Final Report

on

Investigation of the Constitutive Equations for Multi-phase Hypoelastic Materials

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R. D. Snyder

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TABLE OF CONTENTS

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ABSTRACT

13/73

The multi-phase media problem is inherently a statistical one. However, in the continuum mechanics approach to the problem, the heterogeneous multi-phase media is considered to be a macroscopically homogeneous mixture of discrete phases of the constituents. Since the field solution to the problem is intractable, except in very special cases, a practical resolution of the problem is to determine the bulk properties of the mixture in terms of the properties of the constituents and the phase-volume-ratios. With these parameters, the best one can hope to do is to bound the effective bulk properties as has been done for isotropic and anisotropic elastic materials by use of the minimum theorems of classical elasticity.

In this paper the author has initiated the investigation of the multi-phase problem for hypoelastic materials and, more generally, for linear preferred materials. Using classical elasticity as a guide, the author has developed some variational, uniqueness, and minimum theorems analogous to those of classical elasticity. In addition, the possible existence of a deformation energy density function is investigated and discussed. Such a function is derived for two special forms of linear materials. Finally, the simple extensional deformation problem for such linear materials is investigated and discussed using two relatively simple materials as examples.

LIST OF FIGURES

LIST OF SYMBOLS

τ_{ij}	Component of true stress matrix
τ	True stress matrix
e_{ij}	Component of strain matrix
e	Strain matrix
$\bar{\tau}$, $\bar{e}$	Average homogeneous stress and strain matrix, respectively
d_{ij}	Component of rate of deformation matrix
d	Rate of deformation matrix
$\bar{d}$	Average rate of deformation matrix
$\underline{T}$	Surface traction vector
$\underline{n}$	Outward unit normal vector
$\underline{F}$	Generalized body force vector
ρ	Specific density of deformed body
ρ_o	Specific density of undeformed body
$\frac{D(\ )}{Dt}$ or $(\dot{\ })$	Material time derivative
K	Kinetic energy
t	Time
$\mathcal{E}$, ε	Internal energy and internal energy density, respectively
W	Rate of work
$\underline{f}$	Body force per unit mass
U	Extraneous forms of energy
E, e	Deformation energy and deformation energy density, respectively

$\underline{q}$	Heat flux vector
h	Heat source scalar field
S	Surface area
V	Volume
$\underline{v}$	Velocity vector
Σ	Elastic energy density per unit undeformed volume
$\underline{u}$	Displacement vector
Φ	Elastic potential energy
Ψ	Elastic complementary energy
c_i	Phase-volume-ratio
μ	Elastic shear modulus
k	Elastic bulk modulus
E	Young's modulus
γ	Poisson's ratio
Γ	Measure of deformation
x_1, x_2, x_3	Space coordinates (deformed configuration)
X_1, X_2, X_3	Material coordinates (Initial configuration)
ω_{ij}	Component of rotation tensor
$(\hat{\ })$	Jaumann absolute time derivative
$(\tilde{\ })$	Truesdell absolute time derivative
s	Dimensionless stress
Q	Orthogonal transformation matrix
I	Identity matrix
Θ, ξ, ζ	Stress invariants
Π_{Kj}	Piola pseudostress
$\underline{t}$	Stress vector
$\underline{g}$	Base vector

AN INITIAL INVESTIGATION OF THE CONSTITUTIVE
EQUATIONS FOR MULTI-PHASE HYPOELASTIC MATERIALS

I. General Theory of Multi-phase Media.

(a) The problem defined: The multi-phase medium problem may be stated in the following manner: To predict the bulk mechanical (or electrical, thermal, etc.) properties of heterogeneous media in terms of the properties and geometry of the constituents. By heterogeneous medium we mean a mixture of discrete phases at the interface of which continuity of the stress and displacement vectors is assumed so that the aggregate may be regarded an a continuum. It is assumed *a priori* that the properties of the individual phases are known.

The problem as stated above is much too general to have any hope for solution and must therefore be more explicitly defined. To do this, the most logical first step is to restrict the constituents to materials of the same class; e.g., elastic, plastic, viscoelastic, hypoelastic, etc. Also, restricting the number of different phases might be done although this appears to be of limited benefit. Even with these two restrictions, the problem remains extremely formidable and one is naturally led to the other major aspect of the mixture, namely the geometry of the phases.

Generally the geometry of the mixture is assumed to be random, although in some cases it is desirable to consider specific geometries. Furthermore, much of the successful work done to date has been accomplished by restricting the phase-volume-ratios of the constituents; that is, the ratio of the volume of a phase to the volume of the multi-phase medium. In this regard, many investigators have considered the special multi-phase medium called a suspension in which one phase is thought of as matrix in which all the other phases are suspended in the form of inclusions (spheres, ellipsoids, etc.). The suspension may be dilute or finite depending on the PVR of the inclusions.

Various other types of heterogeneous mixtures have been studied in detail by various investigators too numerous to list in this paper. The reader is referred to a paper by Z. Hashin [1][1] in which there appears an excellent bibliography of methods and results of investigations in this field. In this respect it might be well to mention that the problem of field analysis of heterogeneous media is at a much earlier stage of development than the problem of bulk properties of heterogeneous media. This paper makes no attempt at the former problem.

(b) Mathematic formulation: The multi-phase problem outlined in (a) is of such character that one is led to expect that its solution would be of statistical nature. Indeed, much recent work has been devoted to this method of approach. However, most of the results obtained to date on multi-phase media have been arrived at from a continuum mechanics approach; and, since the theory of behavior for hypoelastic media is at a relatively early stage of development, the author has chosen to undertake the development of some basic formulations regarding the behavior of two

1. Numbers in brackets refer to entries in the bibliography.

types of hypoelastic[2] media. It is felt that these results will have two immediate benefits.

(1) The behavior of hypoelastic materials can conceivably shed light on the overall concepts of the behavior of continua in general, ranging from classical elastic media to the viscoelastic and viscous fluid media.

(2) These results are viewed as being necessary to formulate the continuum mechanics approach to the multi-phase hypoelastic medium problem.

In this latter statement, it is fully recognized by the author that the ultimate solution and complete understanding of the behavior of multi-phase hypoelastic materials, as with other multi-phase media, probably lies with a statistical investigation of the problem. This is left for future consideration.

In general, the physical constants of heterogeneous media are random functions of the space coordinates. However, in the continuum theory, the physical constants are assumed to be space independent; that is, the medium is considered to be statistically (or macroscopically) homogeneous. Further, a statistically isotropic medium is one for which the physical constants are assumed to be independent of any rotation or reflection of the body. Consequently, the multi-phase problem can be

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[2]. While one of the media considered in this paper is not hypoelastic, in the strictest sense, it is nonetheless referred to as a hypoelastic medium at this point. The distinction will be made more clearly in Chapter III.

formulated in the following manner: If a representative volume element of a statistically homogeneous and isotropic multi-phase medium is subjected to space constant strains (or strain rates), find the effective stress-strain relation.

$$\bar{\tau}_{ij} = L(\bar{e}_{ij}) \tag{1.1}$$

Or, if the medium is subjected to space constant stresses, find the effective strain-stress relation

$$\bar{e}_{ij} = R(\bar{\tau}_{ij}) \tag{1.1a}$$

where the constants appearing in the operators R and L are called the effective physical constants, of the heterogeneous medium. Clearly the number of constants and the form of the operator will depend upon the physical properties of the phases involved in the mixture. At this point we tacitly assume that there are no body or surface couples acting on the medium and hence the true stress tensor τ will be symmetric.

In this paper it will be more convenient to formulate the problem in terms of the rate of deformation tensor.

$$d_{ij} = 1/2\,(v_{i,j} + v_{j,i}) \tag{1.2}$$

where v_i is the ith component of the velocity vector and the comma denotes covariant differentiation in generalized coordinates or ordinary partial differentiation in rectangular cartesian coordinates with respect to the Eulerian space variables, x_i , i= 1, 2, 3.

For any representative volume element, V, of material, the average deformation rates can be expressed in terms of the velocities on the boundary S of the volume by

$$\bar{d}_{ij} = \frac{1}{V}\int_V d_{ij}\,dV$$

$$= \frac{1}{2V}\int_V (v_{i,j} + v_{j,i})\,dV$$

$$= \frac{1}{2V}\int_S (\overset{o}{v}_i n_j + \overset{o}{v}_j n_i)\,dS \qquad (1.3)$$

where n_i and $\overset{o}{v}_i$ are the cartesian components of the outward normal and surface velocity, respectively. Similarly, the average stresses can be expressed as follows:

$$\bar{\tau}_{ij} = \frac{1}{V}\int_V \tau_{ij}\,dV$$

$$= \frac{1}{2V}\int_V [(\tau_{ik}x_j)_{,k} + (\tau_{kj}x_i)_{,k}]\,dV - \frac{1}{2V}\int_V [\tau_{ik,k}x_j + \tau_{kj,k}x_i]\,dV$$

$$= \frac{1}{2V}\int_S [\tau_{ik}x_j n_k + \tau_{kj}x_i n_k]\,dS - \frac{1}{2V}\int_V [F_i x_j + F_j x_i]\,\rho dV$$

$$= \frac{1}{2V}\int_S [\overset{o}{T}_i x_j + \overset{o}{T}_j x_i]\,dS - \frac{1}{2V}\int_V [F_i x_j + F_j x_i]\,\rho dV \qquad (1.4)$$

where $\overset{o}{T}_i$ and F_i and the cartesian components of the surface tractions and generalized body force, respectively, and ρ is the specific density. In these integrals the representative volume element is assumed to be large in comparison with the phase region size. Also we see that boundary velocities and tractions of the form

$$\overset{\circ}{v}_i = \overset{\circ}{d}_{ij} x_j \tag{1.5}$$

$$\overset{\circ}{T}_i = \overset{\circ}{\tau}_{ij} n_j \tag{1.6}$$

produce space constant strain rate fields $\overset{\circ}{d}_{ij}$ and space constant stress fields $\overset{\circ}{\tau}_{ij}$ if $\underline{F}$ vanishes and are, therefore, suitable for experimental determinations of stress-strain rate relations.

