# On the Stability of Equilibrium of Continuous Systems

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# ON THE STABILITY OF EQUILIBRIUM OF CONTINUOUS SYSTEMS*

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ABSTRACT

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A sufficiency theorem for the stability of a linearly viscoelastic solid subjected to partial follower surface tractions is established. It is shown that an appropriately defined functional metric space must be introduced in order to formulate a well-posed problem. The usual energy method, if applicable, and the Galerkin method, if convergent, yield stability conditions only in a functional space whose metric is defined in an average sense.

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## 1. Introduction

It was shown by R. T. Shield and A. E. Green [1]* that proofs of the commonly used sufficiency theorems for the stability of a linearly elastic continuum are, in general, deficient. It is the aim of this study to indicate, using the stability theorems for partial differential equations given by Zubov [2], that this deficiency arises from the definition of stability of a continuum, and is not directly related to the linearization of the equations of motion governing the elastic continuum.

At the outset, it is shown that the stability of a continuum must necessarily be defined with respect to a metric which measures distance in an infinite-dimensional space. This metric may be postulated in various suitable forms. The equations of the boundary value problem of a continuum, together with an explicitly defined metric, $\rho$, form a functional metric space whose fundamental properties vary depending upon the specification of $\rho$, and thus lead to different stability criteria. In this connection, we shall show that the usual energy methods, if applicable, and the Galerkin method, if convergent, yield stability only with respect to an average metric.

The problem of a linear viscoelastic solid subjected to partial follower surface tractions is treated in detail and a sufficient condition for stability of the continuum with respect to an average metric is established.

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* Numbers in brackets refer to Bibliography at the end of this paper.

## 2. Statement of the Problem

We consider a finite isotropic, homogeneous, linearly viscoelastic solid, bounded by a regular surface S, contained in a volume V. At the time $t = 0$, the solid is in a state of initial stress $\sigma_{ij}$ ; $i,j = 1,2,3$, caused by a system of partial follower surface tractions $p_i$ , applied at the boundary S. By partial follower forces we shall mean forces which follow in a specified manner the deformation of the surface element upon which they are acting and are therefore dependent upon the motion of the system. We shall refer to the state of initial stress of the solid as unperturbed (equilibrium) state and study its possible motions with reference to this state. Furthermore, we shall assume that the perturbed quantities are small (these quantitites will, subsequently, be indicated by a bar – ) so that all terms of order higher than the second may be neglected. The equations of motion of the perturbed solid, referred to a fixed orthogonal Cartesian coordinate system, are [3]

$$\bar{\sigma}_{ij,j} + (\sigma_{jk}\,\bar{u}_{i,k})_{,j} - m\,\ddot{\bar{u}}_i = 0 \qquad \text{in V ,}$$

$$\bar{\sigma}_{ij}\,n_j + \sigma_{jk}\,\bar{u}_{i,k}\,n_j = \tilde{p}_i{}^{*} \qquad \text{on S ,} \quad i,j,k = 1,2,3 , \qquad (1)$$

where m is the mass density, $x_j$ are the coordinates, $\bar{u}_i$ the displacement components measured from the unperturbed state, $n_j$ the components of the unit normal to S, $\tilde{p}_i$ the perturbations of the applied surface tractions. A comma followed by indices k,j indicates differentiation with respect

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*In these equations and in the sequel the repeated indices are summed over the range of their definitions.

to $x_j$ , $x_k$ , and dots denote derivatives with respect to time. We shall assume here that

$$\tilde{p}_i = \alpha(\bar{x})\, p_j\, \bar{u}_{i,j} \qquad \text{on } S \tag{2}$$

where $\alpha(\bar{x}) \equiv \alpha(x_1, x_2, x_3)$ is a parameter which serves to describe the manner in which the surface tractions follow the deformation. If $\alpha \equiv 0$ the system is conservative and for $\alpha \equiv 1$ we have the case of follower force introduced in [3]. We shall consider here the cases where $\alpha(\bar{x})$ is, at least, of class $C^1$ in the region of its definition [4]. The constitutive equations shall be taken in the form

