TECHNICAL NOTE R-92

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Prepared By

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ANALYSIS OF UNSTEADY INCOMPRESSIBLE FLOW OUT OF DIFFERENT TANK CONFIGURATIONS

March 1964

Prepared For

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PROPULSION DIVISION
P & VE LABORATORIES
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ABSTRACT

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An analysis was made in order to find the flow rates and position of liquid levels in tanks when they are drained by gravity. During this process unsteady flow occurs and a given steady flow cannot be assumed any more. Samples for one-tank and two-tank configurations are brought up with and without hydraulic losses in the lines. The results are differential equations which can be solved numerically by a digital computer.

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LIST OF SYMBOLS

A	Cross-sectional area of the tank
a	Cross-sectional area of the tube
C	Integration constant
D	Hydraulic diameter
d	Diameter of the tube
e	Length of the tube
F	External force, inertia force
g	Gravitational constant
H	Initial height in the tank
h	Liquid height in the tank at any time
$\vec{i}$	Unit vector for the x axis
$\vec{j}$	Unit vector for the y axis
$\vec{k}$	Unit vector for the z axis
K	Lateral surface
ℓ	Mean roughness height
m	Mass of the fluid
$\vec{n}$	External normal to the surface element
n	Exponent (constant)
P	Perimeter
p	Pressure
p_o	Tank pressure
p_a	Ambient pressure
Q	Volume flow rate

Re	Reynolds Number
S	Surface
t	Time
V	Volume
v	Velocity
$\vec{v}$	Velocity vector
W	Potential of inertia forces per unit mass
x	Coordinate
y	Coordinate
z	Coordinate

Greek Symbols

α	Constant
β	Constant
ε	Relative roughness
λ	Nondimensional friction coefficient
ν	Kinematic viscosity
π	Total pressure
ρ	Density of the mass
τ	Frictional stress

Subscripts

a	Location at entrance
b	Location at exit
1	Refers to tank 1 and interconnect line
2	Refers to tank 2 and draining line

INTRODUCTION

In order to get a complete and fundamental understanding of the unsteady flow out of the tanks, it is best to start with the governing equation of momentum and continuity and the derivation of these equations. As shown in the following treatment, these governing equations are subject to great simplification and it is pointed out that certain terms in the general equations may be neglected, e.g. when assuming that the hydrodynamic variables are constant over the cross section, comparing first order terms with higher order terms which are negligible and cancelling terms which are much smaller than the main expressions. After obtaining the equations, the application of them on one- and two-tank configurations will follow.

DERIVATION OF THE BASIC EQUATIONS OF FLUID FLOW

The law of momentum for a given mass of fluid

$$m = \iiint_V \rho \, dV$$

is

$$\frac{dm}{dt} \cdot \vec{v} = \sum \vec{F}_i \quad , \tag{1}$$

where $\vec{v}$ denotes the velocity vector and $\sum \vec{F}_i$ denotes the resultant external force on the fluid mass which is equal to the change of the momentum vector $m \cdot \vec{v}$ with respect to time. The momentum vector is

$$m\vec{v} = \iiint_V \rho \, \vec{v} \, dV \quad . \tag{2}$$

Variation of any mass m during flow may be a reason of changes in the density ρ and changes in the volume V which the fluid mass takes in any moment of time.

The expression for changing the mass due to the variation of the density is

$$\iiint_V \frac{\partial \rho}{\partial t} \, dV \quad ,$$

and the change of the mass per unit time due to variation of the volume is

$$\iint_S \rho \, v_n \, dS \quad ,$$

where S is the surface enclosing the volume V and v_n is the normal component of the velocity acting on a given point of the surface. The last term can be changed with the aid of Green's theorem to a volume integral

$$\iint_S \rho \, v_n \, dS = \iiint_V \nabla \cdot \rho \vec{v} \, dV \quad , \tag{3}$$

or with

$$\nabla \cdot \rho \vec{v} = \frac{\partial}{\partial x} (\rho \, v_x) + \frac{\partial}{\partial y} (\rho \, v_y) + \frac{\partial}{\partial z} (\rho \, v_z) \tag{4}$$

$$\iint_S \rho \, v_n \, dS = \iiint_V \left[\frac{\partial}{\partial x} (\rho \, v_x) + \frac{\partial}{\partial y} (\rho \, v_y) + \frac{\partial}{\partial z} (\rho \, v_z) \right] dV \quad . \tag{5}$$

With these expressions the law of momentum is

$$\begin{gathered} \iiint_V \frac{\partial \rho \vec{v}}{\partial t} \, dV + \iint_S \rho \vec{v} \, v_n \, dS = \\ \iiint_V \left[\frac{\partial \rho \vec{v}}{\partial t} + \frac{\partial}{\partial x} (\rho \vec{v} \, v_x) + \frac{\partial}{\partial y} (\rho \vec{v} \, v_y) + \frac{\partial}{\partial z} (\rho \vec{v} v_z) \right] dV = \sum \vec{F}_i \quad . \end{gathered} \tag{6}$$

The resultant external force $\sum \vec{F}$ consists of the inertia force

$$\iiint_V \rho \, \vec{F} \, dV \quad .$$

$\vec{F}$ represents the inertia force per unit mass (actually with the dimensions of acceleration), and the pressure force

$$\iint_S p \, \vec{n} \, dS \quad ,$$

where $\vec{n}$ is the external normal to the surface element dS, p is the absolute pressure. The third force comes from the friction

$$\iint_S \vec{\tau}_n \, dS \quad .$$

$\vec{\tau}_n$ is the frictional stress or vector of the frictional force with a normal $\vec{n}$.

The equation for the law of momentum now yields

$$\iiint_V \left[\frac{\partial \rho \vec{v}}{\partial t} + \frac{\partial}{\partial x} (\rho \vec{v} v_x) + \frac{\partial}{\partial y} (\rho \vec{v} v_y) + \frac{\partial}{\partial z} (\rho \vec{v} v_z) \right] dV$$

$$= \iiint_V \rho \vec{F} \, dV - \iint_S p \vec{n} \, dS + \iint_S \vec{\tau}_n \, dS \quad . \tag{7}$$

The conversion of the surface integrals to a volume integral with the Green's theorem gives

$$\iint_S \vec{\tau} \, dS = \iiint_V \left(\frac{\partial \vec{\tau}_x}{\partial x} + \frac{\partial \vec{\tau}_y}{\partial y} + \frac{\partial \vec{\tau}_z}{\partial z} \right) dV \quad , \tag{8}$$

where $\vec{\tau}_x$, $\vec{\tau}_y$, $\vec{\tau}_z$ are the frictional stresses acting on the surfaces perpendicular to the x, y and z axes, respectively.

Now

$$\iint_S p \vec{n} \, dS = \iiint_V \nabla p \, dV \quad , \tag{9}$$

where the Del operator shows

$$\nabla p = \frac{\partial p}{\partial x} \vec{i} + \frac{\partial p}{\partial y} \vec{j} + \frac{\partial p}{\partial z} \vec{k} \quad .$$

$\vec{i}$, $\vec{j}$, $\vec{k}$ are unit vectors for the x, y, z axes.

