FIRST SEMI-ANNUAL REPORT

NASA Grant No. NGR-35-002-003

"TIME-TEMPERATURE CHARACTERISTICS OF THIN-SKINNED MODELS AS AFFECTED BY THERMOCOUPLE VARIABLES"

by

MILTON B. LARSON
University of North Dakota
Grand Forks, North Dakota

October, 1965

ENGINEERING EXPERIMENT STATION

UNIVERSITY OF NORTH DAKOTA

GRAND FORKS NORTH DAKOTA

FIRST SEMI-ANNUAL REPORT
of the work on
NASA GRANT NO. NGR-35-002-003

"Time-Temperature Characteristics of Thin-Skinned Models as Affected by Thermocouple Variables"

by

Milton B. Larson
University of North Dakota
Grand Forks, North Dakota

October, 1965

The initial report has been delayed approximately six months because of difficulty in obtaining graduate students to work on the project and because of the press of other academic activities. Request for extension has already been made.

As outlined in the proposal, the first part of the problem consists of solving the family of equations

$$\frac{\partial^2 T_1}{\partial r^2} + \frac{1}{r}\frac{\partial T_1}{\partial r} + \frac{Q_1}{k_1 S} = \frac{c_1 \rho_1}{k_1}\frac{\partial T_1}{\partial \theta} \qquad (1)$$

and

$$\frac{\partial^2 T_2}{\partial x^2} = \frac{c_2 \rho_2}{k_2}\frac{\partial T_2}{\partial \theta} \qquad (2)$$

subject to the boundary conditions

$$T_1(r_0,\theta) = T_2(0,\theta) \qquad (a)$$

$$2\pi r_0 S k_1 \frac{\partial T_1}{\partial r}(r_0,\theta) = -\pi r_0^2 k_2 \frac{\partial T_2}{\partial x}(0,\theta) - Q_1 \pi r_0^2 \qquad (b)$$

$$\frac{\partial T_1}{\partial r}(r \to \infty, \theta) = 0 \qquad (c)$$

$$\frac{\partial T_2}{\partial x}(x \to \infty, \theta) = 0 \qquad (d)$$

The initial condition to be satisfied is

$$T_1(r,0) = T_2(x,0) = 0 \tag{e}$$

where

T_1	(r)	is the temperature of the thin skin
T_2	(x)	is the temperature of the wire
θ		is time
Q_1		is the heating rate per unit area of the skin
S		is the skin thickness
c		is the specific heat
ρ		is the density
k		is the thermal conductivity

subscript 1 refers to the skin
subscript 2 refers to the wire.

By means of the following definitions, these equations have been nondimensionalized:

$$R = r/r_0 \qquad \tau_1 = \frac{T_1 k_1 S}{Q_1 r_0^2} \qquad \phi = \frac{\alpha_1 \theta}{r_0^2}$$

$$X = x/r_0 \qquad \tau_2 = \frac{T_2 k_1 S}{Q_1 r_0^2} \quad .$$

The resulting set of equations is

$$\frac{\partial^2 \tau_1}{\partial R^2} + \frac{1}{R}\frac{\partial \tau_1}{\partial R} + 1 = \frac{\partial \tau_1}{\partial \phi} \tag{3}$$

$$\frac{\partial^2 \tau_2}{\partial X^2} = \frac{\alpha_1}{\alpha_2}\frac{\partial \tau_2}{\partial \phi} \tag{4}$$

with boundary conditions

$$\tau_1(1,\phi) = \tau_2(0,\phi) \tag{f}$$

$$2\,\frac{\partial \tau_1}{\partial R}(1,\phi) = -\frac{r_0 k_2}{S k_1}\frac{\partial \tau_2(0,\phi)}{\partial X} - 1 \tag{g}$$

$$\frac{\partial \tau_1}{\partial R}(R\to\infty,\phi) = 0 \tag{h}$$

$$\frac{\partial \tau_2}{\partial X}(X\to\infty,\phi) = 0 \tag{i}$$

and initial condition

$$\tau_1(R,0) = \tau_2(X,0) = 0 \quad . \tag{j}$$

Using the Laplace Transform, the following set of equations is obtained:

$$\frac{d^2 t_1}{dR^2} + \frac{1}{R}\frac{dt_1}{dR} + \frac{1}{s} = s\,t_1 \tag{5}$$

and

$$\frac{d^2 t_2}{dX^2} = \frac{\alpha_1}{\alpha_2}\,s\,t_2 \tag{6}$$

subject to the conditions

$$t_1(1,s) = t_2(0,s) \tag{k}$$

$$2\,\frac{dt_1}{dR}(1,s) = \frac{-k_1 r_0\, dt_2(0,s)}{k_2 S\, dX} - \frac{1}{s} \tag{l}$$

$$\frac{dt_1}{dR}(R\to\infty, s) = 0 \tag{m}$$

$$\frac{dt_2}{dX}(X\to\infty, s) = 0 \tag{n}$$

where t_1 and t_2 are the transforms of τ_1 and τ_2.