$$\bar{\sigma}_{ij} = C_{ijkl}\, \bar{u}_{(k,l)} + C'_{ijkl}\, \dot{\bar{u}}_{(k,l)} \quad ; \quad u_{(k,l)} = \frac{1}{2}(u_{k,l} + u_{l,k}) \, ,$$

$$C_{ijkl} = \lambda\, \delta_{ij}\, \delta_{kl} + 2\mu\, \delta_{ik}\, \delta_{jl} \, ,$$

$$\text{and} \quad C'_{ijkl} = \lambda'\, \delta_{ij}\, \delta_{kl} + 2\mu'\, \delta_{ik}\, \delta_{jl} \tag{3}$$

where $\delta_{ij}$ is the Kronecker delta, $\lambda$ and $\mu$ are Lamé constants, and $\lambda'$ and $\mu'$ are viscous constants corresponding to Lamé constants.

A general solution to the nonself-adjoint mixed initial and boundary value problem (1) cannot, in general, be easily obtained. Therefore, in order to study the stability of this system, we have to resort to some other means and, consequently, we shall not expect to gain as much information concerning stability as we would if we were to construct and evaluate a general solution of the system. As we shall see in the following section, this is by no means a shortcoming. A strong stability criterion, that may be imposed on the system and which could be applied if we were to solve system (1) completely, would be of doubtful interest.

In this connection, we shall consider a certain functional (which, in effect, expresses the energy of the system) and explore the stability of (1) in some appropriate average sense. Furthermore, we shall show that the usual Galerkin method, which reduces the system of partial differential equations (1) to a set of ordinary differential equations, yields the same results as those obtained by a study of the functional mentioned, provided all the series expansions employed converge in an average sense.

To this end, we consider a complete set of normalized eigenvectors, obtained by solving the homogeneous, self-adjoint system deduced from (1) by setting $\sigma_{ij} = C'_{ijkl} = \tilde{p}_i = 0$ , which has the same geometrical boundary conditions as the original problem. Let this set of orthonormal [5] eigenvectors be denoted by $\left\{\varphi_{in}(\bar{x})\right\}$ ; $i = 1,2,3$ , $n = 1,2,\ldots,\infty$ . We shall reduce our original system of partial to a system of ordinary differential equations by expanding $\bar{u}_i$ and its derivatives in terms of these eigenvectors, without any attempt to resolve the question of convergence. In fact, a rigorous proof of convergence of the Galerkin method, as applied to nonself-adjoint linear differential operators, does not, to the best knowledge of the authors, as yet exist. However, some comparison between the results obtained by applying this method to some simple problems and the exact solutions [3], certainly suggests that convergence may be assumed.*

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*The paradox in the problem of flutter of a membrane, as was shown in [3], is not related to the fact that the system is nonself-adjoint.

In our problem, we shall therefore state that if convergence exists (in an average sense at least) then the two methods yield identical results.

Let us now consider the fundamental question concerning stability of a solid.

## 3. Concept of the Stability of a Continuum

The concept of the stability of a state of a dynamic system with a finite number of degrees of freedom has a significant geometrical meaning. We consider a system with r degrees of freedom described by generalized coordinates $q_n$ and generalized velocities $\dot{q}_n$ ; $n = 1,2,\ldots,r$ . For a holonomic and autonomous system, we write the equations of motion as

$$\dot{z}_n = f_n\,(z_1\,,\,z_2\,\ldots,z_{2r}) \quad ; \quad n = 1,2,\ldots,2r \tag{4}$$

