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Research and Analysis Section Tech. Memo. No. 104

Prepared Under Contract NAS8-11255

October 1965

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NOTATION

Term	Definition
$a = \frac{8\pi r d_2 \, KT_o}{I_{cm} \, \Omega^2}$	Dimensionless parameter of the Matheau equation. (see eqs. (4.2-4).
a_x, a_θ, a_z	Axial, circumferential, and normal components of acceleration of the shell element.
$(a)_d$, $(b)_d$, $(c)_d$	Coefficients used in the eigenvalue solutions of $(\bar{\omega}^1_{jo})^2_d$, $(\bar{\omega}^2_{jo})^2_d$.
$(a_{jo})_d, \ldots, (m_{jo})_d$	Coefficients of eqs. (5.4-6). See APPENDIX A.
$(a_{jk})_d, \ldots, (p_{jk})_d$	Coefficients of eqs. (5.4-8). See APPENDIX A.
$(A)_d$, $(B)_d$, $(C)_d$, $(D)_d$	Coefficients used in the eigenvalue solutions of $(\bar{\omega}^1_{jk})^2_d$, $(\bar{\omega}^2_{jk})^2_d$, $(\bar{\omega}^3_{jk})^2_d$.
$A(\theta)_d$, $B(\theta)_d$	Spatial functions $f_{mn}(\bar{\xi},\theta)_d$, $g_{mn}(\bar{\xi},\theta)_d$ $h_{mn}(\bar{\xi},\theta)_d$ evaluated at prescribed limits.
A_i	Boundary value coefficients determined from initial conditions.
c	Subscript used in Galerkin procedure.

NOTATION (continued)

Term	Definition
$C_n^{(s)}$	The n^{th} element in the s^{th} column matrix.
C_n	Constants determined from n^{th} mode shape of a discontinuous beam at the center of mass.
d=1,2	Subscript used to indicate which cylinder of the composite structure is being considered.
$f^*(\bar{\xi},\tau)$, $g^*(\bar{\xi},\tau)$, $h^*(\bar{\xi},\tau)$	Displacement functions in eqs. (5.3-4).
$f_{mn}(\bar{\xi},\theta)_d$, $g_{mn}(\bar{\xi},\theta)_d$, $h_{mn}(\bar{\xi},\theta)_d$	Orthogonal functions of the shell displacements u, v, w, respectively. See eqs. (5.3-2, -3).
$f_1(\bar{x},\theta,t)$, $f_2(\bar{x},\theta,t)$, $f_3(\bar{x},\theta,t)$	Forcing functions due to acceleration components $a_{\bar{x}}$, a_θ, a_z determined for a moving coordinate system. See eqs. (2.2-6).
$F_1(\bar{\xi},\theta,\tau)$, $F_2(\bar{\xi},\theta,\tau)$, $F_3(\bar{\xi},\theta,\tau)$	Dimensionless forcing functions of eqs. (3.0-3).
g	Acceleration of gravity.
h	Thickness of cylindrical shell, sometimes called t.

NOTATION (continued)

Term	Definition
i, j, k, m, n, p, r, s	Indices used in the report.
I_{cm}	Mass moment of inertia of the composite structure about the center of mass cm. Refer to Fig. 1 and eq. (2.2-3).
$(J_1)_d$, $(J_2)_d$, $(J_3)_d$	Convolution integrals of eqs. (6.2-25,-26,-27)
K	Directional control constant.
$\mathcal{L}[\ ]$	Laplace transform notation.
L, L_1, L_2	Half lengths used in the analysis. See Fig. 1.
M	Mass of the composite structure.
M_B	Equivalent mass of the beam.
M_1, N_1, M_2, N_2	Limits on summing indices j_1, k_1, j_2, k_2, respectively.

NOTATION (continued)

Term	Definition
$M_{\bar{x}\bar{x}}$, $M_{\bar{x}\theta}$, $M_{\theta\bar{x}}$, $M_{\theta\theta}$ $N_{\bar{x}\bar{x}}$, $N_{\bar{x}\theta}$, $N_{\theta\bar{x}}$, $N_{\theta\theta}$ $Q_{\bar{x}}$, Q_{θ}	Moment and stress resultants.
n, N	Beam modal index and limit.
$P_{\bar{x}}$, P_{θ}, P_{z}	Distributed force terms of eqs. (2.2-7)
$P_{oo}(\tau)_d$, $P_{ok}(\tau)_d$, $[P_{jo}(\tau)_d]_1$, $[P_{jo}(\tau)_d]_2$	Nonhomogeneous terms of eqs. (5.4-1,-4,-6).
$[\bar{P}_{jo}(\tau)_d]_1$, $[\bar{P}_{jo}(\tau)_d]_2$	Laplace transformed terms.
$q = \frac{\gamma a}{2}$	Dimensionless parameter of the Matheau equation that governs stable solutions of $\psi(\tau)$.
q_A, q_B, q_n	Generalized coordinates of eqs. (4.1-1).
$[Q_{jk}(\tau)_d]_1$, $[Q_{jk}(\tau)_d]_2$, $[Q_{jk}(\tau)_d]_3$	Nonhomogeneous terms of eqs. (5.4-8).
$[\bar{Q}_{jk}(\tau)_d]_1$, $[\bar{Q}_{jk}(\tau)_d]_2$, $[\bar{Q}_{jk}(\tau)_d]_3$	Laplace transformed terms.

NOTATION (continued)

Term	Definition
r	Radius of the cylindrical shell, also used as an index in Section 7.0.
R	Maximum value of the index r.
s	Summing index used in Section 7.0.
S	Maximum value of the summing index s.
t	Real time variable, sometimes used as shell thickness.
$T(t)$	Thrust function in time t.
$T(\tau)$	Thrust function in dimensionless time τ.
T_o	Magnitude of the steady state thrust per unit length applied around the bottom of the shell.
$\bar{T}_o = \frac{T_o}{\rho h L^2 \omega_1^2}$	Dimensionless parameter.
u, v, w	Axial, circumferential, and normal displacements of shell relative to the moving reference frame.
$\bar{u} = \frac{u}{L}, \quad \bar{v} = \frac{v}{L}, \quad \bar{w} = \frac{w}{L}$	Dimensionless displacements.

Term	Definition
$\tilde{u}(\bar{\xi},\theta,\tau)$, $\tilde{v}(\bar{\xi},\theta,\tau)$, $\tilde{w}(\bar{\xi},\theta,\tau)$	Assumed dimensionless displacements used in Galerkin procedure. See eqs. (5.2-1).
u^*, v^*, w^*	Beam contribution to $\tilde{u}$, $\tilde{v}$, $\tilde{w}$.
$(u_{oo})_d$, $(u_{ok})_d$, $(u_{jo})_d$, $(u_{jk})_d$	Initial displacements of $U_{oo}(\tau)_d$, $U_{ok}(\tau)_d$, $U_{jo}(\tau)_d$, $U_{jk}(\tau)_d$.
$U_{ok}(\tau)_d$, $U_{jo}(\tau)_d$, $U_{jk}(\tau)_d$ $V_{ok}(\tau)_d$, $V_{jo}(\tau)_d$, $V_{jk}(\tau)_d$ $W_{ok}(\tau)_d$, $W_{jo}(\tau)_d$, $W_{jk}(\tau)_d$	Generalized coordinates resulting from Galerkin method.
$U_{mn}(\tau)_d$, $V_{mn}(\tau)_d$, $W_{mn}(\tau)_d$	Generalized coordinates of $\bar{u}$, $\bar{v}$, $\bar{w}$.
$\bar{U}_{jo}(\tau)_d$, $\bar{V}_{jo}(\tau)_d$, $\bar{U}_{jk}(\tau)_d$ $\bar{V}_{jk}(\tau)_d$, $\bar{W}_{jk}(\tau)_d$	Transformed quantities.
x, y, z	Coordinates of the shell in axial, circumferential, and radial directions.
X, Y	Reference frame coordinates.
$X_o = \frac{X}{L}$, $Y_o = \frac{Y}{L}$	Dimensionless coordinates.

NOTATION (continued)

Greek Term	Definition
α_r	Stability parameter of the discontinuous beam.
β	Stability parameter of the solution for $\psi(\tau)$.
$\gamma = \frac{T_1}{T_o}$	Ratio of the magnitude of sinusoidal time varying thrust per unit length to T_o.
$(\delta_{jk})_d$	Kronecker delta for d=1,2.
$\Delta(\beta)$	Infinite determinant.
$\Delta(0)$	Value of infinite determinant $\Delta(\beta)$ for $\beta = 0$.
ε_i	Term used in writing the stability equations for the dynamic solution. See eqs. (7.2-5).
Σ	Summation symbol.
η	Dummy variable used in convolution integrals.
θ	Tangential shell coordinate.
$\lambda = \frac{L}{r}$	Dimensionless radius parameter.
$\mu = \frac{\rho L^2 \omega_1^2}{E}$	Dimensionless parameter.
ν	Poisson's ratio of shell material.
$\xi = \frac{\bar{x}}{L}$	Dimensionless axial shell coordinate.

NOTATION (continued)

Greek Term	Definition
π	Constant of value 3.1416.
ρ	Mass per unit volume of the shell material. Assumed to be the same for both cylindrical shells.
$\sigma = \frac{h}{L}$	Dimensionless parameter.
$\tau = \omega_1 t$	Dimensionless time.
$\tau_1 = \frac{\Omega}{2}\frac{\tau}{\omega_1} = \frac{\Omega}{2} t$	Dimensionless time.
ϕ	Mode shape symbol.
ϕ_n	Mode shape of the n^{th} vibration mode of the free-free, composite structure.
$\psi(\tau)$	Rigid body rotation of the composite structure about the center of mass.
ψ_o	Initial value of $\psi(\tau)$.
ω	Circular frequency symbol used in report.
$\omega_1, \omega_2, \omega_3, \ldots, \omega_n$	Lateral bending frequencies of the free-free, discontinuous beam.
$(\bar{\omega}_{ok})_d$, $(\bar{\omega}^m_{jo})_d$, $(\bar{\omega}^m_{jk})_d$	Natural circular frequencies from modal considerations.

Greek Term	Definition
Ω	Circular frequency of the sinusoidal component of the applied frequency.
Ω_i	Solutions of the unstable values of Ω associated with the shell modes.

Numerical Terms	Definition
$\boxed{1}_d, \ldots, \boxed{12}_d$	See eqs. (6.2-22).
$\textcircled{1}_d, \ldots, \textcircled{18}_d$	See eqs. (6.2-23).

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1.0 INTRODUCTION

The dynamic stability of a flexible booster synthesized by two thin-walled cylinders as shown in Figure 1 is investigated. The analysis assumes that the edges of the cylindrical models are attached to rigid, weightless bulkheads incapable of transmitting end moments to the shell. The gimbaled, time-varying thrust, as in the previous analyses of Hill (ref. 1) and Pearson (ref. 2), is assumed to be sinusoidal about some average thrust value. Directional control of the thrust is achieved through the use of a simple proportional feedback system shown in Figure 3b .

The formulation contained in this technical memo was mainly the combined efforts of C. M. Pearson and J. H. Kincaid with other members of the Structural Dynamics Branch participating from time to time. This report represents the progress to date of this portion of the contract work.

2.0 EQUATIONS OF MOTION

2.1 Motion of the Composite Structure

The motion of the composite structure of Figure 1 will be described in the moving coordinate system of Figure 3, where the coordinate system is allowed to move in a plane with the origin always at the center of mass CM of the composite structure.

2.2 Equations of Motion of the Shell Element

The equations of motion of the shell element are as follows:

$$\frac{\partial N_{\bar{x}\bar{x}}}{\partial \bar{x}} + \frac{1}{r}\frac{N_{\theta\bar{x}}}{\partial \theta} + P_{\bar{x}} = \rho h\, a_{\bar{x}} \qquad (a)$$

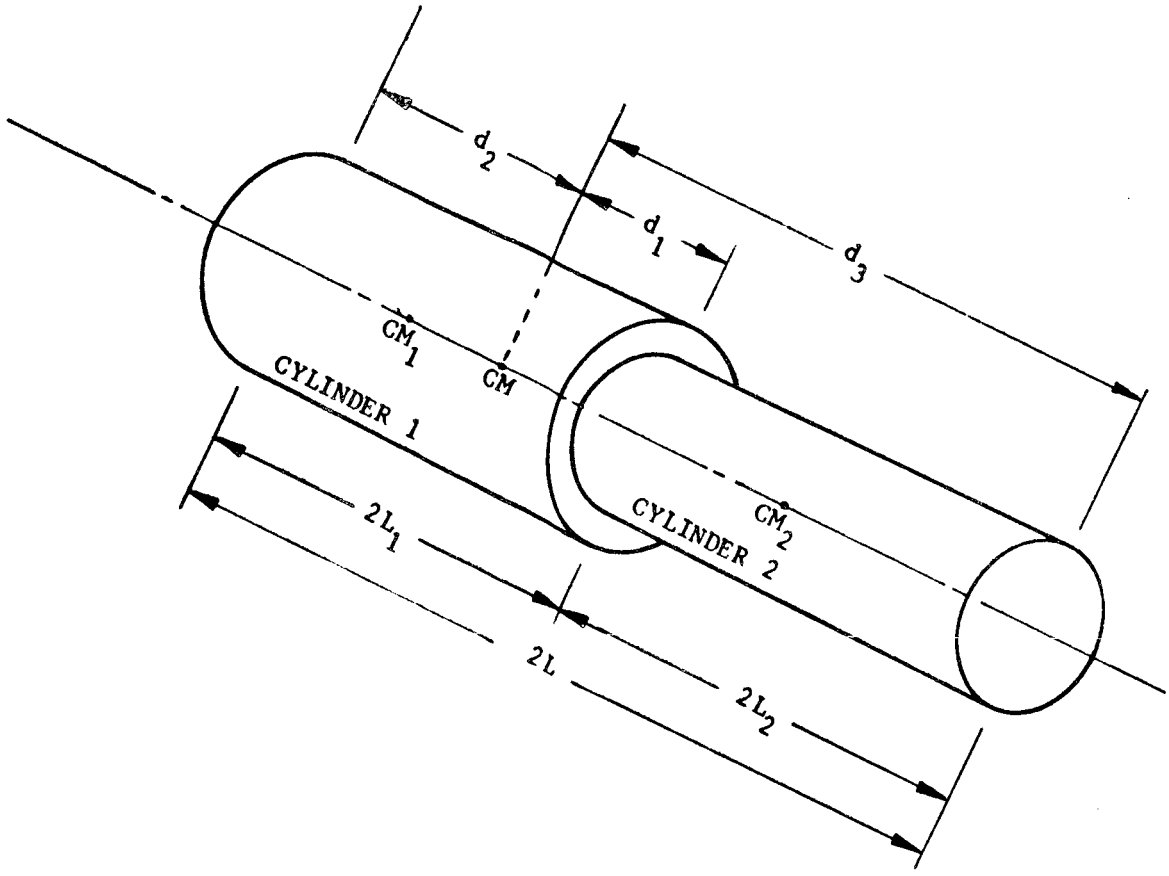
$$\frac{\partial N_{\bar{x}\theta}}{\partial \bar{x}} + \frac{1}{r}\frac{\partial N_{\theta\theta}}{\partial \theta} - \frac{Q_\theta}{r} + P_\theta = \rho h\, a_\theta \qquad (b)$$

$$\frac{\partial Q_{\bar{x}}}{\partial \bar{x}} + \frac{1}{r}\frac{\partial Q_\theta}{\partial \theta} + \frac{1}{r} N_{\theta\theta} + P_z = \rho h\, a_z \qquad (c) \qquad (2.2\text{-}1)$$

$$\frac{\partial M_{\bar{x}\bar{x}}}{\partial \bar{x}} + \frac{1}{r}\frac{\partial M_{\theta\bar{x}}}{\partial \theta} - Q_{\bar{x}} = 0 \qquad (d)$$

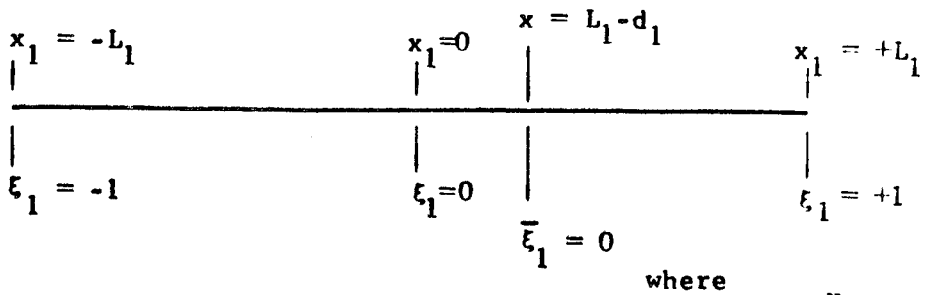
$$\frac{\partial M_{\bar{x}\theta}}{\partial \bar{x}} - \frac{1}{r}\frac{\partial M_{\theta\theta}}{\partial \theta} + Q_\theta = 0 \qquad (e)$$

$$N_{\theta\bar{x}} - N_{\bar{x}\theta} = \frac{1}{r} M_{\theta\bar{x}} \qquad (f)$$



TERM	DEFINITION
CM_1	Center of Mass of Cylinder 1
CM_2	Center of Mass of Cylinder 2
CM	Center of Mass of Composite Structure
d_1	Distance From CM to Right End of Cylinder 1
d_2	Distance From CM to Left End of Cylinder 2
d_3	Distance From CM to Right End of Cylinder 2.
L_1	Half Length of Cylinder 1.
L_2	Half Length of Cylinder 2.
L	Half Length of Composite Structure

Figure 1 COMPOSITE STRUCTURE COMPOSED OF TWO THIN-WALLED CYLINDERS WITH RIGID, WEIGHTLESS BULKHEADS INCAPABLE OF RENDERING END MOMENTS



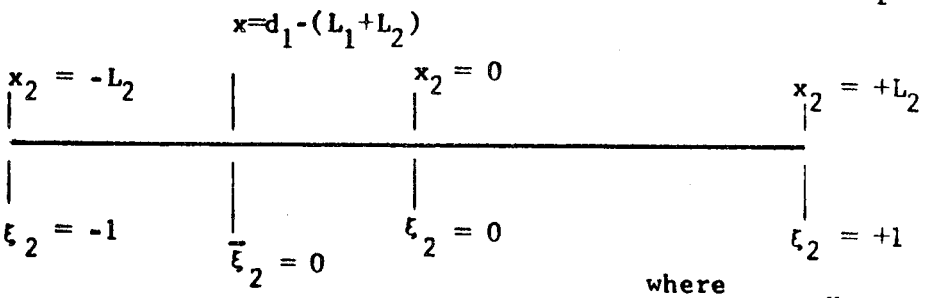
where

$$\xi_1 = \frac{x_1}{L_1}$$

Coordinates of Cylinder 1

$$\bar{\xi}_1 = \xi_1 - r_1$$

$$r_1 = \frac{L_1 - d_1}{L_1}$$



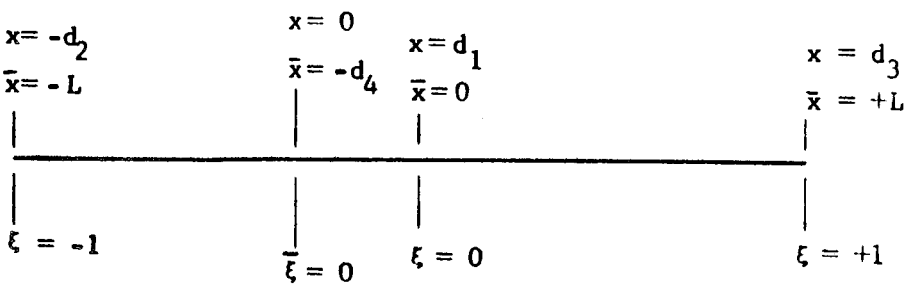
where

$$\xi_2 = \frac{x_2}{L_2}$$

Coordinates of Cylinder 2

$$\bar{\xi}_2 = \xi_2 + r_2$$

$$r_2 = \frac{d_1 - (L_1 + L_2)}{L_2}$$



where

$$\xi = \frac{x}{L}$$

Coordinates of Composite Structure

$$\bar{\xi} = \xi + r$$

$$r = \frac{d_4}{L}$$

Figure 2 DIAGRAM SHOWING THE AXIAL COORDINATE AND THE DIMENSIONLESS AXIAL COORDINATE USED IN THE ANALYSIS

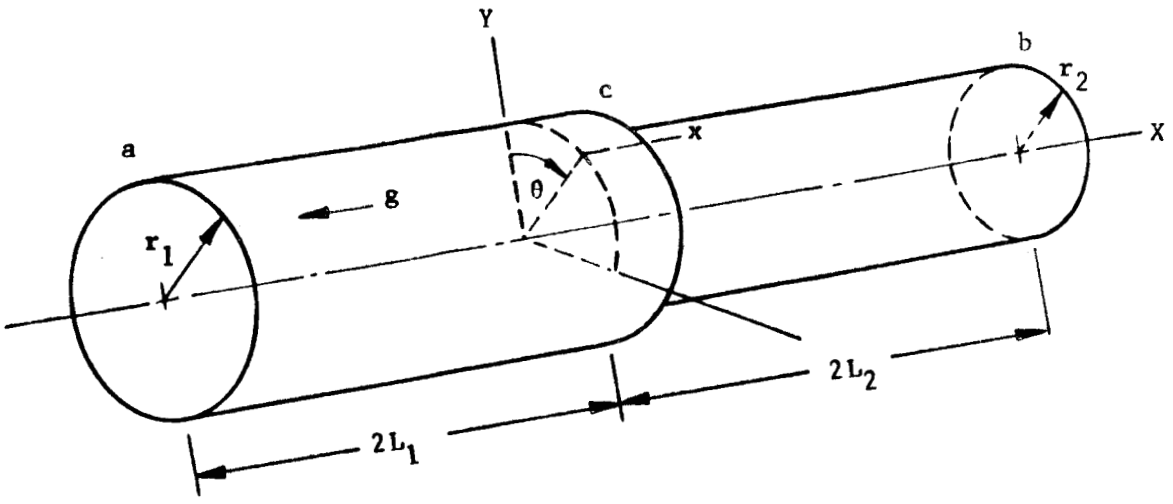


Figure 3a UNDEFLECTED COMPOSITE STRUCTURE IN THE MOVING (X,Y,Z) COORDINATE SYSTEM

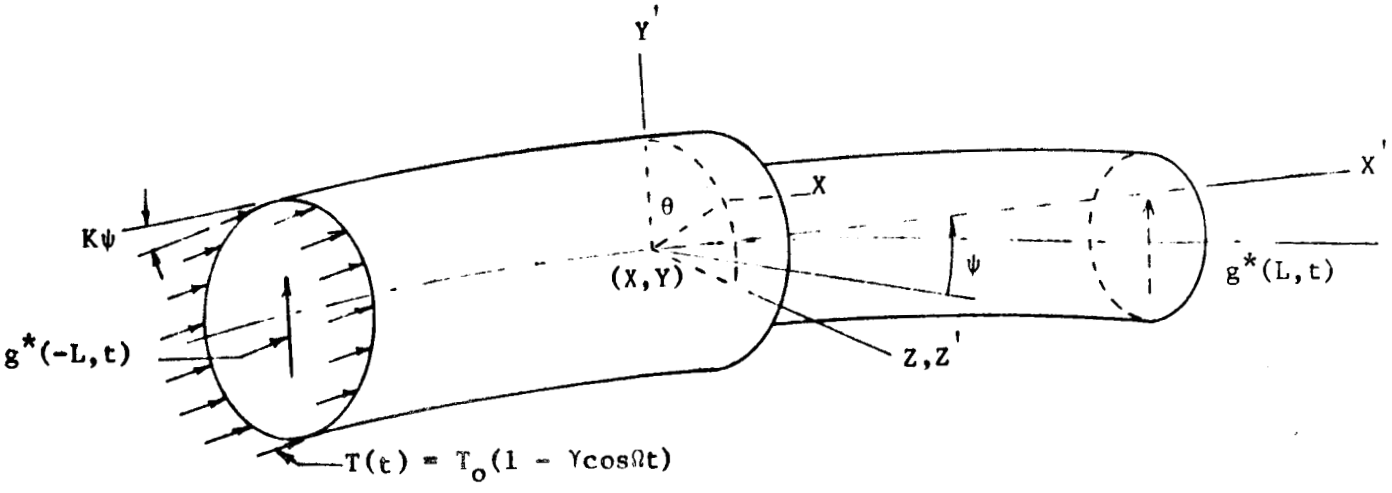


Figure 3b DEFLECTED COMPOSITE STRUCTURE IN THE MOVING (X',Y', Z') COORDINATE SYSTEM WITH ORIGIN ALWAYS AT THE CENTER OF MASS. (NOTE THAT THE X',Y' COORDINATE LINES CONTINUE TO BE IN THE X,Y PLANE).

where ρ is the mass density for both cylindrical shells, h is the thickness considered to be the same for both shells, $a_{\bar{x}}$, a_θ, a_z are the components of acceleration of the shell element in the axial, circumferential, and radial (positive inward) directions, respectively. The terms $N_{\bar{x}\bar{x}}$, $N_{\theta\theta}$, $N_{\bar{x}\theta}$, $N_{\theta\bar{x}}$, $Q_{\bar{x}}$, Q_θ, $M_{\bar{x}\bar{x}}$, $M_{\theta\theta}$, $M_{\bar{x}\theta}$, and $M_{\theta\bar{x}}$ are standard resultants, and $P_{\bar{x}}$, P_θ and P_z are components of the distributed surface forces in the axial, circumferential, and radial directions, respectively.

Using Newton's laws of plane motion we can write (refer to Fig. 3a.)

$$M\ddot{X} = 2\pi R_1 T_o(1 - \gamma\cos\Omega t) \cos (K+1)\psi - Mg \qquad (a)$$

$$M\ddot{Y} = 2\pi R_1 T_o(1 - \gamma\cos\Omega t) \sin (K+1)\psi \qquad (b) \qquad (2.2\text{-}2)$$

$$I_{cm}\ddot{\psi} = -2\pi R_1 T_o(1 - \gamma\cos\Omega t) \sin K\psi \qquad (c)$$

where the assumption is made that the deformation does not change the half length nor the mass moment of inertia of the composite structure, which can be expressed

$$I_{cm} = 2\pi\rho h \left\{ R_1 \left[2L_1\left(\frac{R_1^2}{4} + \frac{L_1^2}{3}\right) + (d_2 - L_1)^3\right] + R_2 \left[2L_2\left(\frac{R_2^2}{4} + \frac{L_2^2}{3}\right) + (d_3 - L_2)^3\right]\right\} \qquad (2.2\text{-}3)$$

where R_1, R_2 are the radii of cylindrical shells 1, 2 and the remaining terms are defined in Fig. (1).

The acceleration components $a_{\bar{x}}$, a_θ, a_z, written in terms of the shell displacements u, v, w and X, Y the coordinates of the reference frame, is written

$$a_{\bar{x}} = \frac{\partial^2 u}{\partial t^2} + \ddot{X}\cos\psi + \ddot{Y}\sin\psi - r\cos\theta\ddot{\psi} - \bar{x}\dot{\psi}^2$$

$$\underline{+ 2\dot{\psi}\left(\frac{\partial v}{\partial t}\sin\theta + \frac{\partial w}{\partial t}\cos\theta\right)} \qquad \text{(a)}$$

$$a_\theta = \frac{\partial^2 v}{\partial t^2} + (\ddot{X}\sin\psi + \ddot{Y}\cos\psi + r\cos\theta\dot{\psi}^2 - \bar{x}\ddot{\psi})\sin\theta$$

$$\underline{- 2\dot{\psi}\frac{\partial u}{\partial t}\sin\theta} \qquad \text{(b)} \qquad (2.2\text{-}4)$$

$$a_z = \frac{\partial^2 w}{\partial t} + (\ddot{X}\sin\psi + \ddot{Y}\cos\psi + r\cos\theta\dot{\psi}^2 - \bar{x}\ddot{\psi})\cos\theta$$

$$\underline{- 2\dot{\psi}\frac{\partial u}{\partial t}\cos\theta} \qquad \text{(c)}$$

The underlined terms are Coriolis terms and will be neglected in the analysis.

Substituting $\ddot{X}$ and $\ddot{Y}$ (from eqs. (2.2-2)) into eqs. (2.2-4) and then substituting these new expressions for $a_{\bar{x}}$, a_θ, a_z into eqs. (2.2-1) and eliminating $Q_{\bar{x}}$, Q_θ we can write

$$\frac{\partial N_{\bar{x}\bar{x}}}{\partial \bar{x}} + \frac{1}{r}\frac{\partial N_{\theta\bar{x}}}{\partial\theta} + P_{\bar{x}} = \rho h\frac{\partial^2 u}{\partial t^2} + f_1(\bar{x},\theta,t) + \rho h g\cos\psi \qquad \text{(a)}$$

$$\frac{1}{r}\frac{\partial N_{\theta\theta}}{\partial\theta} + \frac{\partial N_{\bar{x}\theta}}{\partial\bar{x}} + \frac{1}{r}\frac{\partial M_{\bar{x}\theta}}{\partial\bar{x}} - \frac{1}{r^2}\frac{\partial M_{\theta\theta}}{\partial\theta} + P_\theta$$

$$= \rho h\frac{\partial^2 v}{\partial t^2} + f_2(\bar{x},\theta,t) + \rho h g\sin\psi\sin\theta \qquad \text{(b)} \qquad (2.2\text{-}5)$$

$$\frac{1}{r} N_{\theta\theta} + \frac{1}{r} \frac{\partial^2 M_{\theta\bar{x}}}{\partial \bar{x} \partial \theta} + \frac{\partial^2 M_{\bar{x}\bar{x}}}{\partial \bar{x}^2} - \frac{1}{r} \frac{\partial^2 M_{\bar{x}\theta}}{\partial \bar{x} \partial \theta} + \frac{1}{r} \frac{\partial^2 M_{\theta\theta}}{\partial \theta^2} + P_z$$

$$= \rho h \frac{\partial^2 w}{\partial t^2} + f_3(\bar{x},\theta,t) + \rho h g \sin\psi \cos\theta \qquad (c)$$

where

$$f_1(\bar{x},\theta,t) = \rho h(\ddot{X} \cos\psi + \ddot{Y} \sin\psi - r \cos\theta\ddot{\psi} - \bar{x}\dot{\psi}^2) \qquad (a)$$

$$f_2(\bar{x},\theta,t) = \rho h (\ddot{X} \sin\psi - \ddot{Y} \cos\psi + r \cos\theta\dot{\psi}^2 - \bar{x}\ddot{\psi})\sin\theta \qquad (b) \quad (2.2\text{-}6)$$

$$f_3(\bar{x},\theta,t) = \rho h(\ddot{X} \sin\psi - \ddot{Y} \cos\psi + r \cos\theta\dot{\psi}^2 - \bar{x}\ddot{\psi})\cos\theta \qquad (c)$$

and since only the gimbaled thrust T(t) is considered in the analysis the distributed force terms $P_{\bar{x}}$, P_θ, P_z are written

$$P_{\bar{x}} = T(t) \cos K\psi - \rho g h \cos\psi \qquad (a)$$

$$P_\theta = - [T(t) \sin K\psi + \rho g h \sin\psi] \sin\theta \qquad (b) \quad (2.2\text{-}7)$$

$$P_z = - [T(t) \sin K\psi + \rho g h \sin\psi] \cos\theta \qquad (c)$$

2.3 Equations in Terms of Shell Displacements

Employing the relationships between stress resultants and shell displacements, substituting these relations and the required derivatives into eqs. (2.2-5), making some simplifying assumptions based on consideration of a thin-walled shell with large radius of curvature, we can write

$$\frac{\partial^2 u}{\partial \bar{x}^2} + \frac{(1-\nu)}{2r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{(1+\nu)}{2r}\frac{\partial^2 v}{\partial \bar{x}\partial \theta} - \frac{\nu}{r}\frac{\partial w}{\partial \bar{x}}$$

$$= \frac{(1-\nu^2)}{E}\left[\rho \frac{\partial^2 u}{\partial t^2} + \frac{f_1 - P_{\bar{x}}}{h}\right] \qquad \text{(a)}$$

$$\frac{(1+\nu)}{2r}\frac{\partial^2 u}{\partial \bar{x}\partial \theta} + \frac{(1-\nu)}{2}\frac{\partial^2 v}{\partial \bar{x}^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta} - \frac{1}{r_d^2}$$

$$= \frac{(1-\nu^2)}{E}\left[\rho \frac{\partial^2 v}{\partial t^2} + \frac{f_2 - P_\theta}{h}\right] \qquad \text{(b)} \quad (2.3\text{-}1)$$

$$\frac{\nu}{r}\frac{\partial u}{\partial \bar{x}} + \frac{1}{r^2}\frac{\partial v}{\partial \theta} - \frac{1}{r^2}w - \frac{h^2}{12}\nabla^4 w$$

$$= \frac{(1-\nu^2)}{E}\left[\rho \frac{\partial^2 w}{\partial t^2} + \frac{f_3 - P_z}{h}\right] \qquad \text{(c)}$$

3.0 EQUATIONS OF MOTION IN DIMENSIONLESS FORM

We render eqs. (2.3-1) in dimensionless form by use of the following transformations

$$\bar{\xi} = \frac{\bar{x}}{L} \qquad \tau = \omega_1 t \qquad \lambda = \frac{L}{r}$$

$$\bar{u} = \frac{u}{L} \qquad \bar{v} = \frac{v}{L} \qquad \bar{w} = \frac{w}{L} \tag{3.0-1}$$

$$\sigma = \frac{h}{L}$$

Eqs. (2.3-1) become

$$\frac{\partial^2 \bar{u}}{\partial \bar{\xi}^2} + \frac{(1-\nu)\lambda^2}{2} \frac{\partial^2 \bar{u}}{\partial \theta^2} + \frac{(1+\nu)\lambda}{2} \frac{\partial^2 \bar{v}}{\partial \bar{\xi} \partial \theta} - \nu\lambda \frac{\partial \bar{w}}{\partial \bar{\xi}}$$

$$= (1-\nu^2)\mu \frac{\partial^2 \bar{u}}{\partial \tau^2} + F_1(\bar{\xi}, \theta, \tau) \qquad \text{(a)}$$

$$\frac{(1+\nu)\lambda}{2} \frac{\partial^2 \bar{u}}{\partial \bar{\xi} \partial \theta} + \frac{(1-\nu)}{2} \frac{\partial^2 v}{\partial \bar{\xi}^2} + \lambda^2 \frac{\partial^2 \bar{v}}{\partial \theta^2} - \lambda^2 \frac{\partial \bar{w}}{\partial \theta}$$

$$= (1-\nu^2)\mu \frac{\partial^2 \bar{v}}{\partial \tau^2} + F_2(\bar{\xi}, \theta, \tau) \qquad \text{(b)} \tag{3.0-2}$$

$$\nu\lambda \frac{\partial \bar{u}}{\partial \bar{\xi}} + \lambda^2 \frac{\partial \bar{v}}{\partial \theta} - \lambda^2 \bar{w} - \frac{\sigma^2}{12} \bar{\nabla}^4 \bar{w}$$

$$= (1-\nu^2)\mu \frac{\partial^2 \bar{w}}{\partial \tau^2} + F_3(\bar{\xi}, \theta, \tau) \qquad \text{(c)}$$

where

$$F_1(\bar{\xi},\theta,\tau) = (1-\nu^2)\mu\left[\frac{\bar{T}_o}{2}(1-\gamma\cos\bar{\Omega}\tau)\cos K\psi - \frac{\ddot{\psi}}{\lambda}\cos\theta - \dot{\psi}^2\bar{\xi}\right] \quad (a)$$

$$F_2(\bar{\xi},\theta,\tau) = (1-\nu^2)\mu\left[-\frac{\bar{T}_o}{2}(1-\gamma\cos\bar{\Omega}\tau)\sin K\psi + \frac{\dot{\psi}^2}{\lambda}\cos\theta - \ddot{\psi}\bar{\xi}\right]\sin\theta \quad (b) \quad (3.0\text{-}3)$$

$$F_3(\bar{\xi},\theta,\tau) = (1-\nu^2)\mu\left[-\frac{\bar{T}_o}{2}(1-\gamma\cos\bar{\Omega}\tau)\sin K\psi + \frac{\dot{\psi}^2}{\lambda}\cos\theta - \ddot{\psi}\bar{\xi}\right]\cos\theta \quad (c)$$

and

$$\mu = \frac{\rho L\omega_1^2}{E} \quad (a)$$

$$\bar{T}_o = \frac{T_o}{\rho h\, L^2\, \omega_1^2} \quad (b) \quad (3.0\text{-}4)$$

$$\bar{\nabla}^4 = \frac{\partial^4(\,)}{\partial\bar{\xi}^4} + 2\lambda^2\frac{\partial^4(\,)}{\partial\bar{\xi}^2\partial\theta^2} + \lambda^4\frac{\partial^4(\,)}{\partial\theta^4} \quad (c)$$

4.0 END DISPLACEMENTS AND RIGID BODY ANGLE OF ROTATION

4.1 Method of Determination of the End Displacements

The transverse end displacements $g^*(-L,\tau)$ and $g^*(+L,\tau)$ shown in Figure 1 are determined as follows:

A. The free-free composite structure will be treated as a free-free beam of equal mass and length. A complete analysis of the beam is presented in ref. 11 .

