \psi_m \, ds \cdot g^* \qquad (29c)$$

$$f^{(1,c)}_{nma} = h_o^2 g^{*2} (-A^*B^*)\,\delta_{n,2}\,\delta_{mk} \frac{1}{a_m^2} \int_F \psi_k \frac{\partial^2 \psi_k}{\partial z^{*2}} \psi_m \, ds$$

$$- 2g^*(A^*B^*)\,\delta_{n,2} \frac{1}{a_m^2} \int_F \psi_k \frac{\partial \psi_k}{\partial z^*} \psi_m \, ds \qquad (29d)$$

In order to have periodic first approximations, one must not have nonzero resonant harmonics. Hence $f^{(1,s)}_{1k}$ and $f^{(1,c)}_{1k}$ must vanish. One has

$$f^{(1,s)}_{1,k} = \frac{1}{2} x_o^* B^* (N^3 + N^2)\,\delta_{1,N-1} - 2g^* h_1 B^* = 0 \qquad (30a)$$

$$f^{(1,c)}_{1,k} = \frac{1}{2} x_o^* A^* (-N^3 + N^2)\,\delta_{1,N-1} - 2g^* h_1 A^* = 0 \qquad (30b)$$

As stated under Equation (24), the inphase or out of phase solution is selected for which

$$B^* = 0 \qquad (31a)$$

$$h_1 = \frac{1}{2g^*} \cdot \frac{1}{2} x_o^* (-N^3 + N^2)\,\delta_{1,N-1} = \frac{-x_o^*}{g^*} \delta_{1,N=1} = \frac{-x_o^*}{g^*} \delta_{N,2} \qquad (31b)$$

The discarded solution is

$$A^* = 0 \qquad (32a)$$

$$h_1 = \frac{1}{2g^*} \cdot \frac{1}{2} x_o^* (N^3 + N^2)\,\delta_{1,N-1} = 3 \frac{x_o^*}{g^*} \delta_{N,2} \qquad (32b)$$

For 1/2 subharmonics, $N = 2$. If $A^* \neq 0$, $B^* = 0$, then

$$h_1 = \frac{-x_o^*}{g^*} \leq 0 \qquad g^* = \frac{g}{\omega_k^2 a} \tag{33a}$$

If $B^* \neq 0$, $A^* = 0$, then

$$h_1 = \frac{3x_o^*}{g^*} \geq 0 \tag{33b}$$

If h_2 term is negligible,

$$\omega = \frac{\omega_k}{1 + h_1 \epsilon} \tag{34}$$

In the first case as $\epsilon \longrightarrow 0^+$, $x_o^* \epsilon$ remains a given positive constant, the limiting frequency is higher than ω_k. In particular, the limiting values are in good agreement with experiments (Ref. 15) of a circular cylindrical tank.

The first order solution is completely determined with

$$\zeta_m^{(1)} = \sum_{n=1}^{N+1} \zeta_{n,m}^{(1,s)} \sin(n\tau) + \sum_{n=1}^{N+1} \zeta_{n,m}^{(1,c)} \cos(n\tau) \tag{35}$$

$$\phi_m^{(1)} = \sum_{n=1}^{N+1} \phi_{n,m}^{(1,s)} \sin(n\tau) + \sum_{n=0}^{N+1} \phi_{n,m}^{(1,c)} \cos(n\tau) \tag{36}$$

where

$$\phi_{n,m}^{(1,s)} = \tilde{\phi}_{n,m}^{(1,s)} + \underline{\underline{\delta_{n1}\,\delta_{mk}\,\tilde{f}_{1k}^{(1,s)}}}\,; \quad \tilde{\phi}_{nm}^{(1,s)} = \frac{f_{n,m}^{(1,s)}(1 - \delta_{n1}\delta_{mk})}{g_m^* - n^2} \tag{37a}$$

$$\phi_{n,m}^{(1,c)} = \tilde{\phi}_{n,m}^{(1,c)} + \underline{\underline{\delta_{n1}\,\delta_{mk}\,\tilde{f}_{1k}^{(1,c)}}}\,; \quad \tilde{\phi}_{nm}^{(1,c)} = \frac{f_{nm}^{(1,c)}(1 - \delta_{n1}\delta_{mk})}{g_m^* - n^2} \tag{37b}$$

$$\zeta_{n,m}^{(1,s)} = -\frac{1}{g^*}\Bigg\{ -\frac{f_{nm}^{(1,c)}(1 - \delta_{mk}\delta_{n1})n}{g_m^* - n^2} - \underline{\underline{\tilde{f}_{1,k}^{(1,c)}\,\delta_{mk}\,\delta_{n1}}}$$

$$+ h_1 A^* g^* \delta_{mk}\delta_{n1} - N^2 x_o^* A^* \delta_{mk}(\delta_{n,N+1} - \delta_{n,N-1})\frac{1}{2} + Td_{nm}^{(1,s)} \Bigg\}$$

$$\equiv \tilde{\zeta}_{nm}^{(1,s)} + \frac{1}{g^*}\tilde{f}_{1,k}^{(1,c)}\,\delta_{mk}\delta_{n1} \tag{38a}$$

$$\zeta^{(1,c)}_{n,m} = -\frac{1}{g^*}\left\{\frac{f^{(1,s)}_{nm}(1-\delta_{mk}\delta_{1n})\,n}{g^*_m - n^2} + \underline{\underline{\tilde{f}^{(1,s)}_{1k}\delta_{mk}\delta_{n1}}}\right.$$

$$+\frac{1}{2}\sum_{i=1}^{2}(A^*g^*)^2\left[\frac{1}{a_m^2}\int_F\left(\frac{\partial\psi_k}{\partial x_i}\right)^2 \psi_m\, ds \quad \frac{1}{2}(\delta_{n,0}+\delta_{n,2})\right]$$

$$\left.+\frac{1}{2}\lambda_k^{*2}\left[\frac{1}{a_m^2}\int_F \psi_k\psi_m ds\right](A^*g^*)^2 \quad \frac{1}{2}(\delta_{n,0}+\delta_{n,2}) + Td^{(1,c)}_{nm}\right\}$$

$$+\frac{1}{2}A^{*2}\delta_{m,k}(\delta_{n,2}-\delta_{n,0}) \equiv \tilde{\zeta}^{(1,c)}_{n,m} - \frac{\tilde{f}^{(1,s)}_{1k}}{g^*}\delta_{mk}\delta_{n1} \qquad (38b)$$

$$\tilde{\zeta}^{(1,c)}_{1,k} = 0 \qquad (38c)$$

$$\tilde{\zeta}^{(1,s)}_{1k} = -\frac{1}{g^*}\left\{h_1 A^* g^* + \frac{N^2 x_o^* A^*}{2}\delta_{N,2}\right\} \qquad (38d)$$

$$Td^{(1,s)}_{n,m} = \left(-\frac{1}{2}\delta_{n,0}+\frac{1}{2}\delta_{n,2}\right)A^{*2}\frac{1}{a_m^2}\int_F \psi_k^2\psi_m ds;\ Td^{(1,s)}_{nm} = 0 \qquad (38e, f)$$

$\tilde{f}^{(2,s)}_{1k}$ and $\tilde{f}^{(2,c)}_{1k}$ have to be determined from a third order equation by the condition that the solution is oscillatory and of period 2π, or the coefficients of the first harmonics in the nonhomogeneous terms must vanish.

From $f^{(2,s)}_{1k} = 0$,

$$a^{(2)}_{11}\tilde{f}^{(1,s)}_{1k} + a^{(2)}_{12}\tilde{f}^{(1,c)}_{1k} = R^{(2)}_1 = R^{(2)}_{1a} - \frac{2N^3 - N^2}{2}x_o^*(1-\delta_{N2})\zeta^{(1,c)}_{N-1,k} \qquad (39a)$$

from $f^{(2,c)}_{1,k} = 0$,

$$a^{(2)}_{21}\tilde{f}^{(1,s)}_{1k} + a^{(2)}_{12}\tilde{f}^{(1,c)}_{1k} = R^{(2)}_{2a} - h_2 \cdot 2g^*\zeta^{(0,s)}_{1,k} \qquad (39b)$$

where

$$a^{(2)}_{11} = -4h_1\delta_{N,2} \qquad (40a)$$

$$a^{(2)}_{12} = 0 \qquad (40b)$$

$$a_{21}^{(2)} = 0 \tag{40c}$$

$$a_{22}^{(2)} = 0 \tag{40d}$$

Equation (39a) determines $\tilde{f}_{1k}^{(1,s)}$. Equation (39b) yields h_2 as a function of the constant A^* and tank displacement x_o^*. $f_{1k}^{(1,c)}$ is a constant which will be damped out in a real system; hence, it is assumed to be zero.

The nonhomogeneous terms are

$$R_{1a}^{(2)} = g^* \sum_{\overline{m}=1}^{\infty} \left\{ \sum_{i=1}^{2} \left[\frac{1}{a_k^2} \int_F \frac{\partial \psi_k}{\partial x_i^*} \frac{\partial \psi_{\overline{m}}}{\partial x_i^*} \psi_k ds \right] \cdot \left[-\frac{1}{2} \phi_{2\overline{m}}^{(1,c)} \zeta_{1k}^{(0,s)} + \frac{1}{2} \zeta_{2m}^{(1,s)} \phi_{1k}^{(0,c)} \right] \right\} + \frac{1}{2} \phi_{1k}^{(0,c)} \sum_{\overline{m}=1}^{\infty} \sum_{i=1}^{2} \frac{1}{a_k^2} \int_F \frac{\partial \psi_k}{\partial x_i^*} \frac{\partial \psi_{\overline{m}}}{\partial x_i^*} \psi_k ds \cdot$$

$$\cdot \phi_{2m}^{(1,c)} + 2h_1 g^* \zeta_{1k}^{(1,c)} + \frac{1}{2} \phi_{1k}^{(0,c)} \sum_{\overline{m}=1}^{\infty} \sum_{i=1}^{2} \frac{1}{a_k^2} \int_F \psi_{\overline{m}} \psi_k ds \cdot$$

$$\cdot \lambda_k^* \lambda_m^* \phi_{2,\overline{m}}^{(1,s)} + f_{1ka}^{(2,s)} \tag{41a}$$

$$R_{2a}^{(2)} = -\frac{1}{2} N^2 x_o^* (1 - \delta_{N2}) \zeta_{N-1,k}^{(1,s)} + \delta_{N2} \tilde{\zeta}_{1k}^{(1,s)} \left(-\frac{1}{2} N^2 x_o^* \right)$$

$$+ g^* \sum_{\overline{m}=1}^{\infty} \left\{ \sum_{i=1}^{2} \frac{1}{a_k^2} \int_F \frac{\partial \psi_k}{\partial x_i^*} \frac{\partial \psi_{\overline{m}}}{\partial x_i^*} \psi_k ds \cdot \frac{1}{2} \left[\zeta_{1,k}^{(0,s)} \phi_{2\overline{m}}^{(1,s)} + \phi_{1k}^{(0,c)} \zeta_{2\overline{m}}^{(1,c)} \right] \right\}$$

$$- 2h_1 g^* \tilde{\zeta}_{1k}^{(1,s)} - g^* h_1^2 \zeta_{1k}^{(0,s)} - \frac{1}{2} \sum_{\overline{m}=1}^{\infty} \left\{ \phi_{1,k}^{(0,c)} \phi_{2\overline{m}}^{(1,s)} \sum_{i=1}^{2} \frac{1}{a_k^2} \int_F \frac{\partial \psi_k}{\partial x_i^*} \cdot \right.$$

$$\left. \cdot \frac{\partial \psi_{\overline{m}}}{\partial x_i^*} \psi_k ds \right\} - \frac{1}{2} \sum_{\overline{m}=1}^{\infty} \left\{ \phi_{1,k}^{(0,c)} \phi_{2\overline{m}}^{(1,s)} \lambda_k^* \lambda_{\overline{m}}^* \frac{1}{a_k^2} \int_F \psi_k^2 \psi_{\overline{m}} ds \right\} + f_{1ka}^{(2,c)} \tag{41b}$$

For $N \neq 2$, $h_1 = 0$, $a_{11}^{(2)} = a_{12}^{(2)} = a_{21}^{(2)} = a_{22}^{(2)} = 0$. Both A^* and h_2 are determined by Equation (39b) and Equation (39a). Both $\tilde{f}_{1k}^{(1,s)}$ and $\tilde{f}_{1k}^{(1,c)}$ are set to zero. Since ϵ is still arbitrary, one may use $A^*\epsilon$ as the given liquid amplitude. Further investigation may be necessary to show if this process leads to a valid solution.