The equation for the law of momentum now is

$$\iiint_V \left[\frac{\partial \rho \vec{v}}{\partial t} + \frac{\partial}{\partial x} (\rho \vec{v} v_x) + \frac{\partial}{\partial y} (\rho \vec{v} v_y) + \frac{\partial}{\partial z} (\rho \vec{v} v_z) \right] dV = \iiint_V \rho \vec{F} \, dV$$

$$- \iiint_V \left[\frac{\partial p}{\partial x} \vec{i} + \frac{\partial p}{\partial y} \vec{j} + \frac{\partial p}{\partial z} \vec{k} \right] dV + \iiint_V \left[\frac{\partial \vec{\tau}_x}{\partial x} + \frac{\partial \vec{\tau}_y}{\partial y} + \frac{\partial \vec{\tau}_z}{\partial z} \right] dV \quad . \tag{10}$$

The law of conservation of mass can be written down very easily

$$\frac{dm}{dt} = 0$$

with

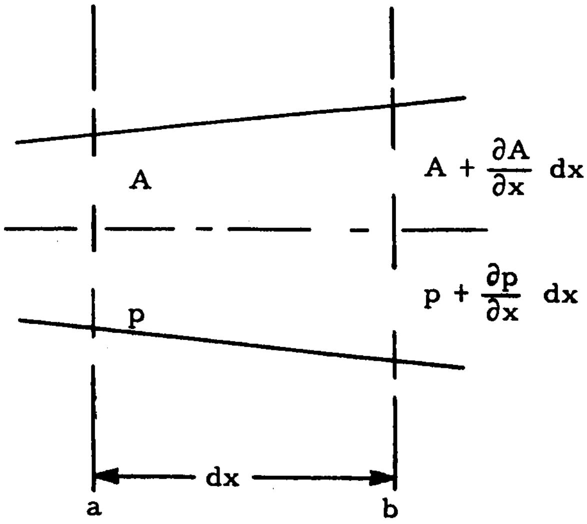
$$m = \iiint_V \rho \, dV \quad ,$$

$$\frac{dm}{dt} = \iiint_V \frac{\partial \rho}{\partial t} \, dV + \iint_S \rho \, v_n \, dS = 0 \quad ,$$

or

$$\iiint_V \frac{\partial \rho}{\partial t} \, dV + \iiint_V \nabla \cdot \rho \vec{v} \, dV = 0 \quad . \tag{11}$$

These integral equations can be applied to a fluid mass which is located between the cross sections a and b. If the element dx is assumed to be very small, the volume occupied by the fluid mass will be A dx. The surface enclosing this volume element consists of the two cross sections A_a and A_b plus the lateral surface K which is equal to the perimeter P times the element dx, assuming a small element dx.



With this restriction the integral can be written

$$\iint_S \rho\, \vec{v}\, v_n\, dS = -\iint_{A_a} \rho\, \vec{v}\, v_x\, dA + \iint_{A_b} \rho\, \vec{v}\, v_x\, dA + \iint_K \rho\, \vec{v}\, v_n\, dS \quad . \tag{12}$$

Similar to that, the integral for the frictional force is

$$\iint_S \vec{\tau}_n\, dS = -\iint_{A_a} \vec{\tau}_x\, dA + \iint_{A_b} \vec{\tau}_x\, dA + \iint_K \vec{\tau}_n\, dS \quad . \tag{13}$$

The pressure force is also for the x-direction

$$\iint_S p\, \vec{n}\, dS = \left(-\iint_{A_a} p\, dA + \iint_{A_b} p\, dA\right) \vec{i} + \iint_K p\, \vec{n}\, dS \quad , \tag{14}$$

where $\vec{i}$ is the unit vector in the x direction. Substituting this into the equation of momentum yields

$$\iiint_V \frac{\partial \rho \vec{v}}{\partial t}\, dV - \iint_{A_a} \rho \vec{v} v_x \, dA + \iint_{A_b} \rho \vec{v} v_x \, dA + \iint_K \rho \vec{v} v_n \, dS$$

$$= \iiint_V \rho \vec{F}\, dV + \left(\iint_{A_a} p \, dA - \iint_{A_b} p \, dA \right) \vec{i} - \iint_K p \vec{n}\, dS - \iint_{A_a} \vec{\tau}_x \, dA \tag{15}$$

$$+ \iint_{A_b} \vec{\tau}_x \, dA + \iint_K \vec{\tau}_n \, dS \quad .$$

Let dx be zero in order to eliminate the volume integral and make the limit condition

$$\lim_{\Delta x \to 0} \frac{\displaystyle\iiint_V \frac{\partial \rho \vec{v}}{\partial t}\, dV}{\Delta x} = \iint_A \frac{\partial \rho \vec{v}}{\partial t}\, dA \quad .$$

The difference between the momentum of entrance and exit can be written at $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{\displaystyle - \iint_{A_a} \rho \vec{v} v_x \, dA + \iint_{A_b} \rho \vec{v} v_x \, dA}{\Delta x} = \frac{\partial}{\partial x} \iint_A \rho \vec{v} v_x \, dA \quad .$$

If the length element goes to zero, the lateral surface K changes to the perimeter P, so that

$$\lim_{\Delta x \to 0} \frac{\displaystyle\iint_K \rho \vec{v} v_n \, dS}{\Delta x} = \int_P \rho \vec{v} v_n \, dP \quad .$$

Also

$$\lim_{\Delta x \to 0} \frac{\iiint\limits_V \rho \vec{F}\, dV}{\Delta x} = \iint\limits_A \rho \vec{F}\, dA \quad .$$

Similar to the resultant momentum the following relation can be obtained by the limit condition.

$$\lim_{\Delta x \to 0} \frac{-\iint\limits_{A_a} p\, dA + \iint\limits_{A_b} p\, dA}{\Delta x} = \frac{\partial}{\partial x} \iint\limits_A p\, dA \quad ,$$

and

$$\lim_{\Delta x \to 0} \frac{-\iint\limits_{A_a} \vec{\tau}_x\, dA + \iint\limits_{A_b} \vec{\tau}_x\, dA}{\Delta x} = \frac{\partial}{\partial x} \iint\limits_A \vec{\tau}_x\, dA \quad .$$

The equation of momentum for the x-projection will be (with the Cauchy formula $\vec{n}\, dS = \cos(n, x)\, dS$

$$\iint\limits_A \frac{\partial \rho\, v_x}{\partial t}\, dA + \frac{\partial}{\partial x} \iint\limits_A \rho\, v_x{}^2\, dA + \int\limits_P \rho\, v_x\, v_n\, dP = \iint\limits_A \rho\, F_x\, dA$$

$$- \frac{\partial}{\partial x} \iint\limits_A p\, dA + \frac{\partial}{\partial x} \iint\limits_A \tau_{xx}\, dA - \lim_{\Delta x \to 0} \frac{\iint\limits_K p \cos(n, x)\, dS}{\Delta x} \qquad (16)$$

$$+ \lim_{\Delta x \to 0} \frac{\iint\limits_K \tau_{nx}\, dS}{\Delta x} \quad .$$

The last two terms must still be evaluated. At first the inertia force per unit mass in x-direction remains to be determined.

$$F_x = \frac{\partial W}{\partial x} \quad ,$$

where W is the potential of inertia forces per unit mass.

$$W = - gx \quad .$$

Substituting into the integral yields

$$\iint\limits_A \rho \; F_x \, dA = \iint\limits_A \rho \, \frac{\partial W}{\partial x} \, dA \quad .$$

The term is a part of the partial derivative

$$\iint\limits_A \frac{\partial \rho \, W \, dA}{\partial x} = \iint\limits_A \rho \, \frac{\partial W}{\partial x} \, dA + \iint\limits_A W \, \frac{\partial \rho}{\partial x} \, dA \quad .$$

So

$$\iint\limits_A \rho \; F_x \, dA = \iint\limits_A \frac{\partial \rho \, W \, dA}{\partial x} - \iint\limits_A W \, \frac{\partial \rho}{\partial x} \, dA \quad . \tag{17}$$

Since

$$\frac{\partial \rho}{\partial x} = 0 \quad ,$$

$$\iint\limits_A \rho \; F_x \, dA = \iint\limits_A \frac{\partial \rho \; W \, dA}{\partial x} = \iint\limits_A \rho \, \frac{\partial W}{\partial x} \, dA \quad .$$

The next step is making the limit condition

$$\lim_{\Delta x \to 0} \frac{\iint\limits_K p \; \cos\,(n, x) \, dS}{\Delta x} \quad .$$

The total pressure will be set equal to

$$\pi = p + \rho\, gx = p - \rho\, W \quad , \tag{18}$$

where W is again the potential of inertia forces per unit mass.