where $z_n = q_n$ ,

$$z_{r+n} = \dot{q}_n \quad ; \quad n = 1,2,\ldots,r \ ,$$

and $f_n(\,\bar{z}\,)$ are bounded, continuous, real functions vanishing for $z_n = 0$ . We assume $f_n$ satisfy all the conditions required for the existence of a single-valued solution for $t > 0$ in the region of the definition of $z_n$ . Furthermore, we represent the state of this dynamic system by a point in a 2r-dimensional Euclidean space, $E_{2r}$ , with coordinates $z_n$ ; $n = 1,2,\ldots,2r$ . The equilibrium state of the system at the origin is said to be stable if for any $\varepsilon > 0$ we can find a $\delta > 0$ depending on $\varepsilon$ only such that when $\sum_{n=1}^{2r} z_n^2 < \delta$ at $t = 0$ , we have $\sum_{n=1}^{2r} z_n^2 < \varepsilon$ for all $t > 0$ . In the opposite case $z_n = 0$ is called unstable [6]. Furthermore, $z_n = 0$ is called asymptotically stable if it is stable and $\lim_{t\to\infty}\left[\sum_{n=1}^{2r} z_n^2\right] \to 0$ .

The above definitions of stability are due to Liapunov [6].

He also supplied the proofs of necessity and sufficiency, employing the notion of distance in the finite-dimensional Euclidean space $E_{2r}$ .

For systems with an infinite number of degrees of freedom (continuous systems) the notion of distance in an infinite dimensional space needs to be introduced, if one wishes to extend Liapunov's concepts to such systems. In this case, we have to be concerned with functionals rather than functions and must explicitly define a measure (metric) of distance of two states of the system and then study the stability of the system with respect to this metric, $\rho$. The metric $\rho$ may be selected in any suitable manner (provided it satisfies three fundamental conditions [7]) so as to fulfill some physical requirements of the problem at hand. It may be desirable, for example, to limit the displacements and the velocities at each point of the solid, in which case we define

$$\rho_1 = \bar{u}_i \, \bar{u}_i + \dot{\bar{u}}_i \, \dot{\bar{u}}_i \quad \text{everywhere in V and on S} .$$

In some other cases, we may wish to restrict the strains as well as the displacements and the velocities at each point of the solid, such that

$$\rho_2 = \bar{u}_i \, \bar{u}_i + \dot{\bar{u}}_i \, \dot{\bar{u}}_i + \bar{u}_{i,j} \, \bar{u}_{i,j} \quad \text{everywhere in V and on S} .$$

For most practical problems, however, it is usually preferable to define $\rho$ in an average sense; for example

$$\rho_3 = \int_V [\, \dot{\bar{u}}_i \, \dot{\bar{u}}_i + \bar{u}_{i,j} \, \bar{u}_{i,j} + \bar{u}_i \, \bar{u}_i \,] dv .$$

We now state the definition of the stability of the initial state of a solid with respect to an explicitly defined metric $\rho$ [2], by appropriately extending the corresponding definition for a finite system.

The initial state of the continuous solid is said to be stable if for a given $\varepsilon > 0$ we can find a $\delta > 0$ depending on $\varepsilon$ only such that when $\rho < \delta$ at $t = 0$ we have $\rho < \varepsilon$ for all $t > 0$. In the opposite case, the initial state is called unstable. Furthermore, the unperturbed state is called asymptotically stable if it is stable and $\lim_{t \to \infty} \rho \to 0$. The sufficiency theorem of stability may now be stated as

Theorem:

> In order that the unperturbed state of system (1) be stable with respect to a metric $\rho$, it is sufficient that there exists, by virtue of the requirements of the boundary value problem (1), a finite, non-increasing functional which is identically equal to zero for $\rho = 0$ and admits an infinitely small upper bound with respect to the metric $\rho$.

The above theorem is an appropriate version of the theorem of stability given by A. A. Movchan [8]. In the sequel we shall use this theorem to establish a sufficiency criterion for the stability of system (1). But let us first discuss some aspects of the definition of stability.