The second approximation (third order terms) has not been completed but can be determined similarly with two constants $\tilde{f}_{1,k}^{(2,c)}$, $\tilde{f}_{1,k}^{(2,s)}$ governed by two conditions of no secular terms.

CONSTRUCTION OF CHARACTERISTIC FUNCTIONS

Method of Separation of Variables

It is easy to show that for rectangular tanks and circular cylindrical tanks, the characteristic functions and eigenvalues are

$$E_{mn} = \cos\left(\frac{mx}{a_o}\right)\cos\left(\frac{ny}{b_o}\right)\cosh\left(\sqrt{\frac{m^2}{a_o^2}+\frac{n^2}{b_o^2}}\,(z+h)\right) \tag{47a}$$

$$\lambda_{mn} = \sqrt{\frac{m^2}{a_o^2}+\frac{n^2}{b_o^2}}\,\tanh\sqrt{\frac{m^2}{a_o^2}+\frac{n^2}{b_o^2}}\,h \tag{47b}$$

and

$$E_{mn} = J_m\left(\bar{\lambda}_{mn}\frac{\rho}{a}\right)\cos(m\theta)\cosh\left(\frac{\bar{\lambda}_{mn}(z+h)}{a}\right) \tag{48a}$$

$$\lambda_{mn} = \frac{\bar{\lambda}_{mn}}{a}\tanh\left(\frac{\bar{\lambda}_{mn}h}{a}\right) \tag{48b}$$

where a_o and b_o are the length and the width of the rectangular tank divided by 2π, and the $\bar{\lambda}_{mn}$ are the roots of $J_m'(\bar{\lambda}_{mn}) = 0$.

These characteristic functions were constructed from the method of separation of variables; they were probably not recognized as characteristic functions. However, it is well known that this method is not suitable for general application.

Integral Equation Method

The characteristic functions satisfy the Laplace equation, and the boundary conditions are given in Equations (9) and (10). The characteristic function on the free surface can be determined by the integral equation, Equation (14), with $z = 0$, provided that the Neumann function on the boundary is known or can be constructed. The Neumann function is singular on the boundary when the field point and the source point coincide. For a spherical tank, a numerical method for constructing the Neumann function was given in Reference 19. The Neumann function constructed was symmetrical and yielded satisfactory eigenvalues. However, Reference 19 finds an addition to a known Neumann function which possesses constant normal derivative on the wetted spherical tank wall. For a general tank, it is easier to construct the required Neumann function directly. It will be seen that the incomplete elliptic functions do not enter the present method, and thus considerable time can be saved for computing matrices of a higher rank. The increase of the rank of the matrix is due to the presence of integrals on the wetted wall which can be eliminated otherwise.

In principle, the addition to a unit sink is to be found numerically to yield the Neumann function. The characteristic functions are calculated next. Then the coupling integrals and the normalizing constants are computed with the numerical knowledge of the characteristic functions at a finite number of points.

Let

$$G = G_o + g(P, Q) \tag{49}$$

$$\frac{\partial G}{\partial n} = k_1 = \frac{-1}{A_F + A_W} \qquad \text{on W and F} \tag{50}$$

W and F are the wetted wall and the free surface, respectively.

Green's theorem states that

$$\phi = \int_{W+F} \left(G \frac{\partial \phi}{\partial n} - \phi \frac{\partial G}{\partial n} \right) ds \tag{51}$$

A new definition of ϕ can be made to absorb a constant term* and yield

$$\phi_\nu = \int_{F_o} G(P, I) \frac{\partial \phi_\nu}{\partial n_I} ds_I = \lambda_\nu \int_{F_o} G(P, I)\, \phi_\nu ds_I \tag{52}$$

where

$$\phi_\nu = \Psi_\nu(x, y, z), \ \phi_\nu(x, y, o) = \psi_\nu(x, y) \tag{52a}$$

For practical purposes, the tank will be assumed to be axial symmetric.

Let

$$\phi_\nu = \bar{\phi}_\nu(r, \sigma)\cos(\nu\theta); \ \psi_\nu = \bar{\psi}_\nu(\rho, \sigma_{F_o})\cos(\nu\theta) \tag{53ab}$$

$$\bar{\phi}_\nu(r, \sigma) = \frac{\lambda_\nu}{(1 + \delta_{o\nu})\pi} \int_0^{2\pi} \int_0^{2\pi} \bar{\phi}_\nu G(r, \sigma, \theta; r', \sigma', \theta') \cos(\nu\theta') d\theta' d\theta \tag{54}$$

Integrating Equation (52) with Equation (53) yields

*As a result, the average value of ϕ_ν over W + F is zero, i.e., $\int \phi_\nu ds = 0$.

$$\overline{\psi}_\nu(\rho) = \frac{\lambda_\nu}{(1+\delta_{o\nu})\pi} \int_0^b \int_0^{2\pi} \int_0^{2\pi} \overline{\psi}_\nu G(\rho, \sigma_{F_o}, \theta; \rho', \sigma_{F_o}, \theta') \cos(\nu'\theta) \cos(\nu\theta) d\theta' d\theta \rho' d\rho'$$

$$\overline{\psi}_\nu(\rho) = \lambda_\nu \int_0^b \overline{H}_\nu(\rho, \rho') \overline{\psi}_\nu(\rho') \rho' d\rho' \tag{55}$$

where

$$\overline{H}_\nu = \overline{H}_{o\nu} + \overline{h}_\nu \tag{56a}$$

$$\overline{H}_{o\nu} = \frac{1}{(1+\delta_{o\nu})\pi} \int_0^{2\pi} \int_0^{2\pi} G_o(\rho, \sigma_{F_o}, \theta; \rho', \sigma_{F_o}, \theta') \cos(\nu\theta) \cos(\nu\theta') d\theta' d\theta \tag{56b}$$

$$\overline{h}_\nu = \frac{1}{(1+\delta_{o\nu})\pi} \int_0^{2\pi} \int_0^{2\pi} g(\rho, \sigma_{F_o}, \theta; \rho', \sigma_{F_o}, \theta') \cos(\nu\theta) \cos(\nu\theta') d\theta' d\theta \tag{56c}$$

Let

$$\phi_\nu^{(n)}(\rho) = \sqrt{\rho}\, \overline{\psi}_\nu(\rho) \tag{57}$$

$\psi_\nu^{(n)}$ is governed by

$$\psi_\nu^{(n)}(\rho) = \lambda_{\nu n} \int_0^b [\overline{H}_\nu(\rho, \rho') \sqrt{\rho, \rho'}]\, \psi_\nu^{(n)}(\rho') d\rho' \tag{58}$$

Analogous to Reference 19, Equation (58) can be approximated by the matrix equation

$$\left(\frac{N}{\lambda_{\nu n} b} \right) \left\{ \phi_\nu^{(n)} \right\} = [A_\nu] \left\{ \phi_\nu^{(n)} \right\} \tag{59}$$

where

$$A_{\nu ik} = H_{\nu ik} + \tilde{C}_{ik} \tag{60}$$

$$\underset{\sim}{H}^{(o)}_{\nu ik} = \frac{1}{2} \left[\frac{1}{4} \sum_{i=4k-3}^{4k} H_{ij}^{(o)} + \frac{1}{4} \sum_{j=4i-3}^{4i} H_{kj}^{(o)} \right] \tag{60a}$$

$$\tilde{C}_{ik} = \frac{1}{2} [C_{ik} + C_{ki}] \tag{60b}$$

$$H_o^{(o)} = \sqrt{r_i r_j}\, \overline{H}_{o_\nu}(r_i, r_j) \tag{60c}$$

First one must evaluate $\bar{H}_{o\nu}$ and then C_{ik}. For simplicity, the sink alone will be contained in G_o. $\bar{\bar{H}}_{o\nu}$ has been expressed in terms of elliptical functions in Appendix G. The integral equation governing the added part h is (Ref. 19)

$$\frac{1}{2}\sqrt{r_P r_Q}\bar{h}_\nu(r_P, r_Q) = -\int_{\bar{W}+\bar{F}_o}[\bar{\mathfrak{J}}_{o\nu}(\bar{I}, \bar{Q})\sqrt{r_Q r_I}]\,[\sqrt{r_P r_I}\bar{H}_{o\nu}(r_P, r_\nu)]\frac{d\bar{s}_I}{r_I}$$

$$-k_1 4\pi^2\delta_{o\nu}\int_{\bar{W}+\bar{F}_o}[\bar{\mathfrak{J}}_{o\nu}(\bar{I}, \bar{Q})\sqrt{r_Q r_I}][\sqrt{r_I r_P}\bar{h}_\nu(\bar{I}, \bar{P})]\frac{d\bar{s}_I}{r_I}$$

$$-\int_{\bar{W}+\bar{F}_o}[\bar{\mathfrak{J}}_{o\nu}(\bar{I}, \bar{Q})\sqrt{r_Q r_I}][\sqrt{r_I r_P}\bar{h}_\nu(\bar{I}, \bar{P})\frac{d\bar{s}_I}{r_I} \qquad (61)$$