A solution of (5) satisfying (m) is

$$t_1 = \frac{1}{s^2} + C_2\, K_0(\sqrt{s}\,R) \quad .$$

A solution of (6) satisfying (n) is

$$t_2 = t_2(0,s)\, e^{-\sqrt{\frac{\alpha_1}{\alpha_2}s}\,X} \quad . \tag{7}$$

Using condition (k)

$$t_2(0,s) = \frac{1}{s^2} + C_2\, K_0\sqrt{s}$$

$$C_2 = \left[t_2(0,s) - \frac{1}{s^2}\right]\frac{1}{K_0\sqrt{s}} \quad .$$

Therefore,

$$t_1 = t_2(0,s)\frac{K_0(\sqrt{s}\,R)}{K_0(\sqrt{s})} + \frac{1}{s^2}\left[1 - \frac{K_0(\sqrt{s}\,R)}{K_0(\sqrt{s})}\right] . \tag{8}$$

Using equations (7) and (8) in condition (ℓ) provides

$$t_2(0,s) = \frac{\sqrt{s}\,K_0(\sqrt{s}) + 2K_1(\sqrt{s})}{s^2\left[2K_1(\sqrt{s}) + B K_0(\sqrt{s})\right]} \tag{9}$$

where

$$B = \frac{r_0}{S}\sqrt{\frac{k_2\, c_2\, \rho_2}{k_1\, c_1\, \rho_1}} .$$

When inversion of this transform is complete the problem is essentially solved.

If in condition (b) the term $Q_1 \pi r_0^2$ is omitted (this represents the energy input to the small area directly behind the wire on the skin), the equation for $t_2(0,s)$ becomes

$$t_2(0,s) = \frac{2K_1(\sqrt{s})}{s^2\left[2K_1(\sqrt{s}) + B K_0(\sqrt{s})\right]} . \tag{10}$$

Using the approach that

$$\mathcal{L}\left\{\int_0^t \mathcal{L}^{-1}(s\bar{v})\,dt\right\} = \frac{1}{s}(s\bar{v})$$

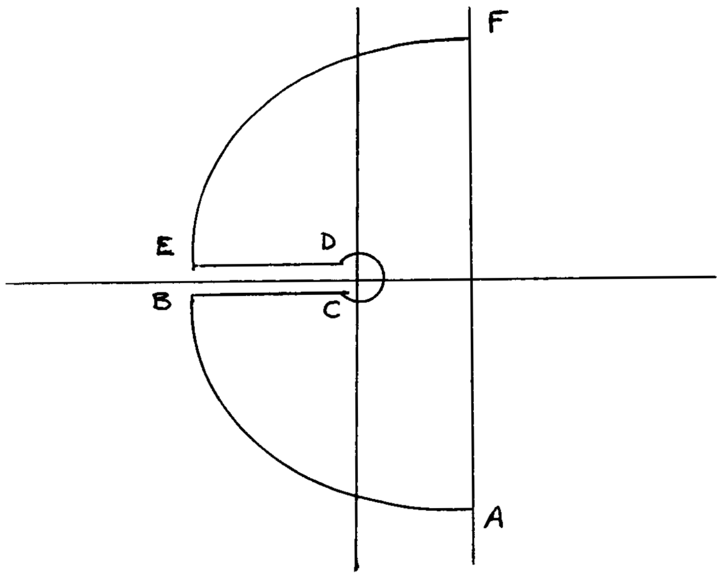
we obtain

$$s\,t_2(0,s) = \frac{K_1(\sqrt{s})}{s\left[\frac{B}{2}K_0(\sqrt{s}) + K_1(\sqrt{s})\right]}$$

and

$$\mathcal{L}^{-1}(s t_2) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{s\phi}\,\frac{K_1(\sqrt{s})}{s\left[\frac{B}{2}K_0(\sqrt{s}) + K_1(\sqrt{s})\right]}\,ds .$$

If the denominator of the integral has no zeros inside the contour shown in the sketch below, the integral above can be replaced by an integral around the contour A-B-C-D-E-F.



