A General Stability Criterion for Feedback Systems

by

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1. Introduction. If a feedback system is described by a system of linear differential equations with constant coefficients, there are a great many techniques for determining whether or not the system is stable [1], [3], [6]. If the coefficients are dependent on time, then stability can be determined by comparing the system to one with constant coefficients [5]. If the system can only be described by nonlinear equations, stability may be determined by various approximations, describing functions or by techniques fitting certain special situations. In some of these cases valid mathematical proof of the results is lacking.

It is the aim of this paper to present two general stability theorems, with proofs, which cover a great many situations and which, even in the linear case with constant coefficients, gives useable new results.

2. Existence, Uniqueness and Stability. We will present the following theorems in a rather general setting, then show the applications.

Let X denote a Banach space with norm denoted by $||\cdot||$. Let F and G be operators on X.

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Definition. *An operator* F *is said to be bounded if there exists a number* K *such that* $||Fx|| \leqq K\,||x||$ *for all* x *in* X.

The smallest of all such numbers K is called the norm of the operator. All such numbers K are bounds for F. In what follows only bounds will be needed. The norm itself will not be necessary. It is known that if FG is a composite operator, F is bounded by K, G is bounded by L, then FG is bounded by KL.

Definition. *An operator* F *is said to satisfy a Lipschitz condition if there exists a number* K *such that* $||Fx - Fy|| \leqq K\,||x - y||$ *for all* x *and* y *in* X.

We are now ready to consider the feedback system $\mathfrak{S}$ illustrated by figure 1. f is the "forcing function" or "input", e is the "error", r is the "response" or "output". F and G are operators on X.

Mathematically $\mathfrak{S}$ can be described by the following equations.

$$f - G(r) = e\,, \qquad F(e) = r\,,$$

or by

$$F(f - G(r)) = r.$$

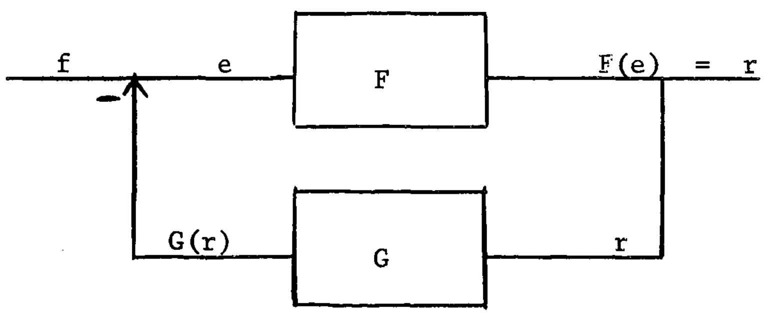

Figure 1. The feedback system $\mathfrak{S}$.

<u>Theorem 1</u>. <u>If</u> F <u>and</u> G <u>satisfy a Lipschitz condition with constants</u> K <u>and</u> L <u>respectively and</u> $KL < 1$, <u>then</u> $F(f - G(r)) = r$ <u>has a unique solution</u> r <u>in</u> X. <u>(See [7])</u>.

Proof. We develop an iterative procedure to give a sequence of elements in X which converge to r. Let

$$r_1 = F(f),$$
$$r_2 = F(f - G(r_1)),$$

and in general

$$r_n = F(g - G(r_{n-1})).$$

Each r_n is in X and

$$\begin{aligned} ||r_n - r_{n-1}|| &= ||F(f - G(r_{n-1})) - F(f - G(r_{n-2}))||, \\ &\leqq K \; ||[f - G(r_{n-1})] - [f - G(r_{n-2})]||, \\ &= K \; ||G(r_{n-1}) - G(r_{n-2})||, \\ &\leqq KL \; ||r_{n-1} - r_{n-2}||. \end{aligned}$$

We see by induction that

$$r_n - r_{n-1} \leqq (KL)^{n-1} \; ||F(f)||$$

which approaches zero as $n \to \infty$ since $KL < 1$. Since X is complete, the Cauchy sequence $\{r_n\}$ converges to an element r in X. Further

$$\begin{aligned} ||r - F(f - G(r))|| &= ||r - r_n + F(f - G(r_{n-1})) - F(f - G(r))|| \\ &\leqq ||r - r_n|| + KL \; ||r_{n-1} - r|| \to 0. \end{aligned}$$

So

$$r = F(f - G(r)).$$

If r and r' are both solutions then

$$||r - r'|| \leqq KL \; ||r - r'|| < ||r - r'||$$

which is a contradiction unless $r = r'$.