Let

$$g_{\nu_{ij}} = \bar{h}_\nu(r_i r_j)\sqrt{r_i r_j} \qquad (62a)\&$$

$$H^{(o)}_{\nu_{ij}} = \sqrt{r_i r_j}\bar{H}_{o\nu}(r_i r_j) \qquad (62b)$$

$$\mathfrak{J}_{\nu_{ji}} = \sqrt{r_i r_j}\;\bar{\mathfrak{J}}_{o\nu}(r_i r_j) \qquad \text{(Note the order of j and i)} \qquad (62c)$$

where $\mathfrak{J}_{o\nu}$ is given in Appendix G. Then, taking b = 1, Equation (61) becomes

$$\frac{1}{2}g_{\nu_{j\ell}} = -\int_0^1 \mathfrak{J}_{\nu_{j\ell}}H^{(o)}_{\nu_{ij}}d\rho_j - \int_{\sigma=\frac{\pi}{2}}^{\pi}\mathfrak{J}_{\nu_{j\ell}}H_{\nu_{ij}}\sqrt{\left(\frac{dr_j}{d\sigma_j}\right)^2 + r_j^2}\,d\sigma_j$$

$$+\delta_{o\nu}\frac{4\pi^2}{A_F + A_W}\int_{\frac{\pi}{2}}^{\pi}H^{(o)}_{\nu_{j\ell}}\sqrt{\left(\frac{dr_j}{d\sigma_j}\right)^2 + r_j^2}\;\sqrt{r_i r_j}\,d\sigma_j$$

$$+\delta_{o\nu}\frac{4\pi^2}{A_F + A_W}\int_0^1 H^{(o)}_{\nu_{j\ell}}\sqrt{r_i r_j}\,d\rho_j$$

$$-\int_0^1 \mathfrak{J}_{\nu_{i\ell}}g_{\nu_{ij}}d\rho_j - \int_{\frac{\pi}{2}}^{\pi}\mathfrak{J}_{\nu_{j\ell}}g_{\nu_{ij}}\sqrt{\left(\frac{dr_j}{d\sigma_j}\right)^2 + (r_j)^2}\,d\sigma_j \qquad (63)$$

---

*The symmetric property of $g_{\nu_{ij}}$ provides a check of the precision and the correctness.

where

$$A_F = \pi ; \; A_W = \int_{\frac{\pi}{2}}^{\pi} 2\pi r \sqrt{\left(\frac{dr}{d\sigma}\right)^2 + r^2}\, d\sigma \tag{63a, b}$$

Equation (63) can be solved by the four point midpoint formula as done in Reference 19.

Let N_F = number of basic intervals on F_o, N_T = total number of basic intervals on $F_o + W$. For $j \leq 4N_F$

$$\rho_j = r_j = \frac{\tilde{\Delta}_j}{8} + (j-1)\frac{\tilde{\Delta}_j}{4} \tag{64aa)*}$$

$$\tilde{\Delta}_j = \frac{1}{N_F} \tag{64ab}$$

$$\mathfrak{F}_{\nu_{i\ell}} = F_{\nu_{i\ell}} \tag{64ac}$$

For $j \geq 4N_F + 1$

$$\tilde{\Delta}_j = \frac{\pi}{2}\,\frac{1}{N_T - N_F} \tag{64ba}$$

$$r_j = r(\sigma_j) \tag{64bb}$$

$$\sigma_j = \frac{1}{8}\tilde{\Delta}_j + (j - 4N_F - 1)\frac{\tilde{\Delta}_j}{4} + \frac{\pi}{2} \tag{64bc}$$

$$\mathfrak{F}_{\nu_{j\ell}} = \mathfrak{J}_{\nu_{j\ell}} \sqrt{\left(\frac{dr_j}{d\sigma_j}\right)^2 + r_j^2} \tag{64bd}$$

For $i \leq N_F$

$$\tilde{\Delta}_i = \frac{1}{N_F} \tag{64ca}$$

$$\rho_i = r_i = \frac{\tilde{\Delta}_i}{2} + (i-1)\tilde{\Delta}_i \tag{64cb}$$

*Subscript j here for integration variables.

For $i \geq N_F + 1$

$$\tilde{\Delta}_i = \frac{\pi}{2} \frac{1}{N_T - N_F} \tag{64da}$$

$$\sigma_i = \frac{\tilde{\Delta}_i}{2} + (i - N_F - 1)\,\tilde{\Delta}_i + \frac{\pi}{2} \tag{64db}$$

$$r_i = r\,(\sigma_i) \tag{64dc}$$

Then Equation (63) can be approximated by the following matrix equation

$$\frac{1}{2}[C] = -[M] - [C][D] \tag{65}$$

The element of C is

$$C_{ik} = g_{\nu_{ik}} \tag{66}$$

The element of [D] is

$$D_{k\ell} = \sum_{j=4k-1}^{4k} \frac{\tilde{\Delta}_j}{4} \mathfrak{J}_{\nu_{i\ell}} \tag{67}$$

The element of M is

$$M_{i\ell} = M_{i\ell}^{(1)} + M_{i\ell}^{(2)} \tag{68}$$

where

$$M_{i\ell}^{(1)} = \sum_{j=1}^{4N_T} \frac{\tilde{\Delta}_j}{4} \mathfrak{J}_{\nu_{ij}} H_{\nu_{ij}}^{(o)} \tag{68a}$$

$$M_{i\ell}^{(2)} = -\,\delta_{o\nu} \frac{4\pi^2}{A_F + A_W} \sum_{i=1}^{4N_T} \frac{\tilde{\Delta}_j}{4} f_{ij} H_{\nu_{\ell j}}^{(o)} \tag{68b}$$

For $j \leq 4N_F$

$$f_{ij} = \sqrt{r_i r_j} \tag{68ba}$$

For $j \geq 4N_F + 1$

$$f_{ij} = \sqrt{\left(\frac{dr_j}{d\sigma_j}\right)^2 + r_j^2} \cdot \sqrt{r_i r_j} \tag{68bb}$$

r_i is given in Equations (64cb) and (64dc). C_{ik} is given by the matrix solution

$$[C] = \frac{-[M]}{\frac{1}{2}[I] + [D]} \tag{69}$$

The symmetric property of C_{ik} should serve as a check of the accuracy and correctness.

After the Neumann function is found, the characteristic functions can be solved numerically from Equation (59). Then the coupling integrals can be calculated numerically. Some notes are given in Appendix I. After the coupling integrals are evaluated, the free surface evaluations can be calculated over one period from its series expansion.

Numerical Finite Difference Method

It is possible to write a relatively general computer program for a domain with irregular boundaries to determine the characteristic functions as well as eigenvalues, (Ref. 20). It is hoped that the integral equation method is more economical.

EXAMPLES OF 1/2 SUBHARMONIC LIQUID SLOSHING

The 1/2 subharmonic oscillations of a liquid in rectangular tank is calculated for comparison with available experiments (Ref. 13). The theoretical results based on three assumed modes ($\bar{m}$ = 0, 1, 2, n = 1) are shown in Figure 2 and are in good agreement with experiments. The disagreement increases with amplitude of liquid motion, as one would expect from a first approximation. The approximate theoretical solution is:

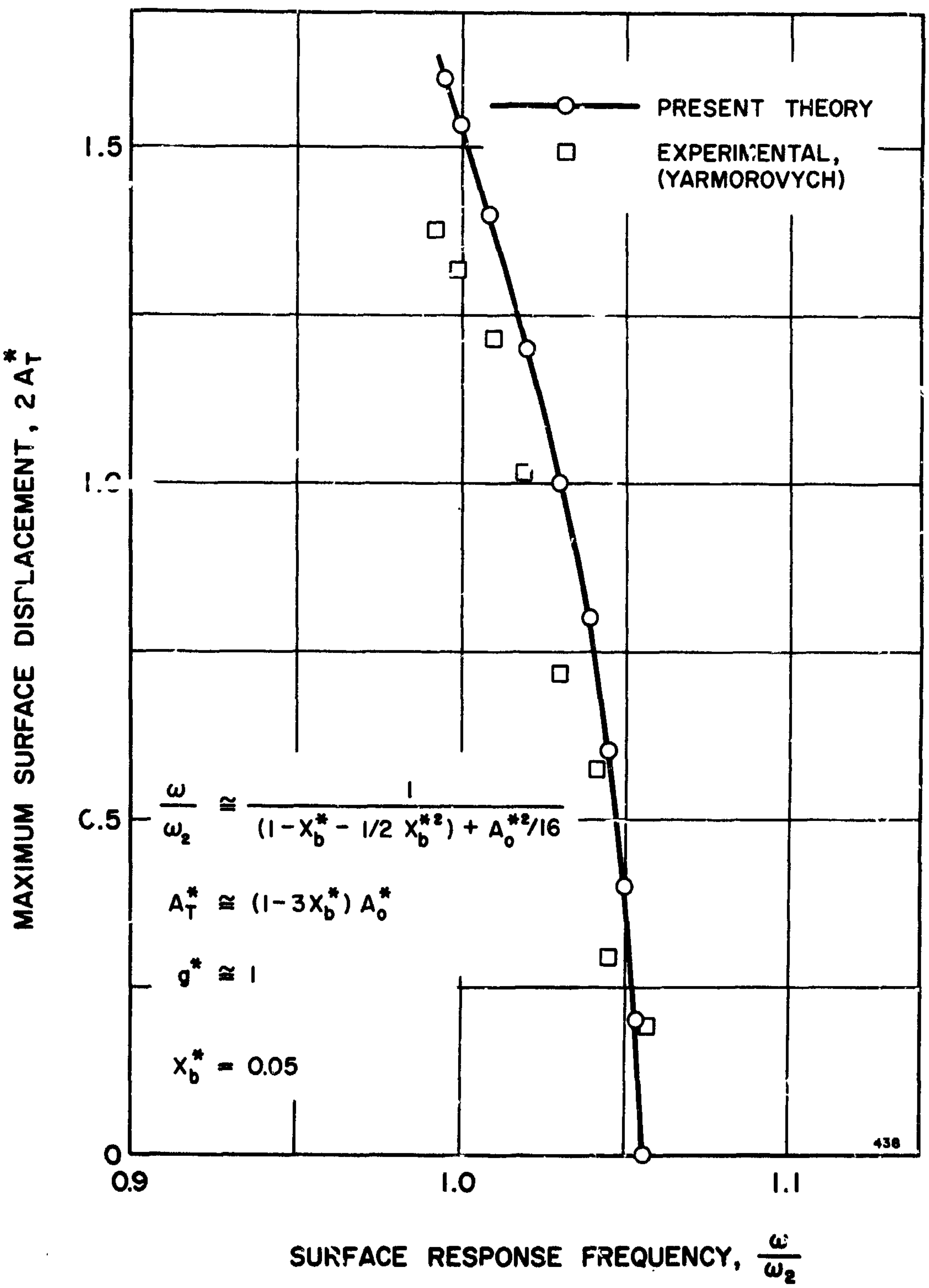

FIG. 2. LIQUID SLOSHING IN A RECTANGULAR TANK — 1/2 SUBHARMONIC

$$\frac{\omega}{\omega_2} \cong \frac{1}{1 - x_b^* - \frac{1}{2}x_b^{*2} + A_o^{*2}/16}$$

$$A_T^* \cong (1 - 3x_b^*)\, A_o^* \qquad ; \qquad A_o^* = A^*\epsilon$$

as k = 2 and g* = 1. x_b^* is the nondimensional amplitude of excitation. A_o^* is the amplitude of the linear term of the first harmonic.. A_T^* is the approximate total liquid amplitude at $\tau = 90°$, which is approximately one half of the maximum amplitude.

In general, the limiting value of zero amplitude 1/2 subharmonic response is given by

$$\frac{\omega_o}{\omega_k} \cong \frac{1}{1 - \frac{x_b^*}{g^*} - \frac{1}{2}\left(\frac{x_b^*}{g^*}\right)^2} \tag{70}$$

The theoretical values based on Equation (70) are compared with experimental values (Ref. 15) in Table 1.