We have not yet been able to show that there are no zeros in the contour in question, but we are continuing to work on this.

The contour integral from A to B and E to F will have zero contribution in the limit as the radius tends to infinity.

For the small circle about the origin, s may be expressed as $s = \rho^2 e^{i\psi}$ while ψ varies from $-\pi$ to π.

$$\lim_{\rho\to 0}\int_{-\pi}^{\pi} \frac{e^{\rho^2 \phi e^{i\psi}} \; K_1(\rho e^{\frac{i\psi}{2}}) \quad i\rho^2 e^{i\psi}}{\rho^2 e^{i\psi}\left[\frac{B}{2} K_0(\rho e^{\frac{i\psi}{2}}) + K_1(\rho e^{\frac{i\psi}{2}})\right]} \, d\psi$$

$$= 2\pi i .$$

For the integral B-C, $\psi = -\pi$

$$= i\int_{\infty}^{0} \frac{e^{\rho^2 e^{-i\pi}\phi}\; K_1(\rho e^{-i\frac{\pi}{2}})\; 2\rho e^{-i\pi}\; d\rho}{\rho^2 e^{-i\pi}\left[\frac{B}{2} K_0(\rho e^{-i\pi/2}) + K_1(\rho e^{-i\pi/2})\right]}$$

$$= i\int_{\infty}^{0} \frac{e^{-\rho^2\phi}\; K_1(-i\rho)\; 2\; d\rho}{\rho\left[\frac{B}{2} K_0(-i\rho) + K_1(-i\rho)\right]}$$

$$= -i\int_{0}^{\infty} \frac{2e^{-\rho^2\phi}\{-i\,Y_1(\rho) - J_1(\rho)\}\, d\rho}{\rho\left[\frac{B}{2}\{i\,J_0(\rho) - Y_0(\rho)\} - \{i\,Y_1(\rho) + J_1(\rho)\}\right]}\,.$$

For the integral over D-E ($\psi = \pi$) we have the negative conjugate of the above. When this is added in

$$= -i\int_{0}^{\infty} 4e^{-\rho^2\phi}\left\{\frac{J_1^2(\rho) + Y_1^2(\rho) + \frac{B}{\pi\rho}}{J_1^2(\rho) + Y_1^2(\rho) + \frac{2B}{\pi\rho} + \frac{B^2}{4}\left[Y_0^2(\rho) + J_0^2(\rho)\right]}\right\} d\rho$$

$$\mathcal{L}^{-1}\left[s\,t_2(0,s)\right] = 1 - \frac{2}{\pi}\int_{0}^{\infty} \frac{e^{-\rho^2\phi}}{\rho}\left\{\quad\right\} d\rho$$

Therefore,

$$\tau_2(0,\phi) = \phi + \frac{2}{\pi}\int_0^\infty \frac{1-e^{-\rho^2\phi}}{\rho^3}\left\{\frac{J_1^2(\rho)+Y_1^2(\rho)+\frac{B}{\pi\rho}}{J_1^2(\rho)+Y_1^2(\rho)+\frac{B^2}{4}\left[Y_0^2(\rho)+J_0^2(\rho)\right]+\frac{2B}{\pi\rho}}\right\}d\rho \tag{11}$$

An inversion of equation (9) has not been attempted at this point. However, solutions of equation (10) suitable for short time have been worked on by use of asymptotic solutions as suggested by Carslaw and Jaeger in Operational Methods in Applied Mathematics pp. 276-270. The result obtained is

$$\tau_2(0,\phi) = \frac{2\phi}{(B+2)\,\Gamma(2)} + \frac{B\,\phi^{3/2}}{\Gamma(5/2)} + \frac{\phi^2\left[-\frac{17}{4}B^2-9B-9\right]}{16\,\Gamma(3)} + \cdots \tag{12}$$

Carslaw and Jaeger provide justification for this approach for small enough time. Physically it would appear that the period of time of interest in the problem would be small enough so that this would be valid. That is, the interest is in the period of time during which the temperature is just starting to rise. However, the magnitude of ϕ calculated for typical values of the parameters and variables is about 1,000. This obviously will prevent the use of equation (12) for any solution. At this time no resolution of this present problem has been reached. At the present time, it would seem that some numerical integration of (11) will provide a satisfactory solution.

Future efforts will be expended in being certain of the validity of previous assumptions and in gaining a perspective as to the type of solution which should be obtained.