The question which now arises is the following: knowing the phase-volume-ratios and the mechanical properties of the individual constitutents, what information can be found in regard to the effective constitutive equations of the form (1.1) or (1.1a) for a macroscopically homogeneous and isotropic multi-phase medium? In most cases the problem is intractable because in order to use our definitions (1.3) and (1.4) in conjunction with (1.1), say, requires a knowledge of the stress-strain rate fields which are usually indeterminate. Thus, we are led to a usually more fruitful formulation of the problem in terms of energy expressions.

(c) <u>Energy relations of continuous media</u>: One of the fundamental axioms of continuum mechanics is the Principle of Conservation of Energy: The time rate of change of kinetic plus internal energy is equal to the sum of the rate of work W of the external forces and all the other energies that enter or leave the system per unit time.

$$\frac{D}{Dt}(K + E) = W + \sum_\alpha U_\alpha \tag{1.7}$$

where K and E are the kinetic and internal energies, respectively, and U_α represents the mechanical equivalent of the αth kind of energy per unit time, e.g., heat energy, electrical energy, chemical energy, etc. We write

$$E = \int_V \varepsilon\rho dV \tag{1.8}$$

where ε is the internal energy density per unit mass.

For a continuous medium

$$K = \frac{1}{2}\int_V \rho v_i v_i dV$$

$$\overset{\circ}{K} = \frac{D}{Dt}\int_V \rho v_i v_i dV$$

$$= \int_V \rho v_i \overset{\circ}{v}_i dV + \frac{1}{2}\int_V v_i v_i \frac{D}{Dt}(\rho dV) \tag{1.9}$$

and

$$\overset{\circ}{E} = \frac{D}{Dt}\int_V \varepsilon\rho dV$$

$$= \int_V \overset{\circ}{\varepsilon}\rho dV + \int_V \varepsilon\frac{D}{Dt}(\rho dV) \tag{1.10}$$

Also, the rate of work done by the surface tranctions and body forces $\underline{f}$ is

$$W = \int_S T_i v_i dS + \int_V f_i v_i \rho dV$$

$$= \int_S \tau_{ij} n_j v_i dS + \int_V f_i v_i \rho dV = \int_V (\tau_{ij} v_i)_{,j} dV + \int_V f_i v_i \rho dV$$

$$= \int_V \tau_{ij,j} v_i dV + \int_V \tau_{ij} d_{ij} dV + \int_V f_i v_i \rho dV \tag{1.11}$$

If we assume that the only other types of energy are heat energies, then,

$$
\begin{aligned}
U &= \int_S q_i n_i \, dS + \int_V \rho h \, dV \\
&= \int_V q_{i,i} \, dV + \int_V \rho h \, dV \qquad (1.12)
\end{aligned}
$$

where $\underline{q}$ is the heat flux vector through S and h is the supply of heat energy per unit mass created by energy sources in the body.

Substituting (1.9), (1.10), (1.11) and (1.12) into (1.7) and rearranging terms, we have

$$
\int_V \left(\frac{1}{2} v_i v_i + \varepsilon\right) \frac{D}{Dt}(\rho dV) + \int_V (\overset{\circ}{\varepsilon}\rho - \tau_{ij} d_{ij} - q_{i,i} - \rho h)\, dV = \int_V (\tau_{ij,j} + f_i \rho - \rho \overset{\circ}{v}_i)\, v_i \, dV
$$

The first integral on the left vanishes by conservation of mass while that on the right vanishes by Euler's equations of motion. Hence,

$$
\begin{aligned}
\overset{\circ}{E} &= \int_V \overset{\circ}{\varepsilon} \rho \, dV \\
&= \int_V (\tau_{ij} d_{ij} + q_{i,i} + \rho h) \, dV \qquad (1.13)
\end{aligned}
$$

where $\int_V \tau_{ij} d_{ij} \, dV$ is sometimes called the stress power, mechanical power, or rate of deformation energy.

(d) <u>Multi-phase hyperelastic media</u>: In order that the reader have a greater appreciation of the results which will be presented in later chapters, it might be well at this point to discuss briefly the multi-phase hyperelastic medium problem. By an ideally elastic medium [2] we mean that (1) no electric, chemical, or thermal changes occur during the deformation process, (2) the body possesses a natural unstressed uniform state to which it will return upon release of external loads, and (3) the minimum energy required to produce a given deformation is available for full recovery. Consequently, the constitutive equations are derivable from an energy density function. (Note: If statement (3) is true only for quasi-static processes, then the medium is termed elastic rather than ideally elastic.)

For a purely mechanical system undergoing a reversible adiabatic deformation, equation (1.7) becomes

$$W = \overset{\circ}{K} + \overset{\circ}{E}$$

or

$$W dt = dK + dE \tag{1.14}$$

As a consequence of statement (3) we see that if the body undergoes some loading and then returns to its initial conditions

$$\oint dE = 0$$

Now, from equation (1.13)

$$\dot{E} = \int_V \overset{\circ}{\varepsilon} \rho \, dV = \int_V \tau_{ij} d_{ij} \, dV$$

or

$$\int_{V_o} \overset{\circ}{\varepsilon} \rho_o \, dV_o = \int_{V_o} \tau_{ij} d_{ij} \frac{\rho_o}{\rho} \, dV_o$$

where V_o and ρ_o are the volume and density of the underformed body, respectively. Hence,

$$\rho\overset{\circ}{\varepsilon} = \tau_{ij} d_{ij} \tag{1.16}$$

If we define Σ as energy density per unit underformed volume, then

$$\Sigma = \rho_o \varepsilon \tag{1.17}$$

so that

$$E = \int_V \varepsilon \rho dV = \int_V \Sigma \frac{\rho_o}{\rho} dV$$
$$= \int_{V_o} \Sigma dV_o$$

Thus, from (1.16) and (1.17) we obtain the well known result

$$\frac{\rho}{\rho_o} \overset{\circ}{\Sigma} = \tau_{ij} d_{ij} \tag{1.18}$$

Now, for a material undergoing <u>infinitesimal</u> elastic deformations and rotations we define the strain by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

where u_i are the cartesian components of the displacement vector. Then

$$\dot{e}_{ij} \doteq \frac{1}{2}(\dot{u}_{i,j} + \overset{\circ}{u}_{j,i}) = d_{ij}$$

Substituting this expression into (1.18) and recalling that ε, and hence Σ is a function of the strains only, we have

$$\frac{\rho}{\rho_o}\frac{\partial \Sigma}{\partial e_{ij}}\dot{e}_{ij} = \tau_{ij}\dot{e}_{ij}$$

Thus for an ideally elastic material

$$\tau_{ij} = \frac{\rho}{2\rho_o}\left(\frac{\partial \Sigma}{\partial e_{ij}} + \frac{\partial \Sigma}{\partial e_{ji}}\right) \tag{1.19}$$

That is, the stresses are derivable from a potential Σ called the strain energy density.

In the case of a linearly elastic material we have constitutive equations of the form

$$\tau_{ij} = C_{ijk\ell}e_{k\ell}$$

or (1.20)

$$e_{ij} = \gamma_{ijk\ell}\tau_{k\ell}$$

where the C's and γ's are possible space functions but are independent of the stress, strain, or time. Hence the strain energy density becomes a quadratic in either the stresses or strain,

$$\Sigma = \tfrac{1}{2}C_{ijk\ell}e_{ij}e_{k\ell}$$

or (1.21)

$$\Sigma = \tfrac{1}{2}\gamma_{ijk\ell}\tau_{ij}\tau_{k\ell}$$

In the investigation of multi-phase elastic medium the minimum energy theorems of elasto statics are used to obtain bounds on the constants appearing in the constitutive equations of the statistically homogeneous heterogeneous mixture. These theorems are stated as follows [3]:

<u>Theorem of Minimum Potential Energy</u>: Of all the kinematically admissible displacements which satisfy the given boundary conditions, those which satisfy the equilibrium equations make the potential energy

$$\Phi = \int_V \Sigma(e)dV - \int_{S_t} T_i u_i dS - \int_V f_i u_i \rho dV$$

an absolute minimum, where S_t is that portion of the surface over which the tractions $\underline{T}$ are prescribed.

<u>Theorem of Minimum Complementary Energy</u>: Of all statically admissible stress fields which satisfy the given boundary conditions, that which satisfies the kinematic conditions make the complementary energy

$$\Psi = \int_V \Sigma(\tau)dV - \int_{S_u} T_i u_i dS$$

an absolute minimum, where S_u is that portion of the surface over which the displacement $\underline{u}$ is prescribed.

In the heterogeneous elastic mixture no restriction is placed on the shapes or sizes of the inclusions of one material (or materials) assumed to be imbedded in a matrix of the other material. The phases themselves are assumed to be homogeneous and isotropic, while it is supposed that the mixture is homogeneous on a macroscopic scale. In the following discussion numerical subscripts will distinguish the phases while quantities without subscripts will refer to a representative volume element of the mixture.

The phase-volume-ratios of the two phases will be denoted by c_1 and c_2, respectively, where $c_1 + c_2 = 1$. Thus by the definition of average stress and strain, equations (1.3) and (1.4), we have the obvious relations:

$$\bar{\tau} = c_1\bar{\tau}_1 + c_2\bar{\tau}_2$$

$$\bar{e} = c_1\bar{e}_1 + c_2\bar{e}_2 \tag{1.22}$$

The stress-strain and strain-stress relations are

$$\tau_1 = L_1 e_1 \qquad e_1 = R_1\tau_1$$

$$\tau_2 = L_2 e_2 \qquad e_2 = R_2\tau_2 \tag{1.23}$$

in the notation of equations (1.1), (1.1a). Hence, since the phases are assumed uniform and isotropic, we have the relations

$$\bar{\tau} = c_1L_1\bar{e}_1 + c_2L_2\bar{e}_2 = L\bar{e}$$

$$\bar{e} = c_1R_1\bar{\tau}_1 + C_2R_2\bar{\tau}_2 = R\bar{\tau} \tag{1.24}$$

Let

$$\bar{e}_1 = A_1\bar{e}\ , \quad \bar{e}_2 = A_2\bar{e}, \text{ with } c_1A_1 + c_2A_2 = I$$

and

$$\bar{\tau}_1 = B_1\bar{\tau} \quad \bar{\tau}_2 = B_2\bar{\tau}, \text{ with } c_1B_1 + c_2B_2 = I$$

where I is the unit matrix. Then from (1.24)

$$L = c_1L_1A_1 + c_2L_2A_2 \qquad (a)$$
$$R = c_1R_1B_1 + c_2R_2B_2 \qquad (b) \qquad (1.26)$$

Therefore, if the average stress or strain in either phase can be found for arbitrary overall values, then the elastic bulk properties of the mixture can easily be found. For a mixture that is macroscopically isotropic only two elastic constants must be found and this may formally be done by choosing two arbitrary independent strain fields at will, such as pure dilatation, pure shear, simple extension, simple shear, etc.