B. The generalized displacement equation used in ref. 11 will be employed in this analysis since solutions for the free-free end conditions are fully detailed.

4.1.1 Generalized Displacement Equation

The generalized displacement

$$g^*(\bar{\xi},\tau) = q_A(\tau) + q_B(\tau)\bar{\xi} + \sum_{n=1}^{N} q_n(\tau)\,\phi_n(\bar{\xi}) \qquad (4.1\text{-}1)$$

where

$q_A(\tau)$ - generalized translation coordinate

$q_B(\tau)$ - generalized rotational coordinate

$q_n(\tau)$ - generalized coordinate associated with $\phi_n(\bar{\xi})$

$\phi_n(\bar{\xi})$ - n^{th} vibrational mode shape of the free-free beam

The expression for the n^{th} vibrational mode shape that is used in this analysis is

$$\phi_n(\bar{\xi}) = \sum_{n=1}^{N} \cosh \frac{\lambda_n}{2} (\bar{\xi} - r + 1) + \cos \frac{\lambda_n}{2} (\bar{\xi} - r + 1)$$

$$- \alpha_n \left[\sinh \frac{\lambda_n}{2} (\bar{\xi} - r + 1) + \sin \frac{\lambda_n}{2} (\bar{\xi} - r + 1)\right] \quad (4.1\text{-}2)$$

The ϕ-function and its respective spatial derivatives is detailed in Appendix C, and the general properties of this function is listed in ref. 9 and ref. 10 .

4.1.2 Displacement Expression and Derivatives

The spatial and time derivatives of the generalized displacement equation written for the coordinate system at the center of mass (the translation coordinate $q_A(\tau)$ can now be disregarded) of the composite structure as follows:

$$g^*(\bar{\xi},\tau) = q_B(\tau)\bar{\xi} + \sum_{n=1}^{N} q_n(\tau)\ \phi_n(\bar{\xi}) \quad (4.1\text{-}3)$$

$$g'^*(\xi,\tau) = q_B(\tau) + \sum_{n=1}^{N} q_n(\tau)\ (\frac{\lambda_n}{2})\ \phi_n'(\bar{\xi}) \quad (4.1\text{-}4)$$

$$g''^*(\bar{\xi},\tau) = \sum_{n=1}^{N} q_n(\tau)\ (\frac{\lambda_n}{2})^2\ \phi_n''(\xi) \quad (4.1\text{-}5)$$

$$g'''^*(\bar{\xi},\tau) = \sum_{n=1}^{N} q_n(\tau)\ (\frac{\lambda_n}{2})^3\ \phi_n'''(\bar{\xi}) \quad (4.1\text{-}6)$$

$$g''''^*(\bar{\xi},\tau) = \sum_{n=1}^{N} q_n(\tau)\ (\frac{\lambda_n}{2})^4 \phi_n''''(\bar{\xi})$$

$$= \sum_{n=1}^{N} q_n(\tau)\ (\frac{\lambda_n}{2})^4 \phi_n(\bar{\xi}) \quad (4.1\text{-}7)$$

$$\dot{g}^{*}(\bar{\xi},\tau) = \dot{q}_B(\tau)\bar{\xi} + \sum_{n=1}^{N} \dot{q}_n(\tau)\ \phi_n(\bar{\xi}) \tag{4.1-8}$$

$$\ddot{g}^{*}(\bar{\xi},\tau) = \ddot{q}_B(\tau)\bar{\xi} + \sum_{n=1}^{N} \ddot{q}_n(\tau)\ \phi_n(\bar{\xi}) \tag{4.1-9}$$

where the generalized coordinates $q_B(\tau)$ and $q_n(\tau)$ are written

$$q_B(\tau) = \sum_{s=-S}^{S} C_B^{(s)}\ e^{i(\alpha_r+s)\bar{\Omega}\tau} \tag{4.1-10}$$

$$q_n(\tau) = \sum_{s=-S}^{S} C_n^{(s)}\ e^{i(\alpha_r+s)\bar{\Omega}\tau} \tag{4.1-11}$$

$$r = 1,2,\ldots,R$$

$$n = 1,2,\ldots,N$$

$$B = N + 1$$

where the notation $C_n^{(s)}$ represents the n^{th} element of s^{th} column matrix

$$C_n^{(s)} = \begin{Bmatrix} C_1 \\ C_2 \\ \cdot \\ \cdot \\ \cdot \\ C_N \\ C_B \end{Bmatrix}^{(s)} \tag{4.1-12}$$

$$B = N + 1$$

and α_r is a stability parameter determined from consideration of beam action (see ref. 11) of the composite structure.

4.2 Method of Determination of the Rigid Body Angle of Rotation, ψ

Rewriting Eq. (2.2-2a) we have

$$\frac{d^2\psi}{dt^2} + \frac{2\pi r d_2}{I_{cm}} T(t) \sin K\psi = 0 \qquad (4.2\text{-}1)$$

where

$$T(t) = T_o(1 - \gamma \cos\Omega t) \qquad (a)$$

$$I_{cm} = 2\pi\rho h \left\{ R_1 \left[2L_1 \left(\frac{R_1^2}{4} + \frac{L_1^2}{4} \right) + (d_2 - L_1)^3 \right] + R_2 \left[2L_2 \left(\frac{R_2^2}{4} + \frac{L_2^2}{4} \right) + (d_3 - L_2)^3 \right] \right\} \qquad (b) \quad (4.2\text{-}2)$$

d_1, d_2, d_3
L_1, L_2, L_3 } distances shown in Figure 1 (c)

Assuming small ψ and changing to time $\tau_1 = \frac{\Omega}{2} t$ we can rewrite eq. (4.2-1)

$$\frac{d^2\psi}{d\tau_1^2} + (a - 2q \cos 2\tau_1)\psi = 0 \qquad (4.2\text{-}3)$$

where the dimensionless parameters a and q are

$$a = \frac{8\pi r d_2 K T_o}{I_{cm}\, \Omega^2}, \qquad q = \frac{\gamma a}{2} \qquad (4.2\text{-}4)$$

Equation (4.2-3) is a particular case of a linear second-order differential equation with periodic coefficients considered as the canonical form of the Matheau equation having different solutions according to the values of the parameters a and q. The theory for the solution of eq. (4.2-3) is

given in refs. 5, 6, 7 , and the complete solution for a similar Matheau equation is given in ref. 3 .

For cylindrical shells approximating the size of a large booster and considering the magnitude of the thrust that they encounter, proper range restrictions on the parameters a and q will be

$$0 < a < 1 \qquad\qquad 0 < q \leq 0.05 \qquad\qquad (4.2\text{-}5)$$

By considering the above range restrictions and by a development similar to that of ref. 3 it is shown that the solution of eq. (4.2-3) can be written

$$\psi(\tau) = \sum_{i=1}^{3} A_i \cos \Omega_i \tau \qquad (4.2\text{-}6)$$

where

$$\tau = \frac{2\omega_1}{\Omega} \tau_1 \qquad (a)$$

$$A_1 = A_2 \frac{q}{a-(\beta-2)^2} \qquad (b)$$

$$A_2 = \frac{\psi_o}{1+q\left[\frac{1}{a-(2+\beta)^2} + \frac{1}{a-(\beta-2)^2}\right]} \qquad (c)$$

$$A_3 = A_2 \frac{q}{a-(2+\beta)^2} \qquad (d) \qquad (4.2\text{-}7)$$

$$\Omega_1 = (2-\beta) \frac{\Omega}{2\omega_1} \qquad (e)$$

$$\Omega_2 = \frac{\beta\Omega}{2\omega_1} \qquad (f)$$

$$\Omega_3 = (2+\beta)\,\frac{\Omega}{2\omega_1} \qquad (g)$$

and

$$\psi_o = \psi(\tau)\Big|_{\tau=o} \qquad \text{(determined from initial conditions)} \quad (a)$$

$$\omega_1 = \left(\frac{\lambda_1}{2L}\right)^2 R_1\sqrt{\frac{E}{2\rho}} \quad \text{(see Appendix C for } \lambda_1\text{)} \qquad (b) \qquad (4.2\text{-}8)$$

and where the stability parameter β can be determined from

$$\cos\beta\pi = \cos\pi a^{\frac{1}{2}} + \frac{\pi q^2}{4a^{\frac{1}{2}}(a-1)}\sin\frac{\pi a^{\frac{1}{2}}}{2} \qquad (4.2\text{-}9)$$

5.0 METHOD OF SOLUTION

5.1 General

Generally, the method of Donnell is used to reduce eqs. (3.0-2) to a more convenient form. In the Donnell method the equations are rearranged into an eighth-order partial differential equation that $\bar{w}$ must satisfy and two fourth-order, partial differential equations that relate $\bar{u}$ to $\bar{w}$ and $\bar{v}$ to $\bar{w}$, respectively. The partial differential equation in $\bar{w}$ is of eighth-order in the spatial variable and sixth order in the time variable, contains rather complicated nonhomogeneous terms, and the solution of the equation for $\bar{w}$ is a lengthy task. An accepted method for the approximate determination of the displacements $\bar{u}$, $\bar{v}$, and $\bar{w}$ is the Galerkin procedure which will be used in this analysis.

5.2 Galerkin Procedure

In accordance with the Galerkin procedure we choose approximating forms for the displacements $\bar{u}$, $\bar{v}$, and $\bar{w}$ as

$$\tilde{u}(\bar{\xi},\theta,\tau) = \sum_{d=1}^{2} G_d(\bar{\xi})\, U_{mn}(\tau)_d\, f_{mn}(\bar{\xi},\theta)_d + u^*(\bar{\xi},\theta,\tau)$$

$$\tilde{v}(\bar{\xi},\theta,\tau) = \sum_{d=1}^{2} G_d(\bar{\xi})\, V_{mn}(\tau)_d\, g_{mn}(\bar{\xi},\theta)_d + v^*(\bar{\xi},\theta,\tau) \qquad (5.2\text{-}1)$$

$$\tilde{w}(\bar{\xi},\theta,\tau) = \sum_{d=1}^{2} G_d(\bar{\xi})\, W_{mn}(\tau)_d\, h_{mn}(\bar{\xi},\theta)_d + w^*(\bar{\xi},\theta,\tau)$$

where the purpose of the functions $f_{mn}(\bar{\xi},\theta)_d$, $g_{mn}(\bar{\xi},\theta)_d$ and $h_{mn}(\bar{\xi},\theta)_d$ are to insure satisfaction of the edge conditions at $\bar{\xi} = a,b,c$ (see Figure 3a).

The asterisked displacement terms $u^*(\bar{\xi},\theta,\tau)$, $v^*(\bar{\xi},\theta,\tau)$, and $w^*(\bar{\xi},\theta,\tau)$, due to the bending action of the cylinder, are superimposed onto the shell action of the cylinder.

Substituting the approximating forms of eqs. (5.2-1) and their required spatial and time derivatives into eqs. (3.0-2a) through (3.0-2c) multiplying the first of the resulting equations by $f_{jk}(\bar{\xi},\theta)_c$, the second by $g_{jk}(\bar{\xi},\theta)_c$, and the third by $h_{jk}(\bar{\xi},\theta)_c$, and forming double integrals one obtains the following system of equations.

$$\int_{L_{1d}}^{L_{2d}} \int_{0}^{2\pi} \sum_{m=o}^{M} \sum_{n=o}^{N}$$

$$\left[\sum_{d=1}^{2} [G_d''(\bar{\xi})\ f_{mn}(\bar{\xi},\theta)_d + 2G_d'(\bar{\xi})\ f_{mn}'(\bar{\xi},\theta)_d + G_d(\bar{\xi})\ f_{mn}''(\bar{\xi},\theta)_d]\ U_{mn}(\tau)_d \right.$$

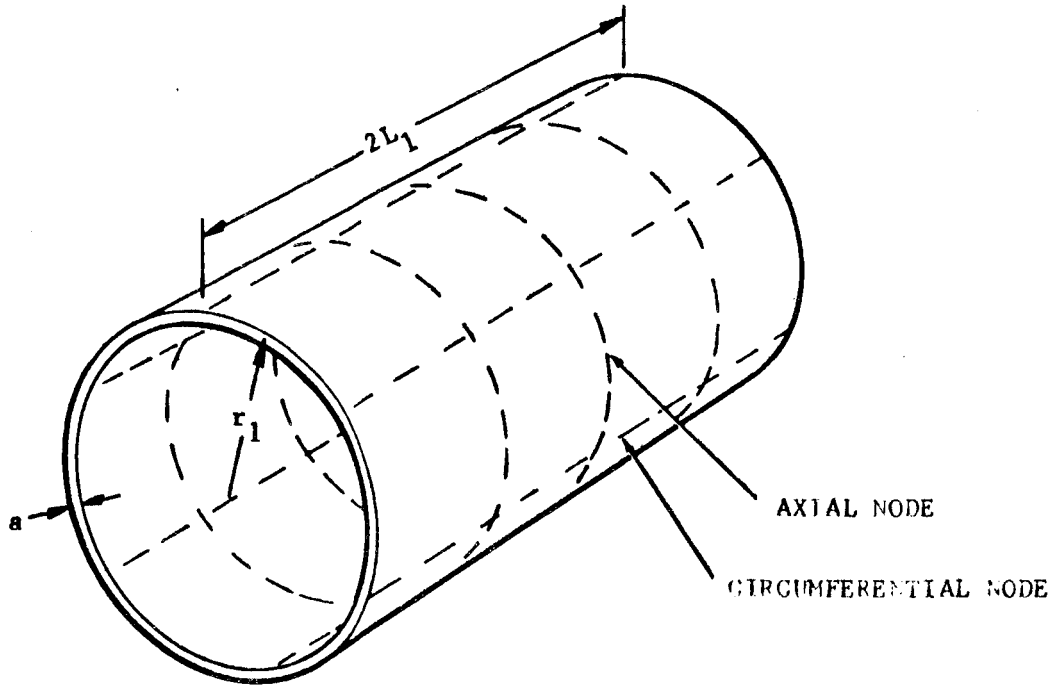
$$\left. + \frac{1-\nu^2}{Eh} \frac{T_o}{2} (1 - \gamma\cos\bar{\Omega}\tau) \right.$$

$$+ \frac{(1-\nu)\lambda^2}{2} \left(\sum_{d=1}^{2} [G_d(\bar{\xi})\ \overset{\infty}{f}_{mn}(\bar{\xi},\theta)_d - \overset{\infty}{A}(\theta)_d - (-1)^d \frac{m\pi}{2} \overset{\infty}{B}(\theta)_d]\ U_{mn}(\tau)_d \right.$$

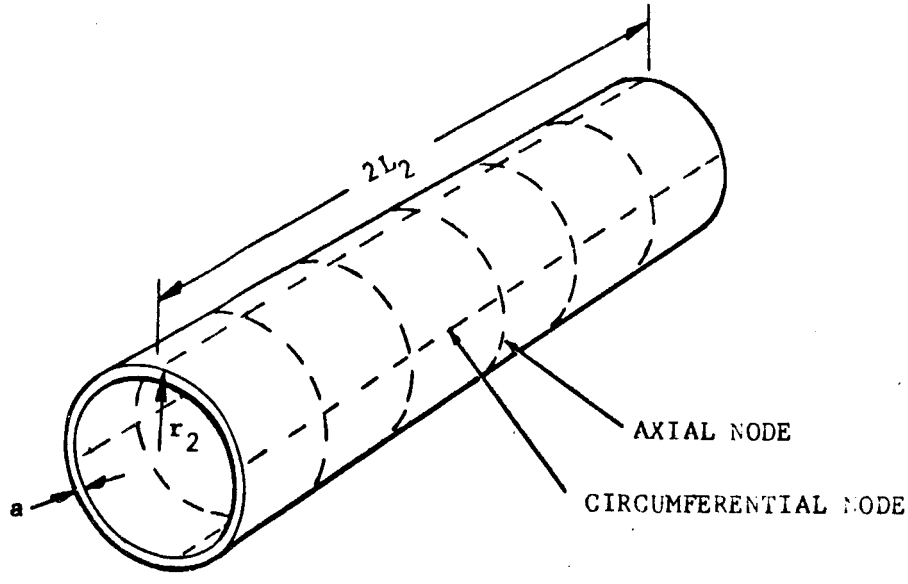
$$\left. + (-1)^d \frac{\nu L}{R} \overset{\infty}{B}(\theta)_d\ [n\ V_{mn}(\tau)_d - W_{mn}(\tau)_d] \right)$$

$$+ \frac{(1+\nu)\lambda}{2} \left(\sum_{d=1}^{2} [G_d'(\bar{\xi})\ \overset{\infty}{g}_{mn}(\bar{\xi},\theta)_d + G_d(\bar{\xi})\ \overset{\infty}{g}_{mn}'(\bar{\xi},\theta)_d]\ V_{mn}(\tau)_d \right.$$

$$\left. + g'^*(\bar{\xi},\tau)\ \cos\theta \right)$$



(a) CYLINDER 1 NODAL ARRANGEMENT FOR $m_1 = 4$, $n_1 = 3$



(b) CYLINDER 2 NODAL ARRANGEMENT FOR $m_2 = 6$, $n_2 = 2$

Figure 4 NODAL ARRANGEMENT OF THE TWO CYLINDERS THAT FORM THE COMPOSITE STRUCTURE SHOWN IN FIGURE 1

$$- \nu\lambda \left(\sum_{d=1}^{2} [G_d'(\bar{\xi})\ h_{mn}(\bar{\xi},\theta)_d + G_d(\bar{\xi})\ h_{mn}'(\bar{\xi},\theta)_d]\ W_{mn}(\tau)_d \right.$$

$$\left. + \ g'^*(\bar{\xi},\tau)\cos\theta \right) \Bigg] \ G_c(\bar{\xi})\ f_{jk}(\bar{\xi},\theta)_c\ d\bar{\xi}d\theta \qquad (5.2\text{-}2)$$

$$= \int_{L_{1d}}^{L_{2d}} \int_{o}^{2\pi} \sum_{m=o}^{M} \sum_{n=o}^{N} \left[(1-\nu^2)\mu \left(\sum_{d=1}^{2} [G_d(\bar{\xi})\ f_{mn}(\bar{\xi},\theta)_d - A(\theta)_d \right.\right.$$

$$- (-1)^d \frac{m\pi}{2} B(\theta)_d]\ \ddot{U}_{mn}(\tau)_d$$

$$\left.\left. + (-1)^d \frac{\nu L}{R} B(\theta)_d\ [n\ \ddot{V}_{mn}(\tau)_d - \ddot{W}_{mn}(\tau)_d] \right) \quad + F_1(\bar{\xi},\theta,\tau) \right]$$

$$G_c(\bar{\xi})\ f_{jk}(\bar{\xi},\theta)_c\ d\bar{\xi}d\theta$$

$$c=1 \begin{cases} j=0,1,2,\ldots,M_1 \\ k=0,1,2,\ldots,N_1 \end{cases} \qquad c=2 \begin{cases} j=0,1,2,\ldots,M_2 \\ k=0,1,2,\ldots,N_2 \end{cases}$$

$$d=1 \begin{cases} m=m_1=0,1,2,\ldots,M_1 \\ n=n_1=0,1,2,\ldots,N_1 \\ L_{1d} = L_{11} = a \\ L_{2d} = L_{21} = c \end{cases} \qquad d=2 \begin{cases} m=m_2=0,1,2,\ldots,M_2 \\ n=n_2=0,1,2,\ldots,N_2 \\ L_{1d} = L_{12} = c \\ L_{2d} = L_{22} = b \end{cases}$$

where

$$A(\theta)_1 = f_{mn}(a,\theta) - f_{mn}(c,\theta)$$

$$A(\theta)_2 = f_{mn}(c,\theta) - f_{mn}(b,\theta)$$

$$B(\theta)_1 = \frac{2L_1}{m\pi L} \left[f_{mn}\left(\frac{d_1-d_2}{L_1},\ \theta\right) - f_{mn}\left(\frac{2d_1}{L_1},\ \theta\right) \right] \qquad (5.2\text{-}3)$$

$$B(\theta)_2 = \frac{2L_2}{m\pi L} \left[f_{mn}\left(\frac{d_1+d_3-L_1}{L_2},\ \theta\right) - f_{mn}\left(\frac{2d_1-L_1}{L_2},\ \theta\right) \right]$$

$$\int_{L_{1d}}^{L_{2d}} \int_{0}^{2\pi} \sum_{m=0}^{M} \sum_{n=0}^{N} \left[\frac{(1+\nu)\lambda}{2} \sum_{d=1}^{2} \left[G_d(\bar{\xi})\, \overset{\circ}{f}_{mn}(\bar{\xi},\theta)_d + G_d(\bar{\xi})\, \overset{\circ}{f}'_{mn}(\bar{\xi},\theta)_d \right] U_{mn}(\tau)_d \right.$$

$$+ \frac{(1-\nu)}{2} \left(\sum_{d=1}^{2} \left[G''_d(\bar{\xi})\, g_{mn}(\bar{\xi},\theta)_d + 2G'_d(\bar{\xi})\, g'_{mn}(\bar{\xi},\theta)_d + G_d(\bar{\xi})\, g''_{mn}(\bar{\xi},\theta)_d \right] V_{mn}(\tau)_d \right.$$

$$\left. \left. + g''^{*}(\bar{\xi},\tau)\, \sin\theta \right) \right.$$

$$\left. + \lambda^2 \sum_{d=1}^{2} G_d(\bar{\xi}) \left[\overset{\circ}{g}_{mn}(\bar{\xi},\theta)_d\, V_{mn}(\tau)_d - \overset{\circ}{h}_{mn}(\bar{\xi},\theta)_d\, W_{mn}(\tau)_d \right] \right]$$

$$G_c(\bar{\xi})\, f_{jk}(\bar{\xi},\theta)_c\, d\bar{\xi} d\theta$$

$$= \int_{L_{1d}}^{L_{2d}} \int_{0}^{2\pi} \sum_{m=0}^{M} \sum_{n=0}^{N} \left[(1-\nu^2)\mu \left(\sum_{d=1}^{2} G_d(\bar{\xi})\, g_{mn}(\bar{\xi},\theta)_d\, \ddot{V}_{mn}(\tau)_d + \ddot{g}^{*}(\bar{\xi},\tau)\, \sin\theta \right) \right.$$

$$\left. + F_2(\bar{\xi},\theta,\tau) \right] G_c(\bar{\xi})\, g_{jk}(\bar{\xi},\theta)_c\, d\bar{\xi} d\theta \qquad (5.2\text{-}4)$$

$$c=1 \begin{cases} j=0,1,2,\ldots,M_1 \\ k=0,1,2,\ldots,N_1 \end{cases} \qquad c=2 \begin{cases} j=0,1,2,\ldots,M_2 \\ k=0,1,2,\ldots,N_2 \end{cases}$$

$$d=1 \begin{cases} m=m_1=0,1,2,\ldots,M_1 \\ n=n_1=0,1,2,\ldots,N_1 \\ L_{1d} = L_{11} = a \\ L_{2d} = L_{21} = c \end{cases} \qquad d=2 \begin{cases} m=m_2=0,1,2,\ldots,M_2 \\ n=n_2=0,1,2,\ldots,N_2 \\ L_{1d} = L_{12} = c \\ L_{2d} = L_{22} = b \end{cases}$$

$$\int_{L_{1d}}^{L_{2d}} \int_{o}^{2\pi} \sum_{m=o}^{M} \sum_{n=o}^{N} \left[\nu\lambda \left(\sum_{d=1}^{2} [G_d'(\bar{\xi})\ f_{mn}(\bar{\xi},\theta)_d + G_d(\bar{\xi})\ f_{mn}'(\bar{\xi},\theta)_d]\ U_{mn}(\tau)_d \right.\right.$$

$$\left. - \frac{1-\nu^2}{Eh} \frac{T_o}{2} (1 - \gamma\cos\bar{\Omega}\tau)\ [1 - (\bar{\xi} - \frac{d_3}{L} + 1)] \right)$$

$$+ \lambda^2 \sum_{d=1}^{2} G_d(\bar{\xi})\ [\overset{\circledast}{g}_{mn}(\bar{\xi},\theta)_d\ V_{mn}(\tau)_d - h_{mn}(\bar{\xi},\theta)_d\ W_{mn}(\tau)_d]$$

$$\left. - \frac{\sigma^2}{12} \bar{\nabla}^4 \left(\sum_{d=1}^{2} [G_d(\bar{\xi})\ h_{mn}(\bar{\xi},\theta)_d\ W_{mn}(\tau)_d] + g^*(\bar{\xi},\tau)\ \cos\theta \right) \right] G_c(\bar{\xi})\ h_{jk}(\bar{\xi},\theta)_c\ d\bar{\xi}d\theta$$

$$= \int_{L_{1d}}^{L_{2d}} \int_{o}^{2\pi} \sum_{m=o}^{M} \sum_{n=o}^{N} \left[(1-\nu^2)\mu \left(\sum_{d=1}^{2} G_d(\bar{\xi})\ h_{mn}(\bar{\xi},\theta)_d\ \ddot{W}_{mn}(\tau)_d + \ddot{g}^*(\bar{\xi},\tau)\ \cos\theta \right) \right.$$

$$\left. + F_3(\bar{\xi},\theta,\tau) \right] G_c(\bar{\xi})\ h_{jk}(\bar{\xi},\theta)_c\ d\bar{\xi}d\theta \qquad (5.2\text{-}5)$$

$$c=1 \begin{cases} j=0,1,2,\ldots,M_1 \\ k=0,1,2,\ldots,N_1 \end{cases} \qquad c=2 \begin{cases} j=0,1,2,\ldots,M_2 \\ k=0,1,2,\ldots,N_2 \end{cases}$$

$$d=1 \begin{cases} m=m_1=0,1,2,\ldots,M_1 \\ n=n_1=0,1,2,\ldots,N_1 \\ L_{1d} = L_{11} = a \\ L_{2d} = L_{21} = d \end{cases} \qquad d=2 \begin{cases} m=m_2=0,1,2,\ldots,M_2 \\ n=n_2=0,1,2,\ldots,N_2 \\ L_{1d} = L_{12} = c \\ L_{2d} = L_{22} = b \end{cases}$$

where in eqs. (5.2-3, 4, 5) the notation is used

$$(') = \frac{\partial()}{\partial\bar{\xi}}\ , \quad (\overset{\circledast}{}) = \frac{\partial()}{\partial\theta}\ , \quad (\dot{}) = \frac{\partial()}{\partial\tau} \qquad (5.2\text{-}6)$$

By requiring $f_{mn}(\bar{\xi},\theta)_d$, $g_{mn}(\bar{\xi},\theta)_d$ and $h_{mn}(\bar{\xi},\theta)_d$ to be orthogonal over the region of the shell we can reduce the expressions to a set of ordinary differential equations in time only.