So

$$\iint_K p \cos(n, x)\, dS = \iint_K \pi \cos(n, x)\, dS + \iint_K \rho\, W \cos(n, x)\, dS \quad . \tag{19}$$

π is assumed at a good approximation to be a mean value over the area, so

$$\iint_K \pi \cos(n, x)\, dS = \pi \iint_K \cos(n, x)\, dS \quad .$$

and the integral $\iint_K \cos(n, x)\, dS$ is equal to $- \frac{\partial A}{\partial x}\, dx$, neglecting small second-order terms.

As Δx is going to zero, the result is

$$\lim_{\Delta x \to 0} \frac{\iint_K \pi \cos(n, x)\, dS}{\Delta x} = - \pi \frac{\partial A}{\partial x} \quad ,$$

where π is an average value around the perimeter P. So far the equation of momentum looks like

$$\iint\limits_A \frac{\partial \rho\ v_x}{\partial x}\ dA + \frac{\partial}{\partial x}\iint\limits_A \rho\ v_x^2\ dA + \int\limits_P \rho\ v_x\ v_n\ dP = \iint\limits_A \frac{\partial \rho\ W}{\partial x}\ dA$$

$$- \frac{\partial}{\partial x}\iint\limits_A p\ dA + \frac{\partial}{\partial x}\iint\limits_A \tau_{xx}\ dA + \pi\,\frac{\partial A}{\partial x} - \lim_{\Delta x \to 0} \frac{\iint\limits_K \rho\ W \cos(n, x)\ dS}{\Delta x}$$

$$+ \lim_{\Delta x \to 0} \frac{\iint\limits_K \tau_{nx}\ dS}{\Delta x} \quad . \tag{20}$$

To eliminate the term

$$\lim_{\Delta x \to 0} \frac{\iint\limits_K \rho\ W \cos(n, x)\ dS}{\Delta x} \quad ,$$

a relation which includes this can be derived from

$$\frac{\partial}{\partial x}\iiint\limits_V \rho\ W\ dV = \iiint\limits_V \frac{\partial \rho\ W}{\partial x}\ dV - \iint\limits_K \rho\ W\,\vec{n}\ dS$$

$$= \iiint\limits_V \frac{\partial \rho\ W}{\partial x}\ dV - \iint\limits_K \rho\ W \cos(n, x)\ dS \quad . \tag{21}$$

Let $\Delta x \to 0$

$$\frac{\partial}{\partial x}\iint\limits_A \rho\ W\ dA = \iint\limits_A \frac{\partial \rho\ W}{\partial x}\ dA - \lim_{\Delta x \to 0} \frac{\iint\limits_K \rho\ W \cos(n, x)\ dS}{\Delta x} \quad . \tag{22}$$

Substitution of this into the equation of momentum results in

$$\iint\limits_A \frac{\partial \rho\, v_x}{\partial x}\, dA + \frac{\partial}{\partial x} \iint\limits_A \rho\, v_x^2\, dA + \int\limits_P \rho\, v_x v_n\, dP = \frac{\partial}{\partial x} \iint\limits_A \rho\, W\, dA$$

$$- \frac{\partial}{\partial x} \iint\limits_A p\, dA + \frac{\partial}{\partial x} \iint\limits_A \tau_{xx}\, dA + \pi \frac{\partial A}{\partial x} + \lim_{\Delta x \to 0} \frac{\iint\limits_K \tau_{nx}\, dS}{\Delta x} \quad . \tag{23}$$

Or with $\pi = p - \rho\, W$

$$\iint\limits_A \frac{\partial \rho\, v_x}{\partial t}\, dA + \frac{\partial}{\partial x} \iint\limits_A \rho\, v_x^2\, dA + \int\limits_P \rho\, v_x\, v_n\, dP = - \frac{\partial}{\partial x} \iint\limits_A \pi\, dA$$

$$+ \pi \frac{\partial A}{\partial x} + \frac{\partial}{\partial x} \iint\limits_A \tau_{xx}\, dA + \lim_{\Delta x \to 0} \frac{\iint\limits_K \tau_{nx}\, dS}{\Delta x} \quad . \tag{24}$$

The frictional force per unit area of lateral surface is τ_o, so at $\Delta x \to 0$ the term changes to

$$\lim_{\Delta x \to 0} \frac{\iint\limits_K \tau_{nx}\, dS}{\Delta x} = - \int\limits_P \tau_o\, dP \quad .$$

P is again the wetted perimeter.

For simplification the relation can be applied

$$\frac{\partial}{\partial t} \iint\limits_A \rho\, v_x\, dA = \iint\limits_A \frac{\partial \rho\, v_x}{\partial t}\, dA + \int\limits_P \rho\, v_x v_n\, dP \quad , \tag{25}$$

which includes two terms of the above equation.

Substituting this into the equation of momentum yields

$$\frac{\partial}{\partial t} \iint\limits_A \rho \, v_x \, dA + \frac{\partial}{\partial x} \iint\limits_A \rho \, v_x^2 \, dA = - \frac{\partial}{\partial x} \iint\limits_A \pi \, dA + \pi \frac{\partial A}{\partial x}$$

$$+ \frac{\partial}{\partial x} \iint\limits_A \tau_{xx} \, dA - \int\limits_P \tau_o \, dP \quad . \tag{26}$$

Since τ_{xx} is small in comparison with p, the term can be neglected. When the integration over the cross section is done, the equation becomes

$$\frac{\partial}{\partial t} \rho \, v_x \, A + \frac{\partial}{\partial x} \rho \, v_x^2 \, A = - \frac{\partial}{\partial x} \pi A + \pi \frac{\partial A}{\partial x} - \int\limits_P \tau_o \, dP \quad . \tag{27}$$

Since

$$\frac{\partial}{\partial x} \pi A = A \frac{\partial}{\partial x} \pi + \pi \frac{\partial A}{\partial x} \quad , \tag{28}$$

the equation of momentum will be

$$\frac{\partial}{\partial t} \rho \, v_x \, A + \frac{\partial}{\partial x} \rho \, v_x^2 \, A = - A \frac{\partial \pi}{\partial x} - \int\limits_P \tau_o \, dP \quad , \tag{29}$$

or with $\pi = p - \rho W = p + \rho g x$, integration over the wetted perimeter P gives the result

$$\frac{\partial \rho \, v_x A}{\partial t} + \frac{\partial}{\partial x} \rho \, v_x^2 \, A = - A \frac{\partial}{\partial x} (p + \rho g x) - \tau_o \, P \quad . \tag{30}$$

This is the equation expressing the law of momentum where v_x, ρ and p are the cross sectional averages of the longitudinal velocity, density and pressure, respectively.