For example, suppose we write the stress-strain relation in the form

$$\tau_{ij} = 2\mu e_{ij} - \frac{2\mu}{3} e_{kk}\delta_{ij} + k\, e_{kk}\delta_{ij} \qquad (1.27)$$

where μ and k are the rigidity and bulk moduli, respectively. Then for a pure dilatation with stress $\bar{\tau}$ and cubical dilatation $\bar{e}_{kk}$ of the overall mixture, equations (1.22), (1.23) and (1.27) yield

$$\bar{\tau} = c_1k_1\bar{e}_{kk_1} + c_2k_2\bar{e}_{kk_2}$$
$$\bar{e}_{kk} = \frac{c_1\bar{\tau}_{kk_1}}{k_1} + \frac{c_2\bar{\tau}_{kk_2}}{k_2}$$

If we define

$$\bar{e}_{kk_1} = a_1\bar{e}_{kk}, \quad \bar{e}_{kk_2} = a_2\bar{e}_{kk}, \text{ with } a_1c_1 + a_2c_2 = 1$$
$$\bar{\tau}_{kk_1} = b_1\bar{\tau}_{kk}, \quad \bar{\tau}_{kk_2} = b_2\bar{\tau}_{kk}, \text{ with } b_2c_2 + b_2c_2 = 1$$

Then

$$k = a_1c_1k_1 + a_2c_2k_2$$

or $\qquad (1.28)$

$$\frac{1}{k} = \frac{b_1c_1}{k_1} + \frac{b_2c_2}{k_2}$$

Similar results in terms of μ can be obtained by letting the second state of strain be one of pure shear.

Results (1.26), or in particular, (1.28), must be viewed as being formal since it would be an extremely difficult experimental problem to measure the average strains in the inclusions. A crude approximate treatment, called the Voigt estimate, assumes that the strain throughout the mixture is uniform; that is, $A_1 = A_2 = I$. Then equation (1.26a) becomes:

$$L_v = c_1L_1 + c_2L_2$$

or, in terms of k and μ

$$k_v = c_1k_1 + c_2k_2$$
$$\mu_v = c_1\mu_1 + c_2\mu_2 \qquad (1.29)$$

Obviously, this approximation is tantamount to a simple volume weighting of the phases.

Another approximation, called the Reuss estimate, assumes that the stress is uniform throughout the mixture; that is, $B_1 = B_2 = I$. Then equation (1.26b) becomes

$$R_R = c_1R_1 + c_2R_2$$

or, in particular

$$\frac{1}{k_R} = \frac{c_1}{k_1} + \frac{c_2}{k_2}$$
$$\frac{1}{\mu_R} = \frac{c_1}{\mu_1} + \frac{c_2}{\mu_2} \qquad (1.30)$$

which is equivalent to a simple weighting of the inverse moduli.

Needless to say, neither assumption is correct since the first would result in discontinuous tractions at the interfaces while the second would result in discontinuous displacements. Both estimates become poor when the phase moduli differ by more than a factor of two.

A more fruitful approach to the multi-phase elastic medium problem stems directly from the previously mentioned energy theorems of elastostatics. For a linear elastic material the strain energy per unit volume is

$$\Sigma = \frac{1}{2}\tau e = \frac{1}{2}\tau(R\tau) = \frac{1}{2}(eL)e \qquad (1.31)$$

Consider any unit volume of the mixture and let it be subjected to prescribed surface displacements that would produce a uniform strain $\bar{e}$ in a homogeneous material. But these are precisely the average strains in the heterogeneous mixture itself since these averages are uniquely determined from the surface displacements alone (see equation (1.3)). Therefore, the total strain energy of the mixture can be evaluated as the integral of $\frac{1}{2}\tau\bar{e}$ since τ is an equilibrium field of stress and the strain field $(e - \bar{e})$ is derived from a displacement field which vanishes on the surface. Hence, we have

$$\Sigma = \frac{1}{2}\bar{\tau}\bar{e}$$

from the definition of $\bar{\tau}$. A similar result can be obtained by prescribing tractions of the kind that would produce a uniform internal stress $\bar{\tau}$ in a homogeneous material. Then if the volume is a representative volume element, we have for either type of boundary condition

$$2\bar{\Sigma} = \bar{\tau}\bar{e} = \bar{\tau}R\bar{\tau} = \bar{e}L\bar{e} \qquad (1.33)$$

Now, according to the theorem of minimum potential energy, the actual strain energy in the mixture does not exceed the energy of any other fictitious (unequilibrated) state of distortion with the same surface displacements. Hence, if we take as the second state a uniform state $\bar{e} = e_1 = e_2$, then

$$\bar{e}L\bar{e} \leq \bar{e}\,(c_1L_1 + c_2L_2)\,\bar{e} \qquad (1.34)$$

Hence, the matrix

$$c_1L_1 + c_2L_2 - L$$ is positive

semi-definite and the restrictions on its components can be determined. In this manner an upper bound on the moduli can be obtained. In the case where the mixture is assumed isotropic, equation (1.34) gives the simple relations

$$k \leq k_V \quad \mu \leq \mu_V \qquad (1.35)$$

in terms of the Voigt estimates (1.29).

Similarly the theorem of minimum complementary energy states that the actual complementary energy does not exceed that of any other fictitions (kinematically incompatible) state of stress with the same surface tractions. Thus, taking the uniform stress field, $\bar{\tau} = \tau_1 = \tau_2$, as the second state of stress yields

$$\bar{\tau} R \bar{\tau} \leq \bar{\tau}(c_1 R_1 + c_2 R_2)\bar{\tau} \tag{1.36}$$

which furnishes upper bounds on the compliances of the mixture material. For the isotropic mixture, these relations are

$$\frac{1}{k} \leq \frac{1}{k_R} \qquad \frac{1}{\mu} \leq \frac{1}{\mu_R}$$

or

$$k \geq k_R \qquad \mu \geq \mu_R \tag{1.37}$$

in terms of the Reuss estimates (1.30).

Recalling that Young's modulus E can be written as

$$\frac{3}{E} = \frac{1}{\mu} + \frac{1}{3k} \tag{1.38}$$

it follows that relations (1.35) and (1.37) can be written as

$$\frac{1}{E_v} \leq \frac{1}{E} \leq \frac{1}{E_R} \tag{1.39}$$

or

$$\left(\frac{c_1}{E_1} + \frac{c_2}{E_2}\right)^{-1} = E_R \leq E \leq E_v \tag{1.40}$$

Generally, these bounds on the elastic constants are rather wide. By making some additional restrictions and employing some rather elaborate variational principles Hashin [4] has obtained a reduced range for the bounds and Hill[11] has shown these bounds on the bulk modulus to be the best possible in terms of only the concentrations and the moduli of the constituents.

(e) Objectives of investigation: With this brief background of the multi-phase medium problem one should now have some concept of the various individual problems involved in the overall problem of multi-phase media. This investigation is an effort to resolve some of these initial problems with regard to a class of continuous media, namely, the hypoelastic[3] material whose mathematical characteristics are discussed in the following chapter of this paper.

First, for two relatively simple forms of the constitutive equations of hypoelasticity, the author will derive an expression for the energy of deformation for the respective materials. Second, a solution will be presented for a simple extensional deformation process for each of the materials. Simple extension was chosen primarily because it is hoped that eventual experimental verification of some of these results can soon be undertaken, and it is anticipated that simple extension offers the best possibility in this respect. Finally, using the theorems of classical elasticity as a guide, some minimum principles and uniqueness theorems for this type of medium will be developed which shed considerable light on the possible solution of the multi-phase medium problem.

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[3] See footnote on Page 3.

II. Hypoelasticity.

(a) The constitutive equations: Hypoelastic theory is an attempt to explain or predict the behavior of materials with short memory, as distinguished from an elastic material which has a perfect memory of its natural state and from a stokesian fluid which has no memory of its past. A hypoelastic material has a memory of the state just passed and thus the theory is most easily formulated in terms of rates. The simplest rate theory takes the form

$$\text{stress rate} = f \ (\text{rate of deformation}) \tag{2.1}$$

While the rate of deformation is a well defined kinematic quantity, there is no unique way to define a stress rate or stress flux. The primary requirement for any defined stress rate tensor for use in the constitutive equation (2.1) is that it should vanish when the rate of deformation tensor vanishes; i.e., that the stress rate tensor should be unaffected by rigid body motions. Certainly the ordinary material time derivative $\frac{D\tau_{ij}}{Dt}$ fails to satisfy this requirement. However, the stress flux first defined by Jaumann [5] as

$$\overset{\frown}{\tau}_{ij} = \overset{\circ}{\tau}_{ij} + \tau_{ik}\omega_{kj} - \tau_{kj}\omega_{ik} \tag{2.2}$$

where ω_{ij} is the rate of rotation tensor

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$$

satisfies this requirement in regard to arbitrary rigid body rotations.

Furthermore, since for a rigid motion $d_{ij} = 0$, $\hat{\tau}_{ij}$ would be unaffected by the presence of terms containing d_{ij} insofar as our requirement of spatial invariance (material objectivity) is concerned. Hence, the various other forms for stress rate derived by Truesdell [6], Oldroyd [16], Rivlin, etc., differ from (2.2) merely by terms which can be incorporated into the right side of our constitutive equation (2.1).