5.3 Edge Conditions

Since the composite structure is considered as having free-free end conditions, the set of edge conditions is as follows:

$$\bar{v}'(\bar{\xi},\theta,\tau)\Big|_{\bar{\xi}=a} = \bar{v}'(\bar{\xi},\theta,\tau)\Big|_{\bar{\xi}=b} = 0 \qquad \text{(a)}$$

$$\bar{w}'(\bar{\xi},\theta,\tau)\Big|_{\bar{\xi}=a} = \bar{w}'(\bar{\xi},\theta,\tau)\Big|_{\bar{\xi}=b} = 0 \qquad \text{(b)}$$

$$M_{\bar{\xi}\bar{\xi}}(\bar{\xi},\theta,\tau)\Big|_{\bar{\xi}=a} = M_{\bar{\xi}\bar{\xi}}(\bar{\xi},\theta,\tau)_{\bar{\xi}=b} = 0 \qquad \text{(c)} \quad (5.3\text{-}1)$$

$$N_{\bar{\xi}\bar{\xi}}(\bar{\xi},\theta,\tau)_{\bar{\xi}=a} = T(\tau) \qquad \text{(d)}$$

$$N_{\bar{\xi}\bar{\xi}}(\bar{\xi},\theta,\tau)_{\bar{\xi}=b} = 0 \qquad \text{(e)}$$

A selection of $f_{mn}(\bar{\xi},\theta)_d$, $g_{mn}(\bar{\xi},\theta)_d$ and $h_{mn}(\bar{\xi},\theta)_d$, under the influence of the shifted step functions $G(\bar{\xi})_d$ for d = 1,2, that assures satisfaction of the edge conditions of eqs. (5.3-1) is as follows:

For d=1 (cylinder 1 of $2L_1$ in length)

$$G_1(\bar{\xi}) = [H(\bar{\xi} + \frac{d_2}{L}) - H(\bar{\xi} - \frac{d_1}{L})] \qquad \text{(a)}$$

$$f_{mn}(\bar{\xi},\theta)_1 = \cos\frac{m_1\pi}{2}\left(\frac{L}{L_1}\bar{\xi} - r_1 + 1\right)\cos n_1\theta \qquad (b)$$

$$g_{mn}(\bar{\xi},\theta)_1 = \sin\frac{m_1\pi}{2}\left(\frac{L}{L_1}\bar{\xi} - r_1 + 1\right)\sin n_1\theta \qquad (c) \quad (5.3\text{-}2)$$

$$h_{mn}(\bar{\xi},\theta)_1 = \sin\frac{m_1\pi}{2}\left(\frac{L}{L_1}\bar{\xi} - r_1 + 1\right)\cos n_1\theta \qquad (d)$$

$$r_1 = \frac{L_1 - d_1}{L_1} \qquad (e)$$

$$m = m_1 = 0,1,2,\ldots,M_1$$

$$n = n_1 = 0,1,2,\ldots,N_1$$

For d=2 (cylinder 2 of $2L_2$ in length)

$$G_2(\bar{\xi}) = \left[H\left(\bar{\xi} - \frac{d_1}{L}\right) - H\left(\bar{\xi} - \frac{d_3}{L}\right)\right] \qquad (a)$$

$$f_{mn}(\bar{\xi},\theta)_2 = \cos\frac{m_2\pi}{2}\left(\frac{L}{L_2}\bar{\xi} + r_2 + 1\right)\cos n_2\theta \qquad (b)$$

$$g_{mn}(\bar{\xi},\theta)_2 = \sin\frac{m_2\pi}{2}\left(\frac{L}{L_2}\bar{\xi} + r_2 + 1\right)\sin n_2\theta \qquad (c) \quad (5.3\text{-}3)$$

$$h_{mn}(\bar{\xi},\theta)_2 = \sin\frac{m_2\pi}{2}\left(\frac{L}{L_2}\bar{\xi} + r_2 + 1\right)\cos n_2\theta \qquad (d)$$

$$r_2 = \frac{d_1 - (L_1 + L_2)}{L_2} \qquad (e)$$

$$m = m_2 = 0,1,2,\ldots,M_2$$

$$n = n_2 = 0,1,2,\ldots,N_2$$

The superimposed displacements from consideration of the loading and the transverse beam action are

$$u^*(\bar{\xi},\theta,\tau) = f^*(\bar{\xi},\tau)$$

$$= -\frac{(1-\nu^2)}{Eh}\frac{N_{\bar{\xi}}\bar{\xi}}{2}[\bar{\xi} - \frac{1}{2}(\bar{\xi} - b + 1)^2] \qquad (a)$$

$$v^*(\bar{\xi},\theta,\tau) = g^*(\bar{\xi},\tau)\sin\theta \qquad (b) \quad (5.3\text{-}4)$$

$$w^*(\bar{\xi},\theta,\tau) = h^*(\bar{\xi},\tau)\cos\theta = g^*(\bar{\xi},\tau)\cos\theta \qquad (c)$$

Using the orthogonal properties of eqs. (5.3-2,-3) the system of equations (5.2-2,-3,-4) reduce to two sets of differential equations, one set for each thin-walled cylinder. Consideration of the indices j and k will then reduce the two sets of equations to differential equations in time τ only, which is the desired objective.

5.4 Consideration of Indices

5.4.1 For d=1, $j=m_1=0$, $k=n_1=0$

d=2, $j=m_2=0$, $k=n_2=0$

$$(1-\nu^2)\mu\,(L_{2d} - L_{1d})\,\ddot{U}_{oo}(\tau)_d = -\int_{L_{1d}}^{L_{2d}} F_1(\bar{\xi},\theta,\tau)d\bar{\xi}$$

$$+ \frac{(1-\nu^2)}{Eh}\frac{T_o}{2}(1 - \gamma\cos\bar{\Omega}\tau)(L_{2d} - L_{1d})$$

$$+ \mu\frac{(1-\nu^2)^2}{Eh}\frac{T_o}{2}\gamma\bar{\Omega}^2\cos\bar{\Omega}\tau[\frac{\bar{\xi}^2}{2} - \frac{1}{6}(\bar{\xi} - b + 1)]\Big|_{L_{1d}}^{L_{2d}} \qquad (5.4\text{-}1)$$

which can be simplified to

$$\ddot{U}_{oo}(\tau)_d = P_{oo}(\tau)_d \tag{5.4-2}$$

where

$$P_{oo}(\tau)_d = -\frac{1}{(L_{2d}-L_{1d})}\ \frac{1}{(1-\nu^2)\mu}\int_{L_{1d}}^{L_{2d}} F_1(\bar{\xi},\theta,\tau)\,d\bar{\xi}$$

$$+\frac{T_o}{2Eh\mu}(1-\gamma\cos\bar{\Omega}\tau)$$

$$+\frac{(1-\nu^2)}{Eh}\frac{T_o}{(L_{2d}-L_{1d})}\ \gamma\bar{\Omega}^2\cos\bar{\Omega}\tau\ \left[\frac{\bar{\xi}^2}{2}-\frac{1}{6}(\bar{\xi}-b+1)\right]\Bigg|_{L_{1d}}^{L_{2d}}$$

(5.4-3)

5.4.2 For $d=1$, $j=m_1=0$, $k=n_1=1,2,\ldots,M_1$

$d=2$, $j=m_2=0$, $k=n_2=1,2,\ldots,N_2$

$$\ddot{U}_{ok}(\tau)_d + \frac{\lambda^2k^2}{2(1+\nu)\mu}\ U_{ok}(\tau)_d = -\frac{1}{\pi(L_{2d}-L_{1d})\mu(1-\nu^2)}$$

$$\left[\int_{L_{1d}}^{L_{2d}}\int_{o}^{2\pi} F_1(\bar{\xi},\theta,\tau)\ G_d(\bar{\tau})\ \cos k\theta\ d\bar{\xi}d\theta\right]$$

$$+\frac{\lambda\delta_{1k}}{2(1+\nu)\mu(L_{2d}-L_{1d})}\int_{L_{1d}}^{L_{2d}} g'^{*}(\bar{\xi},\tau)\ G_d(\bar{\xi})\,d\bar{\xi} \tag{5.4-4}$$

which can be simplified to

$$\ddot{U}_{ok}(\tau)_d + (\omega_{ok})_d^2\ U_{ok}(\tau)_d = P_{ok}(\tau)_d \tag{5.4-5}$$

where $P_{ok}(\tau)_d$ is the r.h.s.of eqs. (5.4-4)

5.4.3 For d=1, $j=m_1=1,2,\ldots,M_1$; $k=n_1=0$

d=2, $j=m_1=1,2,\ldots,M_2$; $k=n_2=0$

$$(a_{jo})_d \ddot{U}_{jo}(\tau)_d + (b_{jo})_d \ddot{W}_{jo}(\tau)_d + (c_{jo})_d U_{jo}(\tau)_d + (d_{jo})_d W_{jo}(\tau)_d = [P_{jo}(\tau)_d]_1 \qquad (a)$$

$$(h_{jo})_d U_{jo}(\tau)_d + (j_{jo})_d \ddot{W}_{jo}(\tau)_d + (k_{jo})_d W_{jo}(\tau)_d = [P_{jo}(\tau)_d]_2 \qquad (b) \qquad (5.4\text{-}6)$$

where

$$[P_{jo}(\tau)_d]_1 = -\frac{1}{2\mu\pi(1-\nu^2)} \int_0^{2\pi} \int_{L_{1d}}^{L_{2d}} F_1(\bar{\xi},\theta,\tau) \cos\frac{j\pi}{2}\left(\frac{L}{L_d}\bar{\xi} + (-1)^d r_d + 1\right) d\bar{\xi}d\theta + \frac{1}{\mu Eh}\frac{T_o}{2}(1 - \gamma\cos\bar{\Omega}\tau)(e_{jo})_d + \frac{(1-\nu^2)}{Eh}\frac{T_o}{2}\gamma\bar{\Omega}^2\cos\bar{\Omega}\tau\,[(f_{jo})_d - (g_{jo})_d] \qquad (a)$$

$$[P_{jo}(\tau)_d]_2 = -\frac{1}{2\mu\pi(1-\nu^2)} \int_0^{2\pi} \int_{L_{1d}}^{L_{2d}} F_3(\bar{\xi},\theta,\tau) \sin\frac{j\pi}{2}\left(\frac{L}{L_d}\bar{\xi} + (-1)^d r_d + 1\right) d\bar{\xi}d\theta + \frac{1}{\mu Eh}\frac{T_o}{2}(1 - \gamma\cos\bar{\Omega}\tau)[(\ell_{jo})_d + (m_{jo})_d] \qquad (b) \qquad (5.4\text{-}7)$$

and the terms $(a_{jo})_d, \ldots, (m_{jo})_d$ are defined in APPENDIX A.

5.4.4 For $d=1$, $j=m_1=1,2,\ldots,M_1$, $k=n_1=1,2,\ldots,N_1$

$d=2$, $j=m_2=1,2,\ldots,M_2$, $k=n_2=1,2,\ldots,N_2$

$$(a_{jk})_d\,\ddot{U}_{jk}(\tau)_d + (b_{jk})_d\,\ddot{V}_{jk}(\tau)_d + (c_{jk})_d\,\ddot{W}_{jk}(\tau)_d + (d_{jk})_d\,U_{jk}(\tau)_d$$

$$+ (e_{jk})_d\,V_{jk}(\tau)_d + (f_{jk})_d\,W_{jk}(\tau)_d = [Q_{jk}(\tau)_d]_1 \qquad \text{(a)}$$

$$(g_{jk})_d\,U_{jk}(\tau)_d + (h_{jk})_d\,\ddot{V}_{jk}(\tau)_d + (j_{jk})_d\,V_{jk}(\tau)_d + (k_{jk})_d\,W_{jk}(\tau)_d$$

$$= [Q_{jk}(\tau)_d]_2 \qquad \text{(b)} \qquad (5.4\text{-}8)$$

$$(\ell_{jk})_d\,U_{jk}(\tau)_d + (m_{jk})_d\,V_{jk}(\tau)_d + (n_{jk})_d\,\ddot{W}_{jk}(\tau)_d + (p_{jk})_d\,W_{jk}(\tau)_d$$

$$= [Q_{jk}(\tau)_d]_3 \qquad \text{(c)}$$

where

$$[Q_{jk}(\tau)_d]_1 = -\frac{1}{\mu\pi(1-\nu^2)} \int_0^{2\pi} \int_{L_{1d}}^{L_{2d}} F_1(\bar{\xi},\theta,\tau)\,\cos\frac{j\pi}{2}\left(\frac{L}{L_d}\,\bar{\xi} + (-1)^d\,r_d + 1\right)$$

$$\cos k\theta\; d\bar{\xi}d\theta$$

$$+ \frac{\lambda\delta_{1k}}{2\mu(1+\nu)} \int_{L_{1d}}^{L_{2d}} g^*(\bar{\xi},\tau)\,\cos\frac{j\pi}{2}\left(\frac{L}{L_d}\,\bar{\xi} + (-1)^d\,r_d + 1\right)d\bar{\xi}$$

$$\text{(a)} \qquad (5.4\text{-}9)$$

$$[Q_{jk}(\tau)_d]_2 = -\frac{1}{\mu\pi(1-\nu^2)} \int_0^{2\pi} \int_{L_{1d}}^{L_{2d}} F_2(\bar{\xi},\theta,\tau) \sin\frac{j\pi}{2}\left(\frac{L}{L_d}\bar{\xi} + (-1)^d r_d + 1\right)$$

$$\sin k\theta \, d\bar{\xi}d\theta$$

$$- \delta_{1k} \int_{L_{1d}}^{L_{2d}} \ddot{g}^*(\bar{\xi},\tau) \sin\frac{j\pi}{2}\left(\frac{L}{L_d}\bar{\xi} + (-1)^d r_d + 1\right)d\bar{\xi}$$

$$+ \frac{\delta_{1k}}{2(1+\nu)\mu} \int_{L_{1d}}^{L_{2d}} {}''g^*(\bar{\xi},\tau) \sin\frac{j\pi}{2}\left(\frac{L}{L_d}\bar{\xi} + (-1)^d r_d + 1\right)d\bar{\xi} \quad (b) \quad (5.4\text{-}9)$$

$$[Q_{jk}(\tau)_d]_3 = -\frac{1}{\mu\pi(1-\nu^2)} \int_0^{2\pi} \int_{L_{1d}}^{L_{2d}} F_3(\bar{\xi},\theta,\tau) \sin\frac{j\pi}{2}\left(\frac{L}{L_d}\bar{\xi} + (-1)^d r_d + 1\right)$$

$$\cos k\theta \, d\bar{\xi} \, d\theta$$

$$- \delta_{1k} \int_{L_{1d}}^{L_{2d}} \ddot{g}^*(\bar{\xi},\tau) \sin\frac{j\pi}{2}\left(\frac{L}{L_d}\bar{\xi} + (-1)^d r_d + 1\right)d\bar{\xi}$$

$$+ \frac{\lambda^2 \delta_{1k}}{\mu(1-\nu^2)} \int_{L_{1d}}^{L_{2d}} g^*(\bar{\xi},\tau) \sin\frac{j\pi}{2}\left(\frac{L}{L_d}\bar{\xi} + (-1)^d r_d + 1\right)d\bar{\xi}$$

$$- \frac{\sigma^2 \delta_{1k}}{12\mu(1-\nu^2)} \int_{L_{1d}}^{L_{2d}} {}''''g^*(\bar{\xi},\tau) \sin\frac{j\pi}{2}\left(\frac{L}{L_d}\bar{\xi} + (-1)^d r_d + 1\right)d\bar{\xi}$$

$$+ \frac{2\sigma^2\lambda^2 \delta_{1k}}{12(1-\nu^2)\mu} \int_{L_{1d}}^{L_{2d}} {}''g^*(\bar{\xi},\tau) \sin\frac{j\pi}{2}\left(\frac{L}{L_d}\bar{\xi} + (-1)^d r_d + 1\right)d\bar{\xi}$$

$$- \frac{\sigma^2\lambda^4\delta_{1k}}{12(1-\nu^2)\mu} \int_{L_{1d}}^{L_{2d}} g^*(\bar{\xi},\tau) \sin\frac{j\pi}{2}\left(\frac{L}{L_d}\bar{\xi} + (-1)^d r_d + 1\right)d\bar{\xi} \quad (c)$$

and the terms $(a_{jk})_d,\ldots,(p_{jk})_d$ are defined in APPENDIX A.

6.0 SOLUTION OF THE ORDINARY DIFFERENTIAL EQUATIONS

6.1 Summary of the Differential Equations

A complete summary of the ordinary differential equations resulting from application of the Galerkin method to eqs. (3.0-2a) through (3.0-2c) is as follows:

$$\ddot{U}_{oo}(\tau)_d = P_{oo}(\tau)_d \tag{6.1-1}$$

$$\ddot{U}_{ok}(\tau)_d + (\omega_{ok})_d^2 \, U_{ok}(\tau) = P_{ok}(\tau)_d \tag{6.1-2}$$

$$(a_{jo})_d \, \ddot{U}_{jo}(\tau) + (b_{jo})_d \, \ddot{W}_{jo}(\tau)_d + (c_{jo})_d \, U_{jo}(\tau)_d + (d_{jo})_d \, W_{jo}(\tau)_d = [P_{jo}(\tau)_d]_1 \quad \text{(a)}$$

(6.1-3)

$$(h_{jo})_d \, U_{jo}(\tau)_d + (j_{jo})_d \, \ddot{W}_{jo}(\tau)_d + (k_{jo})_d \, W_{jo}(\tau)_d = [P_{jo}(\tau)_d]_2 \quad \text{(b)}$$

$$(a_{jk})_d \, \ddot{U}_{jk}(\tau)_d + (b_{jk})_d \, \ddot{V}_{jk}(\tau)_d + (c_{jk})_d \, \ddot{W}_{jk}(\tau)_d + (d_{jk})_d \, U_{jk}(\tau)_d + (e_{jk})_d \, V_{jk}(\tau)_d + (f_{jk})_d \, W_{jk}(\tau)_d = [Q_{jk}(\tau)_d]_1 \quad \text{(a)}$$

$$(g_{jk})_d \, U_{jk}(\tau)_d + (h_{jk})_d \, \ddot{V}_{jk}(\tau)_d + (j_{jk})_d \, V_{jk}(\tau)_d + (k_{jk})_d \, W_{jk}(\tau)_d = [Q_{jk}(\tau)_d]_2 \quad \text{(b)} \tag{6.1-4}$$

$$(\ell_{jk})_d\, U_{jk}(\tau)_d + (m_{jk})_d\, V_{jk}(\tau)_d + (n_{jk})_d\, \ddot{W}_{jk}(\tau)_d + (p_{jk})_d\, W_{jk}(\tau)_d$$

$$= [Q_{jk}(\tau)_d]_3 \qquad (c)$$

where the coefficients $(a_{jo})_d,\ldots,(k_{jo})_d$ and $(a_{jk})_d,\ldots,(n_{jk})_d$ are defined in APPENDIX A and the forcing terms $P_{oo}(\tau)_d$, $P_{ok}(\tau)_d$, $[P_{jo}(\tau)_d]_1$, $[P_{jo}(\tau)_d]_2$, $[Q_{jk}(\tau)_d]_1$, $[Q_{jk}(\tau)_d]_2$ and $[Q_{jk}(\tau)_d]_3$ are given in Section 5.0.

6.2 Solution of the Differential Equations

6.2.1 Solution of Equations (6.1-1)

Since the solution of equations (6.1-1) is non-periodic it is an inadmissible solution in this investigation.

6.2.2 Solution of Equation (6.1-2)

Substituting the expressions for $P_{ok}(\tau)_d$ and observing the subscript values of d=1,2, equations (6.1-2) become

$$\ddot{U}_{ok}(\tau)_1 + (\omega_{ok})_1^2\, U_{ok}(\tau)_1 = (\delta_{1k})_1 \left[\frac{\ddot{\psi}(\tau)}{\lambda} + \frac{\lambda}{2(1+\nu)\mu(c-a)} \int_a^c g'^*(\bar{\xi},\tau)\, d\bar{\xi}\right]$$

$$U_{ok}(\tau)_2 + (\omega_{ok})_2^2\, U_{ok}(\tau)_2 = (\delta_{1k})_2 \left[\frac{\ddot{\psi}(\tau)}{\lambda} + \frac{\lambda}{2(1+\nu)(b-c)} \int_c^b g'^*(\bar{\xi},\tau)\, d\bar{\xi}\right] \qquad (6.2\text{-}1)$$

Substituting $\ddot{\psi}(\tau)$ from Section 4.2 and $g'^*(\bar{\xi},\tau)$ from Section 4.1, and performing Laplace transformations eqs. (6.2-1) become

$$\bar{U}_{ok}(\tau)_1 \left[s^2 + (\omega_{ok})_1^2\right] = s(u_{ok})_1 - \frac{(\delta_{1k})_1}{\lambda} \mathcal{L}\left\{\sum_{i=1}^{I} A_i \Omega_i^2 \cos\Omega_i\tau\right\}$$

$$+ \frac{(\delta_{1k})_1\lambda}{2(1+\nu)\mu} \mathcal{L}\{q_B(\tau)\}$$

$$+ \frac{(\delta_{1k})_1\lambda}{2(1+\nu)\mu(c-a)} \sum_{n=1}^{N} \phi_n(\bar{\xi})\Big|_a^c \mathcal{L}\{q_n(\tau)\} \qquad \text{(a)}$$

(6.2-2)

$$U_{ok}(\tau)_2 \left[s^2 + (\omega_{ok})_2^2\right] = s(u_{ok})_2 - \frac{(\delta_{1k})_2}{\lambda} \mathcal{L}\left\{\sum_{i=1}^{I} A_i \Omega_i \cos\Omega_i\tau\right\}$$

$$+ \frac{(\delta_{1k})_2\lambda}{2(1+\nu)\mu} \mathcal{L}\{q_B(\tau)\}$$

$$+ \frac{(\delta_{1k})_2\lambda}{2(1+\nu)\mu(b-c)} \sum_{n=1}^{N} \phi_n(\bar{\xi})\Big|_c^b \mathcal{L}\{q_n(\tau)\} \qquad \text{(b)}$$

Substituting $q_B(\tau)$ and $q_n(\tau)$ and employing the convolution theorem we arrive at the following

$$U_{ok}(\tau)_1 = (u_{ok})_1 \cos(\omega_{ok})_1\tau + \frac{(\delta_{1k})_1}{\lambda} \sum_{i=1}^{I} \frac{A_i \Omega_i^2}{\Omega_i^2 - (\omega_{ok})_1^2} \left[\cos\Omega_i\tau - \cos(\omega_{ok})_1\tau\right]$$

$$+ \frac{(\delta_{1k})_1\lambda}{2(1+\nu)\mu} \sum_{s=-S}^{+S} \sum_{n=1}^{N} \frac{1}{(\alpha_r+s)^2 \bar{\Omega}^2 - (\omega_{ok})_1^2} \left[C_B^{(s)} + \frac{C_n^{(s)}}{(c-a)} \phi_n(\bar{\xi})\Big|_a^c\right] \left[\cos(\omega_{ok})_1\tau + i\frac{(\alpha_r+s)}{(\omega_{ok})_1} \bar{\Omega} \sin(\omega_{ok})_1\tau - e^{i(\alpha_r+s)\bar{\Omega}\tau}\right] \qquad \text{(a)}$$

(6.2-3)

$$U_{ok}(\tau)_2 = (u_{ok})_2 \cos(\omega_{ok})_2\tau + \frac{(\delta_{1k})_2}{\lambda} \sum_{i=1}^{I} \frac{A_i \Omega_i^2}{\Omega_i^2-(\omega_{ok})_2^2} [\cos\Omega_i\tau - \cos(\omega_{ok})_2\tau]$$

$$+ \frac{(\delta_{1k})_2\lambda}{2(1+\nu)\mu} \sum_{s=-S}^{+S} \sum_{n=1}^{N} \frac{1}{(\alpha_r+s)^2\bar{\Omega}^2-(\omega_{ok})_2^2} \left[C_B^{(s)} + \frac{C_n^{(s)}}{(b-c)} \phi_n(\bar{\xi}) \Big|_c^b \right]$$

$$\left[\cos(\omega_{ok})_2\tau + i \frac{(\alpha_r+s)}{(\omega_{ok})_2} \bar{\Omega} \sin(\omega_{ok})_2\tau - e^{i(\alpha_r+s)\bar{\Omega}\tau} \right] \qquad (b)$$

6.2.3 Solution of Equations (6.1-3)

Performing Laplace transformation techniques on eqs. (6.1-3) and solving for $\bar{U}_{jo}(\tau)_d$ and $\bar{W}_{jo}(\tau)_d$ from the resulting transformation, we can write

$$\bar{U}_{jo}(\tau)_d = \frac{\begin{vmatrix} [\bar{P}_{jo}(\tau)_d]_1 + s[(a_{jo})_d(u_{jo})_d + (b_{jo})_d(w_{jo})_d] & (b_{jo})_d s^2 + (d_{jo})_d \\ [\bar{P}_{jo}(\tau)_d]_2 + s(j_{jo})_d (w_{jo})_d & (j_{jo})_d s^2 + (k_{jo})_d \end{vmatrix}}{\Delta_{jo}(s)_d} \qquad (a)$$

(6.2-4)

$$\bar{W}_{jo}(\tau)_d = \frac{\begin{vmatrix} (a_{jo})_d s^2 + (c_{jo})_d & [\bar{P}_{jo}(\tau)_d]_1 + s[(a_{jo})_d(u_{jo})_d + (b_{jo})_d(w_{jo})_d] \\ (h_{jo})_d & [\bar{P}_{jo}(\tau)_d]_2 + s(j_{jo})_d (w_{jo})_d \end{vmatrix}}{\Delta_{jo}(s)_d} \qquad (b)$$

where

$$\Delta_{jo}(s)_d = \begin{vmatrix} (a_{jo})_d s^2 + (c_{jo})_d & (b_{jo})_d s^2 + (d_{jo})_d \\ (h_{jo})_d & (j_{jo})_d s^2 + (k_{jo})_d \end{vmatrix} = [s^2 + (\bar{\omega}_{jo}^1)_d^2][s^2 + (\bar{\omega}_{jo}^2)_d^2] \qquad (6.2-5)$$

All barred terms in eqs. (6.2-4) represent the Laplace transform of the term and the terms $(u_{jo})_d$ and $(w_{jo})_d$ represent the initial displacements of $U_{jo}(\tau)_d$ and $W_{jo}(\tau)_d$, respectively.

The eigenvalues $(\omega^1_{jo})_d$ and $(\omega^2_{jo})_d$ of eq. (6.2-5) can be determined by

$$\left.\begin{matrix}(\bar{\omega}^1_{jo})_d \\ (\bar{\omega}^2_{jo})_d\end{matrix}\right\} = \frac{(b)_d}{2(a)_d} \mp \sqrt{(b)^2_d - 4(a)_d\,(c)_d} \qquad (6.2\text{-}6)$$

where

$$(a)_d = (a_{jo})_d\,(j_{jo})_d$$

$$(b)_d = (c_{jo})_d\,(j_{jo})_d + (a_{jo})_d\,(k_{jo})_d - (h_{jo})_d\,(b_{jo})_d \qquad (6.2\text{-}7)$$

$$(c)_d = (c_{jo})_d\,(k_{jo})_d - (h_{jo})_d\,(d_{jo})_d$$

Expanding the determinantal expressions of eqs. (6.2-4), rearranging and then applying the convolution theorem we can write

$$U_{jo}(\tau)_d = \mathcal{L}^{-1}\left\{ \bar{U}_{jo}(\tau)_d \right\}$$

$$= \sum_{m=1}^{2} \frac{(-1)^m}{[(\bar{\omega}^1_{jo})^2_d - (\bar{\omega}^2_{jo})^2_d]}$$

$$\cdot \Bigg| (j_{jo})_d \, [(a_{jo})_d \, (u_{jo})_d + (b_{jo})_d \, (w_{jo})_d][(a_3)_d - (\bar{\omega}^m_{jo})^2_d] \cos (\bar{\omega}^m_{jo})_d \tau$$

$$- (b_{jo})_d \, (j_{jo})_d \, (w_{jo})_d \, [(a_2)_d - (\bar{\omega}^m_{jo})^2_d] \cos (\bar{\omega}^m_{jo})_d \tau$$

$$+ \frac{(j_{jo})_d}{(\omega^m_{jo})_d} [(a_3)_d - (\bar{\omega}^m_{jo})^2_d] \int_0^\tau [P_{jo}(\eta)_d]_1 \sin (\bar{\omega}^m_{jo})_d \, (\tau-\eta) d\eta$$

$$- \frac{(b_{jo})_d}{(\omega^m_{jo})_d} [(a_2)_d - (\bar{\omega}^m_{jo})^2_d] \int_0^\tau [P_{jo}(\eta)_d]_2 \sin (\bar{\omega}^m_{jo})_d \, (\tau-\eta) d\eta \Bigg| \qquad (6.2\text{-}8)$$

$d=1,2$

$$\left.\begin{array}{l} j=j_1=1,2,\ldots,M_1 \\ k=k_1=0 \end{array}\right\} \quad d=1$$

$$\left.\begin{array}{l} j=j_2=1,2,\ldots,M_2 \\ k=k_2=0 \end{array}\right\} \quad d=2$$

$$W_{jo}(\tau)_d = \mathcal{L}^{-1}\left\{\bar{W}_{jo}(\tau)_d\right\}$$

$$= \sum_{m=1}^{2} \frac{(-1)^m}{[(\bar{\omega}_{jo}^1)_d^2 - (\bar{\omega}_{jo}^2)_d^2]}$$

$$\cdot \Bigg| (a_{jo})_d\,(j_{jo})_d\,(w_{jo})_d\,[(a_1)_d - (\bar{\omega}_{jo}^m)_d^2]\cos(\bar{\omega}_{jo}^m)_d\tau$$

$$- (h_{jo})_d\,[(a_{jo})_d\,(u_{jo})_d + (h_{jo})_d\,(w_{jo})_d]\cos(\bar{\omega}_{jo}^m)_d\tau$$

$$+ \frac{(a_{jo})_d}{(\bar{\omega}_{jo}^m)_d}[(a_1)_d - (\bar{\omega}_{jo}^m)_d^2]\int_0^\tau [P_{jo}(\eta)_d]_2 \sin(\bar{\omega}_{jo}^m)_d\,(\tau-\eta)\,d\eta$$

$$- \frac{(h_{jo})_d}{(\bar{\omega}_{jo}^m)_d}\int_0^\tau [P_{jo}(\eta)_d]_1 \sin(\bar{\omega}_{jo}^m)_d\,(\tau-\eta)\,d\eta \Bigg| \qquad (6.2\text{-}9)$$