TABLE 1. LIMITING FREQUENCY RATIO OF A CIRCULAR TANK

Mode	m = 0, n = 1		m = 1, n = 1	
x_b^*/g^*	.0242	.0403	.0173	.023
(ω_o/ω_k) theo.	1.024	1.042	1.018	1.022
(ω_o/ω_k) exp.	1.023	1.042	1.014	1.024
h/(2a)	1/4		1	

Preliminary unpublished data of a spherical tank and a sector tank were also compared with the theoretical value, but the agreements were only fair (Table 2). The main difference is most likely caused by damping due to geometry of the tank.

TABLE 2. LIMITING FREQUENCY RATIO FOR A SPHERICAL TANK AND A SECTOR TANK

Tank Shape	Spherical Tank		1/4 Sector Tank	
Mode	m = 0, n = 1		m = 2, n = 1	m = 2, n = 2
x_b^*	0.00853	0.00642	.01364	.01364
ω_2, cps	3.1 (exp)	3.1 (exp)	2.476 (theo)	2.776 (theo)
ω_o, cps	3.16 (exp)	3.125 (exp)	2.52 (exp)	2.845 (exp)
g^*	.2666	.2666	1/3.054	1/3.832
(ω_o/ω_2) theo.	1.032	1.024	1.043	1.058
(ω_o/ω_2) exp.	1.019	1.009	1.017	1.026
Remarks	d = 2a = 7.625" h/d = 1/4		d = 2a = 9.75" h/d = 1	

No example to show the new procedure of numerical construction of the eigenfunctions has been made at present.

CONCLUSIONS AND DISCUSSIONS

A theory to the third order is developed for subharmonic oscillations in an arbitrary axisymmetric tank. This theory is applicable to a regular sector tank if the partitions are sufficiently rigid. The direct suppression effect of the tank wall on the free surface has been neglected. This effect does not exist in the rectangular tank or the cylindrical tank since the side walls are parallel to the z-axis.

For a rectangular tank, the frequency amplitude response is in good agreement with available experiments. In this example, the new procedure of generating characteristic functions through integral equations is not needed; hence, an example, such as a spherical tank, is desirable to examine the new procedure.

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APPENDIX A

Derivation of the Boundary Conditions on F_0

APPENDIX A

Derivation of the Boundary Conditions on F_o

The following relations were employed repeatedly to obtain boundary conditions of equal powers of ϵ on F_o from Equations (6b) and (8).

The product of two infinite series,

$$\sum_{m=0}^{\infty} f_m \cdot \sum_{n=0}^{\infty} g_n = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f_m g_n = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} g_{\ell-m} f_m \qquad \text{(A1)}$$

The product of three infinite series,

$$\sum_{n=0}^{\infty} h_n \cdot \sum_{m=0}^{\infty} f_m \sum_{m_1=0}^{\infty} g_n = \sum_{n=0}^{\infty} h_n \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} g_{\ell-m} f_m$$

$$= \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} h_{k-\ell} \cdot \sum_{m=0}^{\ell} g_{\ell-m} f_m \qquad \text{(A2)}$$

Higher order products can be obtained analogously.

One may either expand every term in Taylor's series from the free surface F to the mean free surface F_o, or collect equal powers on F then recollect them after expanding each term from F to F_o. The latter was employed for collecting terms up to ϵ^4.

Only the nondimensionalized equations are given in the following (length nondimensionalized by a, velocity by ω_k a, acceleration by $\omega_k^2 a$).

The components of the dynamic free surface condition are

$$\zeta_0^* = -\frac{1}{g^*h_o}\frac{\partial\phi_0^*}{\partial\tau} \tag{A3}$$

$$\zeta_1^* = -\frac{1}{g^*h_o^2}\left\{h_o\frac{\partial\phi_1}{\partial\tau} + h_1\left(\frac{\partial\phi_0^*}{\partial\tau} + 2\zeta_0^* g_o^* h_o\right) - N^2\zeta_o^*\cos(N\tau)\,x_o^*\right.$$
$$\left.\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial\phi_0^*}{\partial x_i^*}\right)^2 h_o^2 + \frac{1}{2}\left(\frac{\partial\phi_0}{\partial z^*}\right)^2 h_o + T_{d_1}^*\right\} \tag{A4}$$

$$\zeta_2^* = -\frac{1}{g^*h_o^2}\left\{h_o\frac{\partial\phi_2}{\partial\tau} + h_1\frac{\partial\phi_1}{\partial\tau} + h_2\left(\frac{\partial\phi_0}{\partial\tau} + 2\zeta_0^* g^* h_0\right)\right.$$
$$+ g^*(h_1^2\zeta_o^* + 2h_1h_0\zeta_1^*) - N^2\zeta_1^* x_o^*\cos(N\tau)$$
$$+ \sum_{i=1}^{2}\left[h_1h_0\left(\frac{\partial\phi_0^*}{\partial x_i^*}\right)^2 + h_0^2\frac{\partial\phi_0^*}{\partial x_i^*}\right] + \frac{\partial\phi_0^*}{\partial z^*}\frac{\partial\phi_1^*}{\partial z^*}h_0^2$$
$$\left. + h_1h_0\left(\frac{\partial\phi_0^*}{\partial z^*}\right)^2 + T_{d_2}^*\right\} \tag{A5}$$

$$\zeta_3^* = -\frac{1}{g^*h_0}\left\{h_0\frac{\partial\phi_3^*}{\partial\tau} + h_1\frac{\partial\phi_2^*}{\partial\tau} + h_2\frac{\partial\phi_1^*}{\partial\tau} + h_3\frac{\partial\phi_0^*}{\partial\tau}\right.$$
$$+ g^*[2h_1h_0\zeta_2^* + (2h_2h_0 + h_1)\zeta_1^* + (2h_3h_0 + 2h_1h_2)\zeta_0^*]$$
$$- N\ \zeta^* x^*\cos(N\tau) + \left[(h_2h_0 + \frac{1}{2}h_1^2)\sum_{i=1}^{2}\frac{\partial\phi_0^*}{\partial x_i^*}\frac{\partial\phi_0^*}{\partial x_i^*}\right.$$
$$+ (h_2h_0 + \frac{1}{2}h_1)\left(\frac{\partial\phi_0^*}{\partial z^*}\right)^2 + 2h_1h_0\sum_{i=1}^{2}\left(\frac{\partial\phi_0^*}{\partial x_i^*}\frac{\partial\phi_1^*}{\partial x_i^*}\right.$$
$$\left. + \frac{\partial\phi_0^*}{\partial z^*}\frac{\partial\phi_1^*}{\partial z^*}\right) + h_o^2\sum_{i=1}^{2}\frac{\partial\phi_2^*}{\partial x_i^*}\frac{\partial\phi_0^*}{\partial x_i^*} + \frac{\partial\phi_2^*}{\partial z^*}\frac{\partial\phi_0^*}{\partial z^*}$$
$$\left.\left. + \frac{1}{2}\sum_{i=1}^{2}\frac{\partial\phi_1^*}{\partial x_i^*}\frac{\partial\phi_1^*}{\partial x_i^*} + \frac{1}{2}\left(\frac{\partial\phi_1^*}{\partial z^*}\right)^2\right] + T_{d_3}^*\right\} \tag{A6}$$

where $T_{d_n}^*$'s are given in Appendix B.

The components of the kinematic equations are

$$\frac{\partial \zeta_0^*}{\partial \tau} - h_0 \frac{\partial \phi_0^*}{\partial z^*} = 0 \tag{A7}$$

$$\frac{\partial \zeta_1^*}{\partial \tau} - h_0 \frac{\partial \phi_1^*}{\partial z^*} = - \left\{ - h_1 \frac{\partial \phi_0^*}{\partial z^*} + h_0 \sum_{i=1}^{2} \frac{\partial \phi_0^*}{\partial x_i^*} \frac{\partial \zeta_0^*}{\partial x_i^*} + T_{k_1}^* \right\} \tag{A8}$$

$$\frac{\partial \zeta_2^*}{\partial \tau} - h_0 \frac{\partial \phi_2^*}{\partial z^*} = - \left\{ - h_3 \frac{\partial \phi_1^*}{\partial z^*} - h_2 \frac{\partial \phi_0^*}{\partial z^*} + h_1 \sum_{i=1}^{2} \frac{\partial \phi_0^*}{\partial x_i^*} \frac{\partial \zeta_0^*}{\partial x_i^*} \right.$$
$$\left. + h_2 \sum_{i=1}^{2} \frac{\partial \phi_0^*}{\partial x_i^*} \frac{\partial \zeta_1^*}{\partial x_i^*} + T_{k_2} \right\} \tag{A9}$$

$$\frac{\partial \zeta_3^*}{\partial} - h_0 \frac{\partial \phi_3^*}{\partial z^*} = - \left\{ - h_3 \frac{\partial \phi_0^*}{\partial z^*} - h_2 \frac{\partial \phi_1^*}{\partial z^*} - h_1 \frac{\partial \phi_2^*}{\partial z^*} \right.$$
$$+ h_2 \sum_{i=1}^{2} \frac{\partial \phi_0^*}{\partial x_i^*} \frac{\partial \zeta_0^*}{\partial x_i^*} + h_1 \sum_{i=1}^{2} \left(\frac{\partial \phi_1^*}{\partial x_i^*} \frac{\partial \zeta_0^*}{\partial x_i^*} + \frac{\partial \phi_0^*}{\partial x_i^*} \frac{\partial \phi_1^*}{\partial x_i^*} \right)$$
$$\left. + h_0 \sum_{i=1}^{2} \left(\frac{\partial \phi_2^*}{\partial x_i^*} \frac{\partial \zeta_0^*}{\partial x_i^*} + \frac{\partial \phi_1^*}{\partial x_i^*} \frac{\partial \zeta_1^*}{\partial x_i^*} + \frac{\partial \phi_0^*}{\partial x_i^*} \frac{\partial \zeta_2^*}{\partial x_i^*} \right) + T_{k_3}^* \right\} \tag{A10}$$

where T_{kn}^* 's are given in Appendix C.