It is seen that the stability criteria are highly dependent upon the specification of the metric $\rho$. We may not, therefore, expect to apply a criterion obtained, say, for $\rho_3$ to $\rho_2$ and get like results. The problem which was treated by R. T. Shield and A. E. Green [1] may exemplify this very point. An isotropic, homogeneous, linearly elastic sphere was perturbed by radially symmetric applied infinitesimal disturbances at $t = 0$ and it was shown that the strain at the center of the sphere can become finite for some $t > 0$. Let us show that although this system is unstable with respect to the metric $\rho_2$, it is stable with respect to $\rho_3$. To this end we consider the following functional*

$$H_1 = \frac{1}{2}\left[ \int_V (m\, \dot{\bar{u}}_i\, \dot{\bar{u}}_i + C_{ijkl}\, \bar{u}_{i,j}\, \bar{u}_{k,l})dv \right]$$

whose time derivative is zero by virtue of the equations of motion, and which admits an infinitesimal upper bound with respect to the metric $\rho_3$. From the inequalities [11, 12, 13]

$$C_1 \int_V \bar{u}_i\, \bar{u}_i\, dv \le \int_V \bar{u}_{i,j}\, \bar{u}_{i,j}\, dv ,$$

$$C_2 \int_V \bar{u}_{i,j}\, \bar{u}_{i,j}\, dv \le \int_V C_{ijkl}\, \bar{u}_{i,j}\, \bar{u}_{k,l}\, dv$$

which are valid for all admissible motions of the solid with $C_1$ and $C_2$ being fixed positive constants independent of $\bar{u}_i$, we immediately construct the inequality

$$H_1 \ge K\, \rho_3 \quad \text{for all} \quad t \ge 0 ,$$

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*In reference [9] A. A. Movchan has proved some stability and instability theorems for a linearly elastic solid subjected to conservative forces. See also [10].

where K is also a fixed positive number not dependent on $\bar{u}_i$ . We let $H_1 < K\varepsilon$ and obtain $\rho_3 < \varepsilon$ at $t = 0$ . But $H_1$ is a non-increasing function of time. Therefore $K\varepsilon$ is an upper bound of $H_1$ for all $t \geq 0$ , which implies

$$\rho_3 < \varepsilon \quad \text{for all} \quad t \geq 0 .$$

In [1], the initial disturbances were taken to be

$$u = \frac{u}{r} = \frac{\partial u}{\partial r} = 0 \ , \quad \dot{u} = \frac{2c}{r}\left[ \frac{1}{r} f'(r) - f''(r) \right] ; \ \text{at } t = 0$$

where r measures distances from the center of the sphere, $c = \frac{\lambda + 2\mu}{m}$ and $f(r)$ is given by

$$f(r) = 0 \qquad\qquad 0 \leq r \leq a$$

$$f(r) = \frac{1}{\varepsilon^5 a^5}(r - a)^4 (r - a - 2\varepsilon a) \qquad a \leq r \leq a + 2\varepsilon a$$

$$f(r) = 0 \qquad\qquad a + 2\varepsilon a \leq r .$$

A simple calculation shows that $\rho_3 = O(\varepsilon)$ at $t = 0$ . Furthermore, at $t = a/c$ we have, for $0 \leq r \leq 2\varepsilon a$ ,

$$u = \frac{1}{\varepsilon^5 a^5} r^2 (2\varepsilon a - r)^3 (7r - 6\varepsilon a)$$

which immediately yields $\rho_3 = O(\varepsilon)$ at $t = a/c$ , while the strain at the center of the sphere at this instant is finite:

$$\left[\frac{u}{r}\right]_{r = \varepsilon a} = 1 \ , \quad \left[\frac{\partial u}{\partial r}\right]_{r = \varepsilon a} = 6 \ .$$

In this example, one is able to obtain an exact solution to the differential equations of the boundary value problem. Therefore, one is in the position of requiring as strong a stability criterion as one pleases. We see that the system is not stable with respect to $\rho_2$, although it is stable with respect to $\rho_3$. The important point to note in this connection is that the stability with respect to the metric $\rho_3$ could have been deduced without possessing an explicit solution of the problem.

In most practical problems, the system may well be stable for all practical purposes, while it may not satisfy the point-wise stability conditions with respect to the metrics $\rho_1$ and $\rho_2$. In those cases there may exist a finite number of points in V where an infinitesimal perturbation at $t = 0$ may cause finite, say, strains at these points for some $t > 0$. If the collection of these points forms a set with measure zero, then the stability may exist with respect to the metric $\rho_3$.