For example, Truesdell's stress rate

$$\check{\tau}_{ij} = \mathring{\tau}_{ij} + \tau_{ij} v_{p,p} - \tau_{ip} v_{j,p} - \tau_{pj} v_{i,p} \tag{2.3}$$

differs from Jaumann's by

$$\tau_{ij} v_{p,p} - \tau_{ip} d_{pj} - \tau_{pj} d_{ip}$$

which can clearly be incorporated into the constitutive equation, as will be seen in the subsequent development. For a more detailed discussion of the various tensor time derivatives, see Oldroyd [17] and Thomas [18] and the Appendix of this paper.

The constitutive equation proposed in (2.1) is a special case of the general tensorial relation

$$\hat{s} = f(d,s) \qquad s = \frac{\tau}{2\mu} \text{. . . dimensionless stress} \tag{2.4}$$

$$\mu \text{ . . . shear modulus}$$

where the various quantities are evaluated in some spatial reference frame x_i. Homogeneity and isotropy are implied by the fact that the material coordinates x_i and the material descriptors $\underline{G}_i$ do not appear explicitly in the function indicated in (2.4). Further, if the material is to be spatially invariant, then

$$\dot{q}' = f(d', s')$$

where the primed quantities are now evaluated in any other spatial frame x'_i obtained from the x frame by the full orthogonal group of transformations $Q(t)$ where

$$Q_{ij}Q_{i\ell} = \delta_{j\ell}$$

$$Q_{ji}Q_{\ell i} = \delta_{j\ell}$$

Thus, our invariance requirement becomes

$$Qf(d, s)Q^{-1} = f(QdQ^{-1}, QsQ^{-1})$$

i.e., f is a hemitropic function of d and s. If f is a polynomial in d and s, then Rivlin [7] has shown that

$$f(d,s) = A_{00}I + A_{10}d + A_{20}d^2 + A_{01}s + A_{02}s^2 + A_{11}(sd + ds)$$
$$+ A_{12}(s^2d + ds^2) + A_{21}(d^2s + sd^2) + A_{22}(s^2d^2 + d^2s^2) \qquad (2.5)$$

since, from the Cayley-Hamilton Theorem, all higher powers in s and d can be expressed in terms of the first two powers of s and d. The coefficients $A_{\alpha\beta}$ are analytic scalar functions of the ten invariants

$$D_1 = d_{ii} \qquad D_2 = \frac{1}{2}(d_{ii}d_{jj} - d_{ij}d_{ji}) \qquad D_3 = |d_{ij}|$$

$$T_1 = s_{ii} \qquad T_2 = \frac{1}{2}(s_{ii}s_{jj} - s_{ij}s_{ji}) \qquad T_3 = |s_{ij}|$$

$$M = s_{ij}d_{ji} \qquad N = s_{ij}s_{jk}d_{ki} \qquad P = s_{ij}d_{jk}d_{ki}$$

$$O = s_{ij}s_{jk}d_{k\ell}d_{\ell i} \tag{2.6}$$

The coefficients are further restricted by the <u>First Hypothesis of Hypoelasticity</u>: "No constitutive coefficients of a hypoelastic material shall carry a dimension independent of the dimension of stress. Thus, the physical material moduli shall have the dimension of [stress]$^\alpha$ for some α; and, in particular, this restriction prohibits the moduli from being functions of time which eliminates the 'relaxation effect' associated with time dependent moduli". Hence, equation (2.4) must reduce to the form

$$\overset{\circ}{s}_{ij} = A_{ijk\ell}d_{k\ell} \tag{2.7}$$

where $A_{ijk\ell}$ must be dimensionless functions of stress.

In light of this last statement, it now becomes clear that all terms in (2.5) containing powers of d higher than the first must vanish. Thus

$$A_{20} = A_{21} = A_{22} = 0$$

and A_{10}, A_{11}, A_{12} must be linear in d. Hence, we now write

$$\hat{s} = D_1A_0I + A_1d + D_1A_2s + D_1A_3s^2$$
$$+ MA_4I + A_5(ds + sd) + MA_6s + MA_7s^2$$
$$+ NA_8I + A_9(ds^2 + s^2d) + NA_{10}s + NA_{11}s^2 \qquad (2.8)$$

where the A_i are dimensionless analytic functions of the three stress invariants, T_1, T_2, T_3.

Truesdell has defined various classes of hypoelastic materials which are characterized by the highest degree of s appearing in equation (2.8).

Hypoelastic of grade zero. The right side of (2.8) is independent of s. Hence

$$\check{s} = D_1A_0I + A_1d; \quad A_0, A_1 \quad \text{constants}$$

or if,

$$A_0 = \frac{\lambda}{2\mu}, \quad A_1 = 1, \text{ then}$$

$$2\mu s_{ij} = \lambda d_{kk}\delta_{ij} + 2\mu d_{ij} \qquad (2.9)$$

which is directly similar to the constitutive equations for a linear, isotropic, elastic material.

Hypoelastic of grade one. The right side of (2.8) contains up to the first power of stress. Hence

$$\breve{s} = D_1(\beta_0 + T_1\gamma_0)I + (\beta_1 + T_1\gamma_1)d +$$

$$D_1\beta_2 s + M\beta_4 I + \beta_5(ds + sd) \tag{2.10}$$

with β_0, β_1, β_4, β_5, γ_0, γ_1 being dimensionless constants.

Hypoelastic of grade two: The right side of (2.8) contains up to the second power of stress. Hence

$$\breve{s} = D_1(\beta_0 + T_1\gamma_0 + \alpha_0 T_1^2 + \delta_0 T_2) + (\beta_1 + T_1\gamma_1$$

$$+ \alpha_1 T_1^2 + \delta_1 T_2)d + D_1(\beta_2 + T_1\gamma_2)s + D_1\beta_3 s^2$$

$$+ M(\beta_4 + \gamma_4 T_1)I + (\beta_5 + \gamma_5 T_1)(ds + sd) +$$

$$M\beta_6 s + N\beta_8 I + \beta_9(ds^2 + s^2 d) \tag{2.11}$$

with the β's, γ's and α's and δ's being dimensionless constants.

Note that in establishing all of these constitutive equations, no assumptions have been made in regard to the magnitude of the stresses and strains involved or the time interval over which they occurred and hence are fully general for all types of motion.

(b) Field equations and admissibility: The constitutive equations of hypoelasticity (2.7) are differential equations of the first order, which are not in themselves sufficient to define a hypoelastic material. One must prescribe some conditions, such as initial conditions, which are admissible according to some criteria [8].

Let V be a smooth manifold of material points over which we assign a stress field τ_{ij} and a displacement gradient field $\dfrac{\partial x_i(x)}{\partial X_K}$ subject to the compatibility conditions

$$\frac{\partial\left(\dfrac{\partial x_i}{\partial X_K}\right)}{\partial X_L} = \frac{\partial\left(\dfrac{\partial x_i}{\partial X_L}\right)}{\partial X_K}$$

Such an assignment is called a stress-configuration pair and is denoted by

$$\left\{\tau_{ij}x_{i,K}\right\}$$

Given two such stress-configuration pairs

$$\left\{\tau_{ij},\ x_{i,K}\right\}_1 \quad \left\{\tau_{ij},\ x_{i,K}\right\}_2$$

it may or may not be possible to find a solution of (2.7) for which $\left\{\tau_{ij},\ x_{i,K}\right\}_1$ gives its initial values and $\left\{\tau_{ij},\ x_{i,K}\right\}_2$ gives the final values. However, if we group the stress-configuration pairs into classes, two such pairs being in the same class if it is possible to get from one to the other by a solution of (2.7), then these classes are equivalence classes and are mutually exclusive. Hence, a representation of a hypoelastic material is given by (2.7) and one such equivalence class of stress-configuration pairs.

We now have the basic field equations of hypoelasticity.

(i) Conservation of mass:

$$\overset{\circ}{\rho} + v_{k,k}\,\rho = 0 \qquad \text{I}$$

(ii) Balance of momenta:

$$\tau_{ij,j} + \rho(f_i - \overset{\circ}{v}_i) = 0$$

$$\tau_{ij} = \tau_{ji} \qquad \text{II}$$

(iii) Constitutive equations for isotropic material:

$$\hat{s}_{ij} \text{ or } \breve{s}_{ij} = A_{ijk\ell}\, d_{k\ell}$$

$$s_{ij} = \frac{\tau_{ij}}{2\mu} \qquad \text{III}$$

where $A_{ijk\ell}$ is an isotropic tensor function of the stress; along with an equivalence class of stress-configuration pairs. These equations, of course, reduce to the previously indicated special forms for the various grades of hypoelastic materials.

(iv) Kinematic relations:

$$\hat{\tau}_{ij} = \overset{\circ}{\tau}_{ij} + \tau_{ik}\omega_{kj} - \tau_{kj}\omega_{ik}$$

or other forms of the stress flux.

$$d_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) \qquad \omega_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i}) \qquad \text{IV}$$

In addition we have

(v) Boundary conditions: prescribed tractions and/or velocities (or displacements) on the surface of the body;

(vi) Initial conditions: any initial stress and displacement field consistent with the momenta equations and the compatability conditions and which are admissible for the equivalence class of the hypoelastic material.

In concluding this section of the paper, a few comments regarding the distinguishing features imbedded in the formulation of the hypoelastic theory seem appropriate. First of all, for our hypoelastic material the constitutive equation III is in no sense an approximation; it is completely consistent with the principles of mechanics for all types of motion, and is dynamically admissible for strains and rotations of any magnitude. Furthermore, a relation between stress and strain is the <u>outcome</u> of the theory and the form of this relation will depend upon the manner in which the deformation takes place and the initial conditions of the problem. This will be seen more clearly in the ensuing work.

(c) <u>Preferred Linear Materials</u>: While the foregoing discussion of this chapter was confined to materials with constitutive equations of the form

$$\hat{s} = f(d, s)$$

some extensive work has been done by T. Y. Thomas [10, 12] for materials with constitutive equations of the form

$$\hat{s} = f(\rho, d, s) \qquad (2.12)$$

or

$$\hat{s}_{ij} = f_{ij}(\rho, d, s) \qquad (2.13)$$

where the functions f_{ij} are polynomials in the components s_{ij} and d_{ij} with coefficients depending on the density. Furthermore, Thomas has defined a <u>preferred</u> system as one in which the f_{ij} are <u>linear</u> in d_{ij} and at most <u>quadratic</u> in s_{ij}. Such preferred systems are not too general for application, but are nevertheless of sufficient generality to provide an acceptable description of the dynamical behavior of an important class of materials.