APPENDIX B

Expressions of R_{d_n} in the Dynamic Free Surface Condition

APPENDIX B

Expressions of R_{d_n} in the Dynamic Free Surface Condition

$$\zeta_0^* + \frac{1}{h_0 g^*}\frac{\partial \phi_n}{\partial \tau} = R_{d_n} \tag{B1}$$

ϵ^1:

$$R_{d_0} = 0 \tag{B2}$$

ϵ^2:

$$R_{d_1} = -\frac{1}{h_0^2 g^*}\left\{h_1\left(\frac{\partial \phi_0^*}{\partial \tau} + 2\zeta_0^* g^* h_0\right) - N^2 \zeta_0^* \cos(N\tau)\, x_o^* + \frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial \phi_0}{\partial x_i^*}\right)^2 h_0^2 + \frac{1}{2}\left(\frac{\partial \phi_0^*}{\partial z^*}\right)^2 h_0^2 + T_{d_1}^*\right\} \tag{B3}$$

ϵ^3:

$$R_{d_2} = -\frac{1}{h_0^2 g^*}\left\{h_1\frac{\partial \phi_1}{\partial \tau} + h_2 g^* \zeta_0^* + g^*(h_1^2\zeta_0^* + 2h_1 h_0 \zeta_1^*) - N\,\zeta_1^* x_o^* \cos(N\tau) + \sum_{i=1}^{2}\left[h_1 h_0\left(\frac{\partial \phi_0^*}{\partial x_i^*}\right)^2 + h_0^2\frac{\partial \phi_0^*}{\partial x_i^*}\frac{\partial \phi_1^*}{\partial x_i^*}\right] + \frac{\partial \phi_0^*}{\partial z^*}\frac{\partial \phi_1^*}{\partial z^*}h_0^2 + h_1 h_0\left(\frac{\partial \phi_0^*}{\partial z^*}\right)^2 + T_{d_2}^*\right\} \tag{B4}$$

ϵ^4:

$$\begin{aligned} R_{d_3} = -\frac{1}{g^* h_0^2}\Bigg\{ & h_1\frac{\partial \phi_2^*}{\partial \tau} + h_2\frac{\partial \phi_1^*}{\partial \tau} + h_3\frac{\partial \phi_0^*}{\partial \tau} + g^*[2h_1 h_0 \zeta_2^* + (2h_2 h_0 + h_1^2)\zeta_1^* \\ & + (2h_3 h_0 + 2h_1 h_2)\zeta_0^*] - N^2 \zeta_2 x_o^* \cos(N\tau) \\ & + \left[(h_2 h_0 + \frac{1}{2}h_1^2)\sum_{i=1}^{2}\frac{\partial \phi_0^*}{\partial x_i^*}\frac{\partial \phi_0^*}{\partial x_i^*} + (h_2 h_0 + \frac{1}{2}h_1^2)\left(\frac{\partial \phi_0^*}{\partial z^*}\right)^2\right. \\ & + 2h_1 h_0\left(\sum_{i=1}^{2}\frac{\partial \phi_0}{\partial x_i^*}\frac{\partial \phi_1}{\partial x_i^*} + \frac{\partial \phi_0}{\partial z^*}\frac{\partial \phi_1}{\partial z^*}\right) + h_0^2\left(\sum_{i=1}^{2}\frac{\partial \phi_2}{\partial x_i^*}\cdot\frac{\partial \phi_0}{\partial x_i^*}\right. \\ & \left.\left. + \frac{\partial \phi_2^*}{\partial z^*}\frac{\partial \phi_0^*}{\partial z^*} + \frac{1}{2}\sum_{i=1}^{2}\frac{\partial \phi_1^*}{\partial x_i^*}\frac{\partial \phi_1^*}{\partial x_i^*} + \frac{1}{2}\left(\frac{\partial \phi_1^*}{\partial z^*}\right)^2\right)\right] + T_{d_3}^*\Bigg\} \end{aligned} \tag{B5}$$

where T_{d_n}'s are the added terms due to Taylor's expansion from $z = \zeta$ to $z = 0$.

$$T^*_d = \epsilon^2 T^*_{d_1} + \epsilon^3 T^*_{d_2} + \epsilon^4 T^*_{d_3} + 0(\epsilon^5) \tag{B6}$$

$$T^*_{d_0} = 0 \tag{B7}$$

$$T^*_{d_1} = \zeta^*_o \frac{\partial^2 \phi^*_0}{\partial\tau\partial z^*} h_0 \tag{B8}$$

$$T^*_{d_2} = h \left(\zeta^* \frac{\partial^2 \phi^*_0}{\partial\tau\partial z^*} + \frac{\zeta^{*2}_0}{2} \frac{\partial^3 \phi^*_0}{\partial\tau\partial z^{*2}} \right) + \zeta^*_0 \left(h_0 \frac{\partial^2 \phi^*_1}{\partial\tau\partial z^*} + h_1 \frac{\partial^2 \phi^*_0}{\partial\tau\partial z^*} \right)$$
$$+ \sum_{i=1}^{2} h_0^2 \frac{\partial^2 \phi^*_0}{\partial x^*_i \partial z^*} \frac{\partial \phi^*_0}{\partial x^*_i} \zeta^*_0 + \zeta^*_0 h_0^2 \frac{\partial^2 \phi^*_0}{\partial z^{*2}} \frac{\partial \phi^*_0}{\partial z^*} \tag{B9}$$

$$T^*_{d_3} = h_0 \left(\frac{\zeta^*_2}{2} \frac{\partial^2 \phi^*_0}{\partial\tau\partial z^*} + \zeta^*_0 \zeta^*_1 \frac{\partial^2 \phi^*_0}{\partial\tau\partial z^*} - \frac{1}{6} \zeta^*_0 \sum_{i=1}^{2} \frac{\partial^4 \phi^*_0}{\partial x^{*2}_i \partial\tau\partial z^*} \right)$$
$$+ \zeta^*_1 \left(h_0 \frac{\partial^2 \phi^*_1}{\partial\tau\partial z^*} + h_1 \frac{\partial^2 \phi^*_0}{\partial\tau\partial z^*} \right) + \frac{\zeta_0^2}{2} \left(h_0 \frac{\partial^3 \phi^*_1}{\partial\tau\partial z^{*2}} + h_1 \frac{\partial^3 \phi^*_0}{\partial\tau\partial z^{*2}} \right)$$
$$+ \zeta^*_0 \left(h_0 \frac{\partial^2 \phi^*_2}{\partial\tau\partial z^*} + h_1 \frac{\partial^2 \phi^*_1}{\partial\tau\partial z^*} + h_2 \frac{\partial^2 \phi^*_0}{\partial\tau\partial z^*} \right) + \sum_{i=1}^{2} h_0^2 \zeta^*_1 \frac{\partial^2 \phi^*_0}{\partial x_i \partial z^*} \frac{\partial \phi^*_0}{\partial x^*_i}$$
$$+ h_0^2 \zeta^*_1 \frac{\partial^2 \phi^*_0}{\partial z^{*2}} \frac{\partial \phi^*_0}{\partial z^*} + \sum_{i=1}^{2} \frac{h_0^2}{2} \zeta^{*2}_0 \left(\frac{\partial^3 \phi^*_0}{\partial x^*_i \partial z^{*2}} \frac{\partial \phi^*_0}{\partial x^*_i} + \frac{\partial^2 \phi_0}{\partial x^*_i \partial z^*} \frac{\partial^2 \phi_0}{\partial x^*_i \partial z^*} \right)$$
$$- \frac{h_0^2}{2} \sum_{i=1}^{2} \frac{\partial^3 \phi_0}{\partial x^{*2} \partial z^*} \frac{\partial \phi_0}{\partial z^*} \zeta^{*2}_0 + \frac{h_0^2}{2} \zeta^{*2}_0 \left(\frac{\partial^2 \phi_0}{\partial z^{*2}} \right)^2$$
$$+ h_1 h_0 \sum_{i=1}^{2} \frac{\partial^2 \phi^*_0}{\partial x^*_i \partial z^*} \zeta^*_0 + h_1 h_0 \zeta^*_0 \frac{\partial^2 \phi^*_0}{\partial z^{*2}} + h_0^2 \sum_{i=1}^{2} \left(\frac{\partial \phi^*_0}{\partial x_i} \frac{\partial^2 \phi^*_1}{\partial x^*_i \partial z^*} \right.$$
$$\left. + \frac{\partial \phi^*_1}{\partial x^*_i} \frac{\partial^2 \phi_0}{\partial x^*_i \partial z^*} \right) + h_0^2 \left(\frac{\partial \phi^*_0}{\partial z^*} \frac{\partial^2 \phi^*_1}{\partial z^{*2}} + \frac{\partial \phi^*_1}{\partial z^*} \frac{\partial^2 \phi^*_0}{\partial z^{*2}} \right) \tag{B10}$$

APPENDIX C

Expressions of R_{k_n} in the Kinematic Free Surface Condition

APPENDIX C

Expressions of R_{k_n} in the Kinematic Free Surface Condition

$$\frac{\partial \zeta_n^*}{\partial \tau} - h_0 \frac{\partial \phi_n^*}{\partial z^*} = R_{k_n} \tag{C1}$$

ϵ^1:

$$R_{k0} = 0 \tag{C2}$$

ϵ^2:

$$R_{k1} = - \left\{ - h_1 \frac{\partial \phi_0^*}{\partial z^*} + h_0 \sum_{i=1}^{2} \frac{\partial \phi_0^*}{\partial x_i^*} \frac{\partial \zeta_0^*}{\partial x_i^*} + T_{k1}^* \right\} \tag{C3}$$

ϵ^3:

$$R_{k2} = - \left\{ - h_1 \frac{\partial \phi_1^*}{\partial z^*} - h_2 \frac{\partial \phi_0^*}{\partial z^*} + h_1 \sum_{i=1}^{2} \frac{\partial \phi_0^*}{\partial x^*} \frac{\partial \zeta_0^*}{\partial x^*} + h_0 \sum_{i=1}^{2} \frac{\partial \phi_1^*}{\partial x^*} \frac{\partial \zeta_0^*}{\partial x^*} \right.$$

$$\left. + h_0 \sum_{i=1}^{2} \frac{\partial \phi_0^*}{\partial x_i^*} \frac{\partial \zeta_1}{\partial x_i^*} + T_{k2}^* \right\} \tag{C4}$$

ϵ^4:

$$R_{k3} = - \left\{ - h_3 \frac{\partial \phi_0^*}{\partial z^*} - h_2 \frac{\partial \phi_1^*}{\partial z^*} - h_1 \frac{\partial \phi_2^*}{\partial z^*} + h_2 \sum_{i=1}^{2} \frac{\partial \phi_0}{\partial x_i^*} \frac{\partial \zeta_0^*}{\partial x_i^*} \right.$$

$$+ h_1 \sum_{i=1}^{2} \left(\frac{\partial \phi_1^*}{\partial x_i^*} \frac{\partial \zeta_0^*}{\partial x_i^*} + \frac{\partial \phi_0^*}{\partial x_i^*} \frac{\partial \zeta_1^*}{\partial x_i^*} \right) + h_0 \sum_{i=1}^{2} \left(\frac{\partial \phi_2^*}{\partial x_i^*} \frac{\partial \zeta_0^*}{\partial x_i^*} \right.$$

$$\left. \left. + \frac{\partial \phi_1^*}{\partial x_i^*} \frac{\partial \zeta_1}{\partial x_i^*} + \frac{\partial \phi_0^*}{\partial x_i^*} \frac{\partial \zeta_2^*}{\partial x_i^*} \right) + T_{k3}^* \right\} \tag{C5}$$

$$T_k^* = \epsilon^2 T_{k1}^* + \epsilon^3 T_{k2}^* + \epsilon^4 T_{k3}^* + 0(\epsilon^5) \tag{C6}$$

$$T_{k0}^* = 0 \tag{C7}$$

$$T_{k1}^* = - h_0 \frac{\partial^2 \phi_0}{\partial z^{*2}} \zeta_0^* \tag{C8}$$