The conservation of mass as indicated earlier is equal to

$$\frac{dm}{dt} = 0$$

$$\iiint_V \frac{\partial \rho}{\partial t} \, dV + \iiint_V \nabla \cdot \rho \vec{v} \, dV = 0 \quad ,$$

or

$$\iiint_V \frac{\partial \rho}{\partial t} \, dV + \iint_S \rho \, v_n \, dS = 0 \quad . \tag{31}$$

Similar to the previous relation the integral can be written in x-direction

$$\iint_S \rho \, v_n \, dS = - \iint_{A_a} \rho \, v_x \, dA + \iint_{A_b} \rho \, v_x \, dA + \iint_K \rho \, v_n \, dS \quad , \tag{32}$$

where A_a, A_b and K are the surfaces enclosing the fluid mass. Again let dx be zero.

$$\lim_{\Delta x \to 0} \frac{\iiint_V \frac{\partial \rho}{\partial t} \, dV}{\Delta x} = \iint_A \frac{\partial \rho}{\partial t} \, dA \quad ,$$

$$\lim_{\Delta x \to 0} \frac{- \iint_{A_a} \rho \, v_x \, dA + \iint_{A_b} \rho \, v_x \, dA}{\Delta x} = \frac{\partial}{\partial x} \iint_A \rho \, v_x \, dA \quad ,$$

$$\lim_{\Delta x \to 0} \frac{\iint_K \rho \, v_n \, dS}{\Delta x} = \int_P \rho \, v_n \, dP \quad .$$

P is the wetted perimeter.

The equation of continuity now is

$$\iint\limits_A \frac{\partial \rho}{\partial t}\, dA + \frac{\partial}{\partial x} \iint\limits_A \rho\, v_x\, dA + \int\limits_P \rho\, v_n\, dP = 0 \quad . \tag{33}$$

The first integral can be replaced by the relation

$$\frac{\partial}{\partial t} \iint\limits_A \rho\, dA = \iint\limits_A \frac{\partial \rho}{\partial t}\, dA + \int\limits_P \rho\, v_n\, dP \quad . \tag{34}$$

Substituting this into the above equation delivers

$$\frac{\partial}{\partial t} \iint\limits_A \rho\, dA + \frac{\partial}{\partial x} \iint\limits_A \rho\, v_x\, dA = 0 \quad , \tag{35}$$

and after integration over the cross section

$$\frac{\partial \rho\, A}{\partial t} + \frac{\partial \rho\, v_x\, A}{\partial x} = 0 \quad . \tag{36}$$

This is the equation for the conservation of mass for compressible and incompressible flow.

APPLICATION TO UNSTEADY FLOW PROCESSES

Four different tank configurations will be treated as samples using the equation for the law of momentum.

1. Flow out of a tank through an orifice on the bottom where the hydraulic loss is assumed to be zero.

2. Efflux out of a tank through a tube with hydraulic losses.

3. Two tanks connected by a line. Draining of both tanks through one orifice. Hydraulic losses exist in the interconnect line.

4. Two tanks connected by a line. Draining of both tanks through a tube. Hydraulic losses exist in the interconnect line and draining tube.

Case 1 - The efflux takes place out of a tank through an orifice.

The equation of momentum describing the process without hydraulic losses is

$$\frac{\partial \rho\, v_x A}{\partial t} + \frac{\partial}{\partial x} \rho\, v_x{}^2 A = - A \frac{\partial}{\partial x} (p + \rho g x) \quad . \tag{37}$$

With the substitution of the volume flow rate

$$Q(t) = v_x A \quad ,$$

and dividing by ρA, the equation becomes

$$\frac{\partial}{\partial t} \frac{Q}{A} + \frac{\partial}{\partial x} \frac{Q^2}{A^2} = - \frac{\partial}{\partial x} \left(\frac{p}{\rho} + g x \right) \quad . \tag{38}$$

Let

$$\frac{\partial}{\partial x} \frac{p}{\rho} = 0 \quad .$$

that means $p_o = p_a$, tanks pressure is equal to the ambient pressure.

Integrating with respect to x from x = 0 to x = h yields

$$\frac{dQ}{dt}\int_0^h \frac{dx}{A} + \int_0^h Q^2 \frac{\partial}{\partial x}\frac{1}{A^2}\, dx = gh \quad ,$$

$$\frac{dQ}{dt}\int_0^h \frac{dx}{A} + Q^2 \int_0^h \frac{1}{A}\, d\,\frac{1}{A} = gh \quad ,$$

or

$$\frac{dQ}{dt}\int_0^h \frac{dx}{A} + \frac{Q^2}{2}\left(\frac{1}{a^2} - \frac{1}{A^2}\right) = gh \tag{39}$$

where a is the area of the orifice and A is the area of the tank.

Using the relation for replacing the velocity of the fluid level in the tank by the change of the height per time

$$Q = v_x A = -\frac{dh}{dt} A \quad ,$$

and with the second derivative

$$\frac{dQ}{dt} = \frac{d}{dt}\left(-A\frac{dh}{dt}\right) = -A\frac{d^2h}{dt^2} - \frac{dA}{dt}\frac{dh}{dt} \quad ,$$

the equation becomes

$$\left(-A\frac{d^2h}{dt^2} - \frac{dA}{dt}\frac{dh}{dt}\right)\int_0^h \frac{dh}{A} + \frac{A^2}{2}\left(\frac{1}{a^2} - \frac{1}{A^2}\right)\left(\frac{dh}{dt}\right)^2 = gh \quad . \tag{40}$$

The expression can be rewritten

$$\frac{dA}{dt}\frac{dh}{dt} = \frac{dA}{dt}\frac{dh}{dt}\frac{(dh)}{(dh)} = \frac{dA}{dh}\left(\frac{dh}{dt}\right)^2 \quad .$$

So, the equation will be

$$A \int_0^h \frac{dh}{A} \left[- \frac{d^2h}{dt^2} - \frac{1}{A} \frac{dA}{dh} \left(\frac{dh}{dt}\right)^2 \right] + \frac{1}{2} \left(\frac{A^2}{a^2} - 1 \right) \left(\frac{dh}{dt}\right)^2 = gh \quad . \tag{41}$$

In order to evaluate the second derivative, the operation can be made

$$\frac{d}{dt} \left(\frac{dh}{dt}\right)^2 = 2 \frac{dh}{dt} \frac{d^2h}{dt^2} \quad ,$$

or

$$\frac{d^2h}{dt^2} = \frac{1}{2} \frac{d}{dh} \left(\frac{dh}{dt}\right)^2 \quad .$$

Substituting into the above equation delivers

$$A \int_0^h \frac{dh}{A} \left[- \frac{1}{2} \frac{d}{dh} \left(\frac{dh}{dt}\right)^2 - \frac{1}{A} \frac{dA}{dh} \left(\frac{dh}{dt}\right)^2 \right] + \frac{1}{2} \left(\frac{A^2}{a^2} - 1 \right) \left(\frac{dh}{dt}\right)^2 = gh \quad , \tag{42}$$

or in another form

$$A \int_0^h \frac{dh}{A} \left[- \frac{d}{dh} \left(\frac{dh}{dt}\right)^2 \right] + \left(\frac{dh}{dt}\right)^2 \left(\frac{A^2}{a^2} - 1 \right) - 2 \frac{dA}{dh} \int_0^h \frac{dh}{A} \left(\frac{dh}{dt}\right)^2 = 2gh \quad . \tag{43}$$