Because of the scalar character of the density ρ, it is not necessary to alter any of the previous formulation of this chapter which pertained to hypoelastic materials except to say that the coefficients appearing in the various expressions (2.5),(2.7), (2.8),(2.10) and (2.11) may now be functions of ρ or the dimensionless ratio, $^{\rho}/_{\rho_0}$ Hence in subsequent work it will often be convenient to formulate the discussion in terms of preferred materials.

III. Deformation Energy for Preferred Linear Materials.

(a) General Considerations: Recall that the rate of work done by the external tractions and body forces on a mechanical system composed of a moving volume V(t) of a continuous media is given by

$$W = \frac{D}{Dt}\int_{V(t)} \frac{1}{2}\rho v_i v_i \, dV + \int_{V(t)} \tau_{ij} d_{ij} \, dV \tag{3.1}$$

where the second term on the right represents the time rate of change of the deformation energy which will be denoted by $\overset{\circ}{E}$. Then if

$$\frac{DE}{Dt} = \int_{V(t)} \rho \dot{e} \, dV = \int_{V(t)} \tau_{ij} d_{ij} \, dV$$

it follows that

$$\rho \overset{\circ}{e} = \tau_{ij} d_{ij} \tag{3.2}$$

Using the results of Thomas [9], [10], one can now establish the conditions under which an energy density function e will exist and be a scalar invariant of the tensors τ and d and the density ρ of an isotropic linear media with a constitutive equation of the form

$$\tau_{ij} = f_{ij}(\rho, \tau, d) \tag{3.3}$$

where f_{ij} are hemitropic polynomials linear and homogeneous in d.

Proceeding on the hypothesis that e is a differential scalar invariant, we write

$$\overset{\circ}{e} = \frac{\partial e}{\partial \rho}\frac{D\rho}{Dt} + \frac{\partial e}{\partial \tau_{ij}} \hat{\tau}_{ij} + \frac{\partial e}{\partial d_{ij}} \hat{d}_{ij} = \frac{1}{\rho} \tau_{ij} d_{ij} \tag{3.4}$$

where we have utilized the equality of the absolute time derivative $(\hat{\ })$ and the total time derivative $(\frac{D}{Dt})$ of a scalar invariant of τ and d. But from equation (2.2),

$$\hat{d}_{ij} = \mathring{d}_{ij} + d_{ik}\omega_{kj} - d_{kj}\omega_{ik}$$

Hence, the coefficients of $\hat{d}_{ij}$ in (3.4) must vanish identically, i.e., the invariant e cannot depend on d explicitly.

Now let us define the stress invariants

$$\Theta = \mathrm{tr}\tau = \tau_{ii} \qquad \xi = \mathrm{tr}\tau^2 = \tau_{ij}\tau_{ji} \qquad \zeta = \mathrm{tr}\tau^3 = \tau_{ij}\tau_{jk}\tau_{ki}$$

and thus

$$\frac{\partial\Theta}{\partial\tau_{ij}} = \delta_{ij} \qquad \frac{\partial\xi}{\partial\tau_{ij}} = 2\tau_{ij} \qquad \frac{\partial\zeta}{\partial\tau_{ij}} = 3\tau_{ik}\tau_{kj}$$

Since e is a scalar invariant, we may write

$$e(\rho, \tau) = e(\rho, \Theta, \xi, \zeta)$$

Hence

$$\frac{\partial e}{\partial\tau_{ij}} = \frac{\partial e}{\partial\Theta}\delta_{ij} + \frac{\partial e}{\partial\xi}2\tau_{ij} + \frac{\partial e}{\partial\zeta}3\tau_{ik}\tau_{kj}$$

Also, we recall the equation of continuity

$$\mathring{\rho} + \rho v_{k,k} = \mathring{\rho} + \rho d_{kk} = 0$$

Substituting (3.5) and (3.6) into (3.4) yields

$$-\rho d_{kk}\frac{\partial e}{\partial \rho} + \hat{\tau}_{ii}\frac{\partial e}{\partial \Theta} + 2\tau_{ij}\hat{\tau}_{ij}\frac{\partial e}{\partial \xi} + 3\tau_{ik}\tau_{kj}\hat{\tau}_{ij}\frac{\partial e}{\partial \zeta} = \frac{1}{\rho}\tau_{ij}d_{ij}$$

or in the trace notation

$$-\rho\,\mathrm{tr}d\,\frac{\partial e}{\partial \rho} + \mathrm{tr}\hat{\tau}\frac{\partial e}{\partial \Theta} + 2\,\mathrm{tr}\tau\hat{\tau}\frac{\partial e}{\partial \xi} + 3\,\mathrm{tr}\tau^2\hat{\tau}\frac{\partial e}{\partial \zeta} = \frac{1}{\rho}\mathrm{tr}\tau d \qquad (3.7)$$

which must be satisfied identically if $e(\rho, \tau)$ is to represent the deformation energy per unit mass of the material.

<u>Lemma on Independent Scalars</u>: Suppose the principal stresses τ_i are distinct and let $D_i = d_{ii}$ (no sum) be the diagonal components of d_{ij} relative to a local principal stress coordinate system at some arbitrary point P. Then

$$d_{ii} = \mathrm{tr}d = D_1 + D_2 + D_3$$

$$\tau_{ik}d_{ki} = \mathrm{tr}\tau d = \tau_1 D_1 + \tau_2 D_2 + \tau_3 D_3$$

$$\tau_{ij}\tau_{jk}d_{ki} = \mathrm{tr}\tau^2 d = \tau_1^2 D_1 + \tau_2^2 D_2 + \tau_3^2 D_3$$

Consider this as a system of equations for determining the D_i's. The determinant of the coefficients is

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ \tau_1 & \tau_2 & \tau_3 \\ \tau_1^2 & \tau_2^2 & \tau_3^2 \end{vmatrix} = (\tau_1 - \tau_2)(\tau_2 - \tau_3)(\tau_3 - \tau_1) \neq 0$$

and hence the system will have a solution for arbitrarily assigned values of trd, trτd, $tr\tau^2 d$. We will use this result in what follows.

(b) Preferred material of grade zero: Guided by Thomas [10], let us consider a material having constitutive equations of the form

$$\hat{\tau}_{ij} = \frac{\rho_o}{\rho}\,[\lambda d_{kk}\delta_{ij} + 2\mu\, d_{ij}] \tag{3.8}$$

which we shall call a preferred material of grade zero, substituting this into (3.7) we obtain

$$- \rho trd\,\frac{\partial e}{\partial \rho} + \frac{\rho_o}{\rho}(3\lambda + 2\mu)\,trd\,\frac{\partial e}{\partial \Theta} + \frac{2\rho_o}{\rho}(\lambda trd\; tr\tau + 2\mu tr\tau d)\frac{\partial e}{\partial \xi}$$

$$+ 3\frac{\rho_o}{\rho}(\lambda tr\tau^2\; trd + 2\mu tr\tau^2 d)\,\frac{\partial e}{\partial \rho} = \frac{1}{\rho}\,tr\tau d$$

or

$$P trd + Q tr\tau d + R tr\tau^2 d = 0 \tag{3.9}$$

where

$$P = -\rho\frac{\partial e}{\partial \rho} + \frac{\rho_o}{\rho}(3\lambda + 2\mu)\,\frac{\partial e}{\partial \Theta} + \frac{\rho_o}{\rho}\left(2\lambda tr\tau\frac{\partial e}{\partial \xi} + 3\lambda tr\tau^2\frac{\partial e}{\partial \zeta}\right) \quad \text{(a)}$$

$$Q = 4\,\frac{\rho_o}{\rho}\,\mu\,\frac{\partial e}{\partial \xi} - \frac{1}{\rho} \quad \text{(b)(3.10)}$$

$$R = 6\,\frac{\rho_o}{\rho}\,\mu\,\frac{\partial e}{\partial \zeta} \quad \text{(c)}$$

Now, from the lemma, (3.9) must be satisfied identically in trd, trτd, and $tr\tau^2 d$. Thus

$$P = Q = R = 0$$

From (3.10c) we see immediately that

$$e \equiv e(\rho, \theta, \xi, \zeta) \equiv e(\rho, \theta, \xi)$$

while from (3.10a, b)

$$-\rho\frac{\partial e}{\partial \rho} + \frac{\rho_o}{\rho}(3\lambda + 2\mu)\frac{\partial e}{\partial \theta} + 2\lambda\frac{\rho_o}{\rho}\mathrm{tr}\,\tau\,\frac{\partial e}{\partial \xi} = 0 \qquad (a)$$

$$4\frac{\rho_o}{\rho}\mu\frac{\partial e}{\partial \xi} - \frac{1}{\rho} = 0 \qquad (b) \quad (3.11)$$

If we assume e to be independent of ρ, then

$$\frac{\rho_o}{\rho}(3\lambda + 2\mu)\frac{\partial e}{\partial \theta} + 2\lambda\frac{\rho_o}{\rho}\theta\frac{\partial e}{\partial \xi} = 0$$

$$4\rho_o\mu\frac{\partial e}{\partial \xi} - 1 = 0 \qquad (3.12)$$

Hence

$$e = \frac{1}{4\mu\rho_o}\xi + \Phi(\theta)$$

$$\rho_o(3\lambda + 2\mu)\frac{d\phi}{d\theta} + \frac{2\lambda\rho_o\theta}{4\mu\rho_o} = 0$$

$$\therefore \phi = -\frac{\lambda\theta^2}{4\mu\rho_o(3\lambda + 2\mu)}$$

Thus the deformation energy density for this case is

$$e = \frac{1}{4\mu\rho_o}\left(\xi - \frac{\lambda\theta^2}{3\lambda + 2\mu}\right) \qquad (3.13)$$

This form is similar to that for the deformation energy of a linear isotrope elastic media, and, in fact, from (1.17) we have

$$\Sigma = \frac{1}{4\mu} \left(\xi - \frac{\lambda\theta^2}{3\lambda + 2\mu} \right) \tag{3.14}$$

(c) A preferred material of grade one: Let us now consider a special type of grade one material having a constitutive equation of the following form

$$\hat{\tau}_{ij} = \lambda d_{kk} \delta_{ij} + 2\mu d_{ij} - \frac{1}{2} d_{kk} \tau_{ij} \tag{3.15}$$

This material was chosen because of the particularly interesting form of its energy of deformation density.