$$T_{k2}^* = h_0 \frac{\partial \zeta_0^*}{\partial x_i^*} \frac{\partial^2 \phi_0^*}{\partial x_i^* \partial z^*} \zeta_0 - h_0 \frac{\partial^2 \phi_0^*}{\partial z^{*2}} \zeta_1 - h_0 \frac{\partial^3 \phi_0^*}{\partial z^{*3}} \cdot \frac{\zeta_0^{*2}}{2} \tag{C9}$$

$$
\begin{aligned}
T^*_{k3} = & \sum_{i=1}^{2} \left(h_0 \frac{\partial \zeta^*_0}{\partial x^*_i} \frac{\partial^2 \phi^*_0}{\partial x^*_i \partial z^*} \zeta^*_1 + \frac{1}{2} h_0 \frac{\partial \zeta^*_0}{\partial x^*_i} \frac{\partial^3 \phi^*_0}{\partial x^*_i \partial z^{*2}} \zeta^{*2}_0 \right) \\
& + \sum_{i=1}^{2} \left(h_1 \zeta^*_0 \frac{\partial^2 \phi^*_0}{\partial x^*_i \partial z^*} \frac{\partial \zeta^*_0}{\partial x^*_i} + h_0 \zeta^*_0 \frac{\partial^2 \phi^*_1}{\partial x^*_i \partial z^*} \frac{\partial \zeta^*_0}{\partial x^*_i} + h_0 \frac{\partial^2 \phi^*_0}{\partial x^*_i \partial z^*} \frac{\partial \zeta^*_1}{\partial x^*_i} \zeta^*_0 \right) \\
& - \left[\frac{h_0}{2} \frac{\partial^2 \phi^*_0}{\partial z^{*2}} \zeta^*_2 + h_0 \frac{\partial^3 \phi^*_0}{\partial z^{*3}} \zeta^*_0 \zeta^*_1 + \frac{h_0}{6} \frac{\partial^4 \phi^*_0}{\partial z^{*4}} \zeta^{*3}_0 \right] \\
& - \left[h_1 \frac{\partial^3 \phi^*_0}{\partial z^{*3}} + h_0 \frac{\partial^3 \phi^*_1}{\partial z^{*3}} \right] \frac{\zeta^{*2}_0}{2} - \left[\frac{\partial^2 \phi^*_2}{\partial z^{*2}} h_0 - \frac{\partial^2 \phi^*_1}{\partial z^{*2}} h_1 - \frac{\partial^2 \phi^*_0}{\partial z^{*2}} h_2 \right] \zeta_0
\end{aligned}
\tag{C10}
$$

where T_{kn}'s are the added terms due to expansions from $z = \zeta$ to $z = 0$.

APPENDIX D

Expressions for $f_{nma}^{(2,s)}$ and $f_{nma}^{(2,c)}$

APPENDIX D

Expressions for $f^{(2,s)}_{nma}$ and $f^{(2,c)}_{nma}$

$$
\begin{aligned}
f^{(2,s)}_{n,ma} = -\Big\{ & \phi^{(0,c)}_{1,k} \sum_{\bar{m}=1}^{\infty} \frac{1}{a_m^2} \int_F \psi_k \psi_{\bar{m}} \psi_m \, ds \cdot \sum_{\ell=1}^{N+2} \big(\delta_{n,\ell-1} - \delta_{n,1-\ell} \\
& - \delta_{n,\ell-1}\big) \, \ell \, \zeta^{(1,s)}_{\ell,m} \cdot \frac{1}{2} - \phi^{(0,c)}_{1,k} \sum_{m=1}^{\infty} \frac{1}{a_m^2} \int_F \psi_k \psi_{\bar{m}} \psi_m \, ds \cdot \\
& \sum_{\ell=1}^{N+2} \big(\delta_{n,\ell+1} - \delta_{n,\ell-1} + \delta_{n,1-\ell}\big) \, \zeta^{(1,s)}_{\ell,m} \cdot \frac{1}{2} \\
& - \zeta^{(0,s)}_{1,k} \sum_{\bar{m}=1}^{\infty} \frac{1}{a_m^2} \int_F \psi_k \psi_{\bar{m}} \psi_m \, ds \cdot \lambda_{\bar{m}} \sum_{\ell=1}^{N+2} \frac{\ell}{2} \big(\delta_{n,\ell+1} + \delta_{n,\ell-1} \\
& - \delta_{n,1-\ell}\big) \, \phi^{(1,c)}_{\ell,m} - \zeta^{(0,s)}_{1,k} \phi^{(0,c)}_{1,k} \frac{1}{a_m^2} \int_F \psi_k^2 \psi_m \, ds \cdot \frac{1}{2} \, \delta_{n,2} \\
& - h_0 \zeta^{(0,s)}_{1,k} \sum_{\bar{m}=1}^{\infty} \lambda_{\bar{m}} \frac{1}{a_m^2} \int_F \psi_k \psi_{\bar{m}} \psi_m \, ds \sum^{N+2} \frac{\ell^2}{2} \big(\delta_{n,\ell+1} - \delta_{n,\ell-1} \\
& - \delta_{n,1-\ell}\big) \, \phi^{(1,c)}_{\ell,\bar{m}} - h_1 \zeta^{(0,s)}_{1,k} \phi^{(0,c)}_{1,k} \frac{1}{a_m^2} \int_F \psi_k^2 \psi_m \, ds \cdot \frac{1}{2} \, \delta_{n,2} \\
& - h_0^2 g^* \sum_{\bar{m}=1}^{\infty} \frac{1}{a_m^2} \int_F \frac{\partial^2 \psi_k}{\partial z^{*2}} \psi_{\bar{m}} \psi_m \, ds \sum_{\ell=1}^{N+2} \phi^{(0,c)}_{1,k} \zeta^{(1,s)}_{\ell,\bar{m}} \big(\delta_{n,\ell+1} \\
& + \delta_{n,\ell-1} - \delta_{n,1-\ell}\big) \frac{1}{2} \Big\}
\end{aligned}
\tag{D1}
$$

$$f_{n,ma}^{(2,c)} = -\left\{\phi_{1,k}^{(0,c)} \sum_{\bar{m}=1}^{\infty} \frac{1}{a_m^2} \int_F \psi_k \psi_{\bar{m}} \psi_m ds \sum_{\ell=1}^{N+2} \left(\frac{\delta_{n,\ell-1}+\delta_{n,1-\ell}}{1+\delta_{0n}}\right.\right.$$

$$\left.-\delta_{n,\ell+1}\right) \frac{\ell}{2} \zeta_{\ell\bar{m}}^{(1,c)} - \phi_{1,k}^{(0,c)} \sum_{m=1}^{\infty} \frac{1}{a_m^2} \int_F \psi_k \psi_{\bar{m}} \psi_m ds \sum_{\ell=1}^{N+2} \left(\delta_{n,\ell+1}\right.$$

$$\left.-\frac{\delta_{n,\ell-1}+\delta_{n,1-\ell}}{1+\delta_{0n}}\right) \frac{1}{2} \zeta_{\ell,m}^{(1,s)} + [\zeta_{1,k}^{(0,s)}]^2 \phi_{1,k}^{(0,c)} \frac{1}{a_m^2} \int_F \psi_k^3 \psi_m ds \cdot$$

$$\cdot \frac{3}{8} (\delta_{n,3} - \delta_{n,1}) + \zeta_{1,k}^{(0,s)} \sum_{\bar{m}=1}^{\infty} \frac{1}{a_m^2} \lambda_m \int_F \psi_k \psi_{\bar{m}} \psi_m ds \sum_{\ell=1}^{N+2} \left(\delta_{n,\ell+1}\right.$$

$$\left.+\frac{\delta_{n,\ell-1}+\delta_{n,1-\ell}}{1+\delta_{0n}}\right) \phi_{\ell,\bar{m}}^{(1,s)} + \zeta_{1,k}^{(0,s)} \sum_{\bar{m}=1}^{\infty} \lambda_{\bar{m}} \frac{1}{a_m^2} \int_F \psi_k \psi_m \psi_{\bar{m}} ds \cdot$$

$$\cdot \sum_{\ell=1}^{N+2} \frac{\ell^2}{2} \left(\delta_{n,\ell+1} - \frac{\delta_{n,\ell-1}-\delta_{n,1-\ell}}{1+\delta_{0n}}\right) \phi_{\ell,m}^{(1,s)}$$

$$+[\phi_{1,k}^{(0,c)}]^2 \zeta_{1,k}^{(0,s)} \sum_{i=1}^{2} \frac{1}{a_m^2} \int_F \frac{\partial\psi_k}{\partial x_i^*} \frac{\partial\psi_k}{\partial x_i^*} \psi_k \psi_m ds \cdot \frac{1}{4} (\delta_{n,3} - \delta_{n,1})$$

$$+[\phi_{1,k}^{(0,c)}]^2 \zeta_{1,k}^{(0,s)} \sum_{i=1}^{2} \frac{1}{a_m^2} \int_F \frac{\partial\psi_k}{\partial x_i^*} \frac{\partial\psi_k}{\partial x_i^*} \psi_k \psi_m ds \cdot \frac{1}{4} (\delta_{n,3} - \delta_{n,1})$$

$$+[\phi_{1,k}^{(0,c)}]^2 \zeta_{1,k}^{(0,s)} \sum_{i=1}^{2} \frac{1}{a_m^2} \int_F \frac{\partial\psi_k}{\partial x_i^*} \frac{\partial\psi_k}{\partial x_i^*} \psi_k \psi_m ds \left(\frac{3}{4} \delta_{n,1} + \frac{1}{4} \delta_{n,3}\right)$$

$$+h_0^2 [\phi_{1,k}^{(0,c)}]^2 \zeta_{1,k}^{(0,s)} \frac{1}{a_m^2} \int_F \frac{\partial^2\psi_k}{\partial z^{*2}} \psi_k^2 \psi_m ds \left(\frac{3}{4} \delta_{n,3} + \frac{1}{4} \delta_{n,1}\right)$$

$$+h_0^2 [\zeta_{1,k}^{(0,s)}]^2 \phi_{1,k}^{(0,c)} \frac{1}{a_m^2} \int_F \psi_k \frac{\partial\psi_k}{\partial x_i^*} \frac{\partial\psi_k}{\partial x_i^*} \psi_m ds \left(\frac{1}{4} \delta_{n,1} - \frac{1}{4} \delta_{n,3}\right)$$

$$-h_0^2 g^* \sum_{\bar{m}=1}^{\infty} \frac{1}{a_m^2} \int_F \frac{\partial^2\psi_k}{\partial z^{*2}} \psi_{\bar{m}} \psi_m ds \sum_{\ell=1}^{N+2} \phi_{1,k}^{(0,c)} \zeta_{\ell,m}^{(1,c)} \left(\frac{1}{2} \delta_{n,\ell+1}\right.$$

$$\left.+\frac{\delta_{n,\ell-1}+\delta_{n,1-\ell}}{2(1+\delta_{0n})}\right) - h_0^2 g^* \frac{1}{a_m^2} \int_F \frac{\partial^3\psi_k}{\partial z^{*3}} \psi_k^2 \psi_m ds [\zeta_{1,k}^{(0,s)}]^2 \phi_{1,k}^{(0,c)} \frac{1}{2} \left(\frac{1}{4} \delta_{n,1}\right.$$

$$\left.\left.-\frac{1}{4} \delta_{n,3}\right)\right\} \qquad \text{(D2)}$$

For $n = 1$, there are no first harmonic terms $\tilde{f}_{1,k}^{(1,s)}$, $\tilde{f}_{1,k}^{(1,c)}$ present in Eqs. (D1) & (D2)