The differential equation will be after dividing by the integral

$$\frac{d}{dh} \left(\frac{dh}{dt}\right)^2 + \frac{2 \frac{dA}{dh} \int_0^h \frac{dh}{A} - \left(\frac{A^2}{a^2} - 1 \right)}{A \int_0^h \frac{dh}{A}} \left(\frac{dh}{dt}\right)^2 = - \frac{2g}{A \int_0^h \frac{dh}{A}} h \quad . \tag{44}$$

This linear differential equation can be written in a simple form with

$$\left(\frac{dh}{dt}\right)^2 = \theta \quad ,$$

$$\frac{d\theta}{dh} + \phi_{(h)}\ \theta = \psi_{(h)} \quad , \tag{45}$$

where

$$\phi(h) = \frac{2\,\frac{dA}{dh}\int_0^h \frac{dh}{A} - \left(\frac{A^2}{a^2} - 1\right)}{A\int_0^h \frac{dh}{A}} \quad ,$$

and

$$\psi_{(h)} = -\frac{2g}{A\int_0^h \frac{dh}{A}}\ h \quad .$$

The solution is

$$\theta = e^{-\int \phi_{(h)}\,dh}\left\{C + \int\left[\psi_{(h)}\, e^{\int \phi_{(h)}\,dh}\right]dh\right\} \quad . \tag{46}$$

For evaluating the integration constant C, the boundary condition is:

At

$$t = 0, \qquad h = H$$

and

$$Q = Q_o = -A\,\frac{dh}{dt} \quad .$$

So,

$$\left(\frac{dh}{dt}\right)^2 = C = \frac{Q_o^{\,2}}{A^2}$$

$$\theta = \exp\left(-\int \phi(h)\, dh\right) \left\{ \frac{Q_o^{\,2}}{A^2} - \int \left[\psi(h) \exp\left(\int \phi(h)\, dh\right)\right] dh \right\} \quad . \tag{47}$$

Now the assumption is made that the tank consists mainly of a cylindrical portion with a flat bottom and that means

$$\frac{dA}{dh} = 0 \quad .$$

The coefficients after integration are

$$\phi(h) = \frac{2\,\frac{dA}{dh} \int_0^h \frac{dh}{A} - \left(\frac{A^2}{a^2} - 1\right)}{A \int_0^h \frac{dh}{A}} = -\frac{1}{h}\left(\frac{A^2}{a^2} - 1\right) \quad ,$$

and

$$\psi(h) = -\frac{2g}{A \int_0^h \frac{dh}{A}}\, h = -2g \quad .$$

The differential equation now appears as

$$\left(\frac{dh}{dt}\right)^2 = \exp\left[\int_H^h \left(\frac{A^2}{a^2} - 1\right) \frac{dh}{h}\right] \left\{ \frac{Q_o^{\,2}}{A^2} - \int_H^h 2g \exp\left[-\int_H^h \left(\frac{A^2}{a^2} - 1\right) \frac{dh}{h}\right] dh \right\} \quad . \tag{48}$$

The integrals will be

$$\exp\left[-\left(\frac{A^2}{a^2}-1\right)\int_H^h \frac{dh}{h}\right] = \exp\left[-\left(\frac{A^2}{a^2}-1\right) \ln h \Big|_H^h\right]$$

$$= \exp\left[\ln\left(\frac{h}{H}\right)^{-\left(\frac{A^2}{a^2}-1\right)}\right] = \left(\frac{h}{H}\right)^{-\left(\frac{A^2}{a^2}-1\right)}$$

and

$$\int_H^h 2g \exp\left[-\int_H^h \left(\frac{A^2}{a^2}-1\right)\frac{dh}{h}\right] dh = 2g\int_H^h \left(\frac{h}{H}\right)^{-\left(\frac{A^2}{a^2}-1\right)} dh$$

$$= 2g\, H^{\left(\frac{A^2}{a^2}-1\right)} \int_H^h h^{-\left(\frac{A^2}{a^2}-1\right)} dh = \frac{2g\, H^{\left(\frac{A^2}{a^2}-1\right)}}{-\left(\frac{A^2}{a^2}-1\right)+1}\, h^{-\left(\frac{A^2}{a^2}-1\right)+1} \Bigg|_H^h$$

$$= \frac{2g\, H^{\left(\frac{A^2}{a^2}-1\right)}}{-\left(\frac{A^2}{a^2}-1\right)+1}\left(h^{-\left(\frac{A^2}{a^2}-1\right)+1} - H^{-\left(\frac{A^2}{a^2}-1\right)+1}\right)$$

$$= -\frac{2g\, H^{\left(\frac{A^2}{a^2}-1\right)}\, h^{-\left(\frac{A^2}{a^2}-1\right)+1}}{\left(\frac{A^2}{a^2}-1\right)-1}\left(1-\left(\frac{h}{H}\right)^{\left(\frac{A^2}{a^2}-1\right)-1}\right)$$

With this calculation the equation becomes

$$\left(\frac{dh}{dt}\right)^2 = \left(\frac{h}{H}\right)^{\left(\frac{A^2}{a^2}-1\right)} \left\{\frac{Q_o^2}{A^2} + \frac{2g\,h}{\frac{A^2}{a^2}-2}\left(\frac{h}{H}\right)^{-\left(\frac{A^2}{a^2}-1\right)}\left[1-\left(\frac{h}{H}\right)^{\left(\frac{A^2}{a^2}-2\right)}\right]\right\}$$

or simplified

$$\left(\frac{dh}{dt}\right)^2 = \frac{Q_o^{\,2}}{A^2}\left(\frac{h}{H}\right)^{\left(\frac{A^2}{a^2} - 1\right)} + \frac{2g\,h}{\frac{A^2}{a^2} - 2}\left[1 - \left(\frac{h}{H}\right)^{\left(\frac{A^2}{a^2} - 2\right)}\right] . \tag{49}$$

It can be said that at $t = 0$

$$Q_o = a\sqrt{2g\,H} .$$

The differential equation for the liquid level in the tank will then be

$$\frac{dh}{dt} = \sqrt{\frac{2g\,H}{\frac{A^2}{a^2}}\left(\frac{h}{H}\right)^{\left(\frac{A^2}{a^2} - 2\right)} + \frac{2g\,h}{\frac{A^2}{a^2} - 2}\left[1 - \left(\frac{h}{H}\right)^{\left(\frac{A^2}{a^2} - 2\right)}\right]} . \tag{50}$$

At $t = 0$

$$Q = A\left|\frac{dh}{dt}\right| = a\sqrt{2g\,H}$$

The above differential equation can be programmed very easily in order to solve it numerically by a digital computer to find the liquid level in the tank at any time t.

Case 2 - Efflux out of a tank through a tube with hydraulic losses.

The equation of momentum taking the hydraulic loss into account is

$$\frac{\partial \rho\, v_x A}{\partial t} + \frac{\partial}{\partial x}\rho v_x^{\,2} A = - A\frac{\partial}{\partial x}(p + \rho g x) - \tau_o P , \tag{51}$$

where τ_o is the frictional force per unit area or the skin friction and P is the wetted perimeter.

Since

$$\frac{\tau_o}{\rho\, v_x^{\,2}} = \frac{1}{8}\,\lambda \quad ,$$

where λ is the nondimensional friction coefficient and is a function of

$$\lambda = \lambda\,(\mathrm{Re},\ \varepsilon) \quad ,$$

where ε is the relative roughness

$$\varepsilon = \frac{2\ell}{d} \quad ,$$

ℓ is the mean roughness height and d is the diameter of the tube.