As with ordinary differential equations this method of successive approximations gives a procedure for calculating r as well as an estimate of the error involved.

It is also important to note that this theorem guarantees uniqueness only in X. It is entirely possible to have other solutions not in X and in so doing destroy "stability".

<u>Theorem 2.</u> <u>If</u> F <u>and</u> G <u>are bounded respectively by</u> K <u>and</u> L, $KL < 1$ <u>and</u> $F(f - G(r)) = r$ <u>has a solution</u> r <u>in</u> X, <u>then</u>

$$||r|| \leqq \frac{K}{1 - KL} \; ||f||.$$

<u>(See [7])</u>.

Proof.

$$\begin{aligned} ||r|| &= ||F(f - G(r))||, \\ &\leqq K \; ||f - G(r))||, \\ &\leqq K \; ||f|| + K \; ||G(r)||, \\ &\leqq K \; ||f|| + KL \; ||r||. \end{aligned}$$

Thus

$$||r|| \, [1 - KL] \leqq K \; ||f||$$

and the result follows.

<u>Corollary</u>. <u>If</u> F <u>and</u> G <u>satisfy a Lipschitz condition with constants</u> K <u>and</u> L <u>respectively</u>, $KL < 1$ <u>and</u> $F(o) = 0$, $G(o) = 0$, <u>then</u> $F(f - G(r)) = r$ <u>has a unique solution</u> r <u>in</u> X, <u>and</u>

$$||r|| \leqq \frac{K}{1 - KL} \quad ||f||.$$

(See [7]).

Proof. Since $F(o) = 0$, the Lipschitz condition

$$||Fx - Fy|| < K||x-y||$$

with $y = 0$ shows that K is a bound for F. Similarly L is a bound for G. The results follow from Theorems 1 and 2.

These theorems are valid in any Banach space X. Those most commonly considered are the spaces L^p, $1 \leqq p \leqq \infty$, C, the space of bounded continuous functions, and B the space of bounded functions. (See [4]). In practical applications the functions in these spaces are functions of time, which is permitted to vary from 0 to ∞. If negative times are considered the functions are usually to be zero for $t < 0$.

An important concept associated with feedback systems is the concept of stability. There are many ways of defining stability. The one following is a convenient choice.

<u>Definition</u>. <u>The system</u> $\mathfrak{I}$ <u>is said to be stable if for all forcing functions</u> f <u>in</u> X, <u>the response</u> r <u>is always an element of</u> X.

It is entirely possible for f to be in X with r not. Such systems are unstable.

Let $X(0,\infty)$ be one of the Banach spaces $L^p(0,\infty)$, $1 \leqq p \leqq \infty$, $C(0,\infty)$

or $B(0,\infty)$ with norm $\|\cdot\|$. For any $N > 0$ we denote the norm of $X(0,N)$ by $\|\cdot\|_N$. We then have the following

Stability Theorem 1. *Let f be in $X(0,\infty)$ and let r be in $X(0,N)$ for all $N > 0$. Suppose for all f and g in $X(0,N)$ for all $N > 0$, Fg, Fh, Gg, and Gh are defined and satisfy*

$$\|Fg - Fh\|_N \leqq K \|g - h\|_N$$

$$\|Gg - Gh\|_N \leqq L\|g - h\|_N$$

with $KL < 1$. Then r is in $X(0,\infty)$ and is identical with that of theorem 1.

The proof is identical with the uniqueness part of that of theorem 1.

Stability Theorem 2. *Let f, be in $X(0,\infty)$, and let r be in $X(0,N)$ for all $N > 0$. Suppose for all g in $X(0,N)$ for all $N > 0$, Fg and Gg are defined and satisfy*

$$\|Fg\|_N \leqq K\|g\|_N ,$$

$$\|Gg\|_N \leqq L\|g\|_N ,$$

with $KL < 1$.