APPENDIX E

Expressions for $f_{m,n}^{(2,s)}$ and $f_{m,n}^{(2,c)}$

APPENDIX E

Expressions for $f^{(2,s)}_{m,n}$ and $f^{(2,c)}_{m,n}$

$$f^{(2,s)}_{n,m} = -N^3 x^*_0 \sum_{\ell=1}^{N+1} \zeta^{(1,c)}_{\ell,m} (\delta_{n,N+\ell} + \delta_{n,N-\ell} - \delta_{n,\ell-N}) \frac{1}{2}$$

$$+ N^2 x^*_0 \sum_{\ell=1}^{N+1} \zeta^{(1,c)}_{\ell,m} (\delta_{n,N+\ell} - \delta_{n,N-\ell} + \delta_{n,\ell-N}) \frac{\ell}{2}$$

$$- g^* h_1 \lambda_m \phi^{(1,s)}_{n,m} + \delta_{n,2} g^* \sum_{i=1}^{2} h_1 \frac{1}{a_m^2} \int_F \left(\frac{\partial \psi_k}{\partial x^*_i}\right)^2 \psi_m ds \cdot$$

$$\cdot \frac{1}{2} \phi^{(0,c)}_{1,k} \zeta^{(0,s)}_{1,k} + g^* \zeta^{(0,s)}_{1,k} \sum_{\overline{m}=1}^{\infty} \sum_{i=1}^{2} \frac{1}{a_m^2} \int_F \psi_m \frac{\partial \psi_k}{\partial x^*_i} \frac{\partial \psi_{\overline{m}}}{\partial x^*_i} ds \cdot$$

$$\cdot \sum_{\ell=1}^{N+1} \phi^{(1,c)}_{\ell,\overline{m}} (\delta_{n,\ell+1} - \delta_{n,\ell+1} + \delta_{n,1-\ell}) \frac{1}{2}$$

$$+ g^* \phi^{(0,c)}_{1,k} \sum_{m=1}^{\infty} \sum_{i=1}^{2} \frac{1}{a_m^2} \int_F \psi_m \frac{\partial \psi_k}{\partial x^*_i} \frac{\partial \psi_{\overline{m}}}{\partial x^*_i} ds \cdot \sum_{\ell=1}^{N+1} \zeta^{(1,s)}_{\ell,\overline{m}} (\delta_{n,\ell+1}$$

$$+ \delta_{n,\ell+1} - \delta_{n,1-\ell}) \frac{1}{2} - h_1(-n^2) \phi^{(1,s)}_{n,m} + 2h_1 g^* (+n) \zeta^{(1,c)}_{n,m}$$

$$- \delta_{n,2} 2h_1 \sum_{i=1}^{2} \frac{1}{a_m^2} \int_F \left(\frac{\partial \psi_k}{\partial x^*_i}\right)^2 \psi_m ds \left(-\frac{1}{2}\right) [\phi^{(0,c)}_{1,k}]^2$$

$$- \phi^{(0,c)}_{1,k} \sum_{\overline{m}=1}^{\infty} \sum_{i=1}^{2} \int_F \frac{\partial \psi_k}{\partial x^*_i} \frac{\partial \psi_{\overline{m}}}{\partial x^*_i} \psi_m ds \frac{1}{a_m^2} \cdot \sum_{\ell=1}^{N+1} \phi^{(1,c)}_{\ell,\overline{m}} (-\ell) \frac{1}{2} (\delta_{n,\ell+1}$$

$$+ \delta_{n,\ell-1} - \delta_{n,1-\ell}) - \phi^{(0,c)}_{1,k} \sum_{m=1}^{\infty} \sum_{i=1}^{2} \frac{1}{a_m^2} \int_F \frac{\partial \psi_k}{\partial x^*_i} \frac{\partial \psi_{\overline{m}}}{\partial x^*_i} \psi_m ds \cdot$$

$$\cdot \sum_{\ell=1}^{N+1} \phi^{(1,c)}_{\ell,\overline{m}} \left(-\frac{1}{2}\right) (\delta_{n,\ell+1} - \delta_{n,\ell-1} + \delta_{n,1-\ell})$$

$$- \delta_{n,2} h_1 [\phi^{(0,c)}_{1,k}]^2 \lambda^{*2}_k \left(-\frac{1}{a_{\overline{m}}^2}\right) \int_F \psi_k^2 \psi_m ds$$

$$- \phi_{1,k}^{(0,c)} \sum_{\bar{m}=1}^{\infty} \lambda_k^* \lambda_{\bar{m}}^* \frac{1}{a_m^2} \int_F \psi_m \psi_k \psi_{\bar{m}} ds \sum_{\ell=1}^{N+1} \phi_{\ell,\bar{m}}^{(1,c)} (- \delta_{n,\ell+1}$$

$$+ \delta_{n,\ell-1} - \delta_{n,1-\ell}) \frac{1}{2} - \phi_{1,k}^{(0,c)} \sum_{\bar{m}=1}^{\infty} \lambda_k^* \lambda_m^* \frac{1}{a_m^2} \int_F \psi_m \psi_k \psi_{\bar{m}} ds \sum_{\ell=1}^{N+1} \phi_{\ell,\bar{m}}^{(1,c)}$$

$$- \frac{\ell}{2} \left(\delta_{n,\ell+1} + \delta_{n,\ell-1} - \delta_{n,1-\ell}\right) + f_{n,ma}^{(2,s)} \tag{E1}$$

The last term is given in Appendix D.

$$f_{n,m}^{(2,c)} = - N^3 x_0^* \sum_{\ell=1}^{N+1} \zeta_{\ell,m}^{(1,s)} \frac{1}{2} \left[\frac{\delta_{n,N-\ell} + \delta_{n,\ell-N}}{1+\delta_{0n}} - \delta_{n,N+\ell}\right]$$

$$+ N^2 x_0^* \sum_{\ell=1}^{N+1} \zeta_{\ell,m}^{(1,s)} \frac{\ell}{2} \left[\frac{\delta_{n,N-\ell} + \delta_{n,\ell+N}}{1+\delta_{0n}} + \delta_{n,N+\ell}\right]$$

$$- \delta_{n,1} h_2 2 g^* \zeta_{1,k}^{(0,s)} \delta_{m,k} - g^* h_1 \lambda_m^* \phi_{n,m}^{(1,c)}$$

$$+ g^* \zeta_{1,k}^{(0,s)} \sum_{\bar{m}=1}^{\infty} \sum_{i=1}^{2} \frac{1}{a_m^2} \int_F \left(\frac{\partial \psi_k}{\partial x_i^*} \frac{\partial \psi_{\bar{m}}}{\partial x_i^*}\right) \psi_m ds \sum_{\ell=1}^{N+1} \phi_{\ell,m}^{(1,s)} \cdot$$

$$\cdot \frac{1}{2} \left(\frac{\delta_{n,\ell-1} + \delta_{n,1-\ell}}{1+\delta_{0n}} - \delta_{n,\ell+1}\right)$$

$$+ g^* \phi_{1,k}^{(0,c)} \sum_{\bar{m}=1}^{\infty} \sum_{i=1}^{2} \frac{1}{a_m^2} \int_F \left(\frac{\partial \psi_k}{\partial x_i^*} \frac{\partial \psi_{\bar{m}}}{\partial x_i^*}\right) \psi_m ds \sum_{\ell=1}^{N+1} \zeta_{\ell,\bar{m}}^{(1,c)} \frac{1}{2} \left(\delta_{n,\ell+1}\right.$$

$$\left. + \frac{\delta_{n,\ell-1} + \delta_{n,1-\ell}}{1+\delta_{n0}}\right) - h_1 (-n^2) \phi_{n,m}^{(1,c)} - g^* h_1^2 \zeta_{1,k}^{(0,s)} \delta_{n,1} \delta_{mk}$$

$$- 2 h_1 g^* n \zeta_{nm}^{(1,s)} - \phi_{1,k}^{(0,c)} \sum_{\bar{m}=1}^{\infty} \sum_{i=1}^{2} \frac{1}{a_m^2} \int_F \frac{\partial \psi_k}{\partial x_i^*} \frac{\partial \psi_{\bar{m}}}{\partial x_i^*} \psi_m ds \sum_{\ell=1}^{N+1} \phi_{\ell,\bar{m}}^{(1,s)} \frac{\ell}{2} \left[\delta_{n,\ell+1}\right.$$

$$\left. + \frac{\delta_{n,\ell-1} + \delta_{n,1-\ell}}{1+\delta_{n0}}\right] - \phi_{1,k}^{(0,c)} \sum_{\bar{m}=1}^{\infty} \sum_{i=1}^{2} \frac{1}{a_m^2} \int_F \psi_m \frac{\partial \psi_k}{\partial x_i^*} \frac{\partial \psi_{\bar{m}}}{\partial x_i^*} ds \cdot$$

$$\cdot \sum_{\ell=1}^{N+1} \phi_{\ell,\bar{m}}^{(1,s)} \left(-\frac{1}{2}\right) \left[\frac{\delta_{n,\ell-1} + \delta_{n,1-\ell}}{1+\delta_{n,0}} - \delta_{n,\ell+1}\right]$$

$$- \phi_{1,k}^{(0,c)} \sum_{\bar{m}=1}^{\infty} \lambda_k^* \lambda_{\bar{m}}^* \frac{1}{a_m^2} \int_F \psi_k \psi_{\bar{m}} \psi_m ds \cdot \sum_{\ell=1}^{N+1} \phi_{\ell,\bar{m}}^{(1,s)} \frac{1}{2} \left[\delta_{N,\ell+1} - \frac{\delta_{n,\ell-1} + \delta_{n,1-\ell}}{1 + \delta_{0n}} \right]$$

$$- \phi_{1,k}^{(0,c)} \sum_{\bar{m}=1}^{\infty} \lambda_k^* \lambda_{\bar{m}}^* \frac{1}{a_m^2} \int_F \psi_n \psi_k \psi_{\bar{m}} ds \sum_{\ell=1}^{N+1} \phi_{\ell,\bar{m}}^{(1,s)} \frac{\ell}{2} \left[\delta_{n,\ell+1} + \frac{\delta_{n,\ell-1} + \delta_{n,1-\ell}}{1 + \delta_{0n}} \right] + f_{m,na}^{(2,c)} \quad \text{(E2)}$$

The last term is given in Appendix D.