Introducing

$$Q(t) = v_x\, A \quad ,$$

the equation of momentum will be

$$\frac{\partial \rho\, Q}{\partial t} + \frac{\partial}{\partial x}\,\rho\,\frac{Q^2}{A} = -\,A\,\frac{\partial}{\partial x}\,(p + \rho\, g\, x) - \frac{\lambda}{8}\,\rho\,\frac{Q^2}{A^2}\,P \quad . \tag{52}$$

Dividing by the cross sectional area A, and density ρ, and integrating from x = 0 to x = e of the tube, results in

$$\frac{dQ}{dt}\int_0^e \frac{dx}{A} + Q^2 \int_0^e \frac{1}{A}\, d\,\frac{1}{A} = -\left(\frac{p}{\rho} + g\,h\right)_2 + \left(\frac{p}{\rho} + g\,h\right)_1 - \frac{Q^2}{8}\int_0^e \frac{P}{A^3}\,\lambda\, dx \quad , \tag{53}$$

where the indices 1 and 2 denote the total head at the beginning and the end of the tube, respectively.

It is assumed that the tank is drained only by gravity, $p_o = p_a$. The integration gives

$$\left(\frac{e}{A}\right)\frac{dQ}{dt} + \frac{Q^2}{2}\left(\frac{1}{a^2} - \frac{1}{A^2}\right) = g\,h - \frac{Q^2}{8}\int_0^e \frac{P}{A^3}\,\lambda\, dx \quad , \tag{54}$$

where a is the cross sectional area of the tube and A is the cross sectional area of the tank.

Before solving the integral in the last term, the friction coefficient λ has to be expressed as a function of the Reynolds Number. Experimental results on turbulent fluid motion in smooth pipes for the friction coefficient show expressions made by Nikuradse as

$$\lambda = 0.00332 + 0.221 \ Re^{-0.237}$$

in the range of Re from 10^4 to 10^8 .

Hermann found the formula

$$\lambda = 0.00540 + 0.396 \ Re^{-0.3}$$

for $Re \leq 1.9 \times 10^6$.

In order to make the solution general, the following expression has been chosen

$$\lambda = \alpha + \beta \ Re^{-n} \quad , \tag{55}$$

where α, β, $(-n)$ are constants.

With this the integral is

$$\int_0^e \frac{P}{A^3} \lambda \, dx = \int_0^e \frac{P}{A^3} (\alpha + \beta \ Re^{-n}) \, dx \quad . \tag{56}$$

The Reynolds Number has the form

$$Re = \frac{v_x D}{\nu} = \frac{v_x D}{\nu} \quad ,$$

or with $D = 4 \frac{A}{P}$,

$$Re = \frac{4 \ v_x A}{P \nu} = \frac{4 \ Q}{P \nu} \quad .$$

Introduced into the above

$$\int_0^e \frac{P}{A^3}\,\lambda\,dx = \alpha \int_0^e \frac{P}{A^3}\,dx + \beta\,Q^{-n} \int_0^e \frac{P}{A^3}\left(\frac{4}{P\nu}\right)^{-n} dx\ . \tag{57}$$

The solution of the terms is

$$\alpha \int_0^e \frac{P}{A^3}\,dx = \alpha\,\frac{P}{A^3}\,e\ ,$$

$$Q^{-n}\,\beta \int_0^e \frac{P}{A^3}\left(\frac{4}{P\nu}\right)^{-n} dx = Q^{-n}\,\beta\,\frac{P}{A^3}\left(\frac{4}{P\nu}\right)^{-n} e\ .$$

With these substitutions the equation of momentum will be

$$\frac{e}{A}\,\frac{dQ}{dt} + \frac{Q^2}{2}\left(\frac{1}{a^2} - \frac{1}{A^2}\right) + \frac{Q^2}{8}\left[\alpha\,\frac{P}{A^3}\,e + \beta\,Q^{-n}\,\frac{P}{A^3}\left(\frac{4}{P\nu}\right)^{-n} e\right] = gh\ . \tag{58}$$

In order to get an equation for Q as a function of time, a differentiation with respect to time must be applied

$$\frac{e}{A}\,\frac{d^2Q}{dt^2} + \left(\frac{1}{a^2} - \frac{1}{A^2}\right) Q\,\frac{dQ}{dt} + \frac{\alpha P e}{4A^3}\,Q\,\frac{dQ}{dt} + \frac{\beta P e}{8A^3}\left(\frac{4}{P\nu}\right)^{-n}(2-n)\,Q^{1-n}\,\frac{dQ}{dt} = g\,\frac{dh}{dt}\ . \tag{59}$$

Introducing again

$$Q = -\,A\,\frac{dh}{dt}\ ,$$

the equation yields

$$\frac{e}{A}\,\frac{d^2Q}{dt^2} + \left(\frac{1}{a^2} - \frac{1}{A^2}\right) Q\,\frac{dQ}{dt} + \left[\frac{\alpha P e}{4A^3}\,Q + \frac{\beta P e}{8A^3}\left(\frac{4}{P\nu}\right)^{-n}(2-n)\,Q^{1-n}\right]\frac{dQ}{dt} = -\,\frac{g}{A}\,Q\ . \tag{60}$$

Using the relation

$$\frac{d^2Q}{dt^2} = \frac{d}{dQ} \frac{dQ}{dt} \frac{dQ}{dt} ,$$

and dividing by $\frac{dQ}{dt}$ results in

$$\frac{e}{A} \frac{d}{dQ}\left(\frac{dQ}{dt}\right) + \left(\frac{1}{a^2} - \frac{1}{A^2}\right) Q + \frac{\alpha P e}{4 A^3} Q + \frac{\beta P e}{8 A^3}\left(\frac{4}{P\nu}\right)^{-n} (2-n) Q^{1-n} = - \frac{g}{A} \frac{Q}{\frac{dQ}{dt}} . \tag{61}$$

In an explicit form

$$\frac{d}{dQ}\left(\frac{dQ}{dt}\right) = - \frac{A}{e} Q \left[\frac{g}{A} \frac{1}{\frac{dQ}{dt}} + \left(\frac{1}{a^2} - \frac{1}{A^2}\right) + \frac{\alpha P e}{4 A^3} + \frac{\beta P e}{8 A^3}\left(\frac{4}{P\nu}\right)^{-n} (2-n) Q^{-n}\right] . \tag{62}$$

For high relative roughness the friction coefficient λ, beginning from some fixed value of Re, no longer depends on Re. This can be seen from the experimental results of Nikuradse. In this case the constant β is equal to zero. For solving the above equation a high relative roughness in the pipe will be assumed. Thus the differential equation turns to

$$\frac{d}{dQ}\left(\frac{dQ}{dt}\right) = - \frac{A}{e} Q \left[\frac{g}{A} \frac{1}{\frac{dQ}{dt}} + \left(\frac{1}{a^2} + \frac{1}{A^2}\right) + \frac{\alpha P e}{4 A^3}\right] . \tag{63}$$

A substitution can be made

$$\frac{dQ}{dt} = \xi ,$$

and for simplification the following expressions will be chosen

$$M = \frac{g}{e} ,$$

$$N = - \left[\left(\frac{1}{a^2} - \frac{1}{A^2}\right) \frac{A}{e} + \frac{\alpha P}{4 A^2}\right] .$$

With this the differential equation becomes

$$\frac{d\xi}{dQ} = - M Q \frac{1}{\xi} + N Q \quad . \tag{64}$$

Separation of the variables yields

$$Q\, dQ = \frac{d\xi}{N - \frac{M}{\xi}} = \frac{\xi\, d\xi}{N\xi - M} \quad . \tag{65}$$

In order to know the range of integration the boundary condition has to be found. At $t = 0$, $Q = 0$, $h = H$.