$d=1,2$

$j=j_1=1,2,\ldots,M_1$, $k=k_1=0$ $\Big\}$ $d=1$

$j=j_2=1,2,\ldots,M_2$, $k=k_2=0$ $\Big\}$ $d=2$

where

$$(a_1)_d = \frac{(c_{jo})_d}{(a_{jo})_d} \qquad (a)$$

$$(a_2)_d = \frac{(d_{jo})_d}{(b_{jo})_d} \qquad (b) \qquad (6.2\text{-}10)$$

$$(a_3)_d = \frac{(k_{jo})_d}{(j_{jo})_d} \qquad (c)$$

Substituting for $[P_{jo}(\eta)_d]$ and $[P_{jo}(\eta)_d]_2$ in eqs. (6.2-8) and (6.2-9) and evaluating the convolution integrals, the expressions for $U_{jo}(\tau)_d$ and $W_{jo}(\tau)_d$ are written in final evaluated form as

$$U_{jo}(\tau)_d = \sum_{m=1}^{2} \frac{(-1)^m}{[(\bar{\omega}^1_{jo})^2_d - (\bar{\omega}^2_{jo})^2_d]}$$

$$\left[\left| (j_{jo})_d \left[(a_{jo})_d (u_{jo})_d + (b_{jo})_d (w_{jo})_d \right] \left[(a_3)_d - (\bar{\omega}^m_{jo})^2_d \right] \right.$$

$$\left. - (b_{jo})_d (j_{jo})_d (w_{jo})_d \left[(a_2)_d - (\bar{\omega}^m_{jo})^2_d \right] \right| \cos (\bar{\omega}^m_{jo})_d \tau$$

$$+ \left| \frac{(j_{jo})_d}{(\bar{\omega}^m_{jo})_d} \left[(a_3)_d - (\bar{\omega}^m_{jo})^2_d \right] \left[\frac{1}{\mu Eh} - 1 \right] \frac{T_o}{2} (e_{jo})_d \right.$$

$$\left. - \frac{(b_{jo})_d}{(\bar{\omega}^m_{jo})_d} \left[(a_2)_d - (\bar{\omega}^m_{jo})^2_d \right] \frac{T_o}{2\mu Eh} \left[(m_{jo})_d - \left(\frac{d_3}{L}\right) (r_{jn})_d \right] \right|$$

$$\cdot \left| \frac{1}{2(\bar{\omega}^m_{jo})_d} \left[1 - \cos (\bar{\omega}^m_{jo})_d \tau \right] \right| \qquad \text{(cont'd)}$$

$$+\left\{\frac{(j_{jo})_d}{(\bar{\omega}^m_{jo})_d}\left[(a_3)_d-(\bar{\omega}^m_{jo})^2_d\right]\left[\frac{1}{\mu Eh}-1\right]\frac{T_o\gamma}{2}(e_{jo})_d\right.$$

$$\left.-\frac{(b_{jo})_d}{(\bar{\omega}^m_{jo})_d}\left[(a_2)_d-(\bar{\omega}^m_{jo})^2_d\right]\frac{T_o\gamma}{2\mu Eh}\left[(m_{jo})_d-\frac{d_3}{L}(r_{jn})_d\right]\right\}$$

$$\left\{\frac{(\bar{\omega}^m_{jo})_d}{\bar{\Omega}^2-(\bar{\omega}^m_{jo})^2_d}\left[\cos\bar{\Omega}\tau-\cos(\bar{\omega}^m_{jo})_d\tau\right]\right\}$$

$$+\left\{\frac{(j_{jo})_d}{(\bar{\omega}^m_{jo})_d}\left[(a_3)_d-(\bar{\omega}^m_{jo})^2_d\right](f_{jo})_d+\frac{(b_{jo})_d}{(\bar{\omega}^m_{jo})_d}\left[(a_2)_d-(\bar{\omega}^m_{jo})^2_d\right]\frac{(r_{jn})_d}{2\lambda}\right\}$$

$$\sum_{\substack{i=1\\ i\neq p}}^{I}\sum_{p=1}^{P}A_iA_p\Omega_i\Omega_p\frac{(\bar{\omega}^m_{jo})_d}{2}\left\{\frac{\cos(\Omega_i+\Omega_p)\tau-\cos(\bar{\omega}^m_{jo})_d\tau}{(\Omega_i+\Omega_p)^2-(\bar{\omega}^m_{jo})^2_d}\right.$$

$$\left.-\frac{\cos(\Omega_i-\Omega_p)\tau-\cos(\bar{\omega}^m_{jo})_d\tau}{(\Omega_i-\Omega_p)^2-(\bar{\omega}^m_{jo})^2_d}\right\}$$

$$+\left\{\frac{(j_{jo})_d}{(\bar{\omega}^m_{jo})_d}\left[(a_3)_d-(\bar{\omega}^m_{jo})^2_d\right](f_{jo})_d-\frac{(b_{jo})_d}{(\bar{\omega}^m_{jo})_d}\left[(a_2)_d-(\bar{\omega}^m_{jo})_d\right]\right.$$

$$\left.\frac{T_o}{2\mu Eh}\left[(m_{jo})_d-\frac{d_3}{L}(r_{jn})_d\right]\right\}$$

$$\sum_{i=p=1}^{I}A_i^2\Omega_i^2\left\{\frac{1}{2(\bar{\omega}^m_{jo})_d}\left[1-\cos(\bar{\omega}^m_{jo})_d\tau\right]\right.$$

$$\left.\left.+\frac{(\bar{\omega}^m_{jo})_d}{2\left[(2\Omega_i)^2-(\bar{\omega}^m_{jo})^2_d\right]}\left(\cos 2\Omega_i\tau-\cos(\bar{\omega}^m_{jo})_d\tau\right)\right\}\right] \qquad (6.2\text{-}11)$$

$$\sum_{d=1,2}$$

$$\left.\begin{array}{l}j=j_1=1,2,\ldots,M_1\\ k=k_1=0\end{array}\right\}\ d=1 \qquad \left.\begin{array}{l}j=j_2=1,2,\ldots,M\\ k=k_2=0\end{array}\right\}\ d=2$$

$$\sum_{m=1,2}$$

$$W_{jo}(\tau)_d = \sum_{m=1}^{2} \frac{(-1)^m}{[(\bar{\omega}_{jo}^1)_d^2 - (\bar{\omega}_{jo}^2)_d^2]}$$

$$\left[\left| (a_{jo})_d \, (j_{jo})_d \, (w_{jo})_d \, [(a_1)_d - (\bar{\omega}_{jo}^m)_d^2] - (h_{jo})_d \, [(a_{jo})_d \, (u_{jo})_d \right. \right.$$

$$\left. + (b_{jo})_d \, (w_{jo})_d \right] \Bigg| \cos (\bar{\omega}_{jo}^m)_d \tau$$

$$+ \left| \frac{(a_{jo})_d}{(\bar{\omega}_{jo}^m)_d} \, [(a_1)_d - (\bar{\omega}_{jo}^m)_d^2] \, \frac{T_o}{2\mu Eh} \, [(m_{jo})_d - \frac{d_3}{L} \, (r_{jn})_d] \right.$$

$$\left. - \frac{(h_{jo})_d}{(\bar{\omega}_{jo}^m)_d} \, [\frac{1}{\mu Eh} - 1] \, \frac{T_o}{2} \, (e_{jo})_d \right| \left| \frac{1}{2(\bar{\omega}_{jo}^m)_d} \, [(1 - \cos (\bar{\omega}_{jo}^m)_d \tau] \right|$$

$$+ \left| \frac{(a_{jo})_d}{(\bar{\omega}_{jo}^m)_d} \, [(a_1)_d - (\bar{\omega}_{jo}^m)_d^2] \, \frac{T_o \gamma}{2\mu Eh} \, [(m_{jo})_d - \frac{d_3}{L} \, (r_{jn})_d] \right.$$

$$\left. - \frac{(h_{jo})_d}{(\bar{\omega}_{jo}^m)_d} \, [\frac{1}{\mu Eh} - 1] \, \frac{T_o \gamma}{2} (e_{jo})_d \right|$$

$$\cdot \left| \frac{(\bar{\omega}_{jo}^m)_d}{\bar{\Omega}^2 - (\bar{\omega}_{jo}^m)_d^2} \, [\cos \bar{\Omega} \tau - \cos (\bar{\omega}_{jo}^m)_d \tau] \right|$$

$$- \left| \frac{(a_{jo})_d}{(\bar{\omega}_{jo}^m)_d} \, [(a_1)_d - (\bar{\omega}_{jo}^m)_d^2] \, \frac{(r_{jn})_d}{2} + \frac{(h_{jo})_d}{(\bar{\omega}_{jo}^m)_d} \, (f_{jo})_d \right|$$

$$\sum_{i=1}^{I} \sum_{p=1}^{P} A_i \, A_p \, \Omega_i \, \Omega_p \, \frac{(\bar{\omega}_{jo}^m)_d}{2} \left| \frac{\cos(\Omega_i + \Omega_p)\tau - \cos(\bar{\omega}_{jo}^m)_d \tau}{(\Omega_i + \Omega_p)^2 - (\bar{\omega}_{jo}^m)_d^2} \right.$$

$$\left. - \frac{\cos(\Omega_i - \Omega_p)\tau - \cos(\bar{\omega}_{jo}^m)_d \tau}{(\Omega_i - \Omega_p)^2 - (\bar{\omega}_{jo}^m)_d^2} \right|$$

(cont'd)

$$- \left(\frac{a_{jo}}{(\bar{\omega}^m_{jo})_d} \left[(a_1)_d - (\bar{\omega}^m_{jo})^2_d \right] \frac{(r_{jn})_d}{2} + \frac{(h_{jo})_d}{(\bar{\omega}^m_{jo})_d} (f_{jo})_d \right)$$

$$\sum_{i=p=1}^{I} A_i^2 \Omega_i^2 \left(\frac{1}{2(\bar{\omega}^m_{jo})_d} \left[1 - \cos (\bar{\omega}^m_{jo})_d \tau \right] \right.$$

$$\left. + \frac{(\bar{\omega}^m_{jo})_d}{2[(2\Omega_i)^2 - (\bar{\omega}^m_{jo})^2_d]} (\cos 2\Omega_i \tau - \cos (\bar{\omega}^m_{jo})_d \tau) \right) \Bigg] \qquad (6.2\text{-}12)$$

$d=1,2$

$j=j_1=1,2,\ldots,M_1$, $k=k_1=0$ $\Big\}$ $d=1$

$j=j_2=1,2,\ldots,M_2$, $k=k_2=0$ $\Big\}$ $d=2$

$m=1,2$

The terms $(r_{jn})_d$ are given in APPENDIX B.

6.2.4 Solution of Equations (6.1-4)

Performing Laplace transformation techniques on eqs. (6.1-3) results in

$$[(a_{jk})_d s^2 + (d_{jk})_d] \bar{U}_{jk}(\tau)_d + [(b_{jk})_d s^2 + (e_{jk})_d] \bar{V}_{jk}(\tau)_d$$

$$+ [(c_{jk})_d s^2 + (f_{jk})_d] \bar{W}_{jk}(\tau)_d = [\bar{Q}_{jk}(\tau)_d]_1 + s [(a_{jk})_d (u_{jk})_d$$

$$+ (b_{jk})_d (v_{jk})_d + (c_{jk})_d (w_{jk})_d] \qquad (a)$$

$$(g_{jk})_d\,\bar{U}_{jk}(\tau)_d + [(h_{jk})_d\,s^2 + (j_{jk})_d]\,\bar{V}_{jk}(\tau)_d + (k_{jk})_d\,\bar{W}_{jk}(\tau)_d$$

$$= [\bar{Q}_{jk}(\tau)_d]_2 + s(h_{jk})_d\,(v_{jk})_d \qquad (b) \quad (6.2\text{-}13)$$

$$(\ell_{jk})_d\,\bar{U}_{jk}(\tau)_d + (m_{jk})_d\,\bar{V}_{jk}(\tau)_d + [(n_{jk})_d\,s^2 + (p_{jk})_d]\,\bar{W}_{jk}(\tau)_d$$

$$= [\bar{Q}_{jk}(\tau)_d]_3 + s(n_{jk})_d\,(w_{jk})_d \qquad (c)$$

where the barred terms represent the Laplace transform of the term and $(u_{jk})_d$, $(v_{jk})_d$ and $(w_{jk})_d$ are the initial displacements of $U_{jk}(\tau)_d$, $V_{jk}(\tau)_d$ and $W_{jk}(\tau)_d$, respectively.

Solving for $\bar{U}_{jk}(\tau)_d$, $\bar{V}_{jk}(\tau)_d$ and $\bar{W}_{jk}(\tau)_d$ in eqs. (6.2-13) we have

$$\bar{U}_{jk}(\tau)_d = \frac{\begin{vmatrix} [\bar{Q}_{jk}(\tau)_d]_1 + s[(a_{jk})_d(u_{jk})_d + (b_{jk})_d(v_{jk})_d + (c_{jk})_d(w_{jk})_d] & (b_{jk})_d s^2 + (e_{jk})_d & (c_{jk})_d s^2 + (f_{jk})_d \\ [\bar{Q}_{jk}(\tau)_d]_2 + s(h_{jk})_d(v_{jk})_d & (h_{jk})_d s^2 + (j_{jk})_d & (k_{jk})_d \\ [\bar{Q}_{jk}(\tau)_d]_3 + s(n_{jk})_d(w_{jk})_d & (m_{jk})_d & (n_{jk})_d s^2 + (p_{jk})_d \end{vmatrix}}{\Delta_{jk}(s)_d} \qquad (a)$$

$$\bar{V}_{jk}(\tau)_d = \frac{\begin{vmatrix} (a_{jk})_d s^2+(d_{jk})_d & \{[\bar{Q}_{jk}(\tau)_d]_1+s[(a_{jk})_d(u_{jk})_d & (c_{jk})_d s^2+(f_{jk})_d \\ & +(b_{jk})_d(v_{jk})_d+(c_{jk})_d(w_{jk})_d]\} & \\ (g_{jk})_d & [\bar{Q}_{jk}(\tau)_d]_2+s(h_{jk})_d(v_{jk})_d & (k_{jk})_d \\ (\ell_{jk})_d & [\bar{Q}_{jk}(\tau)_d]_3+s(n_{jk})_d(w_{jk})_d & (n_{jk})_d s^2+(p_{jk})_d \end{vmatrix}}{\Delta_{jk}(s)_d} \qquad (b)$$

(6.2-14)

$$\bar{W}_{jk}(\tau)_d = \frac{\begin{vmatrix} (a_{jk})_d s^2+(d_{jk})_d & (b_{jk})_d s^2+(e_{jk})_d & [\bar{Q}_{jk}(\tau)_d]_1+s[(a_{jk})_d(u_{jk})_d \\ & & +(b_{jk})_d(v_{jk})_d+(c_{jk})_d(w_{jk})_d] \\ (g_{jk})_d & (h_{jk})_d s^2+(j_{jk})_d & [\bar{Q}_{jk}(\tau)_d]_2+s(h_{jk})_d(v_{jk})_d \\ (\ell_{jk})_d & (m_{jk})_d & [\bar{Q}_{jk}(\tau)_d]_3+s(n_{jk})_d(w_{jk})_d \end{vmatrix}}{\Delta_{jk}(s)_d} \qquad (c)$$

where

$$\Delta_{jk}(s)_d = \begin{vmatrix} (a_{jk})_d s^2+(d_{jk})_d & (b_{jk})_d s^2+(e_{jk})_d & (c_{jk})_d s^2+(f_{jk})_d \\ (g_{jk})_d & (h_{jk})_d s^2+(j_{jk})_d & (k_{jk})_d \\ (\ell_{jk})_d & (m_{jk})_d & (n_{jk})_d s^2+(p_{jk})_d \end{vmatrix}$$

$$= [s^2+(\tilde{\omega}^1_{jk})^2_d][s^2+(\tilde{\omega}^2_{jk})^2_d][s^2+(\tilde{\omega}^3_{jk})^2_d] \qquad (6.2\text{-}15)$$

The eigenvalues of eqs. (6.2-15) can be determined by

$$(\bar{\omega}^1_{jk})^2_d + (\bar{\omega}^2_{jk})^2_d + (\bar{\omega}^3_{jk})^2_d = \frac{(B)_d}{(A)_d} \qquad (a)$$

$$(\bar{\omega}^1_{jk})^2_d\ (\bar{\omega}^2_{jk})^2_d + (\bar{\omega}^2_{jk})^2_d\ (\bar{\omega}^3_{jk})^2_d + (\bar{\omega}^3_{jk})^2_d\ (\bar{\omega}^1_{jk})^2_d = \frac{(C)_d}{(A)_d} \qquad (b) \qquad (6.2\text{-}16)$$

$$(\bar{\omega}^1_{jk})^2_d\ (\bar{\omega}^2_{jk})^2_d\ (\bar{\omega}^3_{jk})^2_d = \frac{(D)_d}{(A)_d} \qquad (c)$$

where

$$(A)_d = (a_{jk})_d\ (h_{jk})_d\ (n_{jk})_d \qquad (a)$$

$$\begin{aligned}(B)_d = {} & (a_{jk})_d\ (j_{jk})_d\ (n_{jk})_d + (a_{jk})_d\ (h_{jk})_d\ (p_{jk})_d \\ & + (d_{jk})_d\ (h_{jk})_d\ (n_{jk})_d - (\ell_{jk})_d\ (h_{jk})_d\ (c_{jk})_d \\ & - (g_{jk})_d\ (b_{jk})_d\ (n_{jk})_d \end{aligned} \qquad (b) \qquad (6.2\text{-}17)$$

$$\begin{aligned}(C)_d = {} & (a_{jk})_d\ (j_{jk})_d\ (p_{jk})_d + (d_{jk})_d\ (j_{jk})_d\ (n_{jk})_d \\ & + (d_{jk})_d\ (h_{jk})_d\ (p_{jk})_d + (g_{jk})_d\ (m_{jk})_d\ (c_{jk})_d \\ & + (k_{jk})_d\ (\ell_{jk})_d\ (b_{jk})_d - (k_{jk})_d\ (m_{jk})_d\ (a_{jk})_d \\ & - (\ell_{jk})_d\ (j_{jk})_d\ (c_{jk})_d - (\ell_{jk})_d\ (f_{jk})_d\ (h_{jk})_d \\ & - (g_{jk})_d\ (e_{jk})_d\ (n_{jk})_d - (g_{jk})_d\ (b_{jk})_d\ (p_{jk})_d \end{aligned} \qquad (c)$$

$$(D)_2 = (d_{jk})_d\,(j_{jk})_d\,(p_{jk})_d + (g_{jk})_d\,(m_{jk})_d\,(f_{jk})_d$$

$$+ (k_{jk})_d\,(\ell_{jk})_d\,(e_{jk})_d - (k_{jk})_d\,(m_{jk})_d\,(d_{jk})_d$$

$$- (\ell_{jk})_d\,(j_{jk})_d\,(f_{jk})_d - (g_{jk})_d\,(e_{jk})_d\,(p_{jk})_d \qquad (d)$$

Expanding eqs. (6.2-14) we have

$$\bar{U}_{jk}(\tau)_d = \{[\bar{Q}_{jk}(\tau)_d]_1 + s[(a_{jk})_d\,(u_{jk})_d + (b_{jk})_d\,(v_{jk})_d + (c_{jk})_d\,(w_{jk})_d]\}$$

$$\cdot \{(h_{jk})_d\,(n_{jk})_d\,[s^2+(\alpha_4)_d][s^2+(\alpha_5)_d] - (k_{jk})_d\,(m_{jk})_d\}$$

$$+ \{[\bar{Q}_{jk}(\tau)_d]_2 + s(h_{jk})_d\,(v_{jk})_d\}$$

$$\cdot \{(c_{jk})_d\,(m_{jk})_d\,[s^2+(\alpha_3)_d] - (b_{jk})_d\,(n_{jk})_d\,[s^2+(\alpha_2)_d][s^2+(\alpha_5)_d]\}$$

$$+ \{[\bar{Q}_{jk}(\tau)_d]_3 + s(n_{jk})_d\,(w_{jk})_d\}$$

$$\cdot \{(b_{jk})_d\,(k_{jk})_d\,[s^2+(\alpha_2)_d] - (c_{jk})_d\,(h_{jk})_d\,[s^2+(\alpha_3)_d]$$

$$[s^2+(\alpha_4)_d]\}/\Lambda_{jk}(s)_d \qquad (a)$$

$$\bar{V}_{jk}(\tau)_d = \{[\bar{Q}_{jk}(\tau)_d]_1 + s[(a_{jk})_d\,(u_{jk})_d + (b_{jk})_d\,(v_{jk})_d + (c_{jk})_d\,(w_{jk})_d]\}$$

$$\cdot \{(k_{jk})_d\,(\ell_{jk})_d - (g_{jk})_d\,(n_{jk})_d\,[s^2+(\alpha_5)_d]\}$$

$$+ \{[\bar{Q}_{jk}(\tau)_d]_2 + s(h_{jk})_d\,(v_{jk})_d\}$$

(cont'd)

$$\cdot \{(a_{jk})_d\ (n_{jk})_d[s^2+(\alpha_1)_d][s^2+(\alpha_5)_d] - (c_{jk})_d\ (\ell_{jk})_d\ [s^2+(\alpha_3)_d]\}$$

$$+ \{[\bar{Q}_{jk}(\tau)_d]_3 + s(n_{jk})_d\ (w_{jk})_d\}$$

$$\cdot \{(c_{jk})_d\ (g_{jk})_d\ [s^2+(\alpha_3)_d] - (a_{jk})_d\ (k_{jk})_d\ [s^2+(\alpha_1)_d]\}/\Delta_{jk}(s)_d \qquad \text{(b)}$$

(6.2-18)

$$\bar{W}_{jk}(\tau)_d = \{[\bar{Q}_{jk}(\tau)_d]_1 + s[(a_{jk})_d\ (u_{jk})_d + (b_{jk})_d\ (v_{jk})_d + (c_{jk})_d\ (w_{jk})_d]\}$$

$$\cdot \{(g_{jk})_d\ (m_{jk})_d - (h_{jk})_d\ (\ell_{jk})_d\ [s^2+(\alpha_4)_d]\}$$

$$+ \{[\bar{Q}_{jk}(\tau)_d]_2 + s(h_{jk})_d\ (v_{jk})_d\}$$

$$\cdot \{(b_{jk})_d\ (\ell_{jk})_d\ [s^2+(\alpha_2)_d] - (a_{jk})_d\ (m_{jk})_d\ [s^2+(\alpha_1)_d]\}$$

$$+ \{[\bar{Q}_{jk}(\tau)_d]_3 + s(n_{jk})_d\ (w_{jk})_d\}$$

$$\cdot \{(a_{jk})_d\ (h_{jk})_d\ [s^2+(\alpha_1)_d][s^2+(\alpha_4)_d]$$

$$- (b_{jk})_d\ (g_{jk})_d\ [s^2+(\alpha_2)_d]\}/\Delta_{jk}(s)_d \qquad \text{(c)}$$

where

$$(\alpha_1)_d = \frac{(d_{jk})_d}{(a_{jk})_d} \qquad \text{(a)}$$

$$(\alpha_2)_d = \frac{(e_{jk})_d}{(b_{jk})_d} \qquad \text{(b)} \qquad \text{(6.2-19)}$$

$$(\alpha_3)_d = \frac{(f_{jk})_d}{(c_{jk})_d} \qquad (c)$$

$$(\alpha_4)_d = \frac{(j_{jk})_d}{(h_{jk})_d} \qquad (d)$$

$$(\alpha_5)_d = \frac{(p_{jk})_d}{(n_{jk})_d} \qquad (e)$$

Performing inverse Laplace transform techniques on eqs. (6.2-18) and using the convolution theorem we can write

$$U_{jk}(\tau)_d = \sum_{i=1}^{3} \frac{1}{(D^{im}_{jk})_d} \left[\frac{1}{(\omega^m_{jk})_d} \left(\boxed{1}_d (J_1)_d + \boxed{2}_d (J_2)_d + \boxed{3}_d (J_3)_d \right) + \boxed{4}_d \cos (\omega^m_{jk})_d \tau \right] \qquad (a)$$

$$V_{jk}(\tau)_d = \sum_{i=1}^{3} \frac{1}{(D^{im}_{jk})_d} \left[\frac{1}{(\omega^m_{jk})_d} \left(\boxed{5}_d (J_1)_d + \boxed{6}_d (J_2)_d + \boxed{7}_d (J_3)_d \right) + \boxed{8}_d \cos (\omega^m_{jk})_d \tau \right] \qquad (b) \qquad (6.2\text{-}20)$$

$$W_{jk}(\tau)_d = \sum_{i=1}^{3} \frac{1}{(D^{im}_{jk})_d} \left[\frac{1}{(\omega^m_{jk})_d} \left(\boxed{9}_d (J_1)_d + \boxed{10}_d (J_2)_d + \boxed{11}_d (J_3)_d \right) + \boxed{12}_d \cos (\omega^m_{jk})_d \tau \right] \qquad (c)$$

$$\left.\begin{array}{l} d=1,2 \\ j=j_1=1,2,\ldots,M_1 \\ k=k_1=1,2,\ldots,N_1 \end{array}\right\} d=1 \qquad \left.\begin{array}{l} j=j_2=1,2,\ldots,M_2 \\ k=k_2=1,2,\ldots,N_2 \\ m=1,2,3 \end{array}\right\} d=2$$

where

$$(D_{jk}^{im})_d = \prod_{\substack{i=1 \\ i \neq m}}^{3} [(\bar{\omega}_{jk}^{i})_d^2 - (\bar{\omega}_{jk}^{m})_d^2] \tag{6.2-21}$$

and

$$\boxed{1}_d = [(h_{jk})_d\ (n_{jk})_d\ {}_1(\overset{1}{e}_m)_d - (k_{jk})_d\ (m_{jk})_d\ {}_1(\overset{1}{a}_m)_d]$$

$$\boxed{2}_d = [(c_{jk})_d\ (m_{jk})_d\ {}_1(\overset{1}{c}_m)_d - (b_{jk})_d\ (n_{jk})_d\ {}_1(\overset{2}{e}_m)_d]$$

$$\boxed{3}_d = [(b_{jk})_d\ (k_{jk})_d\ {}_1(\overset{2}{c}_m)_d - (c_{jk})_d\ (h_{jk})_d\ {}_1(\overset{3}{e}_m)_d]$$

$$\boxed{4}_d = [\textcircled{1}_d\ {}_1(\overset{1}{e}_m)_d - \textcircled{2}_d\ {}_1(\overset{1}{a}_m)_d + \textcircled{3}_d\ {}_1(\overset{1}{c}_m)_d - \textcircled{4}_d\ {}_1(\overset{2}{e}_m)_d + \textcircled{5}_d\ {}_1(\overset{2}{c}_m)_d - \textcircled{6}_d\ {}_1(\overset{3}{e}_m)_d]$$

$$\boxed{5}_d = [(k_{jk})_d\ (\ell_{jk})_d\ {}_2(\overset{1}{a}_m)_d - (g_{jk})_d\ (n_{jk})_d\ {}_2(\overset{1}{c}_m)_d]$$

$$\boxed{6}_d = [(a_{jk})_d\ (n_{jk})_d\ {}_2(\overset{1}{e}_m)_d - (c_{jk})_d\ (\ell_{jk})_d\ {}_2(\overset{2}{c}_m)_d]$$

$$\boxed{7}_d = [(c_{jk})_d\ (g_{jk})_d\ {}_2(\overset{3}{c}_m)_d - (a_{jk})_d\ (k_{jk})_d\ {}_2(\overset{4}{c}_m)_d]$$

$$\boxed{8}_d = [\textcircled{7}_d\ {}_2(\overset{1}{a}_m)_d - \textcircled{8}_d\ {}_2(\overset{1}{c}_m)_d + \textcircled{9}_d\ {}_2(\overset{2}{e}_m)_d - \textcircled{10}_d\ {}_2(\overset{2}{c}_m)_d + \textcircled{11}_d\ {}_2(\overset{3}{c}_m)_d - \textcircled{12}_d\ {}_2(\overset{4}{c}_m)_d] \tag{6.2-22}$$

$$\boxed{9}_d\ [(g_{jk})_d\ (m_{jk})_d\ {}_3(\overset{1}{a}_m)_d - (h_{jk})_d\ (\ell_{jk})_d\ {}_3(\overset{1}{c}_m)_d]$$

$$\boxed{10}_d = [(b_{jk})_d\ (\ell_{jk})_d\ 3(\overset{2}{c}_m)_d - (a_{jk})_d\ (m_{jk})_d\ 3(\overset{3}{c}_m)_d]$$

$$\boxed{11}_d = [(a_{jk})_d\ (h_{jk})_d\ 3(\overset{1}{e}_m)_d - (b_{jk})_d\ (g_{jk})_d\ 3(\overset{4}{c}_m)_d]$$

$$\boxed{12}_d = [\,⑬_d\ 3(\overset{1}{a}_m)_d - ⑭_d\ 3(\overset{1}{c}_m)_d + ⑮_d\ 3(\overset{2}{c}_m)_d - ⑯_d\ 3(\overset{3}{c}_m)_d$$

$$+ ⑰_d\ 3(\overset{1}{e}_m)_d - ⑱_d\ 3(\overset{4}{c}_m)_d]$$

$d=1,2$

$$\left.\begin{array}{l} j=j_1=1,2,\ldots,M_1 \\ k=k_1=1,2,\ldots,N_1 \end{array}\right\} d=1$$

$$\left.\begin{array}{l} j=j_2=1,2,\ldots,M_2 \\ k=k_2=1,2,\ldots,N_2 \end{array}\right\} d=2$$

$m=1,2,3$

$$①_d = (h_{jk})_d\ (n_{jk})_d\ [(a_{jk})_d\ (u_{jk})_d + (b_{jk})_d\ (v_{jk})_d + (c_{jk})_d\ (w_{jk})_d]$$

$$②_d = (k_{jk})_d\ (m_{jk})_d\ [(a_{jk})_d\ (u_{jk})_d + (b_{jk})_d\ (v_{jk})_d + (c_{jk})_d\ (w_{jk})_d]$$

$$③_d = (c_{jk})_d\ (h_{jk})_d\ (m_{jk})_d\ (v_{jk})_d$$

$$④_d = (b_{jk})_d\ (h_{jk})_d\ (n_{jk})_d\ (v_{jk})_d$$

$$⑤_d = (b_{jk})_d\ (k_{jk})_d\ (n_{jk})_d\ (w_{jk})_d$$

(cont'd)

$$⑥_d = (c_{jk})_d\ (h_{jk})_d\ (n_{jk})_d\ (w_{jk})_d$$

$$⑦_d = (k_{jk})_d\ (\ell_{jk})_d\ [(a_{jk})_d\ (u_{jk})_d + (b_{jk})_d\ (v_{jk})_d + (c_{jk})_d\ (w_{jk})_d]$$

$$⑧_d = (g_{jk})_d\ (n_{jk})_d\ [(a_{jk})_d\ (u_{jk})_d + (b_{jk})_d\ (v_{jk})_d + (c_{jk})_d\ (w_{jk})_d]$$

$$⑨_d = (a_{jk})_d\ (h_{jk})_d\ (n_{jk})_d\ (v_{jk})_d \qquad (6.2\text{-}23)$$

$$⑩_d = (c_{jk})_d\ (h_{jk})_d\ (\ell_{jk})_d\ (v_{jk})_d$$

$$⑪_d = (c_{jk})_d\ (g_{jk})_d\ (n_{jk})_d\ (w_{jk})_d$$

$$⑫_d = (a_{jk})_d\ (k_{jk})_d\ (n_{jk})_d\ (w_{jk})_d$$

$$⑬_d = (g_{jk})_d\ (m_{jk})_d\ [(a_{jk})_d\ (u_{jk})_d + (b_{jk})_d\ (v_{jk})_d + (c_{jk})_d\ (w_{jk})_d]$$

$$⑭_d = (h_{jk})_d\ (\ell_{jk})_d\ [(a_{jk})_d\ (u_{jk})_d + (b_{jk})_d\ (v_{jk})_d + (c_{jk})_d\ (w_{jk})_d]$$

$$⑮_d = (b_{jk})_d\ (h_{jk})_d\ (\ell_{jk})_d\ (v_{jk})_d$$

$$⑯_d = (a_{jk})_d\ (h_{jk})_d\ (m_{jk})_d\ (v_{jk})_d$$

$$⑰_d = (a_{jk})_d\ (h_{jk})_d\ (n_{jk})_d\ (w_{jk})_d$$

$$⑱_d = (b_{jk})_d\ (g_{jk})_d\ (n_{jk})_d\ (w_{jk})_d$$

and

$$ {}_1({}^1a_m)_d = {}_2({}^1a_m)_d = {}_3({}^1a_m)_d = 1 $$

$$ {}_1({}^1c_m)_d = [(\alpha_3)_d - (\bar{\omega}^m_{jk})^2_d] $$

$$ {}_1({}^2c_m)_d = [(\alpha_2)_d - (\bar{\omega}^m_{jk})^2_d] $$

$$ {}_2({}^1c_m)_d = [(\alpha_5)_d - (\bar{\omega}^m_{jk})^2_d] $$

$$ {}_2({}^2c_m)_d = {}_1({}^1c_m)_d $$

$$ {}_2({}^3c_m)_d = {}_1({}^1c_m)_d $$

$$ {}_2({}^4c_m)_d = [(\alpha_1)_d - (\bar{\omega}^m_{jk})^2_d] \qquad (6.2\text{-}24) $$

$$ {}_3({}^1c_m)_d = [(\alpha_4)_d - (\bar{\omega}^m_{jk})^2_d] $$

$$ {}_3({}^2c_m)_d = {}_1({}^2c_m)_d $$

$$ {}_3({}^3c_m)_d = {}_2({}^4c_m)_d $$

$$ {}_3({}^4c_m)_d = {}_1({}^2c_m)_d $$

$$ {}_1({}^1e_m)_d = [(\alpha_4)_d - (\bar{\omega}^m_{jk})^2_d][(\alpha_5)_d - (\bar{\omega}^m_{jk})^2_d] $$

$$ {}_1({}^2e_m)_d = [(\alpha_2)_d - (\bar{\omega}^m_{jk})^2_d][(\alpha_5)_d - (\bar{\omega}^m_{jk})^2_d] $$

$$_1(e^3_m)_d = [(\alpha_3)_d - (\bar{\omega}^m_{jk})^2_d][(\alpha_4)_d - (\bar{\omega}^m_{jk})^2_d]$$

$$_2(e^1_m)_d = [(\alpha_1)_d - (\bar{\omega}^m_{jk})^2_d][(\alpha_5)_d - (\bar{\omega}^m_{jk})^2_d]$$

$$_3(e^1_m)_d = [(\alpha_1)_d - (\bar{\omega}^m_{jk})^2_d][(\alpha_4)_d - (\bar{\omega}^m_{jk})^2_d]$$

The convolution integrals $(J_1)_d$, $(J_2)_d$, and $(J_3)_d$ of eqs. (6.2-20) are evaluated as follows:

$$(J_1)_d = \int_0^\tau [Q_{jk}(\eta)_d]_1 \sin (\bar{\omega}^m_{jk})_d (\tau-\eta)d\eta$$

$$= -\frac{(\delta_{1k})_d}{\lambda} \sum_{i=1}^{I} (e_{jo})_d A_i^2 \Omega_i^2 \int_0^\tau \cos \Omega_i\eta \sin (\bar{\omega}^m_{jk})_d(\tau-\eta)d\eta$$

$$+ \frac{(\delta_{1k})_d\lambda}{2(1+\nu)\mu} (e_{jo})_d \int_0^\tau q_B(\eta) \sin (\bar{\omega}^m_{jk})_d (\tau-\eta)d\eta$$

$$+ \frac{(\delta_{1k})_d\lambda}{2(1+\nu)\mu} \sum_{n=1}^{N} (z_{jn})_d \left(\frac{\lambda_n}{2}\right) \int_0^\tau q_n(\eta) \sin (\bar{\omega}^m_{jk})_d (\tau-\eta)d\eta \qquad (6.2\text{-}25)$$

$$(J_2)_d = \int_0^\tau [Q_{jk}(\eta)_d]_2 \sin (\bar{\omega}^m_{jk})_d (\tau-\eta)d\eta$$

$$= (\delta_{1k})_d \sum_{i=1}^{I} \left[\frac{T_oK}{2} (r_{jn})_d - (m_{jo})_d \Omega_i^2\right] A_i \int_0^\tau \cos \Omega_i\eta \sin (\bar{\omega}^m_{jk})_d (\tau-\eta)d\eta$$

$$- (\delta_{1k})_d \frac{T_oK\gamma}{2} \sum_{i=1}^{I} (r_{jn})_d A_i \int_0^\tau \cos\bar{\Omega}\eta \cos\Omega_i\eta \sin(\bar{\omega}^m_{jk})_d (\tau-\eta)d\eta$$

$$- \frac{(\delta_{2k})_d}{2\lambda} (r_{jn})_d \sum_{i=p=1}^{I} A_i^2 \Omega_i^2 \int_0^\tau \sin^2\Omega_i\eta \sin(\bar{\omega}^m_{jk})_d (\tau-\eta)d\eta$$