Proceeding in exactly the same manner as for the material of grade zero we obtain

$$\begin{aligned} P &= -\rho\frac{\partial e}{\partial \rho} + (3\lambda + 2\mu - \tfrac{1}{2}\mathrm{tr}\tau) \frac{\partial e}{\partial \theta} + (2\lambda \mathrm{tr}\tau - \mathrm{tr}\tau^2) \frac{\partial e}{\partial \xi} \\ &\quad + (3\lambda \mathrm{tr}\tau^2 - \tfrac{3}{2} \mathrm{tr}\tau^3) \frac{\partial e}{\partial \zeta} \qquad (a) \qquad (3.16) \\ Q &= 4\mu \frac{\partial e}{\partial \xi} - \frac{1}{\rho} \qquad (b) \\ R &= 6\mu \frac{\partial e}{\partial \zeta} \qquad (c) \end{aligned}$$

From (3.16c) we see that

$$e \equiv e\,(\rho, \theta, \xi, \zeta) \equiv e\,(\rho, \theta, \xi)$$

while (3.16a, b) yield

$$\begin{aligned} -\rho\frac{\partial e}{\partial \rho} + (3\lambda + 2\mu - \tfrac{1}{2}\theta) \frac{\partial e}{\partial \theta} + (2\lambda\theta - \xi)\frac{\partial e}{\partial \xi} &= 0 \quad (a) \\ 4\mu \frac{\partial e}{\partial \xi} - \frac{1}{\rho} &= 0 \quad (b) \ (3.17) \end{aligned}$$

If we assume a solution of the form

$$e = \frac{1}{4\mu\rho}[\xi + \phi(\theta)]$$

we find

$$\phi = -\frac{\lambda\theta^2}{3\lambda + 2\mu} \tag{3.18}$$

Hence, for this case, the deformation energy density is

$$e = \frac{1}{4\mu\rho}\left(\xi - \frac{\lambda\theta^2}{3\lambda + 2\mu}\right) \tag{3.19}$$

which is, once again, similar to the form of the strain energy of an isotropic elastic media.

(d) Distortion and volumetric energies: Because of the similar formal character of expressions (3.13) and (3.19) to the energy of deformation for an isotropic elastic media, it is possible to define other analogous expressions. Recall that

$$\rho\dot{e} = \tau_{ij}d_{ij} \tag{3.2}$$

Let us define the stress deviator tensor and rate of deformation deviator tensor by

$$\tau^*_{ij} = \tau_{ij} - \frac{1}{3}\theta\delta_{ij}$$

$$d^*_{ij} = d_{ij} - \frac{1}{3}\mathrm{tr}d\delta_{ij} \tag{3.20}$$

where $\theta = \mathrm{tr}\tau$. Then

$$\tau_{ij} d_{ij} = \tau^*_{ij} d^*_{ij} + \frac{1}{3}\theta \mathrm{tr} d \tag{3.21}$$

and

$$\xi = \tau_{ij}\tau_{ij} = \xi^* + \frac{1}{3}\theta^2 \tag{3.22}$$

where

$$\xi^* = \tau^*_{ij}\tau^*_{ji} = \tau^*_{ij}\tau^*_{ij}$$

since

$$d^*_{kk} = \mathrm{tr}\tau^* = 0$$

If we define

$$\rho \frac{de^*}{dt} = \tau^*_{ij} d^*_{ij} \tag{3.23}$$

$$\rho \frac{d^*e}{dt} = \frac{1}{3}\theta \mathrm{tr} d \tag{3.24}$$

then clearly e can be represented as

$$e = e^* + {}^*e \tag{3.25}$$

In accordance with the usual interpretations used in classical elasticity, we can consider e^* to be the energy per unit mass due to distortion; i.e., change of shape, while the function $^*e$ is considered to be the energy associated solely with the change of volume. We call these quantities the distortion and volumetric energies, respectively.

Recalling relations (3.13) and (3.22) we see that

$$e = \frac{1}{4\mu\rho_o}\left(\xi^* + \frac{2\mu\theta^2}{3(3\lambda + 2\mu)}\right) \tag{3.26}$$

which suggests that we should have

$$e^* = \frac{1}{4\mu\rho_o} \xi^* \tag{3.27}$$

$$^*e = \frac{\theta^2}{6(3\lambda + 2\mu)\rho_o} \tag{3.28}$$

for the preferred material of grade zero.

Using (3.8), (3.27) and the definition of ξ^* we see that

$$\rho \frac{de^*}{dt} = \rho \frac{1}{2\mu\rho_o} \tau^*_{ij} \hat{\tau}^*_{ij}$$

$$= \frac{\rho}{\rho_o} \frac{1}{2\mu} \tau^*_{ij} \frac{\rho_o}{\rho} 2\mu \left(d_{ij} - \frac{1}{3} d_{kk}\right)$$

$$= \tau^*_{ij} d^*_{ij}$$

where the last step follows from the definition of d^*_{ij}. Hence our suggested e^* (3.27) satisfies (3.23) and thus can be taken as the energy of distortion. Similarly, $^*e$, can be taken as the energy of volume change.

In a similar manner for the preferred material of grade one discussed in (c), we have from (3.19)

$$e^* = \frac{1}{4\mu\rho} \xi^* \tag{3.29}$$

$$^*e = \frac{\theta^2}{6(3\lambda + 2\mu)\rho} \tag{3.30}$$

for the distortion and volumetric energies of deformation respectively.

(e) <u>Discussion of energy density forms</u>: It is immediate that the forms (3.13) and (3.19) obtained in the previous work are remarkably similar except for the specific density term ρ_o appearing in (3.13) and the term ρ appearing in (3.19). However, a little reflection shows that this difference could have been anticipated from the assumed constitutive equations. For, from equation (3.2) it is evident that if e does not depend on the specific density ρ, then the constitutive equations must involve ρ in some manner. Likewise, if the constitutive equations do not involve the specific density, then the energy density function must depend on ρ. Of course, it is possible for both the energy density and the constitutive equations to depend on the density . However in general, it would be dimensionally inconsistent for both the energy density and the constitutive equations to be independent of ρ, for if

$$\overset{*}{\tau}_{ij} = A_{ijk\ell}(\Theta, \xi, \zeta)d_{k\ell}$$

and

$$e = e(\Theta, \xi, \zeta)$$

$$= e(\tau_{ij}) \quad \ldots \text{ an invariant function of the stress tensor}$$

then (3.2) yields

$$\rho\overset{\circ}{e} = \tau_{ij}d_{ij}$$

$$\rho(\cancel{\frac{\partial e}{\partial \rho}}^{\,0}\overset{\circ}{\rho} + \frac{\partial e}{\partial \tau_{ij}}\overset{*}{\tau}_{ij}) = \rho\frac{\partial e}{\partial \tau_{ij}}A_{ijk\ell}d_{k\ell} = \tau_{ij}d_{k\ell}$$

Hence

$$\rho\,[\text{function}\ (\tau,d)] = \text{function}\ (\tau,d)$$

which appears to be dimensionally inconsistent, unless the dependence on ρ is merely apparent.

Further comments: Finally, let it be noted that any other definition of stress flux could have been used throughout the entire development of this section so long as its operation on a scalar invariant of τ is equivalent to the total time derivative of the scalar. Our choice of $\hat{\tau}_{ij}$ was purely an arbitrary one of convenience. Also for the energy forms (3.13) and (3.19), the corresponding constitutive equations (3.8) and (3.15), respectively, are not the most general. In fact a far more general class of preferred linear materials can be characterized by each of these energy forms (see Thomas [10]). What we have done here is simply to give a more definite characterization to a simple class of materials whose possible constitutive equations are given by (3.8) and (3.15).

IV. Simple Extensional Deformation.

Simple extensional deformation has been studied by Truesdell [19] and Green [20] for Truesdell's form of a hypoelastic body of grade zero. A detailed discsssion of accelerationless motion, without approximation, has been given by Truesdell while Green has presented a dynamical theory of simple extension. Some of their results and formulation are employed in the work which follows.

(a) <u>A statical theory</u>: Consider a homogeneous deformation field given by

$$\underline{v} = a(t)\underline{x} + \underline{b}(t)$$

In order for the acceleration field to vanish it is necessary and sufficient that the matrices a and b satisfy

$$\frac{da}{dt} + a^2 = 0$$

$$\frac{db}{dt} + ab = 0$$

which have the solutions

$$a = \frac{A}{I + At}$$

$$b = \frac{B}{I + At} \tag{4.1}$$

where A and B are constants a(0) and b(0), respectively.

Furthermore, for any accelerationless motion in the absence of body forces, the equations of motion are satisfied by any spatially constant stress. Also, for a homogeneous deformation and stress field, the equation of continuity and the constitutive equations become ordinary differential equations.

$$\frac{d\rho}{dt} = -\rho d_{k,k}(t)$$

$$\frac{d\tau_{ij}}{dt} = f_{ij}[\rho(t),\ a(t)]$$

Thus, if a(t) is analytic for $|t| < c$, it follows from the theorems on ordinary differential equations that there exists a unique stress field $\tau(t)$ taking an arbitrary value τ_o when $t = 0$, analytic for $|t| < c$. However, if we are to satisfy the dynamical equations exactly, a(t) must be given by (4.1).