Using Equation (54)

$$\frac{e}{A}\frac{dQ}{dt} = g H \quad ,$$

or

$$\frac{dQ}{dt} = \frac{Ag}{e} \cdot H = \xi_o \quad .$$

The integration of the equation now gives

$$\int Q\, dQ = \int_{\xi_o = \frac{Ag}{e} H}^{\xi = \frac{dQ}{dt}} \frac{\xi\, d\xi}{N\xi - M} = \frac{1}{N} \int_{\xi_o}^{\xi} \frac{\xi\, d\xi}{\xi - \frac{M}{N}}$$

$$\frac{Q^2}{2} = \frac{1}{N} \int_{\xi_o}^{\xi} \left[1 - \frac{\frac{M}{N}}{\frac{M}{N} - \xi} \right] d\xi$$

$$\frac{Q^2}{2} = \left| \frac{1}{N}\, \xi + \frac{M}{N^2}\, \ell n \left(\frac{M}{N} - \xi \right) \right|_{\xi_o = \frac{Ag}{e} H}^{\xi = \frac{dQ}{dt}} \tag{66}$$

Substituting the limit yields

$$\frac{Q^2}{2} = \frac{1}{N}\frac{dQ}{dt} + \frac{M}{N^2}\,\ell n\left(\frac{M}{N} - \frac{dQ}{dt}\right) - \frac{1}{N}\frac{Ag}{e}H - \frac{M}{N^2}\,\ell n\left(\frac{M}{N} - \frac{Ag}{e}H\right) . \qquad (67)$$

Introducing the terms again for M and N the following equation

$$\frac{Q^2}{2} = \frac{1}{\left(\frac{1}{A^2} - \frac{1}{a^2}\right)\frac{A}{e} - \frac{\alpha P}{4A^2}}\left\{\frac{dQ}{dt} + \frac{g}{e\left[\left(\frac{1}{A^2} - \frac{1}{a^2}\right)\frac{A}{e} - \frac{\alpha P}{4A^2}\right]}\right.$$

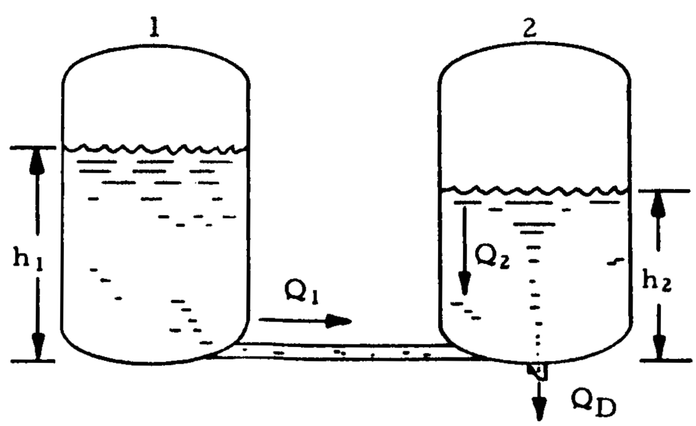
$$\ell n\left[\frac{g}{e\left[\left(\frac{1}{A^2} - \frac{1}{a^2}\right)\frac{A}{e} - \frac{\alpha P}{4A^2}\right]} - \frac{dQ}{dt}\right] - \frac{Ag}{e}H - \frac{g}{e\left[\left(\frac{1}{A^2} - \frac{1}{a^2}\right)\frac{A}{e} - \frac{\alpha P}{4A^2}\right]}$$

$$\left.\ell n\left[\frac{g}{e\left[\left(\frac{1}{A^2} - \frac{1}{a^2}\right)\frac{A}{e} - \frac{\alpha P}{4A^2}\right]} - \frac{Ag}{e}H\right]\right\} \qquad (68)$$

is derived. This is a differential equation which can be solved numerically for Q(t) and h(t) when the relation $Q = -A\frac{dh}{dt}$ is applied.

<u>Case 3</u> - Two tanks connected by a line.

Draining of both tanks through one orifice. Hydraulic losses exist in the interconnect line.



The flow from tank 1 to tank 2 is described by the equation of momentum derived previously

$$\frac{e_1}{A}\frac{dQ_1}{dt} + \frac{Q_1^2}{2}\left(\frac{1}{a_1^2} - \frac{1}{A^2}\right) + \frac{Q_1^2}{8}\left[\alpha_1 \frac{P_1}{A^3} e_1 + \beta_1 Q_1^{-n} \frac{P}{A^3}\left(\frac{4}{P\nu}\right)^{-n} e_1\right] = g(h_1 - h_2) \,, \tag{69}$$

where the index 1 concerns tank 1 and the interconnect line.

Assuming a high relative roughness in the interconnect line, i.e., $\beta = 0$ and using the relations

$$Q_1 = -A\frac{dh_1}{dt}$$

and

$$\frac{dQ_1}{dt} = -A\frac{d^2h_1}{dt^2} \,,$$

the equation becomes

$$-e_1\frac{d^2h_1}{dt^2} + \frac{A^2}{2}\left(\frac{1}{a_1^2} + \frac{1}{A^2}\right)\left(\frac{dh_1}{dt}\right)^2 + \frac{\alpha_1 P_1 e_1}{8A}\left(\frac{dh_1}{dt}\right)^2 = g(h_1 - h_2) \,. \tag{70}$$

This differential equation represents the flow process out of tank 1 through the interconnect line.

In order to solve it a second equation is needed for the unknown h_2. The equation of momentum for tank 2 which only has an orifice as the draining port is

$$\frac{dQ_D}{dt}\int_0^{h_2}\frac{dh}{A} + \frac{Q_D^2}{2}\left(\frac{1}{a_2^2} - \frac{1}{A^2}\right) = gh_2 \,. \tag{71}$$

Q_D is the draining flow and is equal to the sum out of both tanks

$$Q_D = Q_1 + Q_2 \,.$$

Assuming a pure cylindrical tank the integral is

$$\int_0^{h_2} \frac{dh}{A} = \frac{h_2}{A} \quad .$$

Substituting into the above equation yields

$$\left(\frac{dQ_1}{dt} + \frac{dQ_2}{dt}\right)\frac{h_2}{A} + \frac{1}{2}\,(Q_1 + Q_2)^2\left(\frac{1}{a_2{}^2} - \frac{1}{A^2}\right) = g\,h_2 \quad ,$$

or with

$$Q_1 = -\,A\,\frac{dh_1}{dt} \; ; \qquad Q_2 = -\,A\,\frac{dh_2}{dt}$$

$$-\left(\frac{d^2h_1}{dt^2} + \frac{d^2h_2}{dt^2}\right) h_2 + \frac{A^2}{2}\left(\frac{dh_1}{dt} + \frac{dh_2}{dt}\right)^2\left(\frac{1}{a_2{}^2} - \frac{1}{A_2{}^2}\right) = g\,h_2 \quad . \tag{72}$$

The two differential equations in an explicit form for the highest derivative are

$$\frac{d^2h_1}{dt^2} = \left[\frac{A^2}{2e}\left(\frac{1}{a_1{}^2} - \frac{1}{A^2}\right) + \frac{\alpha_1 P_1}{8A}\right]\left(\frac{dh_1}{dt}\right)^2 - \frac{g}{e_1}\,(h_1 - h_2) \quad ,$$