(cont'd)

$$- \frac{(\delta_{2k})_d}{2\lambda} (r_{jn})_d \sum_{\substack{i=1 \\ i\neq p}}^{I} \sum_{p=1}^{P} A_i A_p \Omega_i \Omega_p \int_0^\tau \sin\Omega_i\eta \sin\Omega_p\eta \sin(\bar{\omega}^m_{jk})_d(\tau-\eta)d\eta$$

$$- (\delta_{1k})_d (m_{jo})_d \int_0^\tau q_B(\eta) \sin(\bar{\omega}^m_{jk})_d (\tau-\eta)d\eta$$

$$- (\delta_{1k})_d \sum_{n=1}^{N} (s_{jn})_d \int_0^\tau \ddot{q}_n(\eta) \sin(\bar{\omega}^m_{jk})_d (\tau-\eta)d\eta$$

$$+ \frac{(\delta_{1k})_d}{2(1+\nu)\ \mu} \sum_{n=1}^{N} (x_{jn})_d (\frac{\lambda_n}{2})^2 \int_0^\tau q_n(\eta) \sin(\bar{\omega}^m_{jk})_d(\tau-\eta)d\eta \qquad (6.2\text{-}26)$$

$$(J_3)_d = \int_0^\tau [Q_{jk}(\eta)_d]_3 \sin (\bar{\omega}^m_{jk})_d (\tau-\eta)d\eta$$

$$= (\delta_{1k})_d \sum_{i=1}^{I} [\frac{T_o K}{2} (r_{jn})_d - (m_{jo})_d \Omega_i^2] A_i \int_0^\tau \cos\Omega_i\eta \sin(\bar{\omega}^m_{jk})_d(\tau-\eta)d\eta$$

$$- (\delta_{1k})_d \frac{T_o K Y}{2} (r_{jn})_d \sum_{i=1}^{I} A_i \int_0^\tau \cos\bar{\Omega}\eta \cos\Omega_i\eta \sin(\bar{\omega}^m_{jk})_d(\tau-\eta)d\eta$$

$$- \frac{(\delta_{2k})_d}{2\lambda} (r_{jn})_d \sum_{i=p=1}^{I} A_i^2 \Omega_i^2 \int_0^\tau \sin^2\Omega_i\eta \sin(\bar{\omega}^m_{jk})_d(\tau-\eta)d\eta$$

$$- \frac{(\delta_{2k})_d}{2} (r_{jn})_d \sum_{\substack{i=1 \\ i\neq p}}^{I} \sum_{p=1}^{P} A_i A_p \Omega_i \Omega_p \int_0^\tau \sin\Omega_i\eta \sin\Omega_p\eta \sin(\bar{\omega}^m_{jk})_d(\tau-\eta)d\eta$$

$$- (\delta_{1k})_d (m_{jo})_d \int_0^\tau \ddot{q}_B(\eta) \sin(\bar{\omega}^m_{jk})_d (\tau-\eta)d\eta$$

$$- (\delta_{1k})_d \sum_{n=1}^{N} (s_{jn})_d \int_0^\tau \ddot{q}_n(\eta) \sin(\bar{\omega}^m_{jk})_d (\tau-\eta)d\eta$$

$$+ \frac{(\delta_{1k})_d \lambda^2}{\mu(1-\nu^2)} [1 - \frac{\sigma^2\lambda^2}{12}] (m_{jo})_d \int_0^\tau q_B(\eta) \sin(\bar{\omega}^m_{jk})_d (\tau-\eta)d\eta$$

(cont'd)

$$+\frac{(\delta_{1k})_d}{\mu(1-\nu^2)}\sum_{n=1}^{N}\left[\left(\lambda^2-\frac{\sigma^2}{12}\left[\lambda^4+(\frac{\lambda_n}{2})^4\right]\right)(s_{jn})_d+\frac{\sigma^2\lambda^2}{6}(\frac{\lambda_n}{2})^2(x_{jn})_d\right]\int_0^{\tau}q_n(\eta)\sin(\bar{\omega}^m_{jk})_d(\tau-\eta)d\eta \qquad (6.2\text{-}27)$$

$d=1,2$

$\left.\begin{matrix}j=j_1=1,2,\ldots,M_1\\ k=k_1=1,2,\ldots,N_1\end{matrix}\right\}d=1$

$\left.\begin{matrix}j=j_2=1,2,\ldots,M_2\\ k=k_2=1,2,\ldots,N_2\end{matrix}\right\}d=2$

$m=1,2,3$

Substituting for $q_B(\eta)$ and $q_n(\eta)$ in eqs. (6.2-25, 26, 27) and evaluating all remaining integrals, the convolution integrals $(J_1)_d$, $(J_2)_d$, and $(J_3)_d$ in their final evaluated form are written

$$(J_1)_d=\int_0^{\tau}[Q_{jk}(\eta)_d]_1\sin(\bar{\omega}^m_{jk})_d(\tau-\eta)d\eta$$

$$=-\frac{(\delta_{1k})_d}{\lambda}\sum_{i=1}^{I}(e_{jo})_d A_i^2\Omega_i^2\left(-\frac{(\bar{\omega}^m_{jk})_d}{\Omega_i^2-(\bar{\omega}^m_{jk})_d^2}\left[\cos\Omega_i\tau-\cos(\bar{\omega}^m_{jk})_d\tau\right]\right)$$

$$+\frac{(\delta_{1k})_d(\bar{\omega}^m_{jk})_d\lambda}{2(1+\nu)\mu}(e_{jo})_d\left[\sum_{s=-S}^{S}\frac{C_B^{(s)}}{(\alpha_r+s)^2\bar{\Omega}^2-(\bar{\omega}^m_{jk})_d^2}\right.$$

$$\cdot\left.\left(\cos(\bar{\omega}^m_{jk})_d+i\frac{(\alpha_r+s)}{(\bar{\omega}^m_{jk})_d}\bar{\Omega}\sin(\bar{\omega}^m_{jk})_d\tau-e^{i(\alpha_r+s)\bar{\Omega}\tau}\right)\right]$$

$$+\frac{(\delta_{1k})_d(\bar{\omega}^m_{jk})_d\lambda}{2(1+\nu)\mu}\sum_{n=1}^{N}(z_{jn})_d(\frac{\lambda_n}{2})\left[\sum_{s=-S}^{S}\frac{C_n^{(s)}}{(\alpha_r+s)^2\bar{\Omega}^2-(\bar{\omega}^m_{jk})_d^2}\right.$$

(cont'd)

$$\cdot \left[\cos(\bar{\omega}^m_{jk})_d \tau + i \frac{(\alpha_r+s)}{(\bar{\omega}^m_{jk})_d} \bar{\Omega} \sin (\bar{\omega}^m_{jk})_d - e^{i(\alpha_r+s)\bar{\Omega}\tau} \right] \right] \qquad (6.2\text{-}28)$$

$$d=1,2$$

$$\left.\begin{array}{l} j=j_1=1,2,\ldots,M_1 \\ k=k_1=1,2,\ldots,N_1 \end{array}\right\} d=1$$

$$\left.\begin{array}{l} j=j_2=1,2,\ldots,M_2 \\ k=k_2=1,2,\ldots,N_2 \end{array}\right\} d=2$$

$$m=1,2,3$$

$$(J_2)_d = \int_0^\tau [O_{jk}(\eta)_d]_2 \sin (\bar{\omega}^m_{jk})_d (\tau-\eta) d\eta$$

$$= (\delta_{1k})_d \sum_{i=1}^{I} [\frac{T_o K}{2} (r_{jn})_d - (m_{jo})_d \Omega_i^2] A_i$$

$$\cdot \left(- \frac{(\bar{\omega}^m_{jk})_d}{\Omega_i^2 - (\bar{\omega}^m_{jk})_d} [\cos\Omega_i\tau - \cos(\bar{\omega}^m_{jk})_d\tau] \right)$$

$$- (\delta_{1k})_d \frac{T_o K^\gamma}{2} \sum_{i=1}^{I} (r_{jn})_d A_i$$

$$\cdot \left(- \frac{(\bar{\omega}^m_{jk})_d}{2[(\bar{\Omega}+\Omega_i)^2 - (\bar{\omega}^m_{jk})_d^2]} [\cos(\bar{\Omega}+\Omega_i)\tau - \cos(\bar{\omega}^m_{jk})_d\tau] \right.$$

$$\left. - \frac{(\bar{\omega}^m_{jk})_d}{2[(\bar{\Omega}-\Omega_i)^2 - (\bar{\omega}^m_{jk})_d^2]} [\cos(\bar{\Omega}-\Omega_i)\tau - \cos(\bar{\omega}^m_{jk})_d\tau] \right)$$

$$- \frac{(\delta_{2k})_d}{2\lambda} (r_{jn})_d \sum_{i=p=1}^{I} A_i^2\Omega_i^2 \left(\frac{1}{2(\bar{\omega}^m_{jk})_d} [1 - \cos(\bar{\omega}^m_{jk})_d\tau] \right.$$

(cont'd)

$$+ \frac{(\bar{\omega}^m_{jk})_d}{2[(2\Omega_i)^2 - (\bar{\omega}^m_{jk})_d^2]} \left[\cos 2\Omega_i \tau - \cos(\bar{\omega}^m_{jk})_d \tau\right] \Bigg)$$

$$- \frac{(\delta_{2k})_d}{2\lambda} (r_{jn})_d \sum_{\substack{i=1 \\ i\neq p}}^{I} \sum_{p=1}^{P} A_i A_p \Omega_i \Omega_p \quad \frac{(\bar{\omega}^m_{jk})_d}{2} \left[\frac{\cos(\Omega_i+\Omega_p)\tau - \cos(\bar{\omega}^m_{jk})_d \tau}{(\Omega_i+\Omega_p)^2 - (\omega^m_{jk})_d^2} \right.$$

$$\left. - \frac{\cos(\Omega_i-\Omega_p)\tau - \cos(\bar{\omega}^m_{jk})_d \tau}{(\Omega_i-\Omega_p)^2 - (\bar{\omega}^m_{jk})_d^2} \right]$$

$$- (\delta_{1k})_d (m_{jo})_d \left[- \frac{(\bar{\omega}^m_{jk})_d \; C_B^{(s)}}{(\alpha_r+s)^2 \bar{\Omega}^2 - (\bar{\omega}^m_{jk})_d^2} \Bigg(\cos(\bar{\omega}^m_{jk})_d \tau \right.$$

$$\left. + i \frac{(\alpha_r+s)}{(\bar{\omega}^m_{jk})_d} \bar{\Omega} \sin(\bar{\omega}^m_{jk})_d \tau - e^{i(\alpha_r+s)\bar{\Omega}\tau} \Bigg) \right]$$

$$+ (\delta_{1k})_d \sum_{n=1}^{N} \sum_{s=-S}^{S} \left(\frac{(x_{jn})_d}{2(1+\nu)\mu} \left(\frac{\lambda_n}{2}\right)^2 + (s_{jn})_d (\alpha_r+s)^2 \bar{\Omega}^2 \right)$$

$$\cdot \left[\frac{C_n^{(s)} \; (\bar{\omega}^m_{jk})_d}{(\alpha_r+s)^2 \bar{\Omega}^2 - (\bar{\omega}^m_{jk})_d^2} \Bigg(\cos(\bar{\omega}^m_{jk})_d \tau \right.$$

$$\left. + i \frac{(\alpha_r+s)}{(\bar{\omega}^m_{jk})_d} \bar{\Omega} \sin(\bar{\omega}^m_{jk})_d \tau - e^{i(\alpha_r+s)\bar{\Omega}\tau} \Bigg) \right] \qquad (6.2\text{-}29)$$

$$d=1,2$$

$$\left.\begin{array}{l} j=j_1=1,2,\ldots,M_1 \\ k=k_1=1,2,\ldots,N_1 \end{array}\right\} d=1$$

$$\left.\begin{array}{l} j=j_2=1,2,\ldots,M_2 \\ k=k_2=1,2,\ldots,N_2 \end{array}\right\} d=2$$

$$m=1,2,3$$

$$(J_3)_d = \int_0^\tau [Q_{jk}(\eta)_d]_3 \sin (\bar{\omega}^m_{jk})_d (\tau-\eta)d\eta$$

$$= (\delta_{1k})_d \sum_{i=1}^{I} \left[\frac{T_o K}{2} (r_{jn})_d - (m_{jo})_d \Omega_i^2\right] A_i$$

$$\cdot \left(- \frac{(\bar{\omega}^m_{jk})_d}{\Omega_i^2 - (\bar{\omega}^m_{jk})_d^2} [\cos\Omega_i\tau - \cos(\bar{\omega}^m_{jk})_d\tau] \right)$$

$$- (\delta_{1k})_d \frac{T_o K\gamma}{2} (r_{jn})_d \sum_{i=1}^{I} A_i$$

$$\cdot \left(- \frac{(\bar{\omega}^m_{jk})_d}{2[(\bar{\Omega}+\Omega_i)^2 - (\bar{\omega}^m_{jk})_d^2]} [\cos(\bar{\Omega}+\Omega_i)\tau - \cos(\bar{\omega}^m_{jk})_d\tau] \right.$$

$$\left. - \frac{(\bar{\omega}^m_{jk})_d}{2[(\bar{\Omega}-\Omega_i)^2 - (\bar{\omega}^m_{jk})_d^2]} [\cos(\bar{\Omega}-\Omega_i)\tau - \cos(\bar{\omega}^m_{jk})_d\tau] \right)$$

$$- (\delta_{2k})_d \frac{(r_{jn})_d}{2\lambda} \sum_{i=p=1}^{I} A_i^2\Omega_i^2 \left(\frac{1}{2(\bar{\omega}^m_{jk})_d} [1 - \cos(\bar{\omega}^m_{jk})_d\tau] \right.$$

$$\left. + \frac{(\bar{\omega}^m_{jk})_d}{2[(2\Omega_i)^2 - (\bar{\omega}^m_{jk})_d^2]} [\cos 2\Omega_i\tau - \cos(\bar{\omega}^m_{jk})_d\tau] \right)$$

$$- (\delta_{2k})_d \frac{(r_{jn})_d}{2\lambda} \sum_{\substack{i=1 \\ i\neq p}}^{I} \sum_{p=1}^{p} A_i A_p \Omega_i\Omega_p \frac{(\bar{\omega}^m_{jk})_d}{2} \left[\frac{\cos(\Omega_i+\Omega_p)\tau - \cos(\bar{\omega}_{jk})_d\tau}{(\Omega_i+\Omega_p)^2 - (\bar{\omega}^m_{jk})_d^2} \right.$$

$$\left. - \frac{\cos(\Omega_i-\Omega_p)\tau - \cos(\bar{\omega}^m_{jk})_d\tau}{(\Omega_i-\Omega_p)^2 - (\bar{\omega}^m_{jk})_d^2} \right]$$

$$+ (\delta_{1k})_d \sum_{s=-S}^{S} \left(\frac{\lambda^2}{\mu(1-\nu^2)} \left[1 - \frac{\sigma^2\lambda^2}{12}\right] + (\alpha_r+s)\bar{\Omega}^2 \right) (m_{jo})_d$$

(cont'd)

$$\cdot\left[\frac{C_B^{(s)}\ (\bar{\omega}_{jk}^m)_d}{(\alpha_r+s)^2\bar{\Omega}^2 - (\bar{\omega}_{jk}^m)_d^2}\left(\cos(\bar{\omega}_{jk}^m)_d\tau\right.\right.$$

$$\left.\left. + i\,\frac{(\alpha_r+s)}{(\bar{\omega}_{jk}^m)_d}\,\bar{\Omega}\,\sin\,(\bar{\omega}_{jk}^m)_d\tau - e^{i(\alpha_r+s)\bar{\Omega}\tau}\right)\right]$$

$$+ (\delta_{1k})_d \sum_{n=1}^{N}\sum_{s=-S}^{S}\left\{\frac{1}{\mu(1-\nu^2)}\left[\left(\lambda^2 - \frac{\sigma^2}{12}\left[\lambda^4 + (\tfrac{\lambda n}{2})^4\right]\right)(s_{jn})_d\right.\right.$$

$$\left.\left. + \frac{\sigma^2\lambda^2}{6}\,(\tfrac{\lambda n}{2})^2\,(x_{jn})_d\right] + (\alpha_r+s)^2\bar{\Omega}^2(s_{jn})_d\right\}$$

$$\cdot\left[\frac{C_n^{(s)}\ (\bar{\omega}_{jk}^m)_d}{(\alpha_r+s)^2\bar{\Omega}^2 - (\bar{\omega}_{jk}^m)_d^2}\left(\cos(\bar{\omega}_{jk}^m)_d\tau\right.\right.$$

$$\left.\left. + i\,\frac{(\alpha_r+s)}{(\bar{\omega}_{jk}^m)_d}\,\bar{\Omega}\,\sin(\bar{\omega}_{jk}^m)_d\tau - e^{i(\alpha_r+s)\bar{\Omega}\tau}\right)\right] \qquad (6.2\text{-}30)$$

$$d=1,2$$

$$\left.\begin{array}{l} j=j_1=1,2,\ldots,M_1 \\ k=k_1=1,2,\ldots,N_1 \end{array}\right\} \quad d=1$$

$$\left.\begin{array}{l} j=j_2=1,2,\ldots,M_2 \\ k=k_2=1,2,\ldots,N_2 \end{array}\right\} \quad d=2$$

$$m=1,2$$

The terms $(r_{jn})_d,\ldots,(z_{jn})_d$ in eqs. (6.2-28, 29, 30) are given in APPENDIX B.

7.0 STABILITY ANALYSIS OF THE COMPOSITE STRUCTURE

7.1 Inspection

Inspection of the displacement equations $U_{0k}(\tau)_d$, $U_{j0}(\tau)_d$, $W_{j0}(\tau)_d$ and terms in the convolution integrals $(J_1)_d$, $(J_2)_d$ and $(J_3)_d$ shows that certain denominator frequency factors, if equated to zero, would render the displacements unbounded. The solutions of the factors will be the unstable values of the thrust frequency.

7.2 Explanation of Subscript Notation

The three expressions for Ω_i (defined earlier in Matheau equation section) are written

$$\Omega_1 = (i - \beta/2)\frac{\Omega}{\omega_1} \tag{7.2-1}$$

$$\Omega_2 = \frac{\beta\Omega}{2\omega_1} \tag{7.2-2}$$

$$\Omega_3 = (1 + \beta/2)\frac{\Omega}{\omega_1} \tag{7.2-3}$$

where

$$\beta = \beta(\Omega) \qquad \text{(a)}$$

$$\omega_1 = \left(\frac{\lambda_1}{2L}\right)^2 R\sqrt{\frac{E}{2\rho}} \qquad \text{(b)} \tag{7.2-4}$$

$$R = R_1 \qquad \text{(c)}$$

$$\lambda_1^2 = 22.373 \qquad \text{(d)}$$

Using the parameter

$$\varepsilon_i = \begin{cases} -\frac{1}{2} \; ; \; i = 1 \\ \frac{1}{2} \; ; \; i = 2 \\ \frac{1}{2} \; ; \; i = 3 \end{cases} \tag{7.2-5}$$

equations (7.2-1) through (7.2-3) can be written concisely as

$$\Omega_i = [(i-2)^2 + \epsilon_i \beta] \frac{\Omega}{\omega_1} \qquad (7.2\text{-}6)$$

$$i = 1, 2, 3$$

7.3 Stability Equations

Using the notation of equations (7.2-6) and referring to eqs. (6.2-3, -11, -12, -25, -26, -27) the stability equations, and source of the equation, is presented as follows:

1. From $U_{0k}(\tau)_d$

$$\Omega_i = [(i-2)^2 + \epsilon_i \beta] \frac{\Omega}{\omega_1} = (\bar{\omega}_{0k})_d \qquad (7.3\text{-}1)$$

$$d = 1, 2$$
$$i = 1, 2, 3$$
$$k = k_1 = 1,2,\ldots,N_1;\ d=1$$
$$k = k_2 = 1,2,\ldots,N_2;\ d=2$$

2. From $U_{0k}(\tau)_d$

$$[(\alpha_r + s)\bar{\Omega}]^2 = (\bar{\omega}_{0k})_d^2 \qquad (7.3\text{-}2)$$

$$\bar{\Omega} = \frac{\Omega}{\omega_1} = \left| \frac{(\bar{\omega}_{0k})_d}{\alpha_r + s} \right| \qquad (7.3\text{-}3)$$

$$d = 1,2$$
$$k = k_1 = 1,2,\ldots,N_1;\ d=1$$
$$k = k_2 = 1,2,\ldots,N_2;\ d=2$$
$$r = 1,2,\ldots,R$$
$$s = -S \text{ to } +S$$

3. From $U_{j0}(\tau)_d$, $W_{j0}(\tau)_d$

$$\bar{\Omega}^2 = (\bar{\omega}_{j0}^{m})_d^2 \tag{7.3-4}$$

$$\frac{\Omega}{\omega_1} = (\bar{\omega}_{j0}^{m})_d \tag{7.3-5}$$

$d = 1,2$

$j = j_1 = 1,2,\ldots,M_1$; $d=1$

$j = j_2 = 1,2,\ldots,M_2$; $d=2$

$m = 1,2$

4. From $U_{j0}(\tau)_d$, $W_{j0}(\tau)_d$

$$(2\Omega_i)^2 - (\bar{\omega}_{j0}^{m})_d^2 = 0 \tag{7.3-6}$$

$$2\Omega_i = (\bar{\omega}_{j0}^{m})_d \tag{7.3-7}$$

$$2[(i-2)^2 + \epsilon_i\beta]\frac{\Omega}{\omega_1} = (\bar{\omega}_{j0}^{m})_d \tag{7.3-8}$$

$d = 1,2$

$i = 1,2,3$

$j = j_1 = 1,2,\ldots,M_1$; $d=1$

$j = j_2 = 1,2,\ldots,M_2$; $d=2$

$m = 1,2$

5. From $U_{j0}(\tau)_d$, $W_{j0}(\tau)_d$

$$(\Omega_i+\Omega_p)^2 - (\bar{\omega}_{j0}^{m})_d^2 = 0 \tag{7.3-9}$$

$$\Omega_i + \Omega_p = (\bar{\omega}_{j0}^{m})_d \tag{7.3-10}$$

$$\left|(i-2)^2 + (p-2)^2 + (\epsilon_i+\epsilon_p)\beta\right| \frac{\Omega}{\omega_1} = (\bar{\omega}^m_{j0})_d \qquad (7.3\text{-}11)$$

$d = 1,2$

$i = 1,2,3$

$j = j_1 = 1,2,\ldots,M_1;\ d=1$

$j = j_2 = 1,2,\ldots,M_2;\ d=2$

$m = 1,2$

6. From $U_{j0}(\tau)_d$, $W_{j0}(\tau)_d$

$$(\Omega_i-\Omega_p)^2 - (\bar{\omega}^m_{j0})_d^2 = 0 \qquad (7.3\text{-}12)$$

$$\Omega_i - \Omega_p = (\bar{\omega}^m_{j0})_d \qquad (7.3\text{-}13)$$

$$\left|(i-2)^2 - (p-2)^2 + (\epsilon_i-\epsilon_p)\beta\right| \frac{\Omega}{\omega_1} = (\bar{\omega}^m_{j0})_d \qquad (7.3\text{-}14)$$

$d = 1,2$

$i = 1,2,3$

$j = j_1 = 1,2,\ldots,M_1;\ d=1$

$j = j_2 = 1,2,\ldots,M_2;\ d=2$

$m = 1,2$

7. From $(J_1)_d$, $(J_2)_d$, $(J_3)_d$

$$\Omega_i^2 - (\bar{\omega}^m_{jk})_d^2 = 0 \qquad (7.3\text{-}15)$$

$$\Omega_i = (\bar{\omega}^m_{jk})_d \qquad (7.3\text{-}16)$$

$$[(i-2)^2 + \epsilon_i\beta]\frac{\Omega}{\omega_1} = (\bar{\omega}^m_{jk})_d \qquad (7.3\text{-}17)$$

$d = 1,2$

$i = 1,2,3$

$$\left.\begin{array}{l} j = j_1 = 1,2,\ldots,M_1 \\ k = k_1 = 1,2,\ldots,N_1 \end{array}\right\} \quad d=1$$

$$\left.\begin{array}{l} j = j_2 = 1,2,\ldots,M_2 \\ k = k_2 = 1,2,\ldots,N_2 \end{array}\right\} \quad d=2$$

$$m = 1,2,3$$

8. From $(J_1)_d$, $(J_2)_d$, $(J_3)_d$

$$(\bar{\Omega}+\Omega_i)^2 - (\bar{\omega}_{jk}^{m})_d^2 = 0 \tag{7.3-18}$$

$$\bar{\Omega} + \Omega_i = (\bar{\omega}_{jk}^{m})_d \tag{7.3-19}$$

$$\left|(i-2)^2 + \epsilon_i\beta + 1\right| \frac{\Omega}{\omega_1} = (\bar{\omega}_{jk}^{m})_d \tag{7.3-20}$$

$$d = 1,2$$

$$i = 1,2,3$$

$$\left.\begin{array}{l} j = j_1 = 1,2,\ldots,M_1 \\ k = k_1 = 1,2,\ldots,N_1 \end{array}\right\} \quad d=1$$

$$\left.\begin{array}{l} j = j_2 = 1,2,\ldots,M_2 \\ k = k_2 = 1,2,\ldots,N_2 \end{array}\right\} \quad d=2$$

$$m = 1,2,3$$

9. From $(J_1)_d$, $(J_2)_d$, $(J_3)_d$

$$(\bar{\Omega}-\Omega_i)^2 - (\bar{\omega}_{jk}^{m})_d^2 = 0 \tag{7.3-21}$$

$$(\bar{\Omega}-\Omega_i) = (\bar{\omega}_{jk}^{m})_d \tag{7.3-22}$$

$$\left|(i-2)^2 + \epsilon_i\beta - 1\right| \frac{\Omega}{\omega_1} = (\bar{\omega}_{jk}^{m})_d \tag{7.3-23}$$

$$d = 1,2$$

$$i = 1,2,3$$

$$\left.\begin{array}{l} j = j_1 = 1,2,\ldots,M_1 \\ k = k_1 = 1,2,\ldots,N_1 \end{array}\right\} \quad d=2$$

$$\left.\begin{array}{l} j = j_2 = 1,2,\ldots,M_2 \\ k = k_2 = 1,2,\ldots,N_2 \end{array}\right\} \quad d=2$$
$$m = 1,2,3$$

10. From $(J_1)_d$, $(J_2)_d$, $(J_3)_d$

$$(\Omega_i+\Omega_p)^2 - (\bar{\omega}_{jk}^{m})_d^2 = 0 \tag{7.3-24}$$

$$\Omega_i + \Omega_p = (\bar{\omega}_{jk}^{m})_d \tag{7.3-25}$$

$$\left|(i-2)^2 + (p-2)^2 + (\Omega_i+\Omega_p)\beta\right| \frac{\Omega}{\omega_1} = (\bar{\omega}_{jk}^{m})_d \tag{7.3-26}$$

$$d = 1,2$$
$$i = 1,2,3$$
$$\left.\begin{array}{l} j = j_1 = 1,2,\ldots,M_1 \\ k = k_1 = 1,2,\ldots,N_1 \end{array}\right\} \quad d=1$$
$$\left.\begin{array}{l} j = j_2 = 1,2,\ldots,M_2 \\ k = k_2 = 1,2,\ldots,N_2 \end{array}\right\} \quad d=2$$
$$m = 1,2,3$$
$$p = 1,2,3$$

11. From $(J_1)_d$, $(J_2)_d$, $(J_3)_d$

$$(\Omega_i-\Omega_p)^2 - (\bar{\omega}_{jk}^{m})_d^2 = 0 \tag{7.3-27}$$

$$\Omega_i - \Omega_p = (\bar{\omega}_{jk}^{m})_d \tag{7.3-28}$$

$$\left|(i-2)^2 - (p-2)^2 + (\varepsilon_i-\varepsilon_p)\beta\right| \frac{\Omega}{\omega_1} = (\bar{\omega}_{jk}^{m})_d \tag{7.3-29}$$

$$d = 1,2$$
$$i = 1,2,3$$

$$\left.\begin{aligned} j &= j_1 = 1,2,\ldots,M_1 \\ k &= k_1 = 1,2,\ldots,N_1 \end{aligned}\right\} \quad d=1$$

$$\left.\begin{aligned} j &= j_2 = 1,2,\ldots,M_2 \\ k &= k_2 = 1,2,\ldots,N_2 \end{aligned}\right\} \quad d=2$$

$$m = 1,2,3$$

$$p = 1,2,3$$

12. From $(J_1)_d$, $(J_2)_d$, $(J_3)_d$

$$(2\Omega_i)^2 - (\tilde{\omega}_{jk}^{m})_d^2 = 0 \tag{7.3-30}$$

$$2\Omega_i = (\tilde{\omega}_{jk}^{m})_d \tag{7.3-31}$$

$$2[(1-2)^2 + \epsilon_i\beta]\frac{\Omega}{\omega_1} = (\tilde{\omega}_{jk}^{m})_d \tag{7.3-32}$$

$$d = 1,2$$

$$i = 1,2,3$$

$$\left.\begin{aligned} j &= j_1 = 1,2,\ldots,M_1 \\ k &= k_1 = 1,2,\ldots,N_1 \end{aligned}\right\} \quad d=1$$

$$\left.\begin{aligned} j &= j_2 = 1,2,\ldots,M_2 \\ k &= k_2 = 1,2,\ldots,N_2 \end{aligned}\right\} \quad d=2$$

$$m = 1,2,3$$

13. From $(J_1)_d$, $(J_2)_d$, $(J_3)_d$

$$[(\alpha_r+s)\bar{\Omega}]^2 - (\tilde{\omega}_{jk}^{m})_d^2 = 0 \tag{7.3-33}$$

$$(\alpha_r+s)\bar{\Omega} = (\tilde{\omega}_{jk}^{m})_d \tag{7.3-34}$$

$$\bar{\Omega} = \frac{\Omega}{\omega_1} = \left| \frac{(\tilde{\omega}_{jk}^{m})_d}{\alpha_r + s} \right| \tag{7.3-35}$$

$$
\begin{aligned}
&d = 1,2 \\
&\left.\begin{aligned} j &= j_1 = 1,2,\ldots,M_1 \\ k &= k_1 = 1,2,\ldots,N_1 \end{aligned}\right\} \quad d=1 \\
&\left.\begin{aligned} j &= j_2 = 1,2,\ldots,M_2 \\ k &= k_2 = 1,2,\ldots,N_2 \end{aligned}\right\} \quad d=2 \\
&m = 1,2,3 \\
&r = 1,2,\ldots,R \\
&s = -S \text{ to } +S
\end{aligned}
$$

7.4 Summary of Stability Equations

Solutions of the stability equations of Section 7.3 will render the unstable values of the thrust frequency. A complete summary of the stability equations are given below.

No.	Equation No.
1.	(7.3-1)
2.	(7.3-3)
3.	(7.3-5)
4.	(7.3-8)
5.	(7.3-11)
6.	(7.3-14)
7.	(7.3-17)
8.	(7.3-20)
9.	(7.3-23)
10.	(7.3-26)
11.	(7.3-29)
12.	(7.3-32)
13.	(7.3-35)

8.0 COMMENTS

8.1 General

The report represents a continuation in the analysis of the dynamic structural behavior of a large rocket booster (see Fig. 1 for the model used in this analysis) subjected to longitudinal excitations. In consideration of the model used in this analysis, certain deflection forms are assumed and by application of the well-known Galerkin method two shell solutions result, one for each shell.

8.2 Natural Frequencies

The natural frequencies of the free-free, thin-walled, circular, cylindrical shells as modified by the relative coupling effect of each cylinder upon the other, with a comparison of the natural frequencies of an earlier single cylinder analysis of ref. 3 , is fully detailed in Appendix D.

8.3 Unstable Values of the Thrust Frequency, Ω

Referring to 7.0 STABILITY ANALYSIS of this report one can see that stability equations are used to determine the unstable values of the thrust frequency for certain longitudinal and circumferential modal configurations of each circular, cylindrical shell.

8.4 Recommendation for Future Investigation

Use of the assumed deflection shapes and the Galerkin method has rendered solutions involving undisturbed (or nondistorted) modal patterns in the model. An improved model consisting of thin-wall cylinder - conical

frustum - cylinder combination should be considered. While a model of this type is still an idealization of an actual vehicle, it would possess the following improvements over the previously considered model of cylinder - bulkhead - cylinder combination.

A. A more accurate synthesization of a missile profile would be achieved.

B. Consideration of the end-rigidity effect of the conical frustum in matching boundary conditions would improve the shell analysis.

C. A better understanding of the relative coupling effect of certain shell geometries in altering the vibrational mode shapes of their adjacent members in a composite structure would result.