The metric $\rho_3$ seems to be more appealing also from a purely mathematical point of view. In this regard, let us note that the series expansion of a piecewise continuous function in a finite domain is an approximation in a mean square sense and not a pointwise representation. The following discussion will, therefore, be devoted to the stability of system (1) with respect to the metric $\rho_3$.

## 4. Analysis of Stability

We consider a functional H given by

$$H = \frac{1}{2}\left\{ \int_V [m\, \dot{\bar{u}}_i\, \dot{\bar{u}}_i + C_{ijkl}\, \bar{u}_{i,j}\, \bar{u}_{k,l} + (1-\alpha)\, \sigma_{jk}\, \bar{u}_{i,k}\, \bar{u}_{i,j}]\, dv + \right.$$

$$\left. + 2\int_0^t \int_V [C'_{ijkl}\, \dot{\bar{u}}_{i,j}\, \dot{\bar{u}}_{k,l} - \sigma_{jk}\, (\alpha\, \bar{u}_{i,j})_{,k}\, \dot{\bar{u}}_i]\, dv dt \right\} \qquad (5)$$

and note that, from the requirements of the boundary value problem (1), H is a continuous functional which vanishes identically at the initial unperturbed state of the solid, $\rho_3 = 0$ . The total time derivative of H is

$$\frac{dH}{dt} = \int_V [m\, \ddot{\bar{u}}_i\, \dot{\bar{u}}_i + C_{ijkl}\, \dot{\bar{u}}_{i,j}\, \bar{u}_{k,l} + (1-\alpha)\, \sigma_{jk}\, \bar{u}_{i,k}\, \dot{\bar{u}}_{i,j} +$$

$$+ C'_{ijkl}\, \dot{\bar{u}}_{i,j}\, \dot{\bar{u}}_{k,l} - \sigma_{jk}\, (\alpha\, \bar{u}_{i,j})_{,k}\, \dot{\bar{u}}_i]\, dv\ . \qquad (6)$$

But we have

$$\int_V [C_{ijkl}\, \bar{u}_{k,l} + C'_{ijkl}\, \dot{\bar{u}}_{k,l}]\, \dot{\bar{u}}_{i,j}\, dv = \int_S \bar{\sigma}_{ij}\, n_j\, \dot{\bar{u}}_i\, dS -$$

$$- \int_V \bar{\sigma}_{ij,j}\, \dot{\bar{u}}_i\, dv\ ,$$

and

$$\int_V \left\{ (1-\alpha)\, \sigma_{jk}\, \bar{u}_{i,k}\, \dot{\bar{u}}_{i,j} - \sigma_{jk}\, (\alpha\, \bar{u}_{i,j})_{,k}\, \dot{\bar{u}}_i \right\} dv = -$$

$$- \int_V [\sigma_{jk}\, \bar{u}_{i,k}]_{,j}\, \dot{\bar{u}}_i\, dv + \int_S [\sigma_{jk}\, \bar{u}_{i,k}\, n_j - \alpha\, p_k\, \bar{u}_{i,k}]\, \dot{\bar{u}}_i\, dS\ ,$$

where in the last reduction we have used the fact that for the unperturbed state we have

$$\sigma_{ij,j} = 0 \quad \text{in } V \qquad \text{and} \qquad \sigma_{jk}\, n_k = p_j \quad \text{on } S\ .$$

Equation (5) now becomes

$$\frac{dH}{dt} = \int_V \left\{ m\, \ddot{\bar{u}}_i - \bar{\sigma}_{ij,j} - (\sigma_{jk}\, \bar{u}_{i,k})_{,j} \right\} \dot{\bar{u}}_i \, dv \, +$$

$$+ \int_S \left[ \bar{\sigma}_{ij}\, n_j + \sigma_{jk}\, \bar{u}_{i,k}\, n_j - \alpha\, p_j\, \bar{u}_{i,j} \right] \dot{\bar{u}}_i \, dS \tag{7}$$