Simple homogeneous extension is described by

$$a(t) = \left(d_{ij}(t)\right) = K(t)\begin{bmatrix} -\sigma(t) & 0 & 0 \\ 0 & -\sigma(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$b = 0 \qquad\qquad \omega_{ij} = 0 \qquad\qquad (4.2)$$

Hence, for accelerationless homogeneous extension

$$K(t) = \frac{K_o}{1 + K_o t} \qquad\qquad (a)$$

$$\sigma(t) = \sigma_o \frac{1 + K_o t}{1 - K_o \sigma_o t} \qquad (b)$$

$$K_o = K(0) \qquad \sigma_o = \sigma(0) \qquad (c) \qquad (4.3)$$

b. Simple extension of preferred material of grade zero:

Consider the system of field equations

$$\frac{\partial \rho}{\partial t} + (\rho v_k)_{,k} = 0 \qquad (4.4)$$

$$\tau_{ij,j} = 0 \qquad (4.5)$$

$$\hat{\tau}_{ij} = \frac{\rho_o}{\rho}(2\mu d_{ij} + \lambda \delta_{ij} d_{kk}) \qquad (4.6)$$

$$d_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) \qquad \omega_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i}) \qquad (4.7)$$

along with (4.2) and (.4.3). The equation of continuity becomes

$$\frac{d\rho}{dt} = -\rho K(t)\,[1 - 2\sigma(t)] \qquad (4.8)$$

or, for accelerationless motion

$$\rho = \rho_o \frac{1}{(1 + K_o t)(1 - K_o \sigma_o t)^2} \qquad (4.9)$$

We seek a solution of the form

$$(\tau_{ij}) = \begin{bmatrix} P(t) & 0 & 0 \\ 0 & P(t) & 0 \\ 0 & 0 & T(t) \end{bmatrix}$$

where P and T are functions of time alone. For this choice the equations of motion (4.5) are satisfied identically and we have merely to satisfy the constitutive equations (4.6). Recalling the definition of the ($\check{}$) operator we have

$$\frac{dP}{dt} = \frac{\rho_o}{\rho}\left[2\mu(-K\,\sigma) + \lambda(K - 2\sigma\,K)\right] \qquad (a)$$

$$\frac{dT}{dt} = \frac{\rho_o}{\rho}\left[2\mu\,K + \lambda(K - 2\sigma K)\right] \qquad (b) \qquad (4.11)$$

Let

$$\lambda = \frac{2\mu\gamma}{1 - 2\gamma} \qquad (4.12)$$

where γ is Poisson's ratio of classical elasticity. The equations (4.11) become

$$\frac{dP}{dt} = \frac{\rho_o}{\rho}\, 2\mu K\, \frac{\gamma - \sigma}{1 - 2\gamma} \qquad (a)$$

$$\frac{dT}{dt} = \frac{\rho_o}{\rho}\, 2\mu K\left[\frac{1 - \gamma - 2\gamma\sigma}{1 - 2\gamma}\right] \qquad (b) \qquad (4.13)$$

For a solution of this system to satisfy the equations of accelerationless motion (equilibrium,) it is necessary and sufficient that σ and K be given by (4.3). Hence, no solution for which P = 0 for all time is possible; that is, tensile stress alone is insufficient to produce simple statical extension. This conclusion agrees with finite elasticity theory and does not contradict classical elasticity theory since, if $\sigma(0) = \gamma$, then

$$\left.\frac{dP}{dt}\right]_{t=0} = 0$$

which implies that if the initial contraction ratio σ_o is equal to Poisson's ratio and the initial cross stress P(0) is zero, the cross stress P will remain very nearly zero at least for small deformations from the reference state at t = 0.

For the exact statical solution, recall that

$$v_x = -\sigma(t)\,K(t)\,x$$

$$v_y = -\sigma(t)\,K(t)\,y$$

$$v_z = -K(t)\,z$$

Or, using (4.3)

$$v_x = -\frac{\sigma_o K_o}{1 - \sigma_o K_o t}\,x$$

$$v_y = -\frac{\sigma_o K_o}{1 - \sigma_o K_o t}\,y$$

$$v_z = \frac{K_o}{1 + K_o t}\,z$$

which, when integrated yield

$$x = X(1 - \sigma_o K_o t)$$

$$y = Y(1 - \sigma_o K_o t)$$

$$z = Z(1 + K_o t) \tag{4.14}$$

Hence, it is reasonable to take $K_o t$ as a dimensionless measure of deformation, say Γ. Further, let us take $\sigma_o = \gamma$. Then, substitution of (4.9) into (4.13) yields

$$\frac{dP}{d\Gamma} = -2\mu\ (1 - \gamma\Gamma)\ \Gamma\gamma \frac{(1 + \gamma)}{1 - 2\gamma}$$

$$P = -2\mu \frac{\gamma(1 + \gamma)}{1 - 2\gamma} \left(\frac{\Gamma^2}{2} - \gamma \frac{\Gamma^3}{3} \right) \tag{4.15}$$

and

$$\frac{dT}{d\Gamma} = \frac{2\mu}{1 - 2\gamma}\ [(1 - \gamma\Gamma)^2\ (1 - \gamma)$$

$$- 2\gamma^2(1 - \gamma\Gamma)(1 + \Gamma)]$$

$$= 2\mu(1 + \gamma)[1 - 2\gamma\Gamma \frac{(1 - \gamma)}{(1 - 2\gamma)} + \gamma^2\Gamma^2 \frac{1}{1 - 2\gamma}]$$

$$T = 2\mu(1 + \gamma)[\Gamma - \gamma\Gamma^2 \frac{(1 - \gamma)}{(1 - 2\gamma)} + \frac{1}{3}\gamma^2\Gamma^3 \frac{1}{1 - 2\gamma}] \tag{4.16}$$

where we have set $T(0) = P(0) = 0$. It is significant to note that no linear term appears in the expression for the cross stress (4.15) which bears out our earlier remark.

If, on the other hand, we choose to ignore the kinematic requirement (4.3b) on σ and set $\sigma \equiv \gamma$, we would, if effect, ignore the inertia of the material. (This will be more clearly seen in a dynamical analysis of the problem which follows later.) Then by choosing $K(t)$ according to (4.3a) we have from (4.13),

$$P \equiv 0$$

$$\frac{dT}{d\Gamma} = 2\mu(1 + \gamma)(1 - \gamma\Gamma)^2$$

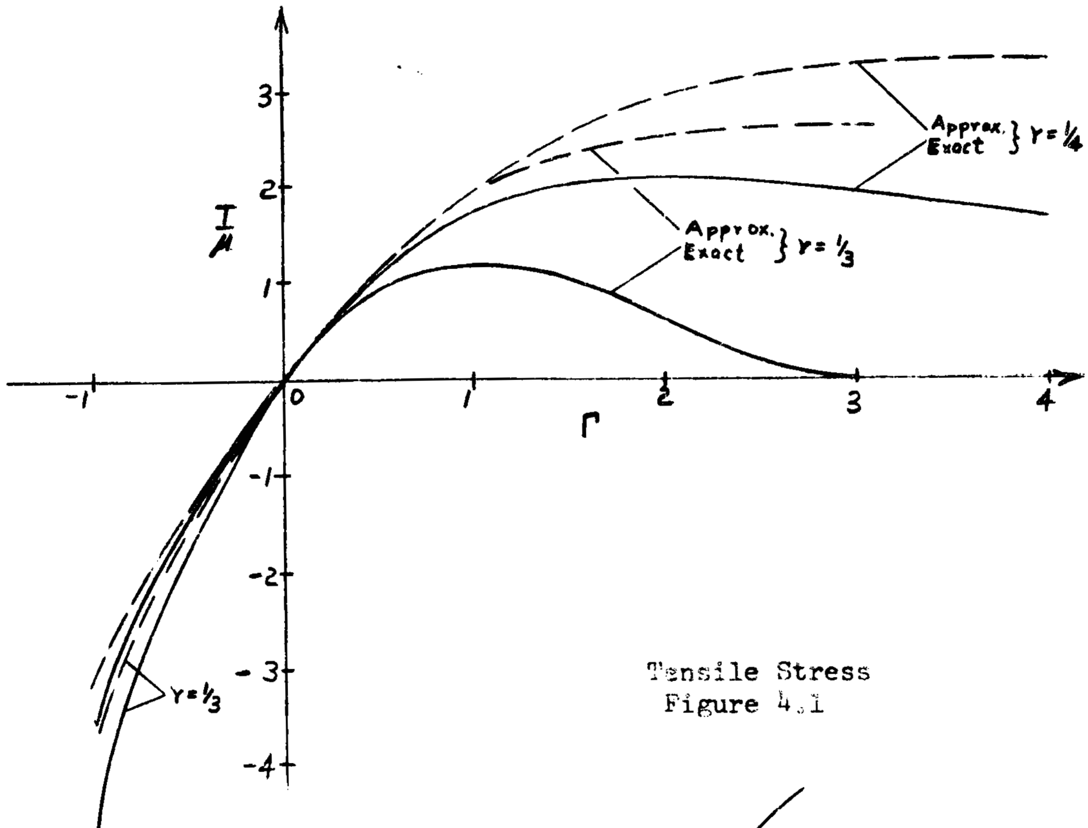
$$T_a = 2\mu(1 + \gamma)(\Gamma - \gamma\Gamma^2 + \frac{1}{3}\gamma^2\Gamma^3) \qquad (4.17)$$

Observe that this approximate solution for T agrees with the exact solution in the linear terms and the difference in the non-linear terms is dependent on the value of γ. Furthermore, we see from (4.9) that the values of Γ must lie within