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APPENDIX A

SPECIAL TERMS APPEARING IN THE REPORT

A-I. General Statement

Certain terms in the second order differential equations in this report are quite lengthy. A complete summary of these important terms are listed in this appendix.

A-II. Cylinder 1 Coefficients

A-II-1. For $j_1 = 1,2,\ldots,M$; $k_1 = 0$

The coefficients used in eqs. (5.4-6a, -6b) for d=1 are as follows:

$$(a_{j0})_1 = \left\{\frac{c-a}{2} + \frac{L_1}{2j_1\pi L}\left[\sin j_1\pi\left(\frac{2d_1}{L_1}\right) - \sin j_1\pi\left(\frac{d_1-d_2}{L_1}\right)\right]\right\}$$

$$+ \left(\frac{2L_1}{j_1\pi L}\right)\left[\sin\frac{j_1\pi}{2}\left(\frac{2d_1}{L_1}\right) - \sin\frac{j_1\pi}{2}\left(\frac{d_1-d_2}{L_1}\right)\right]$$

$$\cdot \left\{\left(\frac{L_1}{L} - 1\right)\left[\cos\frac{j_1\pi}{2}\left(\frac{d_1-d_2}{L_1}\right) - \cos\frac{j_1\pi}{2}\left(\frac{2d_1}{L_1}\right)\right]\right\} \qquad \text{(A-1)}$$

$$(b_{j0})_1 = \left(\frac{\nu L}{R}\right)\left(\frac{2L_1}{j_1\pi L}\right)^2\left[\sin\frac{j_1\pi}{2}\left(\frac{2d_1}{L_1}\right) - \sin\frac{j_1\pi}{2}\left(\frac{d_1-d_2}{L_1}\right)\right]$$

$$\cdot \left[\cos\frac{j_1\pi}{2}\left(\frac{d_1-d_2}{L_1}\right) - \cos\frac{j_1\pi}{2}\left(\frac{2d_1}{L_1}\right)\right] \qquad \text{(A-2)}$$

$$(c_{j0})_1 = \frac{1}{\mu(1-\nu^2)} \left(\frac{j_1\pi L}{2L_1}\right) \left[\sin j_1\pi\left(\frac{d_1-d_2}{L_1}\right) + \left(\frac{j_1\pi L}{2L_1}\right) \left\{\frac{c-a}{2} + \left(\frac{L_1}{2j_1\pi L}\right) \left[\sin j_1\pi\left(\frac{2d_1}{L_1}\right) - \sin j_1\pi \frac{d_1-d_2}{L_1}\right]\right\}\right] \quad \text{(A-3)}$$

$$(d_{j0})_1 = \frac{\nu\lambda}{\mu(1-\nu^2)} \left[\frac{1}{2} \sin j_1\pi\left(\frac{d_1-d_2}{L_1}\right) + \left(\frac{j_1\pi L}{2L_1}\right) \left\{\frac{c-a}{2} + \left(\frac{L_1}{2j_1\pi L}\right) \left[\sin j_1\pi\left(\frac{2d_1}{L_1}\right) - \sin j_1\pi\left(\frac{d_1-d_2}{L_1}\right)\right]\right\}\right] \quad \text{(A-4)}$$

$$(e_{j0})_1 = \left(\frac{2L_1}{j_1\pi L}\right) \left[\sin \frac{j_1\pi}{2} \left(\frac{2d_1}{L_1}\right) - \sin \frac{j_1\pi}{2} \left(\frac{d_1-d_2}{L_1}\right)\right] \quad \text{(A-5)}$$

$$(f_{j0})_1 = \left(\frac{2L_1}{j_1\pi L}\right)^2 \left[\cos \frac{j_1\pi}{2} \left(\frac{2d_1}{L_1}\right) + \frac{j_1\pi L}{2L_1} \left(\frac{d_1}{L}\right) \sin \frac{j_1\pi}{2} \left(\frac{2d_1}{L_1}\right) - \cos \frac{j_1\pi}{2} \left(\frac{d_1-d_2}{L_1}\right) + \frac{j_1\pi L}{2L_1} \left(\frac{d_2}{L}\right) \sin \frac{j_1\pi}{2} \left(\frac{d_1-d_2}{L_1}\right)\right] \quad \text{(A-6)}$$

$$\begin{aligned}(g_{j0})_1 = \Bigg[&\left(\frac{2L_1}{j_1\pi L}\right) \frac{1}{2} \left(\frac{d_1}{L} - \frac{d_3}{L} + 1\right)^2 \sin \frac{j_1\pi}{2} \left(\frac{2d_1}{L_1}\right) \\ &+ \left(\frac{2L_1}{j_1\pi L}\right)^2 \left(\frac{d_1}{L} - \frac{d_3}{L} + 1\right) \cos \frac{j_1\pi}{2} \left(\frac{2d_1}{L_1}\right) - \left(\frac{2L_1}{j_1\pi L}\right)^3 \sin \frac{j_1\pi}{2} \left(\frac{2d_1}{L_1}\right) \\ &- \left(\frac{2L_1}{j_1\pi L}\right) \frac{1}{2} \left(-\frac{d_2}{L} - \frac{d_3}{L} + 1\right)^2 \sin \frac{j_1\pi}{2} \left(\frac{d_1-d_2}{L_1}\right) \\ &- \left(\frac{2L_1}{j_1\pi L}\right)^2 \left(-\frac{d_2}{L} - \frac{d_3}{L} + 1\right) \cos \frac{j_1\pi}{2} \left(\frac{d_1-d_2}{L_1}\right) \\ &+ \left(\frac{2L_1}{j_1\pi L}\right)^3 \sin \frac{j_1\pi}{2} \left(\frac{d_1-d_2}{L_1}\right)\Bigg] \end{aligned} \quad \text{(A-7)}$$

$$(h_{j0})_1 = -\frac{\nu\lambda}{\mu(1-\nu^2)}\left\{\frac{1}{2}\sin j_1\pi\left(\frac{d_1-d_2}{L_1}\right) - \left(\frac{j_1\pi L}{2L_1}\right)\left\{\frac{c-a}{2} - \left(\frac{L_1}{2j_1\pi L}\right)\right.\right.$$
$$\left.\left.\left[\sin j_1\pi\left(\frac{2d_1}{L_1}\right) - \sin j_1\pi\left(\frac{d_1-d_2}{L_1}\right)\right]\right\}\right] \qquad \text{(A-8)}$$

$$(j_{j0})_i = \left\{\frac{c-a}{2} - \left(\frac{L_1}{2j_1\pi L}\right)\left[\sin j_1\pi\left(\frac{2d_1}{L_1}\right) - \sin j_1\pi\left(\frac{d_1-d_2}{L_1}\right)\right]\right\} \qquad \text{(A-9)}$$

$$(k_{j0})_1 = \frac{1}{\mu(1-\nu^2)}\left\{\left[\frac{\sigma^2}{12}\left(\frac{j_1\pi L}{2L_1}\right)^4 + \lambda^2\right]\left\{\frac{c-a}{2} - \frac{L_1}{2j_1\pi L}\left[\sin j_1\pi\left(\frac{2d_1}{L_1}\right)\right.\right.\right.$$
$$\left.\left.\left. - \sin j_1\pi\left(\frac{d_1-d_2}{L_1}\right)\right]\right\} - \frac{\sigma^2}{12}\left(\frac{j_1\pi L}{2L_1}\right)^3\left[2\sin j_1\pi\left(\frac{d_1-d_2}{L_1}\right)\right]\right\} \qquad \text{(A-10)}$$

$$(\ell_{j0})_1 = \frac{2L_1}{j_1\pi L}\left(\frac{d_3}{L}\right)\left[\cos\frac{j_1\pi}{2}\left(\frac{2d_1}{L_1}\right) - \cos\frac{j_1\pi}{2}\left(\frac{d_1-d_2}{L_1}\right)\right] \qquad \text{(A-11)}$$

$$(m_{j0})_1 = \left(\frac{2L_1}{j_1\pi L}\right)^2\left[\sin\frac{j_1\pi}{2}\left(\frac{2d_1}{L_1}\right) - \frac{j_1\pi L}{2L_1}\left(\frac{d_1}{L}\right)\cos\frac{j_1\pi}{2}\left(\frac{2d_1}{L_1}\right)\right.$$
$$\left. - \sin\frac{j_1\pi}{2}\left(\frac{d_1-d_2}{L_1}\right) - \frac{j_1\pi L}{2L_1}\left(\frac{d_2}{L}\right)\cos\frac{j_1\pi}{2}\left(\frac{d_1-d_2}{L_1}\right)\right] \qquad \text{(A-12)}$$

A-II-2. For $j_1 = 1,2,\ldots,M_1$; $k_1 = 1,2,\ldots,N_1$

The coefficients used in eqs. (5.4-8) for d=1 are as follows:

$$(a_{jk})_1 = \left\{ \frac{c-a}{2} + \frac{L_1}{2j_1\pi L} \left[\sin j_1\pi \left(\frac{2d_1}{L_1}\right) - \sin j_1\pi \left(\frac{d_1-d_2}{L_1}\right)\right] \right.$$

$$+ \left(\frac{2L_1}{j_1\pi L}\right) \left[\sin \frac{j_1\pi}{2} \left(\frac{2d_1}{L_1}\right) - \sin \frac{j_1\pi}{2} \left(\frac{d_1-d_2}{L_1}\right)\right]$$

$$\left. \cdot \left\{ \left(\frac{L_1}{L} - 1\right) \left[\cos \frac{j_1\pi}{2} \left(\frac{d_1-d_2}{L_1}\right) - \cos \frac{j_1\pi}{2} \left(\frac{2d_1}{L_1}\right)\right] \right\} \right] \qquad \text{(A-13)}$$

$$(b_{jk})_1 = \left(\frac{\nu L}{R}\right) \left(\frac{2L_1}{j_1\pi L}\right)^2 k_1 \left[\sin \frac{j_1\pi}{2} \left(\frac{2d_1}{L_1}\right) - \sin \frac{j_1\pi}{2} \left(\frac{d_1-d_2}{L_1}\right)\right]$$

$$\cdot \left[\cos \frac{j_1\pi}{2} \left(\frac{2d_1}{L_1}\right) - \cos \frac{j_1\pi}{2} \left(\frac{d_1-d_2}{L_1}\right)\right] \qquad \text{(A-14)}$$

$$(c_{jk})_1 = \left(\frac{\nu L}{R}\right) \left(\frac{2L_1}{j_1\pi L}\right)^2 \left[\sin \frac{j_1\pi}{2} \left(\frac{2d_1}{L_1}\right) - \sin \frac{j_1\pi}{2} \left(\frac{d_1-d_2}{L_1}\right)\right]$$

$$\cdot \left[\cos \frac{j_1\pi}{2} \left(\frac{d_1-d_2}{L_1}\right) - \cos \frac{j_1\pi}{2} \left(\frac{2d_1}{L_1}\right)\right] \qquad \text{(A-15)}$$

$$(d_{jk})_1 = \frac{1}{\mu(1-\nu^2)} \left\{ \left(\frac{j_1\pi L}{2L_1}\right) \left| \sin j_1\pi \left(\frac{d_1-d_2}{L_1}\right) + \left(\frac{j_1\pi L}{2L_1}\right) \left\{ \frac{c-a}{2} + \frac{L_1}{2j_1\pi L} \right.\right.\right.$$

$$\cdot \left[\sin j_1\pi \left(\frac{2d_1}{L_1}\right)\right.$$

$$\left.\left.\left. - \sin j_1\pi \left(\frac{d_1-d_2}{L_1}\right)\right]\right\}\right| + \frac{(1-\nu)}{2} \lambda^2 k_1^2 \left| \left\{ \frac{c-a}{2} + \frac{L_1}{2j_1\pi L} \left[\sin j_1\pi \left(\frac{2d_1}{L_1}\right)\right.\right.\right.$$

$$\left.\left.\left. - \sin j_1\pi \left(\frac{d_1-d_2}{L_1}\right)\right]\right\} + \left(\frac{2L_1}{j_1\pi L}\right) \left[\sin \frac{j_1\pi}{2} \left(\frac{2d_1}{L_1}\right) - \sin \frac{j_1\pi}{2} \left(\frac{d_1-d_2}{L_1}\right)\right]$$

$$\left.\left.\left. \cdot \left\{ \left(\frac{L_1}{L} - 1\right) \left[\cos \frac{j_1\pi}{2} \left(\frac{d_1-d_2}{L_1}\right) - \cos \frac{j_1\pi}{2} \left(\frac{2d_1}{L_1}\right)\right] \right\} \right| \right\} \right] \qquad \text{(A-16)}$$

$$(e_{jk})_1 = \frac{1}{\mu(1-\nu^2)} \left\{ \left[\frac{(1-\nu)}{2} \lambda^2 \left(\frac{\nu L}{R}\right) \left(\frac{2L_1}{j_1\pi L}\right)^2 k_1^3 \left[\sin \frac{j_1\pi}{2} \left(\frac{2d_1}{L_1}\right) - \sin \frac{j_1\pi}{2} \left(\frac{d_1-d_2}{L_1}\right)\right] \right.\right.$$

$$\cdot \left[\cos \frac{j_1\pi}{2} \left(\frac{d_1-d_2}{L_1}\right) - \cos \frac{j_1\pi}{2} \left(\frac{2d_1}{L_1}\right)\right]$$

$$- \frac{(1+\nu)}{2} \lambda k_1 \left(\frac{1}{2} \sin j_1\pi \left(\frac{d_1-d_2}{L_1}\right) + \left(\frac{j_1\pi L}{2L_1}\right) \left\{ \frac{c-a}{2} + \frac{L_1}{2j_1\pi L} \right.\right.$$

$$\left.\left.\left.\left. \cdot \left[\sin j_1\pi \left(\frac{2d_1}{L_1}\right) - \sin j_1\pi \left(\frac{d_1-d_2}{L_1}\right)\right] \right\} \right) \right] \right\} \qquad \text{(A-17)}$$

$$(f_{jk})_1 = \frac{1}{\mu(1-\nu^2)} \left\{ \left[\frac{(1-\nu)}{2} \lambda^2 \left(\frac{\nu L}{R}\right) \left(\frac{2L_1}{j_1\pi L}\right)^2 k_1^2 \left[\sin \frac{j_1\pi}{2} \left(\frac{2d_1}{L_1}\right) - \sin \frac{j_1\pi}{2} \left(\frac{d_1-d_2}{L_1}\right)\right] \right.\right.$$

$$\cdot \left[\cos \frac{j_1\pi}{2} \left(\frac{d_1-d_2}{L_1}\right) - \cos \frac{j_1\pi}{2} \left(\frac{2d_1}{L_1}\right)\right]$$

$$+ \nu\lambda \left(\frac{1}{2} \sin j_1\pi \left(\frac{d_1-d_2}{L_1}\right) + \frac{j_1\pi L}{2L_1} \left\{ \frac{c-a}{2} + \frac{L_1}{2j_1\pi L} \left[\sin j_1\pi \left(\frac{2d_1}{L_1}\right) \right.\right.\right.$$

$$\left.\left.\left.\left. - \sin j_1\pi \left(\frac{d_1-d_2}{L_1}\right)\right] \right\} \right) \right] \right\} \qquad \text{(A-18)}$$

$$(g_{jk})_1 = \frac{\lambda k_1}{2(1-\nu)\mu} \left\{ \left[\frac{1}{2} \sin j_1\pi \left(\frac{d_1-d_2}{L_1}\right) - \left(\frac{j_1\pi L}{2L_1}\right) \left\{ \frac{c-a}{2} - \frac{L_1}{2j_1\pi L} \left[\sin j_1\pi \left(\frac{2d_1}{L_1}\right) \right.\right.\right.\right.$$

$$\left.\left.\left.\left. - \sin j_1\pi \left(\frac{d_1-d_2}{L_1}\right)\right] \right\} \right] \right\} \qquad \text{(A-19)}$$

$$(h_{jk})_1 = \left\{ \left[\frac{c-a}{2} - \frac{L_1}{2j_1\pi L} \left[\sin j_1\pi \left(\frac{2d_1}{L_1}\right) - \sin j_1\pi \left(\frac{d_1-d_2}{L_1}\right)\right] \right] \right\} \qquad \text{(A-20)}$$

$$(j_{jk})_1 = \frac{1}{\mu(1-\nu^2)} \left\{ \left[\frac{(1-\nu)}{2} \left(\frac{j_1\pi L}{2L_1}\right)^2 + \lambda^2 k_1^2 \right] \left\{ \frac{c-a}{2} - \frac{L_1}{2j_1\pi L} \left[\sin j_1\pi \left(\frac{2d_1}{L_1}\right) - \sin j_1\pi \left(\frac{d_1-d_2}{L_1}\right) \right] \right\} - \left(\frac{1-\nu}{2}\right) \left(\frac{j_1\pi L}{2L_1}\right) \left[\sin j_1\pi \left(\frac{d_1-d_2}{L_1}\right) \right] \right\} \quad \text{(A-21)}$$

$$(k_{jk})_1 = -\frac{\lambda^2 k_1}{\mu(1-\nu^2)} \left\{ \frac{c-a}{2} - \frac{L_1}{2j_1\pi L} \left[\sin j_1\pi \left(\frac{2d_1}{L_1}\right) - \sin j_1\pi \left(\frac{d_1-d_2}{L_1}\right) \right] \right\} \quad \text{(A-22)}$$

$$(\ell_{jk})_1 = \frac{\nu\lambda}{\mu(1-\nu^2)} \left[\left(\frac{j_1\pi L}{2L_1}\right) \left\{ \frac{c-a}{2} - \frac{L_1}{2j_1\pi L} \left[\sin j_1\pi \left(\frac{2d_1}{L_1}\right) - \sin j_1\pi \left(\frac{d_1-d_2}{L_1}\right) \right] \right\} - \frac{1}{2} \sin j_1\pi \left(\frac{d_1-d_2}{L_1}\right) \right] \quad \text{(A-23)}$$

$$(m_{jk})_1 = \frac{\lambda^2 k_1}{\mu(1-\nu^2)} \left\{ \left(\frac{L_1}{2j_1\pi L}\right) \left[\sin j_1\pi \left(\frac{2d_1}{L_1}\right) - \sin j_1\pi \left(\frac{d_1-d_2}{L_1}\right) \right] - \frac{c-a}{2} \right\} \quad \text{(A-24)}$$

$$(n_{jk})_1 = \left\{ \frac{c-a}{2} - \left(\frac{L_1}{2j_1\pi L}\right) \left[\sin j_1\pi \left(\frac{2d_1}{L_1}\right) - \sin j_1\pi \left(\frac{d_1-d_2}{L_1}\right) \right] \right\} \quad \text{(A-25)}$$

$$(p_{jk})_1 = \frac{1}{\mu(1-\nu^2)} \left\{ \left\{ \frac{\sigma^2}{12} \left[\lambda^4 k_1^4 + 2\lambda^2 k_1^2 \left(\frac{j_1\pi L}{2L_1}\right)^2 + \left(\frac{j_1\pi L}{2L_1}\right)^4 \right] + \lambda^2 \right\} \cdot \left\{ \frac{c-a}{2} - \frac{L_1}{2j_1\pi L} \left[\sin j_1\pi \left(\frac{2d_1}{L_1}\right) - \sin j_1\pi \left(\frac{d_1-d_2}{L_1}\right) \right] \right\} - \frac{\sigma^2}{12} (4) \left(\frac{j_1\pi L}{2L_1}\right) \left[\left(\frac{j_1\pi L}{2L_1}\right)^2 + \lambda^2 k_1^2 \right] \left[\frac{1}{2} \sin j_1\pi \left(\frac{d_1-d_2}{L_1}\right) \right] \right\} \quad \text{(A-26)}$$

A-III. Cylinder 2 Coefficients

A-III-1. For $j_2 = 1,2,\ldots,M_2$; $k_2 = 0$

The coefficients used in eqs. (5.4-6) for d=2 are as follows:

$$(a_{j0})_2 = \left\{\frac{b-c}{2} + \frac{L_2}{2j_2\pi L}\left[\sin j_2\pi\left(\frac{d_1+d_3-L_1}{L_2}\right) - \sin j_2\pi\left(\frac{2d_1-L_1}{L_2}\right)\right]\right\}$$

$$+ \left(\frac{2L_2}{j_2\pi L}\right)\left[\sin\frac{j_2\pi}{2}\left(\frac{d_1+d_3-L_1}{L_2}\right) - \sin\frac{j_2\pi}{2}\left(\frac{2d_1-L_1}{L_2}\right)\right]$$

$$\cdot \left\{\left(\frac{L_2}{L} - 1\right)\left[\cos\frac{j_2\pi}{2}\left(\frac{2d_1-L_1}{L_2}\right) - \cos\frac{j_2\pi}{2}\left(\frac{d_1+d_3-L_1}{L_2}\right)\right]\right\} \qquad \text{(A-27)}$$

$$(b_{j0})_2 = \left(\frac{\nu L}{R}\right)\left(\frac{2L_2}{j_2\pi L}\right)^2\left[\sin\frac{j_2\pi}{2}\left(\frac{d_1+d_3-L_1}{L_2}\right) - \sin\frac{j_2\pi}{2}\left(\frac{2d_1-L_1}{L_2}\right)\right]$$

$$\cdot \left[\cos\frac{j_2\pi}{2}\left(\frac{2d_1-L_1}{L_2}\right) - \cos\frac{j_2\pi}{2}\left(\frac{d_1+d_3-L_1}{L_2}\right)\right] \qquad \text{(A-28)}$$

$$(c_{j0})_2 = \frac{1}{\mu(1-\nu^2)}\left(\frac{j_2\pi L}{2L_2}\right)\left\{\sin j_2\pi\left(\frac{2d_1-L_1}{L_2}\right) + \left(\frac{j_2\pi L}{2L_2}\right)\left\{\frac{b-c}{2} + \frac{L_2}{2j_2\pi L}\right.\right.$$

$$\left.\left.\cdot \left[\sin j_2\pi\left(\frac{d_1+d_3-L_1}{L_2}\right) - \sin j_2\pi\left(\frac{2d_1-L_1}{L_2}\right)\right]\right\}\right\} \qquad \text{(A-29)}$$

$$(d_{j0})_2 = \frac{\nu\lambda}{\mu(1-\nu^2)}\left\{\frac{1}{2}\sin j_2\pi\left(\frac{2d_1-L_1}{L_2}\right) + \frac{j_2\pi L}{2L_2}\left\{\frac{b-c}{2} + \frac{L_2}{2j_2\pi L}\right.\right.$$

$$\left.\left.\cdot \left[\sin j_2\pi\left(\frac{d_1+d_3-L_1}{L_2}\right) - \sin j_2\pi\left(\frac{2d_1-L_1}{L_2}\right)\right]\right\}\right\} \qquad \text{(A-30)}$$

$$(e_{j0})_2 = \left(\frac{2L_2}{j_2\pi L}\right)\left[\sin\frac{j_2\pi}{2}\left(\frac{d_1+d_3-L_1}{L_2}\right) - \sin\frac{j_2\pi}{2}\left(\frac{2d_1-L_1}{L_2}\right)\right] \tag{A-31}$$

$$(f_{j0})_2 = \left(\frac{2L_2}{j_2\pi L}\right)^2\left[\cos\frac{j_2\pi}{2}\left(\frac{d_1+d_3-L_1}{L_2}\right) + \frac{j_2\pi L}{2L_2}\left(\frac{d_3}{L}\right)\sin\frac{j_2\pi}{2}\left(\frac{d_1+d_3-L_1}{L_2}\right)\right.$$
$$\left. - \cos\frac{j_2\pi}{2}\left(\frac{2d_1-L_1}{L_2}\right) - \frac{j_2\pi L}{2L_2}\left(\frac{d_1}{L}\right)\sin\frac{j_2\pi}{2}\left(\frac{2d_1-L_1}{L_2}\right)\right] \tag{A-32}$$

$$(g_{j0})_2 = \left[\left(\frac{2L_2}{j_2\pi L}\right)\frac{1}{2}(1)^2\sin\frac{j_2\pi}{2}\left(\frac{d_1+d_3-L_1}{L_2}\right)\right.$$
$$+ \left(\frac{2L_2}{j_2\pi L}\right)^2(1)\cos\frac{j_2\pi}{2}\left(\frac{d_1+d_3-L_1}{L_2}\right)$$
$$- \left(\frac{2L_2}{j_2\pi L}\right)^3\sin\frac{j_2\pi}{2}\left(\frac{d_1+d_3-L_1}{L_2}\right)$$
$$- \left(\frac{2L_1}{j_2\pi L}\right)\frac{1}{2}\left(\frac{d_1}{L} - \frac{d_3}{L} + 1\right)^2\sin\frac{j_2\pi}{2}\left(\frac{2d_1-L_1}{L_2}\right)$$
$$- \left(\frac{2L_2}{j_2\pi L}\right)^2\left(\frac{d_1}{L} - \frac{d_3}{L} + 1\right)\cos\frac{j_2\pi}{2}\left(\frac{2d_1-L_1}{L_2}\right)$$
$$\left. + \left(\frac{2L_2}{j_2\pi L}\right)^3\sin\frac{j_2\pi}{2}\left(\frac{2d_1-L_1}{L_2}\right)\right] \tag{A-33}$$

$$(h_{j0})_2 = -\frac{\nu\lambda}{\mu(1-\nu^2)}\left\{\left[\frac{1}{2}\sin j_2\pi\left(\frac{2d_1-L_1}{L_2}\right) - \left(\frac{j_2\pi L}{2L_2}\right)\left\{\frac{b-c}{2} - \frac{L_2}{2j_2\pi L}\right.\right.\right.$$
$$\left.\left.\left. \cdot\left[\sin j_2\pi\left(\frac{d_1+d_3-L_1}{L_2}\right) - \sin j_2\pi\left(\frac{2d_1-L_1}{L_2}\right)\right]\right\}\right] \tag{A-34}$$

$$(j_{j0})_2 = \left\{ \frac{b-c}{2} - \frac{L_2}{2j_2\pi L} \left[\sin j_2\pi \left(\frac{d_1+d_3-L_1}{L_2} \right) - \sin j_2\pi \left(\frac{2d_1-L_1}{L_2} \right) \right] \right\} \qquad \text{(A-35)}$$

$$(k_{j0})_2 = \frac{1}{\mu(1-\nu^2)} \left\{ \left[\frac{\sigma^2}{12} \left(\frac{j_2\pi L}{2L_2} \right)^4 + \lambda^4 \right] \left\{ \frac{b-c}{2} - \frac{L_2}{2j_2\pi L} \left[\sin j_2\pi \left(\frac{d_1+d_3-L_1}{L_2} \right) - \sin j_2\pi \left(\frac{2d_1-L_1}{L_2} \right) \right] \right\} - \frac{\sigma^2}{12} \left(\frac{j_2\pi L}{2L_2} \right)^3 \left[2 \sin j_2\pi \left(\frac{2d_1-L_1}{L_2} \right) \right] \right\} \qquad \text{(A-36)}$$

$$(\ell_{j0})_2 = \left(\frac{2L_2}{j_2\pi L} \right) \left(\frac{d_3}{L} \right) \left[\cos \frac{j_2\pi}{2} \left(\frac{d_1+d_3-L_1}{L_2} \right) - \cos \frac{j_2\pi}{2} \left(\frac{2d_1-L_1}{L_2} \right) \right] \qquad \text{(A-37)}$$

$$(m_{j0})_2 = \left(\frac{2L_2}{j_2\pi L} \right)^2 \left[\sin \frac{j_2\pi}{2} \left(\frac{d_1+d_3-L_1}{L_2} \right) - \frac{j_2\pi L}{2L_2} \left(\frac{d_3}{L} \right) \cos \frac{j_2\pi}{2} \left(\frac{d_1+d_3-L_1}{L_2} \right) - \sin \frac{j_2\pi}{2} \left(\frac{2d_1-L_1}{L_2} \right) + \frac{j_2\pi L}{2L_2} \left(\frac{d_1}{L} \right) \cos \frac{j_2\pi}{2} \left(\frac{2d_1-L_1}{L_2} \right) \right] \qquad \text{(A-38)}$$

A-III-2. For $j_2 = 1,2,\ldots,M_2$; $k_2 = 1,2,\ldots,N_2$

The coefficients used in eqs. (5.4-8) for d=2 are as follows:

$$(a_{jk})_2 = \left\{ \left[\frac{b-c}{2} + \frac{L_2}{2j_2\pi L} \left[\sin j_2\pi \left(\frac{d_1+d_3-L_1}{L_2} \right) - \sin j_2\pi \left(\frac{2d_1-L_1}{L_2} \right) \right] + \left(\frac{2L_2}{j_2\pi L} \right) \left[\sin \frac{j_2\pi}{2} \left(\frac{d_1+d_3-L_1}{L_2} \right) - \sin \frac{j_2\pi}{2} \left(\frac{2d_1-L_1}{L_2} \right) \right] \cdot \left\{ \left(\frac{L_2}{L} - 1 \right) \left[\cos \frac{j_2\pi}{2} \left(\frac{2d_1-L_1}{L_2} \right) - \cos \frac{j_2\pi}{2} \left(\frac{d_1+d_3-L_1}{L_2} \right) \right] \right\} \right] \right\} \qquad \text{(A-39)}$$

$$(b_{jk})_2 = \left(\frac{\nu L}{R}\right)\left(\frac{2L_2}{j_2\pi L}\right)^2 k_2 \left[\sin\frac{j_2\pi}{2}\left(\frac{d_1+d_3-L_1}{L_2}\right) - \sin\frac{j_2\pi}{2}\left(\frac{2d_1-L_1}{L_2}\right)\right]$$

$$\cdot \left[\cos\frac{j_2\pi}{2}\left(\frac{d_1+d_3-L_1}{L_2}\right) - \cos\frac{j_2\pi}{2}\left(\frac{2d_1-L_1}{L_2}\right)\right] \qquad \text{(A-40)}$$

$$(c_{jk})_2 = \left(\frac{\nu L}{R}\right)\left(\frac{2L_2}{j_2\pi L}\right)^2 \left[\sin\frac{j_2\pi}{2}\left(\frac{d_1+d_3-L_1}{L_2}\right) - \sin\frac{j_2\pi}{2}\left(\frac{2d_1-L_1}{L_2}\right)\right]$$

$$\cdot \left[\cos\frac{j_2\pi}{2}\left(\frac{2d_1-L_1}{L_2}\right) - \cos\frac{j_2\pi}{2}\left(\frac{d_1+d_3-L_1}{L_2}\right)\right] \qquad \text{(A-41)}$$

$$(d_{jk})_2 = \frac{1}{\mu(1-\nu^2)}\left\{\left[\left(\frac{j_2\pi L}{2L_2}\right)^2 + \frac{(1-\nu)}{2}\lambda^2 k_2^2\right]\left\{\frac{b-c}{2} + \frac{L_2}{2j_2\pi L}\left[\sin j_2\pi\left(\frac{d_1+d_3-L_1}{L_2}\right)\right.\right.\right.$$

$$\left.\left.\left. - \sin j_2\pi\left(\frac{2d_1-L_1}{L_2}\right)\right]\right\}$$

$$+ \frac{(1-\nu)}{2}\lambda^2 k_2^2\left(\frac{2L_2}{j_2\pi L}\right)\left[\sin\frac{j_2\pi}{2}\left(\frac{d_1+d_3-L_1}{L_2}\right) - \sin\frac{j_2\pi}{2}\left(\frac{2d_1-L_1}{L_2}\right)\right]$$

$$\left.\cdot \left\{\left(\frac{L_2}{L} - 1\right)\left[\cos\frac{j_2\pi}{2}\left(\frac{2d_1-L_1}{L_2}\right) - \cos\frac{j_2\pi}{2}\left(\frac{d_1+d_3-L_1}{L_2}\right)\right]\right\}\right] \qquad \text{(A-42)}$$

$$(e_{jk})_2 = \frac{1}{\mu(1-\nu^2)}\left\{\left[\frac{1-\nu}{2}\lambda^2\left(\frac{\nu L}{R}\right)\left(\frac{2L_2}{j_2\pi L}\right)^2 k_2^3\left[\sin\frac{j_2\pi}{2}\left(\frac{d_1+d_3-L_1}{L_2}\right)\right.\right.\right.$$

$$\left.\left.\left. - \sin\frac{j_2\pi}{2}\left(\frac{2d_1-L_1}{L_2}\right)\right]\left[\cos\frac{j_2\pi}{2}\left(\frac{d_1+d_3-L_1}{L_2}\right) - \cos\frac{j_2\pi}{2}\left(\frac{2d_1-L_1}{L_2}\right)\right]\right.\right.$$

$$\left.\left. - \frac{(1+\nu)}{2}\lambda k_2\left\{\frac{1}{2}\sin j_2\pi\left(\frac{2d_1-L_1}{L_2}\right) + \left(\frac{j_2\pi L}{2L_2}\right)\left\{\frac{b-c}{2} + \frac{L_2}{2j_2\pi L}\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\cdot \left[\sin j_2\pi\left(\frac{d_1+d_3-L_1}{L_2}\right) - \sin j_2\pi\left(\frac{2d_1-L_1}{L_2}\right)\right]\right\}\right\}\right] \qquad \text{(A-43)}$$