which is identically equal to zero by virtue of equations (1) for all admissible perturbed motions of the solid. Moreover, if H is a positive definite functional, then it admits an infinitely small upper bound with respect to $\rho_3$ . To show this we let $|\bar{u}_i| < \sqrt{\varepsilon}$ , $|\dot{\bar{u}}_i| < \sqrt{\varepsilon}$ and $|\bar{u}_{i,j}| < \sqrt{\varepsilon}$ at $t = 0$ , and obtain

$$\rho_3 < 15\, V\, \varepsilon \qquad \text{at } t = 0\ .$$

Then, as $H > 0$, we have

$$H < K\, \varepsilon = \delta \qquad \text{at } t = 0\ ,$$

where K is a positive constant. But $\delta$ is an upper bound of H for all $t > 0$ , as H is a non-increasing function of time. Therefore, if H is a positive definite functional, then all the requirements of the sufficiency theorem are fulfilled and we have the following theorem

Theorem:

For a linearly viscoelastic solid subjected to a set of partial follower forces to be stable with respect to the metric $\rho_3$, it is sufficient that the functional H given by equation (5) be a positive definite quantity for admissible perturbed motions of the solid about the state of initial stress.

Let us note that the requirement of H being a positive definite functional may imply a stronger stability condition than is given by $\rho_3$. This touches then upon the question of the necessary conditions which will not be dealt with here.

From the above discussion we may conclude that the commonly used energy methods yield stability criteria with respect to an average metric $\rho_3$. Therefore we may not, by any means, expect to retrieve any more information than is retained after this averaging process. This conclusion is also valid for most approximate methods such as the Ritz, the Galerkin and other methods, where we use some averaging processes to reduce the system of partial to a set of ordinary differential equations. We shall explore this point further in the sequel, but let us make first another remark regarding system (1) and functional H. We let solution of (1) be of a form $\bar{u}_i = \psi_i(\bar{x})\, e^{pt}$ and obtain from (5)

$$H = e^{2pt} \left\{ \frac{1}{2} \int_V \left[ p^2 m\, \psi_i\, \psi_i + C_{ijkl}\, \psi_{i,j}\, \psi_{k,l} + (1-\alpha)\, \overset{\circ}{\sigma}_{jk}\, \psi_{i,k}\, \psi_{i,j} \right] dv + \right.$$

$$\left. + \int_V \left[ p\, C'_{ijkl}\, \psi_{i,j}\, \psi_{k,l} - \overset{\circ}{\sigma}_{jk}\, (\alpha\, \psi_{i,j})_{,k}\, \psi_i \right] dv \right\} . \qquad (8)$$

If we substitute $\bar{u}_i = \psi_i e^{pt}$ into equations (1), we obtain an eigenvalue problem with eigenvalues p . From equation (8) we may conclude that, for H to be a non-increasing function of time, p must have a non-positive real part.

We now reduce equations (1) to a set of ordinary differential equations. We assume that $\bar{u}_i$ and its derivatives can be expanded in terms of the complete set of eigenvectors $\left\{ \varphi_{in} ( \bar{x} ) \right\}$ ; i = 1,2,3 , n = 1,2,...,∞, such that

$$\int_V \left| \bar{u}_i \bar{u}_i - \sum_{n=1}^{N} \varphi_{in} \varphi_{in} q_n^2(t) \right| dv < \varepsilon_1 , \int_V \left| \dot{\bar{u}}_i \dot{\bar{u}}_i - \sum_{n=1}^{N} \varphi_{in} \varphi_{in} \dot{q}_n^2(t) \right| dv < \varepsilon_2 ,$$

$$\int_V \left| \bar{u}_{i,j} \bar{u}_{k,l} - \sum_{n=1}^{N} \sum_{m=1}^{N} \varphi_{in,j} \varphi_{km,l} q_n(t) q_m(t) \right| dv < \varepsilon_3 ,$$

$$\int_V \left| \bar{u}_{i,jk} \dot{\bar{u}}_i - \sum_{n=1}^{N} \sum_{m=1}^{N} \varphi_{in,jk} \varphi_{im} q_n(t) \dot{q}_m(t) \right| dv < \varepsilon_4$$

and

$$\int_V \left| \dot{\bar{u}}_{i,j} \dot{\bar{u}}_{k,l} - \sum_{n=1}^{N} \sum_{m=1}^{N} \varphi_{in,j} \varphi_{km,l} \dot{q}_n(t) \dot{q}_m(t) \right| dv < \varepsilon_5$$

$$i,j,k,l = 1,2,3 , \qquad (9)$$

for some N > M , where M is a large positive number depending on $\varepsilon_i$ ; i = 1,2,...,5 in the above inequalities and $\varepsilon_i$ may be made as small as we please by selecting M sufficiently large. For such an M, equation