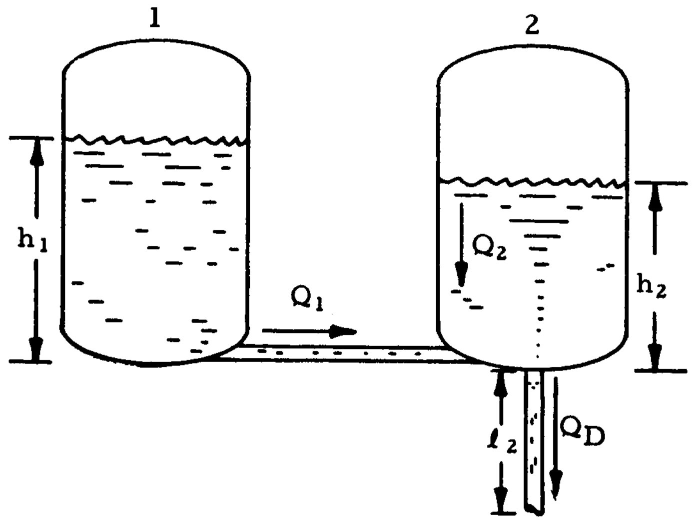
and

$$\frac{d^2h_2}{dt^2} = \frac{A^2}{2}\left(\frac{dh_1}{dt} + \frac{dh_2}{dt}\right)^2\left(\frac{1}{a_2{}^2} - \frac{1}{A^2}\right)\frac{1}{h_2} - \frac{d^2h_1}{dt^2} - g \quad . \tag{73}$$

These equations can be solved numerically for determination of the liquid levels in the tanks at any time t.

Case 4 - Two tanks connnected by a line.

Draining of both tanks through a tube. Hydraulic losses exist in the interconnect line and the draining tube on tank 2.



The equation of momentum for tank 1 again is

$$\frac{e_1}{A}\frac{dQ_1}{dt} + \frac{Q_1^2}{2}\left(\frac{1}{a_1^2} - \frac{1}{A^2}\right) + \frac{Q_1^2}{8}\left[\alpha_1 \frac{P_1}{A^3} e_1 + \beta_1 Q_1^{-n} \frac{P_1}{A^3}\left(\frac{4}{P\nu}\right)^{-n} e_1\right] = g(h_1 - h_2)\ . \tag{74}$$

Differentiating the equation with respect to time and substituting

$$\frac{dh_1}{dt} = -\frac{Q_1}{A}\ , \quad \frac{dh_2}{dt} = -\frac{Q_2}{A}$$

gives

$$\begin{aligned}\frac{e_1}{A}\frac{d^2Q_1}{dt^2} + \left(\frac{1}{a_1^2} - \frac{1}{A^2}\right) Q_1 \frac{dQ_1}{dt} + \frac{\alpha_1 P_1 e_1}{4A^3} Q_1 \frac{dQ_1}{dt} \\ + \frac{\beta_1 P_1 e_1}{8A^3}\left(\frac{4}{P_1 \nu}\right)^{-n} (2-n)\, Q_1^{1-n} \frac{dQ_1}{dt} = \frac{g}{A}(Q_2 - Q_1)\ .\end{aligned} \tag{75}$$

The flow process out of tank 2 is described by the following equation

$$\frac{e_2}{A}\frac{dQ_D}{dt} + \frac{Q_D^2}{2}\left(\frac{1}{a_2^2} - \frac{1}{A^2}\right) + \frac{Q_D^2}{8}\left[\frac{\alpha_2 P_2 e_2}{A^3} + \beta_2 Q_D^{-n}\left(\frac{4}{P_2 \nu}\right)^{-n} \frac{P_2 e_2}{A^3}\right] = g h_2\ . \tag{76}$$

The draining flow is

$$Q_D = Q_1 + Q_2$$

Substitution results in

$$\frac{e_2}{A}\left(\frac{dQ_1}{dt} + \frac{dQ_2}{dt}\right) + \frac{1}{2}(Q_1 + Q_2)^2\left(\frac{1}{a_2{}^2} - \frac{1}{A^2}\right) + (Q_1 + Q_2)^2\,\frac{\alpha_2 P_2 e_2}{8 A^3}$$
$$+ (Q_1 + Q_2)^2 (Q_1 + Q_2)^{-n}\,\frac{\beta_2 P_2 e_2}{8 A^3}\left(\frac{4}{P_2 \nu}\right)^{-n} = g h_2 \quad . \tag{77}$$

Differentiating with respect to time yields

$$\frac{e_2}{A}\left(\frac{d^2 Q_1}{dt^2} + \frac{d^2 Q_2}{dt^2}\right) + \frac{1}{2}\left(\frac{1}{a_2{}^2} - \frac{1}{A^2}\right) 2(Q_1 + Q_2)\left(\frac{dQ_1}{dt} + \frac{dQ_2}{dt}\right)$$
$$+ \frac{\alpha_2 P_2 e_2}{8 A^3}\, 2(Q_1 + Q_2)\left(\frac{dQ_1}{dt} + \frac{dQ_2}{dt}\right) \tag{78}$$
$$+ \frac{\beta_2 P_2 e_2}{8 A^3}\left(\frac{4}{P_2 \nu}\right)^{-n} \cdot (2-n)(Q_1 + Q_2)^{1-n}\left(\frac{dQ_1}{dt} + \frac{dQ_2}{dt}\right) = \frac{g\, dh_2}{dt} = -\frac{g}{A} Q_2 \quad .$$

So, the two differential equations are in an explicit form for the highest derivatives.

$$\frac{d^2 Q_1}{dt^2} = \frac{g}{e_1}(Q_2 - Q_1) - \left[\frac{A}{e_1}\left(\frac{1}{a_1{}^2} - \frac{1}{A^2}\right) Q_1 + \frac{\alpha_1 P_1}{4 A^2} Q_1\right.$$
$$\left. + \frac{\beta_1 P_1}{8 A^2}\left(\frac{4}{P_1 \nu}\right)^{-n}(2 - n) Q_1{}^{1-n}\right]\frac{dQ_1}{dt} \quad ,$$

and

$$\frac{d^2 Q_2}{dt^2} = -\frac{g}{e_2} Q_2 - \left[\frac{A}{e_2}\left(\frac{1}{a_2{}^2} - \frac{1}{A^2}\right)(Q_1 + Q_2) + \frac{\alpha_2 P_2}{4 A^2}(Q_1 + Q_2)\right.$$
$$\left. + \frac{\beta_2 P_2}{8 A^2}\left(\frac{4}{P_2 \nu}\right)^{-n}(2-n)(Q_1 + Q_2)^{1-n}\right]\left(\frac{dQ_1}{dt} + \frac{dQ_2}{dt}\right) - \frac{d^2 Q_1}{dt^2} \quad . \tag{79}$$

A numerical solution is the only method for solving the equation for obtaining the flow, Q, and the liquid level, h, in the tanks as a function of time.

RESULTS

An analysis of unsteady flow out of tanks was made and the derived equations were applied to four samples with one- and two-tank configurations. The results are still differential equations for the flow rate or the liquid level in the tank which must be solved numerically by a digital computer. The tank shape is assumed to be pure cylindrical with flat tops and flat bottoms.

CONCLUSIONS

When the pressure force is much larger in comparison to the force due to gravity the quasi-steady flow equations will hold true and the unsteady flow terms can be neglected (References 1, 2 and 3). But when only gravity is acting on the liquid, the unsteady flow equations must be applied. The derived differential equations for unsteady flow of a viscous or inviscous fluid out of tanks through a draining tube by which the tanks can be emptied simultaneously are also valid for any other tank system.

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