$$(f_{jk})_2 = \frac{1}{\mu(1-\nu^2)} \left[\frac{(1-\nu)}{2} \lambda^2 \left(\frac{\nu L}{R}\right) \left(\frac{2L_2}{j_2\pi L}\right)^2 k_2^2 \left[\sin \frac{j_2\pi}{2} \left(\frac{d_1+d_3-L_1}{L_2}\right) - \sin \frac{j_2\pi}{2} \left(\frac{2d_1-L_1}{L_2}\right)\right] \cdot \left[\cos \frac{j_2\pi}{2} \left(\frac{2d_1-L_1}{L_2}\right) - \cos \frac{j_2\pi}{2} \left(\frac{d_1+d_3-L_1}{L_2}\right)\right] + \nu\lambda \left\{ \frac{1}{2} \sin j_2\pi \left(\frac{2d_1-L_1}{L_2}\right) + \frac{j_2\pi L}{2L_2} \left\{ \frac{b-c}{2} + \frac{L_2}{2j_2\pi L} \left[\sin \frac{j_2\pi}{2} \left(\frac{d_1+d_3-L_1}{L_2}\right) - \sin \frac{j_2\pi}{2} \left(\frac{2d_1-L_1}{L_2}\right)\right]\right\}\right\}\right] \quad \text{(A-44)}$$

$$(g_{jk})_2 = \frac{\lambda k_2}{2(1-\nu)\mu} \left[\frac{1}{2} \sin j_2\pi \left(\frac{2d_1-L_1}{L_2}\right) - \left(\frac{j_2\pi L}{2L_2}\right) \left\{ \frac{b-c}{2} - \frac{L_2}{2j_2\pi L} \cdot \left[\sin j_2\pi \left(\frac{d_1+d_3-L_1}{L_2}\right) - \sin j_2\pi \left(\frac{2d_1-L_1}{L_2}\right)\right]\right\}\right] \quad \text{(A-45)}$$

$$(h_{jk})_2 = \left\{ \frac{b-c}{2} - \frac{L_2}{2j_2\pi L} \left[\sin j_2\pi \left(\frac{d_1+d_3-L_1}{L_2}\right) - \sin j_2\pi \left(\frac{2d_1-L_1}{L_2}\right)\right]\right\} \quad \text{(A-46)}$$

$$(j_{jk})_2 = \frac{1}{\mu(1-\nu^2)} \left[\left[\frac{(1-\nu)}{2} \left(\frac{j_2\pi L}{2L_2}\right)^2 + \lambda^2 k_2^2\right] \left\{ \frac{b-c}{2} - \frac{L_2}{2j_2\pi L} \left[\sin j_2\pi \left(\frac{d_1+d_3-L_1}{L_2}\right) - \sin j_2\pi \left(\frac{2d_1-L_1}{L_2}\right)\right]\right\} - \frac{(1-\nu)}{2} \left(\frac{j_2\pi L}{2L_2}\right) \left[\sin j_2\pi \left(\frac{2d_1-L_1}{L_2}\right)\right]\right] \quad \text{(A-47)}$$

$$(k_{jk})_2 = -\frac{\lambda^2 k_2}{\mu(1-\nu^2)} \left\{ \frac{b-c}{2} - \frac{L_2}{2j_2\pi L} \left[\sin j_2\pi \left(\frac{d_1+d_3-L_1}{L_2}\right) - \sin j_2\pi \left(\frac{2d_1-L_1}{L_2}\right)\right]\right\} \quad \text{(A-48)}$$

$$(\ell_{jk})_2 = \frac{\nu\lambda}{\mu(1-\nu^2)} \left[\left(\frac{j_2\pi L}{2L_2}\right) \left\{ \frac{b-c}{2} - \frac{L_2}{2j_2\pi L} \left[\sin j_2\pi \left(\frac{d_1+d_3-L_1}{L_2}\right) - \sin j_2\pi \left(\frac{2d_1-L_1}{L_2}\right) \right] \right\} - \frac{1}{2} \sin j_2\pi \left(\frac{2d_1-L_1}{L_2}\right) \right] \quad \text{(A-49)}$$

$$(m_{jk})_2 = \frac{\lambda^2 k_2}{\mu(1-\nu^2)} \left\{ \frac{L_2}{2j_2\pi L} \left[\sin j_2\pi \left(\frac{d_1+d_3-L_1}{L_2}\right) - \sin j_2\pi \left(\frac{2d_1 - L_1}{L_2}\right) \right] - \frac{b-c}{2} \right\} \quad \text{(A-50)}$$

$$(n_{jk})_2 = \left\{ \frac{b-c}{2} - \frac{L_2}{2j_2\pi L} \left[\sin j_2\pi \left(\frac{d_1+d_3-L_1}{L_2}\right) - \sin j_2\pi \left(\frac{2d_1-L_1}{L_2}\right) \right] \right\} \quad \text{(A-51)}$$

$$(p_{jk})_2 = \frac{1}{\mu(1-\nu^2)} \left[\left\{ \frac{\sigma^2}{12} \left[\lambda^4 k_2^4 + 2\lambda^2 k_2^2 \left(\frac{j_2\pi L}{2L_2}\right)^2 + \left(\frac{j_2\pi L}{2L_2}\right)^4 \right] + \lambda^2 \right\} \cdot \left\{ \frac{b-c}{2} - \frac{L_2}{2j_2\pi L} \left[\sin j_2\pi \left(\frac{d_1+d_3-L_1}{L_2}\right) - \sin j_2\pi \left(\frac{2d_1-L_1}{L_2}\right) \right] \right\} - \frac{\sigma^2}{12} (4) \left(\frac{j_2\pi L}{2L_2}\right) \left[\left(\frac{j_2\pi L}{2L_2}\right)^2 + \lambda^2 k_2^2 \right] \left[\frac{1}{2} \sin j_2\pi \left(\frac{2d_1-L_1}{L_2}\right) \right] \right\} \quad \text{(A-52)}$$

APPENDIX B

COEFFICIENT INTEGRALS

B-I General Statement

Certain integrals appear as coefficients in the second order differential equations in this report. A summary of these coefficient integrals appear in this appendix. The ϕ-functions and their respective derivatives that appear in the coefficient integrals are listed in Appendix C.

B-II Cylinder 1 Coefficient Integrals

$$(r_{jn})_1 = \left(\frac{2L_1}{j_1\pi L}\right)\left[\cos\frac{j_1\pi}{2}\left(\frac{d_1-d_2}{L_1}\right) - \cos\frac{j_1\pi}{2}\left(\frac{2d_1}{L_1}\right)\right] \tag{B-1}$$

$$(s_{jn})_1 = \int_a^c \phi_n(\bar{\xi}) \sin\frac{j_1\pi}{2}\left(\frac{L}{L_1}\bar{\xi} - r_1 + 1\right)d\bar{\xi} \tag{B-2}$$

$$(t_{jn})_1 = \int_a^c \phi_n'(\bar{\xi}) \sin\frac{j_1\pi}{2}\left(\frac{L}{L_1}\bar{\xi} - r_1 + 1\right)d\bar{\xi} \tag{B-3}$$

$$(x_{jn})_1 = \int_a^c \phi_n''(\bar{\xi}) \sin\frac{j_1\pi}{2}\left(\frac{L}{L_1}\bar{\xi} - r_1 + 1\right)d\bar{\xi} \tag{B-4}$$

$$(y_{jn})_1 = \int_a^c \phi_n'''(\bar{\xi}) \sin\frac{j_1\pi}{2}\left(\frac{L}{L_1}\bar{\xi} - r_1 + 1\right)d\bar{\xi} \tag{B-5}$$

$$(z_{jn})_1 = \int_a^c \phi_n'(\bar{\xi}) \cos \frac{j_1\pi}{2} \left(\frac{L}{L_1}\bar{\xi} - r_1 + 1\right) d\bar{\xi} \qquad \text{(B-6)}$$

$$j = j_1 = 1,2,\ldots,M_1$$

$$n = 1,2,\ldots,N$$

B-III Cylinder 2 Coefficient Integrals

$$(r_{jn})_2 = \left(\frac{2L_2}{j_2\pi L}\right) \left[\cos \frac{j2\pi}{2} \left(\frac{2d_1 - L_1}{L_2}\right) - \cos \frac{j2\pi}{2} \left(\frac{d_1 + d_3 - L_1}{L_2}\right)\right] \qquad \text{(B-7)}$$

$$(s_{jn})_2 = \int_c^b \phi_n(\bar{\xi}) \sin \frac{j2\pi}{2} \left(\frac{L}{L_2}\bar{\xi} + r_2 + 1\right) d\bar{\xi} \qquad \text{(B-8)}$$

$$(t_{jn})_2 = \int_c^b \phi_n'(\bar{\xi}) \sin \frac{j2\pi}{2} \left(\frac{L}{L_2}\bar{\xi} + r_2 + 1\right) d\bar{\xi} \qquad \text{(B-9)}$$

$$(x_{jn})_2 = \int_c^b \phi_n''(\bar{\xi}) \sin \frac{j2\pi}{2} \left(\frac{L}{L_1}\bar{\xi} + r_2 + 1\right) d\bar{\xi} \qquad \text{(B-10)}$$

$$(y_{jn})_2 = \int_c^b \phi_n'''(\bar{\xi}) \sin \frac{j2\pi}{2} \left(\frac{L}{L_2}\bar{\xi} + r_2 + 1\right) d\bar{\xi} \qquad \text{(B-11)}$$

$$(z_{jn})_2 = \int_c^b \phi_n'(\bar{\xi}) \cos \frac{j2\pi}{2} \left(\frac{L}{L_2}\bar{\xi} + r_2 + 1\right) d\bar{\xi} \qquad \text{(B-12)}$$

$$j = j_2 = 1,2,\ldots,M_2$$

$$n = 1,2,\ldots,N$$

B-IV Evaluation of Cylinder 1 Integrals

By substituting the ϕ-functions of Appendix C into equations (B-2) through (B-6) the coefficient integrals of Cylinder 1 may be evaluated. The evaluated integrals are presented below.

$$(s_{jn})_1 = \int_a^c \phi_n(\bar{\xi}) \sin \frac{j_1\pi}{2} \left(\frac{L}{L_1}\bar{\xi} - r_1 + 1\right) d\bar{\xi}$$

$$= \frac{2}{(\bar{B_{jn}})_1} \left(\left(\frac{j_1\pi L}{L_1}\right) \left\{ \cos \left(\frac{j_1\pi d_1}{L_1}\right) - \cos \left[\frac{j_1\pi(d_1-d_2)}{2L_1}\right] \right\} \right.$$

$$\left[(\bar{A_{jn}})_1 \left\{ \cosh (Arg)_c - \alpha_n \sinh (Arg)_c - \cosh (Arg)_a \right.\right.$$

$$\left. + \alpha_n \sinh (Arg)_a \right\}$$

$$+ (A^+_{jn})_1 \left\{ - \cos (Arg)_c + \alpha_n \sin (Arg)_c + \cos (Arg)_a \right.$$

$$\left.\left. - \alpha_n \sin (Arg)_a \right\} \right]$$

$$+ \lambda_n \left\{ \sin \left(\frac{j_1\pi d_1}{L_1}\right) - \sin \left[\frac{j_1\pi(d_1-d_2)}{2L_1}\right] \right\}$$

$$\left[(\bar{A_{jn}})_1 \left\{ - \sinh (Arg)_c + \alpha_n \cosh (Arg)_c + \sinh (Arg)_a \right.\right.$$

$$\left. - \alpha_n \cosh (Arg)_a \right\}$$

$$\left. + (A^+_{jn})_1 \left\{ - \sin (Arg)_c - \alpha_n \cos (Arg)_c + \sin (Arg)_a + \alpha_n \cos (Arg)_a \right\} \right] \Bigg)$$

$$j = j_1 = 1,2,\ldots,M_1 \qquad \text{(B-13)}$$

$$n = 1,2,\ldots,N$$

$$(t_{jn})_1 = \int_a^c \phi_n'(\bar{\xi}) \sin \frac{j_1\pi}{2} \left(\frac{L}{L_1}\bar{\xi} - r_1 + 1\right) d\bar{\xi}$$

$$= \frac{2}{(B_{jn}^-)_1} \Bigg(\left(\frac{j_1\pi L}{L_1}\right) \left\{ \cos\left(\frac{j_1\pi d_1}{L_1}\right) - \cos\left[\frac{j_1\pi(d_1-d_2)}{2L_1}\right] \right\}$$

$$\left[(A_{jn}^-)_1 \left\{ \sinh (Arg)_c - \alpha_n \cosh (Arg)_c - \sinh (Arg)_a + \alpha_n \cosh (Arg)_a \right\} \right.$$

$$\left. + (A_{jn}^+)_1 \left\{ \sin (Arg)_c + \alpha_n \cos (Arg)_c - \sin (Arg)_a - \alpha_n \cos (Arg)_a \right\} \right]$$

$$+ \lambda_n \left\{ \sin\left(\frac{j_1\pi d_1}{L_1}\right) - \sin\left[\frac{j_1\pi(d_1-d_2)}{2L_1}\right] \right\}$$

$$\left[(A_{jn}^-)_1 \left\{ - \cosh (Arg)_c + \alpha_n \sinh (Arg)_c + \cosh (Arg)_a - \alpha_n \sinh (Arg)_a \right\} \right.$$

$$+ (A_{jn}^+)_1 \left\{ - \cos (Arg)_c + \alpha_n \sin (Arg)_c + \cos (Arg)_a \right.$$

$$\left. \left. \left. - \alpha_n \sin (Arg)_a \right\} \right] \right) \qquad \text{(B-14)}$$

$$j=j_1=1,2,\ldots,M_1$$

$$n=1,2,\ldots,N$$

$$(x_{jn})_1 = \int_a^c \phi_n''(\bar{\xi}) \sin \frac{j_1\pi}{2} \left(\frac{L}{L_1}\bar{\xi} - r_1 + 1\right) d\bar{\xi}$$

$$= \frac{2}{(B_{jn}^-)_1} \Bigg(\left(\frac{j_1\pi L}{L_1}\right) \left\{ \cos\left(\frac{j_1\pi d_1}{L_1}\right) - \cos\left[\frac{j_1\pi(d_1-d_2)}{2L_1}\right] \right\}$$

$$\left[(A_{jn}^-)_1 \left\{ \cosh (Arg)_c - \alpha_n \sinh (Arg)_c - \cosh (Arg)_a + \alpha_n \sinh (Arg)_a \right\} \right.$$

$$\left. + (A_{jn}^+)_1 \left\{ \cos (Arg)_c - \alpha_n \sin (Arg)_c - \cos (Arg)_a + \alpha_n \sin (Arg)_a \right\} \right]$$

$$+ \lambda_n \left\{ \sin \frac{j_1 \pi d_1}{L_1} - \sin \left[\frac{j_1 \pi (d_1 - d_2)}{2L_1}\right] \right\}$$

$$\left[(A^-_{jn})_1 \left\{ - \sinh (Arg)_c + \alpha_n \cosh (Arg)_c + \sinh (Arg)_a \right.\right.$$

$$\left. - \alpha_n \cosh (Arg)_a \right\}$$

$$+ (A^+_{jn})_1 \left\{ \sin (Arg)_c + \alpha_n \cos (Arg)_c - \sin (Arg)_a \right.$$

$$\left.\left. - \alpha_n \cos (Arg)_a \right\} \right] \Bigg) \qquad \text{(B-15)}$$

$$j = j_1 = 1,2,\ldots,M_1$$

$$n = 1,2,\ldots,N$$

$$(y_{jn})_1 = \int_a^c \phi_n'''(\bar{\xi}) \sin \frac{j_1 \pi}{2} \left(\frac{L}{L_1} \bar{\xi} - r_1 + 1\right) d\bar{\xi}$$

$$= \frac{2}{(B^-_{jn})_1} \Bigg(\left(\frac{j_1 \pi L}{L_1}\right) \left\{ \cos \left(\frac{j_1 \pi d_1}{L_1}\right) - \cos \left[\frac{j_1 \pi (d_1 - d_2)}{2L_1}\right] \right\}$$

$$\left[(A^-_{jn})_1 \left\{ \sinh (Arg)_c - \alpha_n \cosh (Arg)_c - \sinh (Arg)_a + \alpha_n \cosh (Arg)_a \right\} \right.$$

$$\left. + (A^+_{jn})_1 \left\{ - \sin (Arg)_c - \alpha_n \cos (Arg)_c + \sin (Arg)_a + \alpha_n \cos (Arg)_a \right\} \right]$$

$$+ \lambda_n \left\{ \sin \left(\frac{j_1 \pi d_1}{L_1}\right) - \sin \left[\frac{j_1 \pi (d_1 - d_2)}{2L_1}\right] \right\}$$

$$\left[(A^-_{jn})_1 \left\{ - \cosh (Arg)_c + \alpha_n \sinh (Arg)_c + \cosh (Arg)_a - \alpha_n \sinh (Arg)_a \right\} \right.$$

$$\left. + (A^+_{jn})_1 \left\{ \cos (Arg)_c - \alpha_n \sin (Arg)_c - \cos (Arg)_a + \alpha_n \sin (Arg)_a \right\} \right] \Bigg)$$

(B-16)

$$j = j_1 = 1,2,\ldots,M_1$$

$$n = 1,2,\ldots,N$$

$$(z_{jn})_1 = \int_a^c \phi_n'(\bar{\xi}) \cos \frac{j_1\pi}{2} \left(\frac{L}{L_1}\bar{\xi} - r_1 + 1\right) d\bar{\xi}$$

$$= \frac{2}{(B^-_{jn})_1} \Bigg(\left(\frac{j_1\pi L}{L_1}\right) \left\{ \sin \left(\frac{j_1\pi d_1}{L_1}\right) - \sin \left[\frac{j_1\pi(d_1-d_2)}{2L_1}\right] \right\}$$

$$\Bigg[(A^-_{jn})_1 \left\{ - \sinh (Arg)_c + \alpha_n \cosh (Arg)_c + \sinh (Arg)_a - \alpha_n \cosh (Arg)_a \right\}$$

$$+ (A^+_{jn})_1 \left\{ - \sin (Arg)_c - \alpha_n \cos (Arg)_c + \sin (Arg)_a + \alpha_n \cos (Arg)_a \right\} \Bigg]$$

$$+ \lambda_n \left\{ \cos \left(\frac{j_1\pi d_1}{L_1}\right) - \cos \left[\frac{j_1\pi(d_1-d_2)}{2L_1}\right] \right\}$$

$$\Bigg[(A^-_{jn})_1 \left\{ - \cosh (Arg)_c + \alpha_n \sinh (Arg)_c + \cosh (Arg)_a - \alpha_n \sinh (Arg)_a \right\}$$

$$+ (A^+_{jn})_1 \left\{ - \cos (Arg)_c + \alpha_n \sin (Arg)_c + \cos (Arg)_a - \alpha_n \sin (Arg)_a \right\} \Bigg] \Bigg)$$

$$j = j_1 = 1,2,\ldots,M_1 \qquad \text{(B-17)}$$

$$n = 1,2,\ldots,N$$

where

$$(A^-_{jn})_1 = \left[\lambda_n^2 - \left(\frac{j_1\pi L}{L_1}\right)^2\right] \qquad \text{(a)}$$

$$(A^+_{jn})_1 = \left[\lambda_n^2 + \left(\frac{j_1\pi L}{L_1}\right)^2\right] \qquad \text{(b)}$$

$$(B^-_{jn})_1 = \left[\left(\frac{j_1\pi L}{L_1}\right)^4 - \lambda_n^4\right] \qquad \text{(c)} \qquad \text{(B-18)}$$

$$(Arg)_c = \frac{\lambda_n}{2} \left[\frac{L-(d_4-d_1)}{L}\right] \qquad \text{(d)}$$

$$(Arg)_a = \frac{\lambda_n}{2} \left[\frac{L-(d_2+d_4)}{L}\right] \qquad \text{(e)}$$

$$j = j_1 = 1,2,\ldots,M_1$$

$$n = 1,2,\ldots,N$$

and λ_n and α_n are given in Appendix C.

B-V. Evaluation of Cylinder 2 Integrals

By substituting the ϕ-functions of Appendix C into equations (B-7) through (B-12) the coefficient integrals of Cylinder 2 may be evaluated. The evaluated integrals are presented below.

$$(s_{jn})_2 = \int_c^b \phi_n(\bar{\xi}) \sin \frac{j_2\pi}{2} \left(\frac{L}{L_2}\bar{\xi} + r_2 + 1\right) d\bar{\xi}$$

$$= \frac{2}{(\bar{B}_{jn})_2} \Bigg(\frac{j_2\pi L}{L_2} \left\{ \cos\left[\frac{j_2\pi(2d_1-L_1)}{2L_2}\right] - \cos\left[\frac{j_2\pi(d_1+d_3-L_1)}{2L_2}\right] \right\}$$

$$\left[(\bar{A}_{jn})_2 \left\{ \cosh(\text{Arg})_b - \alpha_n \sinh(\text{Arg})_b - \cosh(\text{Arg})_c + \alpha_n \sinh(\text{Arg})_c \right\} \right.$$

$$\left. + (A^+_{jn})_2 \left\{ -\cos(\text{Arg})_b + \alpha_n \sin(\text{Arg})_b + \cos(\text{Arg})_c - \alpha_n \sin(\text{Arg})_c \right\} \right]$$

$$+ \lambda_n \left\{ \sin\left[\frac{j_2\pi(2d_1-L_1)}{2L_2}\right] - \sin\left[\frac{j_2\pi(d_1+d_3-L_1)}{2L_2}\right] \right\}$$

$$\left[(\bar{A}_{jn})_2 \left\{ -\sinh(\text{Arg})_b + \alpha_n \cosh(\text{Arg})_b + \sinh(\text{Arg})_c - \alpha_n \cosh(\text{Arg})_c \right\} \right.$$

$$\left. + (A^+_{jn})_2 \left\{ -\sin(\text{Arg})_b - \alpha_n \cos(\text{Arg})_b + \sin(\text{Arg})_c + \alpha_n \cos(\text{Arg})_c \right\} \right] \Bigg)$$

$$j = j_2 = 1,2,\ldots,M_2 \qquad \text{(B-19)}$$

$$n = 1,2,\ldots,N$$

$$(t_{jn})_2 = \int_c^b \phi_n'(\bar{\xi}) \sin \frac{j_2\pi}{2} \left(\frac{L}{L_2}\bar{\xi} + r_2 + 1\right) d\bar{\xi}$$

$$= \frac{2}{(\bar{B}_{jn})_2} \Bigg(\left(\frac{j_2\pi L}{L_2}\right) \left\{ \cos\left[\frac{j_2\pi(2d_1-L_1)}{2L_2}\right] - \cos\left[\frac{j_2\pi(d_1+d_3-L_1)}{2L_2}\right] \right\}$$

$$\left[(A_{jn}^{-})_2 \left\{ \sinh (Arg)_b - \alpha_n \cosh (Arg)_b - \sinh (Arg)_c + \alpha_n \cosh (Arg)_c \right\}\right.$$

$$\left. + (A_{jn}^{+})_2 \left\{ \sin (Arg)_b + \alpha_n \cos (Arg)_b - \sin (Arg)_c - \alpha_n \cos (Arg)_c \right\} \right]$$

$$+ \lambda_n \left\{ \sin \left[\frac{j_2\pi(2d_1-L_1)}{2L_2}\right] - \sin \left[\frac{j_2\pi(d_1+d_3-L_1)}{2L_2}\right] \right\}$$

$$\left[(A_{jn}^{-})_2 \left\{ - \cosh (Arg)_b + \alpha_n \sinh (Arg)_b + \cosh (Arg)_c - \alpha_n \sinh (Arg)_c \right\}\right.$$

$$\left. \left. + (A_{jn}^{+})_2 \left\{ - \cos (Arg)_b + \alpha_n \sin (Arg)_b + \cos (Arg)_c - \alpha_n \sin (Arg)_c \right\} \right] \right)$$

$$j = j_2 = 1,2,\ldots,M_2 \qquad \text{(B-20)}$$

$$n = 1,2,\ldots,N$$

$$(x_{jn})_2 = \int_c^b \phi_n''(\bar{\xi}) \sin \frac{j_2\pi}{2} \left(\frac{L}{L_2}\bar{\xi} + r_2 + 1\right) d\bar{\xi}$$

$$= \frac{2}{(B_{jn}^{-})_2} \left(\left(\frac{j_2\pi L}{L_2}\right) \left\{ \cos \left[\frac{j_2\pi(2d_1-L_1)}{2L_2}\right] - \cos \left[\frac{j_2\pi(d_1+d_3-L_1)}{2L_2}\right] \right\} \right.$$

$$\left[(A_{jn}^{-})_2 \left\{ \cosh (Arg)_b - \alpha_n \sinh (Arg)_b - \cosh (Arg)_c + \alpha_n \sinh (Arg)_c \right\}\right.$$

$$\left. + (A_{jn}^{+})_2 \left\{ \cos (Arg)_b - \alpha_n \sin (Arg)_b - \cos (Arg)_c + \alpha_n \sin (Arg)_c \right\} \right]$$

$$+ \lambda_n \left\{ \sin \left[\frac{j_2\pi(2d_1-L_1)}{2L_2}\right] - \sin \left[\frac{j_2\pi(d_1+d_3-L_1)}{2L_2}\right] \right\}$$

$$\left[(A_{jn}^{-})_2 \left\{ - \sinh (Arg)_b + \alpha_n \cosh (Arg)_b + \sinh (Arg)_c - \alpha_n \cosh (Arg)_c \right\}\right.$$

$$\left. \left. + (A_{jn}^{+})_2 \left\{ \sin (Arg)_b + \alpha_n \cos (Arg)_b - \sin (Arg)_c - \alpha_n \cos (Arg)_c \right\} \right] \right)$$

$$j = j_2 = 1,2,\ldots,M_2 \qquad \text{(B-21)}$$

$$n = 1,2,\ldots,N$$

$$(y_{jn})_2 = \int_c^b \phi_n'''(\bar{\xi}) \sin \frac{j_2\pi}{2} \left(\frac{L}{L_2}\bar{\xi} + r_2 + 1\right) d\bar{\xi}$$

$$= \frac{2}{(B_{jn}^-)_2} \Bigg(\left(\frac{j_2\pi L}{L_2}\right) \left\{ \cos\left[\frac{j_2\pi(2d_1 - L_1)}{2L_2}\right] - \cos\left[\frac{j_2\pi(d_1+d_3-L_1)}{2L_2}\right] \right\}$$

$$\left[(A_{jn}^-)_2 \left\{ \sinh(\text{Arg})_b - \alpha_n \cosh(\text{Arg})_b - \sinh(\text{Arg})_c + \alpha_n \cosh(\text{Arg})_c \right\} \right.$$

$$\left. + (A_{jn}^+)_2 \left\{ -\sin(\text{Arg})_b - \alpha_n \cos(\text{Arg})_b + \sin(\text{Arg})_c + \alpha_n \cos(\text{Arg})_c \right\} \right]$$

$$+ \lambda_n \left\{ \sin\left[\frac{j_2\pi(2d_1 - L_1)}{2L_2}\right] - \sin\left[\frac{j_2\pi(d_1+d_3-L_1)}{2L_2}\right] \right\}$$

$$\left[(A_{jn}^-)_2 \left\{ -\cosh(\text{Arg})_b + \alpha_n \sinh(\text{Arg})_b + \cosh(\text{Arg})_c - \alpha_n \sinh(\text{Arg})_c \right\} \right.$$

$$\left. + (A_{jn}^+)_2 \left\{ \cos(\text{Arg})_b - \alpha_n \sin(\text{Arg})_b - \cos(\text{Arg})_c + \alpha_n \sin(\text{Arg})_c \right\} \right] \Bigg)$$

$$j = j_2 = 1,2,\ldots,M_2 \qquad \text{(B-22)}$$

$$n = 1,2,\ldots,N$$

$$(z_{jn})_2 = \int_c^b \phi_n'(\bar{\xi}) \cos \frac{j_2\pi}{2} \left(\frac{L}{L_2}\bar{\xi} + r_2 + 1\right) d\bar{\xi}$$

$$= \frac{2}{(B_{jn}^-)_2} \Bigg(\left(\frac{j_2\pi L}{L_2}\right) \left\{ \sin\left[\frac{j_2\pi(2d_1 - L_1)}{2L_2}\right] - \sin\left[\frac{j_2\pi(d_1+d_3-L_1)}{2L_2}\right] \right\}$$

$$\left[(A_{jn}^-)_2 \left\{ -\sinh(\text{Arg})_b + \alpha_n \cosh(\text{Arg})_b + \sinh(\text{Arg})_c - \alpha_n \cosh(\text{Arg})_c \right\} \right.$$

$$\left. + (A_{jn}^+)_2 \left\{ -\sin(\text{Arg})_b - \alpha_n \cos(\text{Arg})_b + \sin(\text{Arg})_c + \alpha_n \cos(\text{Arg})_c \right\} \right]$$

$$+ \lambda_n \left\{ \cos\left[\frac{j_2\pi(2d_1 - L_1)}{2L_2}\right] - \cos\left[\frac{j_2\pi(d_1+d_3-L_1)}{2L_2}\right] \right\}$$

$$\left[(A_{jn}^-)_2 \left\{ -\cosh(\text{Arg})_b + \alpha_n \sinh(\text{Arg})_b + \cosh(\text{Arg})_c - \alpha_n \sinh(\text{Arg})_c \right\} \right.$$

$$+ (A^{+}_{jn})_2 \left\{ - \cos (Arg)_b + \alpha_n \sin (Arg)_b + \cos (Arg)_c \right.$$

$$\left. - \alpha_n \sin (Arg)_c \right\} \Bigg] \Bigg) \qquad \text{(B-23)}$$

$$j = j_2 = 1,2,\ldots,M_2$$

$$n = 1,2,\ldots,N$$

where

$$(A^{-}_{jn})_2 = [\lambda_n^2 - (\frac{j_2 \pi L}{L_2})^2] \qquad \text{(a)}$$

$$(A^{+}_{jn})_2 = [\lambda_n^2 + (\frac{j_2 \pi L}{L_2})^2] \qquad \text{(b)}$$

$$(B^{-}_{jn})_2 = [(\frac{j_2 \pi L}{L_2})^4 - \lambda_n^4] \qquad \text{(c)} \qquad \text{(B-24)}$$

$$(Arg)_b = \frac{\lambda_n}{2} [\frac{L-(d_4-d_3)}{L}] \qquad \text{(d)}$$

$$(Arg)_c = \frac{\lambda_n}{2} [\frac{L-(d_4-d_1)}{L}] \qquad \text{(e)}$$

$$j = j_2 = 1,2,\ldots,M_2$$

$$n = 1,2,\ldots,N$$

and λ_n and α_n are given in Appendix C.