(7) reduces to

$$\sum_{m=1}^{N} \left\{ \ddot{q}_m + \sum_{n=1}^{N} c_{mn} \dot{q}_n + \omega_m^2 \sum_{n=1}^{N} (\delta_{mn} + b_{mn}) q_n \right\} \dot{q}_m = 0$$

where

$$b_{mn} = \frac{1}{\omega_m^2} \left[ \int_V (1 - \alpha) \sigma_{jk} \varphi_{in,j} \varphi_{im,k} \, dv - \int_V \sigma_{jk} (\alpha \varphi_{in,j})_{,k} \varphi_{im} \, dv \right]$$

and

$$c_{mn} = \int_V C'_{ijkl} \varphi_{kn,l} \varphi_{im,j} \, dv \tag{10}$$

In obtaining (10), in addition to the Gauss theorem we have also utilized the fact that $\{ \varphi_{in} \}$ are solutions to

$$C_{ijkl} \varphi_{kn,lj} + m \omega_n^2 \varphi_{in} = 0 \quad \text{in } V ,$$

$$C_{ijkl} \varphi_{kn,l} n_j = 0 \quad \text{on } S ,$$

$$\int_V m \varphi_{in} \varphi_{im} \, dv = \delta_{mn} .$$

For $\dot{q}_m$ ; $m = 1,2,\ldots,N$ not identically zero, equations (10) yield

$$\ddot{q}_m + \sum_{n=1}^{N} c_{mn} \dot{q}_n + \omega_m^2 \sum_{n=1}^{N} (\delta_{mn} + b_{mn}) q_n = 0 , \quad m = 1,2,\ldots,N , \tag{10'}$$

which is a system of non-self-adjoint, ordinary differential equations.

Similarly, H reduces to

$$\bar{H} = \frac{1}{2} \sum_{m=1}^{N} \Big\{ \Big[ \dot{q}_m^2 + \omega_m^2 \, q_m^2 \Big] + \sum_{n=1}^{N} a_{mn} \, q_n \, q_m +$$

$$+ 2 \int_0^t \sum_{n=1}^{N} \Big[ c_{mn} \, \dot{q}_n \, \dot{q}_m + \bar{b}_{mn} \, q_n \, \dot{q}_m \Big] dt \Big\} ,$$

where $a_{mn} = \int_V (1 - \alpha) \, \sigma_{jk} \, \varphi_{in,j} \, \varphi_{im,k} \, dv$ ,

$$\bar{b}_{mn} = - \int_V \sigma_{jk} \, (\alpha \, \varphi_{in,j})_{,k} \, \varphi_{im} \, dv ,$$

and
$$b_{mn} = \frac{1}{\omega_m^2} (a_{mn} + \bar{b}_{mn}) . \tag{11}$$

For a positive definite H in a region $\rho_3 < R$ ; $R > 0$ , we can find an M such that $\bar{H}$ is also a positive definite quantity within a ring $\bar{R}_1 < \bar{\rho}_3 < \bar{R}$ , where $\bar{\rho}_3$ is defined by

$$\bar{\rho}_3 = \sum_{n=1}^{N} (q_n^2 + \dot{q}_n^2)$$ in a 2N-dimensional Euclidean space. Moreover, $\bar{R}_1$ is dependent only upon $\varepsilon_i$ in inequalities (9) and may be made as small as we please by choosing M large enough. From the stability theorem we therefore conclude that, for system (1) to be stable with respect to the metric $\rho_3$ , it is sufficient that $\bar{H}$ be a positive definite quantity. But $\bar{H}$ vanishes for $\bar{\rho}_3 = 0$ and $\frac{d\bar{H}}{dt}$ is identically equal to zero along any path satisfying equations (10′). Therefore, by Liapunov's stability theorem [6], system (10′) is stable when $\bar{H}$

is a positive definite quantity, and likewise when H is a positive definite quantity.