APPENDIX F

Evaluation of $\overline{H}_{o\nu}(\overline{P}, \overline{I})$

APPENDIX F

Evaluation of $\overline{H}_{o\nu}(\overline{P},\overline{I})$

Let G_o be the unit sink

$$G_o(P,I) = \frac{1}{4\pi R_{PI}} = \frac{1}{4\pi\sqrt{r^2 + r'^2 - 2rr'\cos\tilde{\gamma}}} = G_o(r,\sigma,\theta;r',\sigma',\theta') \quad \text{(F1)}$$

$$\cos\tilde{\gamma} = \cos\sigma\cos\sigma' + \sin\sigma\sin\sigma'\cos\gamma \quad \text{(F2)}$$

$$\gamma = \theta - \theta' \quad \text{(F3)}$$

$$\overline{H}_{o\nu}(\overline{P},\overline{I}) = \frac{1}{(1+\delta_{o\nu})\pi}\int_0^{2\pi}\int_0^{2\pi} G_o(\rho,\rho';\cos\gamma;\sigma\,\sigma')\cos(\nu\theta)\cos(\nu'\theta)\,d\theta'\,d\theta$$

$$= \int_0^{2\pi} G_o(\rho,\rho';\cos\gamma;\sigma,\sigma')\cos(\nu\gamma)\,d\gamma \quad \text{(F4)}$$

Let $\beta = \frac{\gamma}{2}$, $\beta' = \frac{\pi}{2} - \beta$ and change the other primed quantities to variables with subscripts I and the unprimed to variables with subscript P.

$$\overline{H}_{o\nu}(\overline{P},\overline{I}) = \frac{1}{\pi}\int_0^{\pi/2} \frac{\cos(2\nu\beta)\,d\beta}{[r^2 + r'^2 - 2rr'\cos(\sigma-\sigma') + 4rr'\sin\sigma\sin\sigma'\sin^2\beta]^{1/2}}$$

$$- \frac{1}{\pi}\int_0^{\pi/2} \frac{(-1)^\nu\cos(2\nu\beta')\,d\beta'}{R_{t1}\sqrt{1 - q_1^2\sin^2\beta'}} \quad \text{(F5)}$$

where

$$R_{t1} = \sqrt{r_P^2 + r_I^2 - r_P r_I\cos(\sigma_P + \sigma_I)} \quad \text{(F6)}$$

$$q_1 = \frac{\sqrt{4r_P r_I\sin\sigma_P\sin\sigma_I}}{R_{t1}} \quad \text{(F7)}$$

From formula No. 484 of Reference 21, one has

$$\cos(2\nu\beta) = \sum_{m=0}^{\nu} a_{\nu m} \sin^{2m}\beta \tag{F8}$$

where

$$a_{\nu o} = 1 \qquad \text{for } m = 0 \tag{F9a}$$

$$a_{\nu m} = \frac{(-1)^m \, 2^{2m-1} (\nu + m - 1)!}{(2m)! \; (\nu - m)!} \qquad \text{for } m \geq 1 \tag{F9b}$$

Using Equation (F8),

$$\overline{H}_{o\nu} = \frac{1}{R_{t1}\pi} \sum_{\mu=1}^{\nu} a_{\nu\mu} V_\mu(q_1) \tag{F10}$$

where

$$V_\mu(q_1) = \int_0^{2\pi} \frac{\sin^{2\mu}\beta}{\sqrt{1 - q_1^2 \sin^2\beta}} \, d\beta \tag{F11}$$

Using the reduction formula on p. 59-60 of Reference 22

$$V_\mu = \frac{1}{(2\mu - 1) q_1^2} \left\{ (2\mu - 2)(1 + q_1^2) V_{\mu-1} - (2\mu - 3) V_{\mu-2} \right\} \tag{F12a}$$

$$V_o = K(q_1) = \int_0^{\pi/2} \frac{d\beta}{\sqrt{1 - q_1^2 \sin^2\beta}} \tag{F12b}$$

$$V_1 = \frac{1}{q_1^2} \left[E(q_1) - K(q_1) \right] \equiv D(q_1) \tag{F12c}$$

As $q_1 \to 0$, the above formula (F12a) appears singular, but it should be finite, so a precision problem would be encountered for q_1 small. However, the binomial expansion of the integral leads to

$$V_\mu = \sum_{n=o}^{\infty} C_n^{-1/2} q_1^{2n} (-1)^n \frac{\sqrt{\pi}}{2} \frac{\Gamma(n + \mu + 1/2)}{\Gamma(n + \mu + 1)} \tag{F13}$$

where the binomial coefficient (No. 189, Ref. 21)

$$C_n^{-\frac{1}{2}} = \frac{-\frac{1}{2}(-\frac{1}{2}-1) - - - - (-\frac{1}{2}-n+1)}{n!} = \frac{1 \cdot 3 \cdot 5 - - - - - (2n-1)}{2 \cdot 4 \cdot 6 - - - - - (2n)}(-1)^n \quad n \geq 1 \tag{F14a}$$

$$C_o^{-\frac{1}{2}} = 1 \tag{F14b}$$

and the quotient

$$\frac{\sqrt{\pi}}{2} \frac{\Gamma(n+\mu+\frac{1}{2})}{\Gamma(n+\mu+1)} = \frac{1 \cdot 3 \cdot 5 - - - (2\mu + 2n - 1)}{2 \cdot 4 \cdot 6 - - - (2\mu + 2n)} \frac{\pi}{2} \tag{F15}$$

Equation (F13) can be employed for q_1 small.

APPENDIX G

Evaluation of $\bar{F}_{o}$ $(\bar{Q}, \bar{I})$

APPENDIX G

Evaluation of $\overline{F}_{o\nu}$ $(\overline{Q}, \overline{I})$

The definition of $\overline{F}_{o\nu}$ is

$$\bar{\mathcal{F}}_{o\nu}(\bar{Q}, \bar{I}) = \int_0^{2\pi} \frac{\partial G_o}{\partial n_I}(r_Q, r_I, \cos\gamma, \sigma_I, \sigma_Q) \cos(\nu\gamma) d\gamma \tag{G1}$$

which is nonsymmetric.

For an axial symmetric tank

$$\frac{\partial G_o}{\partial n_I} = \cos(n_I, r_I) \frac{\partial G_o}{\partial r_I} + \cos(n_I, \sigma_I) \frac{\partial G_o}{r_I \partial \sigma} \tag{G2}$$

$$\cos(n, r) = \frac{1}{\sqrt{1 + \left(\frac{dr}{r d\sigma}\right)^2}} \tag{G3}$$

$$\cos(n, \sigma) = \frac{-\frac{1}{r}\frac{dr}{d\sigma}}{\sqrt{1 + \left(\frac{dr}{r d\sigma}\right)^2}} \tag{G4a}$$

$$R_{t2} = \sqrt{r_Q^2 + r_I^2 - 2 r_Q r_I \cos(\sigma_Q + \sigma_I)} \tag{G4b}$$

$$R_{t3} = \sqrt{1 + \left(\frac{dr_I}{r_I d\sigma_I}\right)^2} \tag{G4c}$$

$$q_2 = \sqrt{4 r_Q r_I \sin\sigma_Q \sin\sigma_I} / R_{t2} \tag{G4d}$$

$$\beta = \frac{\pi}{2} - \frac{\sigma}{2} \tag{G4e}$$

From Equations (G1) and (G2), by integration, it is found that

$$\bar{\mathcal{J}}_{o\nu}(\bar{I}, \bar{Q}) = -\frac{1}{4\pi}\int_0^{2\pi} \frac{g_1\cos(\nu\sigma) + g_2\cos^2\beta\cos(\nu\sigma)}{[1 - q_2^2\sin^2\beta]^{3/2}}\, d\sigma$$

$$= -\frac{1}{\pi}\left\{g_1 \sum_{m=0}^{\nu} a_{\nu m}U_m(q_2) + g_2 \sum_{m=0}^{\nu} a_{\nu m}\left[V_m(q_2) + (1 - q_2^2)\, U_m(q_2)\right]\right\} \tag{G5}$$

where

$$g_1 = [r_I^2 - r_I r_Q\cos(\sigma_I - \sigma_Q) - r_I\frac{dr_I}{d\sigma_I}\sin(\sigma_I - \sigma_Q)]/(R_{t3}R_{t2}^3) \tag{G6a}$$

$$g_2 = [2r_I r_Q \sin\sigma_I\sin\sigma_Q - 2r_I\frac{dr_I}{d\sigma_I}\sin\sigma_I\sin\sigma_Q]/(R_{t3}R_{t2}^3) \tag{G6b}$$

$V_m(q_2)$ is defined in the previous Appendix, replacing q_1 to q_2

$$U_m(q_2) = \int_0^{\pi/2} \frac{\sin^{2m}\beta\, d\beta}{(1 - q_2^2\sin^2\beta)^{3/2}} = \frac{1}{q_1^2}[U_{m-1}(q_2) - V_{m-1}(q_2)] \tag{G6c}$$

$$U_o(q_2) = \frac{E(q_2)}{1 - q_2^2}, \quad V_o = K(q_2) \tag{G6d}$$

from reduction formulas in Reference 22.

It is noted that $q_2 = 1$ is only an apparent singularity which is finite when multiplied by g_1 or $(1 - q_2^2)$ in Equation (G5).

In order to have higher precision as $q_2 \to 0$, one may use binomial expansion again,

$$U_m(q_2) = \sum_{m=0}^{\infty} C_n^{-\frac{3}{2}}\, q_2^{2n}(-1)^n \frac{\sqrt{\pi}}{2}\frac{\Gamma(n + m + \frac{1}{2})}{\Gamma(n + m + 1)} \tag{G7}$$

$$C_n^{-\frac{3}{2}} = \frac{-\frac{3}{2}\left(-\frac{3}{2}-1\right) \; --- \; \left(-\frac{3}{2}-n+1\right)}{n!}$$

$$= \frac{1 \cdot 3 \cdot 5 ---- (2n+1)}{2 \cdot 4 \cdot 6 \qquad (2n)} (-1)^n \tag{G7a}$$

$$C_o^{-\frac{3}{2}} = 1 \tag{G7b}$$

ERRATA - September 21, 1965

The following corrections are applicable to Technical Report No. 5 Contract No. NAS8-11045, SwRI Project No. 02-1391, titled "Subharmonic Oscillations in an Arbitrary Axisymmetric Tank Resulting from Axial Excitation," (September 1965):

1. Following page 5 before "Method of Perturbation," the following paragraph was inadvertently left out:

> When the frequency is not given, the amplitude of the linear first harmonics can be a prescribed constant. The frequency response curve is determined by varying this constant.

2. In the last term in Equation (10), Ψ_m should be ψ_m.

3. In Equation (20a), ϕ^*_{0i} should be ϕ^*_0.

4. In Equation (25a), ψ_m should be ψ_k.

5. In Equations (38e, f), the first $Td^{(1,s)}_{n,m}$ should be $Td^{(1,c)}_{n,m}$.

6. In Equation (41a), the $\zeta^{(1,s)}_{2m}$ in the 2nd line should be $\zeta^{(1,s)}_{2,\overline{m}}$, the $\phi^{(1,c)}_{2m}$ on the 3rd line should be $\phi^{(1,c)}_{2,\overline{m}}$ and λ^*_m on the 4th line should be $\lambda^*_{\overline{m}}$.

7. In Equation (60c), $H^{(o)}_o$ should be $H^{(o)}_{o_{ij}}$.

8. In Equation (61), the $\bar{\mathfrak{J}}_{o\nu}$ on the second line should be $\bar{H}_{o\nu}$.