Then r is in $X(0,\infty)$ and

$$\|r\| \leqq \frac{K}{1 - KL} \|f\|.$$

Proof. We have

$$F(f - G(r)) = r.$$

Thus $\|r\|_N \leqq K \|f - G(r)\|_N$,

$$\leqq K \|f\|_N + KL \|r\|_N ,$$

and

$$||r||_N \leqq [K/(1 - KL)] \; ||f||_N .$$

Letting $N \to \infty$ completes the proof.

We finally remark that these theorems can be generalized to multiloop systems with similar results.

3. Applications

Inverse Differential Operators. The most commonly encountered feedback systems are usually expressed in terms of linear differential equations with constant coefficients. In this case the operators F and G are expressed as inverse differential operators, $F = f_1(D)/f_2(D)$, $G = g_1(D)/g_2(D)$, where f_1, f_2, g_1 and g_2 are polynomials in $D = \frac{d}{dt}$, $\deg f_1 \leqq \deg f_2$, $\deg g_1 \leqq \deg g_2$. These operators are uniquely defined by requiring that $Ff(0)$ and $Gf(0)$ be zero along with an appropriate number of derivatives as long as $f(t)$ is a suitable function.

Another more suitable representation of F and G is by the use of one sided Green's functions [5]. It is known that there exist unique functions $\phi(t)$ and $\Gamma(t)$ so that

$$Ff(t) = \int_0^t \phi(t - \tau) f(\tau)d\tau \quad \text{and} \quad Gf(t) = \int_0^t \Gamma(t - \tau)f(\tau)d\tau .$$

Using this last representation, if all of the zeros of $f_2(z)$ and $g_2(z)$ lie in the left half plane it can be shown that when X is L^1, L^2, L^∞, B or C, the norms of F and G are $\int_0^\infty |\phi(t)|dt$ and $\int_0^\infty |\Gamma(t)|dt$ respectively.

As an example let $F = \frac{1}{D+a}$, $a > 0$. Then if f is suitable, $Ff(t) = \int_0^t e^{-a(t-u)} f(u)du$. The norm of F is a^{-1} in L^1, L^2, L^∞, B or C.

If $F = \frac{1}{D+a_1} \cdot \frac{1}{D+a_2} \cdots \frac{1}{D+a_n}$, then $[a_1, a_2, \cdots, a_n]^{-1}$ is a bound for F in L^1, L^2, L^∞, B or C.

As a corollary to the preceeding discussion we have the following.

__Theorem__. __Let__ $X = L^1, L^2, L^\infty, B, C$. __Let__ $\mathfrak{F}$ __be a simple feedback system with__ $F = K/(D+a_1) \cdots (D+a_n)$, $a_i > 0$, $G = 1$. __Then__ $\mathfrak{F}$ __is stable if__ $|K| < a_1 a_2 \ldots a_n$.

If we consider the feedback system where $F = \frac{1}{D+a}$, $G = 1$ and determine stability by any other method, we find that the system is stable provided $|K| < a$ and possibly unstable if $|K| > a$. Thus the result is the "best possible". Actually this system is stable for $K > -a$ and unstable otherwise.

Similar results may be found for linear differential systems with variable coefficients. In this case the functions ϕ and Γ are functions of two variables t and τ.

$$Ff(t) = \int_0^t \phi(t,\tau)f(\tau)d\tau, \quad \text{and} \quad Gf(t) = \int_0^t \Gamma(t,\tau)f(\tau)d\tau. \quad \text{(See [5])}.$$

__Time Lags__. A time lag operator L_τ defined by $L_\tau f(t) = f(t-\tau)$ can be represented by $e^{-\tau D}$. It has norm 1 in all those Banach spaces mentioned. The results of the previous section may be restated with time lags with no change in the results.

Certain Nonlinear Operators. There are a great many nonlinear operators such as saturation, dead zone, hysteresis, backlash which frequently occur alone or in any possible combination. If these nonlinear elements are considered separately and a graph of output versus input is made, there usually is a point on the graph so that a line joining it to the origin has maximum slope K. It is easy to see that K is a bound for that operator. (See [3]).

We note in passing that the "describing functions" used to describe these phenomena have K as a maximum value.

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