The study of stability of the system of linear homogeneous ordinary differential equations (10) is, however, a classical mathematical problem. For the stability of (10′), it is necessary and sufficient that the roots of the characteristic equation of (10′) have non-positive real parts. However, the study of the function $\bar{H}$, which in fact is a statement of the energy of the system, can provide us with a better insight into the physical behavior of the system. We shall consider this aspect in detail in another study and merely note here that there exist two distinct modes of instability of system (1). One is characterized by divergent motion or the existence of an adjacent equilibrium configuration, the other by flutter or the existence of an amplified oscillation. Divergent motion may occur if, for a virtual (static) displacement of the system, the work of the applied forces equals the change in the strain energy of the system, namely

$$\delta \int_V \frac{1}{2} \left[ C_{ijkl}\, \bar{u}_{i,j}\, \bar{u}_{k,l} + (1 - \alpha)\, \sigma_{jk}\, \bar{u}_{i,k}\, \bar{u}_{i,j} \right] dv -$$

$$- \int_V \sigma_{jk} \left(\alpha\, \bar{u}_{i,j}\right)_{,k} \delta\bar{u}_i \, dv = 0 \;,$$

or equivalently

$$\delta \int_V \frac{1}{2} \left[ C_{ijkl} \bar{u}_{i,j} \bar{u}_{k,l} + \sigma_{jk} \bar{u}_{i,k} \bar{u}_{i,j} \right] dv -$$

$$- \int_S \alpha \, p_j \, \bar{u}_{i,j} \, \delta\bar{u}_i \, dS = 0 \qquad (12)$$

where $\delta$ is the variational symbol.

Let us now assume that $\alpha$ is function of a real parameter $\gamma$ ; $-\infty < \gamma < +\infty$ , in addition to $x_1$ , $x_2$ , and $x_3$ ; $\alpha \equiv \alpha(x_1 , x_2 , x_3 ; \gamma)$ . Moreover, we consider a proportional loading $\beta \, p_j(\bar{x})$ , where $\beta$ is a finite, dimensionless, real number; $0 \leq \beta < \infty$ . In this way, the plane of $\beta$-$\gamma$ is divided into regions of stability and instability by equation (12). The effect of the linear viscosity (equation (3)), in this case, is to make the stability regions a closed set (except, possibly, for a set with measure zero; a finite number of isolated points in this plane.).

The limiting condition for the flutter of system (1), by contrast, is obtained when

$$H_3 = \int_0^{\frac{2\pi}{\omega}} \int_V \left[ C'_{ijkl} \dot{u}_{i,j} \dot{u}_{k,l} - \sigma_{jk} (\alpha \, \bar{u}_{i,j})_{,k} \dot{u}_i \right] dvdt = 0$$

where $\omega$ is the frequency of steady state oscillation of the solid about its unperturbed state. The motion of the solid decays if $H_3 > 0$ and amplifies if $H_3 < 0$ .

## 5. Concluding Remarks

In conclusion it should be emphasized that the sufficiency theorem for the stability of a linearly viscoelastic solid subjected to partial follower surface tractions advanced in this study has been established only with respect to a particular functional metric space. This work, then, in effect, is an illustration of the indispensability of an explicitly defined metric and no attempt was made to establish a necessity theorem. Nor was the question raised as to the convergence of the Galerkin method as applied to non-self-adjoint operators. Likewise, the important problem of the possible role of nonlinearity of various sources was deliberately excluded. The two different types of loss of stability (divergent motion and flutter), possible in the presence of follower forces, were mentioned only briefly and will be treated in detail for a general finite system in a separate investigation. The destabilizing effect of linear viscous damping in a continuous system subjected to nonconservative forces will also be discussed elsewhere.

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