APPENDIX C

REPRESENTATION OF THE ϕ-FUNCTION AND DERIVATIVES

C-I ϕ -Functions for the Free-Free Case

The ϕ-function and its respective derivative for the free-free case is summarized below.

$$\sum_{n=1}^{N} \phi_n(\bar{\xi}) = \sum_{n=1}^{N} \cosh \frac{\lambda_n}{2} (\bar{\xi}-r+1) + \cos \frac{\lambda_n}{2} (\bar{\xi}-r+1)$$

$$- \alpha_n \left[\sinh \frac{\lambda_n}{2} (\bar{\xi}-r+1) + \sin \frac{\lambda_n}{2} (\bar{\xi}-r+1)\right] \qquad (C-1)$$

$$\sum_{n=1}^{N} \frac{d\phi_n(\bar{\xi})}{d\bar{\xi}} = \sum_{n=1}^{N} \frac{\lambda_n}{2} \phi_n'(\bar{\xi})$$

$$= \sum_{n=1}^{N} \frac{\lambda_n}{2} \left\{\sinh \frac{\lambda_n}{2} (\bar{\xi}-r+1) - \sin \frac{\lambda_n}{2} (\bar{\xi}-r+1)\right.$$

$$\left. - \alpha_n \left[\cosh \frac{\lambda_n}{2} (\bar{\xi}-r+1) + \cos \frac{\lambda_n}{2} (\bar{\xi}-r+1)\right]\right\} \qquad (C-2)$$

$$\sum_{n=1}^{N} \frac{d^2\phi_n(\bar{\xi})}{d\bar{\xi}} = \sum_{n=1}^{N} \left(\frac{\lambda_n}{2}\right)^2 \phi_n''(\bar{\xi})$$

$$= \sum_{n=1}^{N} \left(\frac{\lambda_n}{2}\right)^2 \left\{\cosh \frac{\lambda_n}{2} (\bar{\xi}-r+1) - \cos \frac{\lambda_n}{2} (\bar{\xi}-r+1)\right.$$

$$\left. - \alpha_n \left[\sinh \frac{\lambda_n}{2} (\bar{\xi}-r+1) - \sin \frac{\lambda_n}{2} (\bar{\xi}-r+1)\right]\right\} \qquad (C-3)$$

$$\sum_{n=1}^{N} \frac{d^3\phi_n(\bar{\xi})}{d\bar{\xi}} = \sum_{n=1}^{N} \left(\frac{\lambda_n}{2}\right)^3 \phi_n'''(\bar{\xi})$$

$$= \sum_{n=1}^{N} \left(\frac{\lambda_n}{2}\right)^3 \left\{ \sinh \frac{\lambda_n}{2}(\bar{\xi}-r+1) + \sin \frac{\lambda_n}{2}(\bar{\xi}-r+1) \right.$$

$$\left. - \alpha_n \left[\cosh \frac{\lambda_n}{2}(\bar{\xi}-r+1) - \cos \frac{\lambda_n}{2}(\bar{\xi}-r+1)\right] \right\} \qquad (C\text{-}4)$$

$$\sum_{n=1}^{N} \frac{d^4\phi_n(\bar{\xi})}{d\bar{\xi}} = \sum_{n=1}^{N} \left(\frac{\lambda_n}{2}\right)^4 \phi_n''''(\bar{\xi}) = \sum_{n=1}^{N} \left(\frac{\lambda_n}{2}\right)^4 \phi_n(\bar{\xi}) \qquad (C\text{-}5)$$

C-II Argument Terms

Referring to Figs. 2 and 3, it can be seen that the argument in Equations (C-1) thru (C-5), at certain distances along the axial coordinate will be as follows:

Argument in Equations (C-1) thru (C-5)

$$(\bar{\xi}-r+1) = (\bar{\xi} - \frac{d_4}{L} + 1) \qquad (C\text{-}6)$$

At $\bar{\xi} = a = -\frac{d_2}{L}$, $(\bar{\xi} - \frac{d_4}{L} + 1) = \frac{L - (d_2+d_4)}{L}$ (C-7)

At $\bar{\xi} = c = \frac{d_1}{L}$, $(\bar{\xi} - \frac{d_4}{L} + 1) = \frac{L - (d_4-d_1)}{L}$ (C-8)

At $\bar{\xi} = b = \frac{d_3}{L}$, $(\bar{\xi} - \frac{d_4}{L} + 1) = \frac{L - (d_4-d_3)}{L}$ (C-9)

At $\bar{\xi} = 0$, $(\bar{\xi} - \frac{d_4}{L} + 1) = \frac{L-d_4}{L}$ (C-10)

C-III Evaluation of the ϕ-Functions at the Limits

Since it is necessary to have the ϕ values at the limits before we can evaluate the coefficient integrals in APPENDIX B, these expressions are listed below.

C-III-1 Evaluated at limits a and c.

$$\sum_{n=1}^{N} \phi_n(\bar{\xi}) \Big|_a^c = \sum_{n=1}^{N} \cosh \frac{\lambda_n}{2}\left[\frac{L-(d_4-d_1)}{L}\right] + \cos \frac{\lambda_n}{2}\left[\frac{L-(d_4-d_1)}{L}\right]$$

$$- \alpha_n \left\{ \sinh \frac{\lambda_n}{2}\left[\frac{L-(d_4-d_1)}{L}\right] + \sin \frac{\lambda_n}{2}\left[\frac{L-(d_4-d_1)}{L}\right] \right\}$$

$$- \cosh \frac{\lambda_n}{2}\left[\frac{L-(d_2+d_4)}{L}\right] - \cos \frac{\lambda_n}{2}\left[\frac{L-(d_2+d_4)}{L}\right]$$

$$+ \alpha_n \left\{ \sinh \frac{\lambda_n}{2}\left[\frac{L-(d_2+d_4)}{L}\right] + \sin \frac{\lambda_n}{2}\left[\frac{L-(d_2+d_4)}{L}\right] \right\} \qquad \text{(C-11)}$$

$$\sum_{n=1}^{N} \phi_n'(\bar{\xi}) \Big|_a^c = \sum_{n=1}^{N} \sinh \frac{\lambda_n}{2}\left[\frac{L-(d_4-d_1)}{L}\right] - \sin \frac{\lambda_n}{2}\left[\frac{L-(d_4-d_1)}{L}\right]$$

$$- \alpha_n \left\{ \cosh \frac{\lambda_n}{2}\left[\frac{L-(d_4-d_1)}{L}\right] + \cos \frac{\lambda_n}{2}\left[\frac{L-(d_4-d_1)}{L}\right] \right\}$$

$$- \sinh \frac{\lambda_n}{2}\left[\frac{L-(d_2+d_4)}{L}\right] + \sin \frac{\lambda_n}{2}\left[\frac{L-(d_2+d_4)}{L}\right]$$

$$+ \alpha_n \left\{ \cosh \frac{\lambda_n}{2}\left[\frac{L-(d_2+d_4)}{L}\right] + \cos \frac{\lambda_n}{2}\left[\frac{L-(d_2+d_4)}{L}\right] \right\} \qquad \text{(C-12)}$$

$$\sum_{n=1}^{N} \phi_n''(\bar{\xi}) \Big|_a^c = \sum_{n=1}^{N} \cosh \frac{\lambda_n}{2}\left[\frac{L-(d_4-d_1)}{L}\right] - \cos \frac{\lambda_n}{2}\left[\frac{L-(d_4-d_1)}{L}\right]$$

$$- \alpha_n \left\{ \sinh \left[\frac{L - (d_4 - d_1)}{L} \right] - \sin \frac{\lambda_n}{2} \left[\frac{L - (d_4 - d_1)}{L} \right] \right\}$$

$$- \cosh \frac{\lambda_n}{2} \left[\frac{L - (d_2 + d_4)}{L} \right] + \cos \frac{\lambda_n}{2} \left[\frac{L - (d_2 + d_4)}{L} \right]$$

$$+ \alpha_n \left\{ \sinh \frac{\lambda_n}{2} \left[\frac{L - (d_2 + d_4)}{L} \right] - \sin \frac{\lambda_n}{2} \left[\frac{L - (d_2 + d_4)}{L} \right] \right\} \qquad \text{(C-13)}$$

$$\sum_{n=1}^{N} \phi_n'''(\bar{\xi}) \Big|_a^c = \sum_{n=1}^{N} \sinh \frac{\lambda_n}{2} \left[\frac{L - (d_4 - d_1)}{L} \right] + \sin \frac{\lambda_n}{2} \left[\frac{L - (d_4 - d_1)}{L} \right]$$

$$- \alpha_n \left\{ \cosh \frac{\lambda_n}{2} \left[\frac{L - (d_4 - d_1)}{L} \right] - \cos \frac{\lambda_n}{2} \left[\frac{L - (d_4 - d_1)}{L} \right] \right\}$$

$$- \sinh \frac{\lambda_n}{2} \left[\frac{L - (d_2 + d_4)}{L} \right] - \sin \frac{\lambda_n}{2} \left[\frac{L - (d_2 + d_4)}{L} \right]$$

$$+ \alpha_n \left\{ \cosh \frac{\lambda_n}{2} \left[\frac{L - (d_2 + d_4)}{L} \right] - \cos \frac{\lambda_n}{2} \left[\frac{L - (d_2 + d_4)}{L} \right] \right\} \qquad \text{(C-14)}$$

$$\sum_{n=1}^{N} \phi_n''''(\bar{\xi}) \Big|_a^c = \sum_{n=1}^{N} \phi_n(\bar{\xi}) \Big|_a^c \qquad \text{(C-15)}$$

C-III-2 Evaluated at limits c and b

$$\sum_{n=1}^{N} \phi_n(\bar{\xi}) \Big|_c^b = \sum_{n=1}^{N} \cosh \frac{\lambda_n}{2} \left[\frac{L - (d_4 - d_3)}{L} \right] + \cos \frac{\lambda_n}{2} \left[\frac{L - (d_4 - d_3)}{L} \right]$$

$$- \alpha_n \left\{ \sinh \frac{\lambda_n}{2} \left[\frac{L - (d_4 - d_3)}{L} \right] + \sin \frac{\lambda_n}{2} \left[\frac{L - (d_4 - d_3)}{L} \right] \right\}$$

$$- \cosh \frac{\lambda_n}{2}\left[\frac{L - (d_4-d_1)}{L}\right] - \cos \frac{\lambda_n}{2}\left[\frac{L - (d_4-d_1)}{L}\right]$$

$$+ \alpha_n \left\{\sinh \frac{\lambda_n}{2}\left[\frac{L - (d_4-d_1)}{L}\right] + \sin \frac{\lambda_n}{2}\left[\frac{L - (d_4-d_1)}{L}\right]\right\} \qquad \text{(C-16)}$$

$$\sum_{n=1}^{N} \phi_n'(\bar{\xi}) \Big|_c^b = \sum_{n=1}^{N} \sinh \frac{\lambda_n}{2}\left[\frac{L - (d_4-d_3)}{L}\right] - \sin \frac{\lambda_n}{2}\left[\frac{L - (d_4-d_3)}{L}\right]$$

$$- \alpha_n \left\{\cosh \frac{\lambda_n}{2}\left[\frac{L - (d_4-d_3)}{L}\right] + \cos \frac{\lambda_n}{2}\left[\frac{L - (d_4-d_3)}{L}\right]\right\}$$

$$- \sinh \frac{\lambda_n}{2}\left[\frac{L - (d_4-d_1)}{L}\right] + \sin \frac{\lambda_n}{2}\left[\frac{L - (d_4-d_1)}{L}\right]$$

$$+ \alpha_n \left\{\cosh \frac{\lambda_n}{2}\left[\frac{L - (d_4-d_1)}{L}\right] + \cos \frac{\lambda_n}{2}\left[\frac{L - (d_4-d_1)}{L}\right]\right\} \qquad \text{(C-17)}$$

$$\sum_{n=1}^{N} \phi_n''(\bar{\xi}) \Big|_c^b = \sum_{n=1}^{N} \cosh \frac{\lambda_n}{2}\left[\frac{L - (d_4-d_3)}{L}\right] - \cos \frac{\lambda_n}{2}\left[\frac{L - (d_4-d_3)}{L}\right]$$

$$- \alpha_n \left\{\sinh \frac{\lambda_n}{2}\left[\frac{L - (d_4-d_3)}{L}\right] - \sin \frac{\lambda_n}{2}\left[\frac{L - (d_4-d_3)}{L}\right]\right\}$$

$$- \cosh \frac{\lambda_n}{2}\left[\frac{L - (d_4-d_1)}{L}\right] + \cos \frac{\lambda_n}{2}\left[\frac{L - (d_4-d_1)}{L}\right]$$

$$+ \alpha_n \left\{\sinh \frac{\lambda_n}{2}\left[\frac{L - (d_4-d_1)}{L}\right] - \sin \frac{\lambda_n}{2}\left[\frac{L - (d_4-d_1)}{L}\right]\right\} \qquad \text{(C-18)}$$

$$\sum_{n=1}^{N} \phi_n'''(\bar{\xi}) \Big|_c^b = \sum_{n=1}^{N} \sinh \frac{\lambda_n}{2}\left[\frac{L - (d_4-d_3)}{L}\right] + \sin \frac{\lambda_n}{2}\left[\frac{L - (d_4-d_3)}{L}\right]$$

$$- \alpha_n \left\{ \cosh \frac{\lambda_n}{2} \left[\frac{L - (d_4 - d_3)}{L} \right] - \cos \frac{\lambda_n}{2} \left[\frac{L - (d_4 - d_3)}{L} \right] \right\}$$

$$- \sinh \frac{\lambda_n}{2} \left[\frac{L - (d_4 - d_1)}{L} \right] - \sin \frac{\lambda_n}{2} \left[\frac{L - (d_4 - d_1)}{L} \right]$$

$$+ \alpha_n \left\{ \cosh \frac{\lambda_n}{2} \left[\frac{L - (d_4 - d_1)}{L} \right] - \cos \frac{\lambda_n}{2} \left[\frac{L - (d_4 - d_1)}{L} \right] \right\} \quad \text{(C-19)}$$

$$\sum_{n=1}^{N} \phi_n''''(\bar{\xi}) \Big|_c^b = \sum_{n=1}^{N} \phi_n(\bar{\xi}) \Big|_c^b \quad \text{(C-20)}$$

C-IV Values of λ_n and α_n

The values of λ_n and α_n appearing in eqs. (C-11) through (C-14) and (C-16) through (C-19) have been calculated (see Young, ref. 10) for the free-free beam and are presented here for purposes of computation of the ϕ-functions for a given cylinder.

n	λ_n	α_n
1.	4.7300	0.9825
2.	7.8532	1.0007
3.	10.9956	0.9999
4.	14.1377	1.0000
5.	17.2787	0.9999

For $n>5$, the values of λ_n and α_n are approximated by

$$\lambda_n \cong (2n-1)\frac{\pi}{2}$$

$$\alpha_n \cong 1.0$$

APPENDIX D

COMPARISON OF THE NATURAL FREQUENCIES OF THE SINGLE CYLINDER AND THE NATURAL FREQUENCIES OF THE COMPOSITE CYLINDER

D-I General Statement

A comparison of the natural frequencies of the single cylinder analysis (see ref. 3) and the natural frequencies of the composite cylinder analysis in this report must be made if the relative coupling effect of joined cylinders is to be investigated.

D-II Natural Frequency Equations for the Single Cylinder

D-II-1 For j=1, 2, ... M; k = 0

The natural frequency equations for the case of j = 1, 2, ... M; and k = 0 are written

$$\left(\omega_{j0}^{1}\right)^2 = \frac{b}{2a} - \frac{1}{2a}\sqrt{b^2 - 4ac} \tag{D-1}$$

$$\left(\omega_{j0}^{2}\right)^2 = \frac{b}{2a} + \frac{1}{2a}\sqrt{b^2 - 4ac} \tag{D-2}$$

where (see p 60 of ref. 3)

$$a = 1 \tag{D-3}$$

$$b = C_{11}^{j0} + C_{22}^{j0} = \frac{j^2\pi^2}{4(1-\nu^2)\mu} + \frac{192\lambda^2 + j^4\pi^4\sigma^2}{192(1-\nu^2)\mu} \tag{D-4}$$

$$c = C_{11}^{j0}\,C_{22}^{j0} - \left(C_{12}^{j0}\right)^2 = \left(\frac{j^2\pi^2}{4(1-\nu^2)\mu}\right)\left(\frac{192\lambda^2 + j^4\pi^4\sigma^2}{192(1-\nu^2)\mu}\right) - \left(\frac{\nu\lambda j\pi}{2(1-\nu^2)\mu}\right)^2 \tag{D-5}$$

D-II-2 For $j = 1, 2, \ldots, M$; $k = 1, 2, \ldots, N$

The natural frequency equations for the case of $j = 1, 2, \ldots M$ and $k = 1, 2, \ldots, N$ are written (see p. 62 of ref. 3)

$$\left(\omega_{jk}^{1}\right)^2 + \left(\omega_{jk}^{2}\right)^2 + \left(\omega_{jk}^{3}\right)^2 = C_{11}^{jk} + C_{22}^{jk} + C_{33}^{jk} \tag{D-6}$$

$$\begin{aligned}\left(\omega_{jk}^{1}\right)^2\left(\omega_{jk}^{2}\right)^2 + \left(\omega_{jk}^{2}\right)^2\left(\omega_{jk}^{3}\right)^2 + \left(\omega_{jk}^{3}\right)^2\left(\omega_{jk}^{1}\right)^2 &= \left(C_{22}^{jk} C_{33}^{jk} - C_{32}^{jk} C_{23}^{jk}\right) \\ &+ \left(C_{11}^{jk} C_{33}^{jk} - C_{31}^{jk} C_{13}^{jk}\right) \\ &+ \left(C_{11}^{jk} C_{22}^{jk} - C_{21}^{jk} C_{12}^{jk}\right)\end{aligned} \tag{D-7}$$

$$\begin{aligned}\left(\omega_{jk}^{1}\right)^2\left(\omega_{jk}^{2}\right)^2\left(\omega_{jk}^{3}\right)^2 &= C_{11}^{jk} C_{22}^{jk} C_{33}^{jk} + C_{21}^{jk} C_{32}^{jk} C_{13}^{jk} + C_{31}^{jk} C_{12}^{jk} C_{23}^{jk} \\ &- C_{31}^{jk} C_{22}^{jk} C_{13}^{jk} - C_{21}^{jk} C_{12}^{jk} C_{33}^{jk} - C_{11}^{jk} C_{32}^{jk} C_{23}^{jk}\end{aligned} \tag{D-8}$$

D-III Natural Frequency Equations For the Composite Structure Formed by Two Cylinders

D-III-1 Natural Frequency Equations For Cylinder 1

D-III-1.1 For $j = j_1 = 1, 2, \ldots M_1$; $k = k_1 = 0$

$$\left(\bar{\omega}_{j0}^{1}\right)_1^2 = \frac{(b)_1}{2(a)_1} - \frac{1}{2(a)_1}\sqrt{(b)_1^2 - 4(a)_1(c)_1} \tag{D-9}$$

$$\left(\bar{\omega}_{j0}^{2}\right)_1^2 = \frac{(b)_1}{2(a)_1} + \frac{1}{2(a)_1}\sqrt{(b)_1^2 - 4(a)_1(c)_1} \tag{D-10}$$

where

$$(a)_1 = (a_{j0})_1 (j_{j0})_1 \tag{D-11}$$

$$(b)_1 = (c_{j0})_1 (j_{j0})_1 + (a_{j0})_1 (k_{j0})_1 - (b_{j0})_1 (h_{j0})_1 \tag{D-12}$$

$$(c)_1 = (c_{j0})_1 (k_{j0})_1 - (d_{j0})_1 (h_{j0})_1 \tag{D-13}$$

where $(a_{j0})_1$, $(b_{j0})_1$, etc., are defined in APPENDIX A.

D-III-1.2 For $j = j_1 = 1, 2, \ldots, M_1$; $k = k_1 = 1, 2, \ldots, N_1$

$$\left(\bar{\omega}_{jk}^{1}\right)_1^2 + \left(\bar{\omega}_{jk}^{2}\right)_1^2 + \left(\bar{\omega}_{jk}^{3}\right)_1^2 = \frac{(B)_1}{(A)_1} \tag{D-14}$$

$$\left(\bar{\omega}_{jk}^{1}\right)_1^2 \left(\bar{\omega}_{jk}^{2}\right)_1^2 + \left(\bar{\omega}_{jk}^{2}\right)_1^2 \left(\bar{\omega}_{jk}^{3}\right)_1^2 + \left(\bar{\omega}_{jk}^{1}\right)_1^2 \left(\bar{\omega}_{jk}^{1}\right)_1^2 = \frac{(C)_1}{(A)_1} \tag{D-15}$$

$$\left(\bar{\omega}_{jk}^{1}\right)_1^2 \left(\bar{\omega}_{jk}^{2}\right)_1^2 \left(\bar{\omega}_{jk}^{3}\right)_1^2 = \frac{(D)_1}{(A)_1} \tag{D-16}$$

where

$$(A)_1 = (a_{jk})_1 (h_{jk})_1 (n_{jk})_1 \tag{D-17}$$

$$\begin{aligned}(B)_1 = {} & (a_{jk})_1 (j_{jk})_1 (n_{jk})_1 + (a_{jk})_1 (h_{jk})_1 (p_{jk})_1 \\ & + (d_{jk})_1 (h_{jk})_1 (n_{jk})_1 - (\ell_{jk})_1 (h_{jk})_1 (C_{jk})_1 \\ & - (g_{jk})_1 (b_{jk})_1 (n_{jk})_1\end{aligned} \tag{D-18}$$

$$(C)_1 = (a_{jk})_1 (j_{jk})_1 (p_{jk})_1 + (d_{jk})_1 (j_{jk})_1 (n_{jk})_1$$

$$+ (d_{jk})_1 (h_{jk})_1 (p_{jk})_1 + (g_{jk})_1 (m_{jk})_1 (c_{jk})_1$$

$$+ (k_{jk})_1 (\ell_{jk})_1 (b_{jk})_1 - (k_{jk})_1 (m_{jk})_1 (a_{jk})_1$$

$$- (\ell_{jk})_1 (j_{jk})_1 (c_{jk})_1 - (\ell_{jk})_1 (f_{jk})_1 (h_{jk})_1$$

$$- (g_{jk})_1 (e_{jk})_1 (n_{jk})_1 - (g_{jk})_1 (b_{jk})_1 (p_{jk})_1 \qquad \text{(D-19)}$$

$$(D)_1 = (d_{jk})_1 (j_{jk})_1 (p_{jk})_1 + (g_{jk})_1 (m_{jk})_1 (f_{jk})_1$$

$$+ (k_{jk})_1 (\ell_{jk})_1 (e_{jk})_1 - (k_{jk})_1 (m_{jk})_1 (d_{jk})_1$$

$$- (\ell_{jk})_1 (j_{jk})_1 (f_{jk})_1 - (g_{jk})_1 (e_{jk})_1 (p_{jk})_1 \qquad \text{(D-20)}$$

where $(a_{jk})_1$, $(b_{jk})_1$, etc., are defined in APPENDIX A

D-III-2 Natural Frequency Equations For Cylinder 2

D-III-2.1 For $j = j_2 = 1, 2, \ldots, M_2$; $k = k_2 = 0$ (D-21)

$$\left(\bar{\omega}_{j0}^{1}\right)_2^2 = \frac{(b)_2}{2(a)_2} - \frac{1}{2(a)_2} \sqrt{(b)_2^2 - 4(a)_2 (c)_2}$$

$$\left(\bar{\omega}_{j0}^{2}\right)_2^2 = \frac{(b)_2}{2(a)_2} + \frac{1}{2(a)_2} \sqrt{(b)_2^2 - 4(a)_2 (c)_2} \qquad \text{(D-22)}$$

where

$$(a)_2 = (a_{j0})_2 (j_{j0})_2 \qquad \text{(D-23)}$$

$$(b)_2 = (c_{j0})_2 (j_{j0})_2 + (a_{j0})_2 (k_{j0})_2 - (h_{j0})_2 (b_{j0})_2 \qquad \text{(D-24)}$$

$$(c)_2 = (c_{j0})_2 (k_{j0})_2 - (h_{j0})_2 (d_{j0})_2 \qquad \text{(D-25)}$$

where $(a_{j0})_2$, $(b_{j0})_2$, etc., are defined in APPENDIX A.

D-III-2.2 For $j = j_2 = 1, 2, \ldots, M_2$; $k = k_2 = 1, 2, \ldots, N_2$

$$\left(\bar{\omega}^1_{jk}\right)_2^2 + \left(\bar{\omega}^2_{jk}\right)_2^2 + \left(\bar{\omega}^3_{jk}\right)_2^2 = \frac{(B)_2}{(A)_2} \qquad \text{(D-26)}$$

$$\left(\bar{\omega}^1_{jk}\right)_2^2 \left(\bar{\omega}^2_{jk}\right)_2^2 + \left(\bar{\omega}^2_{jk}\right)_2^2 \left(\bar{\omega}^3_{jk}\right)_2^2 + \left(\bar{\omega}^3_{jk}\right)_2^2 \left(\bar{\omega}^1_{jk}\right)_2^2 = \frac{(C)_2}{(A)_2} \qquad \text{(D-27)}$$

$$\left(\bar{\omega}^1_{jk}\right)_2^2 \left(\bar{\omega}^2_{jk}\right)_2^2 \left(\bar{\omega}^3_{jk}\right)_2^2 = \frac{(D)_2}{(A)_2} \qquad \text{(D-28)}$$

where

$$(A)_2 = (a_{jk})_2 (h_{jk})_2 (n_{jk})_2 \qquad \text{(D-29)}$$

$$\begin{aligned}(B)_2 = {} & (a_{jk})_2 (j_{jk})_2 (n_{jk})_2 + (a_{jk})_2 (h_{jk})_2 (p_{jk})_2 \\ & + (d_{jk})_2 (h_{jk})_2 (n_{jk})_2 - (\ell_{jk})_2 (h_{jk})_2 (c_{jk})_2 \\ & - (g_{jk})_2 (b_{jk})_2 (n_{jk})_2\end{aligned} \qquad \text{(D-30)}$$

$$\begin{aligned}(C)_2 = {} & (a_{jk})_2 (j_{jk})_2 (p_{jk})_2 + (d_{jk})_2 (j_{jk})_2 (n_{jk})_2 \\ & + (d_{jk})_2 (h_{jk})_2 (p_{jk})_2 + (g_{jk})_2 (m_{jk})_2 (c_{jk})_2 \\ & + (k_{jk})_2 (\ell_{jk})_2 (b_{jk})_2 - (k_{jk})_2 (m_{jk})_2 (a_{jk})_2\end{aligned}$$

$$- (\ell_{jk})_2(j_{jk})_2(c_{jk})_2 - (\ell_{jk})_2(f_{jk})_2(h_{jk})_2$$

$$- (g_{jk})_2(e_{jk})_2(n_{jk})_2 - (g_{jk})_2(b_{jk})_2(p_{jk})_2 \qquad \text{(D-31)}$$

$$(D)_1 = (d_{jk})_2(j_{jk})_2(p_{jk})_2 + (g_{jk})_2(m_{jk})_2(f_{jk})_2$$

$$+ (k_{jk})_2(\ell_{jk})_2(e_{jk})_2 - (k_{jk})_2(m_{jk})_2(d_{jk})_2$$

$$- (\ell_{jk})_2(j_{jk})_2(f_{jk})_2 - (g_{jk})_2(e_{jk})_2(p_{jk})_2 \qquad \text{(D-32)}$$

where $(a_{jk})_2$, $(b_{jk})_2$, etc., are defined in APPENDIX A.

D-IV Limiting Cases

The lengths of cylinder 1 and cylinder 2 will be reduced to zero, respectively, and the special terms of APPENDIX A. are evaluated for these two limiting cases. The natural frequencies of the composite structure should reduce to the natural frequencies of the single cylinder in the limiting case.

D-IV-1 Limiting Case 1, $L_2 = 0$

Referring to Figure 1 on page 3 it can be seen that when $L_2 = 0$, the center of mass shifts to the center of cylinder 1, thus, $d_1 = d_2 = L_1 = L$. For this limiting case the terms of equations (D-11) thru (D-13) are

$$(a_{j0})_1 = L_1 \qquad \text{(D-33)}$$

$$(b_{j0})_1 = 0 \qquad \text{(D-34)}$$

$$(c_{j0})_1 = \left(\frac{j\pi}{2}\right)^2 \frac{1}{(1-\nu^2)\mu} \qquad \text{(D-35)}$$

$$(d_{j0})_1 = \frac{\nu\lambda L}{(1-\nu^2)\mu}\left(\frac{j\pi}{2}\right) \tag{D-36}$$

$$(h_{j0})_1 = \frac{\nu\lambda L}{(1-\nu^2)\mu}\left(\frac{j\pi}{2}\right) \tag{D-37}$$

$$(j_{j0})_1 = L \tag{D-38}$$

$$(k_{j0}) = \frac{\nu\lambda L}{(1-\nu^2)\mu}\left[\frac{\sigma^2}{12}\left(\frac{j\pi}{2}\right)^4 + \lambda^2\right] \tag{D-39}$$

Substituting equations (D-11) thru (D-13) into equations (D-9) and (D-10), it can be seen that the natural frequency equations of cylinder 1 become just that of the single cylinder (equations (D-1) and (D-2)), i.e.,

$$\left(\bar{\omega}^1_{j0}\right)_1^2 \longrightarrow \left(\omega^1_{j0}\right)^2 \tag{D-40}$$

$$\left(\bar{\omega}^2_{j0}\right)_1^2 \longrightarrow \left(\omega^2_{j0}\right)^2 \tag{D-41}$$

Similarly, for the case $j = 1, 2, \ldots, M_1$; $k = 1, 2, \ldots, N_1$, equations (D-17) thru (D-20) are evaluated and substituted into equations (D-14) thru (D-16). A comparison of equations (D-14) thru (D-16) with equations (D-6) thru (D-8), respectively, shows that the natural frequency equations of cylinder 1 become just that of the single cylinder, i.e.,

$$\left(\bar{\omega}^1_{jk}\right)_1^2 + \left(\bar{\omega}^2_{jk}\right)_1^2 + \left(\bar{\omega}^3_{jk}\right)_1^2 \longrightarrow \left(\omega^1_{jk}\right)^2 + \left(\omega^2_{jk}\right)^2 + \left(\omega^3_{jk}\right)^2 \tag{D-42}$$

$$\left(\bar{\omega}^1_{jk}\right)_1^2\left(\bar{\omega}^2_{jk}\right)_1^2 + \left(\bar{\omega}^2_{jk}\right)_1^2\left(\bar{\omega}^3_{jk}\right)_1^2 + \left(\bar{\omega}^3_{jk}\right)_1^2\left(\bar{\omega}^1_{jk}\right)_1^2 \longrightarrow \left(\omega^1_{jk}\right)^2\left(\omega^2_{jk}\right)^2 + \left(\omega^2_{jk}\right)^2\left(\omega^3_{jk}\right)^2 + \left(\omega^3_{jk}\right)^2\left(\omega^1_{jk}\right)^2 \tag{D-43}$$

$$\left(\bar{\omega}_{jk}^{1}\right)_1^2 \left(\bar{\omega}_{jk}^{2}\right)_1^2 \left(\bar{\omega}_{jk}^{3}\right)_1^2 \longrightarrow \left(\omega_{jk}^{1}\right)^2 \left(\omega_{jk}^{2}\right)^2 \left(\omega_{jk}^{2}\right)^2 \tag{D-44}$$

which renders

$$\left(\bar{\omega}_{jk}^{1}\right)_1 \longrightarrow \omega_{jk}^{1} \tag{D-45}$$

$$\left(\bar{\omega}_{jk}^{2}\right)_1 \longrightarrow \omega_{jk}^{2} \tag{D-46}$$

$$\left(\bar{\omega}_{jk}^{3}\right)_1 \longrightarrow \omega_{jk}^{3} \tag{D-47}$$

D-IV-2 Limiting Case 2, $L_1 = 0$

Referring to Figure 1 on page 3 it can be seen that when $L_1=0$, the center of mass shifts to the center of cylinder 2, thus, $d_1 = d_3 = L_2 = L$. For this limiting case the terms of equations (D-23) thru (D-25) are determined and the natural frequency equations (D-21) and (D-22) are evaluated. A comparison of equations (D-21) and (D-22) with equations (D-1 and (D-2) shows that for the limiting case $L_1 = 0$, the natural frequency equations of cylinder 2 become just that of the single cylinder, i.e.,

$$\left(\bar{\omega}_{j0}^{1}\right)_2 \longrightarrow \omega_{j0}^{1} \tag{D-48}$$

$$\left(\bar{\omega}_{j0}^{2}\right)_2 \longrightarrow \omega_{j0}^{2} \tag{D-49}$$

Similarly, it can be shown that the natural frequency equations (D-42) thru (D-44) for cylinder 2 become just that of the single cylinder equations (D-6) thru (D-8), i.e.,

$$\left(\bar{\omega}_{jk}^{1}\right)_2^2 + \left(\bar{\omega}_{jk}^{2}\right)_2^2 + \left(\bar{\omega}_{jk}^{3}\right)_2^2 \longrightarrow \left(\omega_{jk}^{1}\right)^2 + \left(\omega_{jk}^{2}\right)^2 + \left(\omega_{jk}^{3}\right)^2 \tag{D-50}$$

$$\left(\bar{\omega}_{jk}^{1}\right)_2^2 \left(\bar{\omega}_{jk}^{2}\right)_2^2 + \left(\bar{\omega}_{jk}^{2}\right)_2^2 \left(\bar{\omega}_{jk}^{3}\right)_2^2$$

$$+ \left(\bar{\omega}_{jk}^{3}\right)_2^2 \left(\bar{\omega}_{jk}^{1}\right)_2^2 \longrightarrow \left(\omega_{jk}^{1}\right)^2 \left(\omega_{jk}^{2}\right)^2$$

$$+ \left(\omega_{jk}^{2}\right)^2 \left(\omega_{jk}^{3}\right)^2 + \left(\omega_{jk}^{3}\right)^2 \left(\omega_{jk}^{1}\right)^2 \tag{D-51}$$

$$\left(\bar{\omega}_{jk}^{1}\right)_2^2 \left(\bar{\omega}_{jk}^{2}\right)_2^2 \left(\bar{\omega}_{jk}^{3}\right)_2^2 \longrightarrow \left(\omega_{jk}^{1}\right)^2 \left(\omega_{jk}^{2}\right)^2 \left(\omega_{jk}^{3}\right)^2 \tag{D-52}$$

which renders

$$\left(\bar{\omega}_{jk}^{1}\right)_2 \longrightarrow \omega_{jk}^{1} \tag{D-53}$$

$$\left(\bar{\omega}_{jk}^{2}\right)_2 \longrightarrow \omega_{jk}^{2} \tag{D-54}$$

$$\left(\bar{\omega}_{jk}^{3}\right)_2 \longrightarrow \omega_{jk}^{3} \tag{D-55}$$