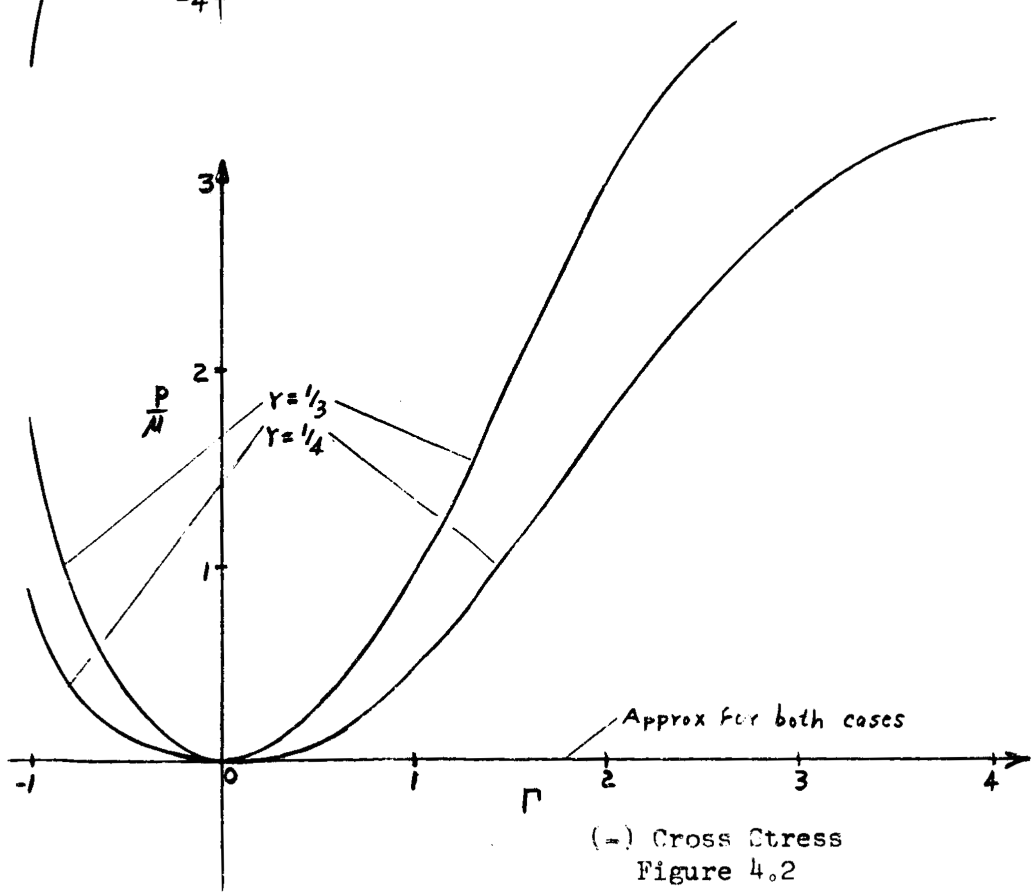
$$-1 < \Gamma < \frac{1}{\gamma}$$

since the volume is annulled for these limiting values. The details of these results are illustrated more clearly in the figures on the next page.

Extension of Grade Zero



Tensile Stress
Figure 4.1



(–) Cross Stress
Figure 4.2

While the exact solution of this statical extension problem is interesting and enlightening, a dynamical theory of extension sheds additional light on the problem.

Consider the field equations (4.4), (4.6) and (4.7) along with the equations of motion for zero body force

$$\tau_{ij,j} = \rho\left(\frac{\partial v_i}{\partial t} + v_{i,j}v_j\right) \qquad (4.18)$$

In addition let us take the homogeneous velocity field

$$v_x = \dot{\eta}x, \quad v_y = \dot{\eta}y, \quad v_z = \dot{\alpha}z$$

where η and α are monotonic functions of t alone. Hence,

$$(d_{ij}) = \begin{bmatrix} \dot{\eta} & 0 & 0 \\ 0 & \dot{\eta} & 0 \\ 0 & 0 & \dot{\alpha} \end{bmatrix}, \quad \omega_{ij} = 0 \qquad (4.19)$$

and since $\rho = \rho(t)$, the equation of continuity yields

$$\rho = \rho_o e^{-(\alpha + 2\eta)} \qquad (4.20)$$

provided $\rho_o = \rho(0)$ and $\alpha(0) = \eta(0) = 0$.

We seek a solution of the form

$$(\tau_{ij}) = \begin{bmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & T \end{bmatrix}$$

The equations of motion (4.18) yield

$$\frac{\partial P}{\partial x} = \rho(\ddot{\eta} + \dot{\eta}^2)x = \rho_o(\ddot{\eta} + \dot{\eta}^2)x\, e^{-(\alpha + 2\eta)} \quad \text{(a)}$$

$$\frac{\partial T}{\partial z} = \rho(\ddot{\alpha} + \dot{\alpha}^2)z = \rho_o(\ddot{\alpha} + \dot{\alpha}^2)z\, e^{-(\alpha + 2\eta)} \quad \text{(b)} \quad (4.21)$$

while the constitutive equations (4.6) yield

$$\frac{\partial P}{\partial t} + (\frac{\partial P}{\partial x}x + \frac{\partial P}{\partial y}y)\dot{\eta} + \frac{\partial P}{\partial z}z\dot{\alpha} = \frac{\rho_o}{\rho}[2\mu\dot{\eta} + \lambda(2\dot{\eta} + \dot{\alpha})]$$

$$= [2\mu\dot{\eta} + \lambda(2\dot{\eta} + \dot{\alpha})]e^{(\alpha + 2\eta)} \quad \text{(a)}$$

$$\frac{\partial T}{\partial t} + (\frac{\partial T}{\partial x}x + \frac{\partial T}{\partial y}y)\dot{\eta} + \frac{\partial T}{\partial z}z\dot{\alpha} = [2\mu\dot{\alpha} + \lambda(2\dot{\eta} + \dot{\alpha})]e^{-(\alpha + 2\eta)}$$

$$\text{(b)} \quad (4.22)$$

Let

$$P = P_o(t) + P_1(x, y, z, t)$$

$$T = T_o(t) + T_1(x, y, z, t)$$

Then the constitutive equations become

$$\left.\begin{aligned}
\frac{dP_o}{dt} &= [2\mu\dot{\eta} + \lambda(2\dot{\eta} + \dot{\alpha})]e^{(\alpha + 2\eta)} && \text{(a)}\\
\frac{dT_o}{dt} &= [2\mu\dot{\alpha} + \lambda(2\dot{\eta} + \dot{\alpha})]e^{(\alpha + 2\eta)} && \text{(b)}\\
\frac{\partial P_1}{\partial t} &+ (\frac{\partial P_1}{\partial x}x + \frac{\partial P_1}{\partial y}y)\dot{\eta} + \frac{\partial P_1}{\partial z}z\dot{\alpha} = 0 && \text{(c)}\\
\frac{\partial T_1}{\partial t} &+ (\frac{\partial T_1}{\partial x}x + \frac{\partial T_1}{\partial y}y)\dot{\eta} + \frac{\partial T_1}{\partial z}z\dot{\alpha} = 0 && \text{(d)}
\end{aligned}\right] \quad (4.23)$$

Recalling equation (4.12), equations (4.23a, b) can be written as

$$\frac{dP_o}{dt} = 2\mu[\dot{\eta} + \frac{\gamma}{1 - 2\gamma}(2\dot{\eta} + \dot{\alpha})]e^{(\alpha + 2\eta)} \qquad \text{(a)}$$

$$\frac{dT_o}{dt} = 2\mu[\dot{\alpha} + \frac{\gamma}{1 - 2\gamma}(2\dot{\eta} + \dot{\alpha})]e^{(\alpha + 2\eta)} \qquad \text{(b)} \qquad (4.24)$$

We seek a solution such that $P_o \equiv 0$. This is possible provided $\eta = -\gamma\alpha$. Then

$$\begin{aligned} \frac{dT_o}{dt} &= 2\mu[\dot{\alpha} + \gamma\dot{\alpha}]e^{(1 - 2\gamma)\alpha} \\ &= 2\mu\dot{\alpha}(1 + \gamma)e^{(1 - 2\gamma)\alpha} \end{aligned} \qquad (4.25)$$

And

$$\begin{aligned} \frac{\partial P_1}{\partial t} - \gamma\dot{\alpha}\left(\frac{\partial P_1}{\partial x}x + \frac{\partial P_1}{\partial y}y\right) + \dot{\alpha}\frac{\partial P_1}{\partial z}z &= 0 \\ \frac{\partial T_1}{\partial t} - \gamma\dot{\alpha}\left(\frac{\partial T_1}{\partial x}x + \frac{\partial T_1}{\partial y}y\right) + \dot{\alpha}\frac{\partial T_1}{\partial z}z &= 0 \end{aligned} \qquad (4.26)$$

The general solution of equations (4.26) is

$$P_1 = f(xe^{\gamma\alpha}, ye^{\gamma\alpha}, ze^{-\alpha})$$

$$T_1 = g(xe^{\gamma\alpha}, ye^{\gamma\alpha}, ze^{-\alpha})$$

where f and g are arbitrary functions of their arguments and of class c^1. Inspection of the equations of motion (4.21) leads us to choose

$$P_1 = \frac{Ax^2}{2}e^{2\gamma\alpha}$$

$$T_1 = \frac{Bz^2}{2}e^{-2\alpha}$$

These will satisfy (4.21) provided

$$\frac{A}{\rho_o} = (-\gamma\ddot{\alpha} + \gamma^2\dot{\alpha}^2)e^{-\alpha} \qquad (a)$$

$$\frac{B}{\rho_o} = (\ddot{\alpha} + \dot{\alpha}^2)e^{(1+2\gamma)\alpha}$$

Once again, we see that tensile stress alone is not sufficient to produce the deformation field of <u>homogeneous</u> simple extension, for the only solution corresponding to $A = B = 0$ is the null velocity field of $\dot{\alpha} = 0$. If, on the other hand, we neglect the inertia terms in the equations of motion and seek a solution for which T is the only non-zero stress, then we are led once again to equations (4.24). It can be easily verified that when α conforms to $K(t)$ of equation (4.3a), equation (4.24b) is identical to equation (4.17) and hence we recover the approximate statical solution.

It may be possible to select $\dot{\alpha}$ in such a manner so as to correspond closely to the exact dynamical solution of the problem. However, it is apparent that no <u>homogeneous</u> stress and velocity field will produce simple extensional deformation for this material.

(c) <u>Simple statical extension of material of grade one</u>: Consider the system of field equations (4.4), (4.5), (4.7) and

$$\hat{\tau}_{ij} = 2\mu d_{ij} + \lambda\delta_{ij}d_{kk} - \frac{1}{2}d_{kk}\tau_{ij} \qquad (4.28)$$

along with equation (4.12). Then using the notation of (4.2) and (4.10) the constitutive equations become

$$\frac{dP}{dt} = 2\mu K \frac{\gamma - \sigma}{1 - 2\gamma} - \frac{1}{2}(1 - 2\sigma)KP \qquad (a)$$

$$\frac{dT}{dt} = 2\mu\left(\frac{1 - \gamma - 2\gamma\sigma}{1 - 2\gamma}\right) K - \frac{1}{2}(1 - 2\sigma)KT \qquad (b) \qquad (4.29)$$

As with the material of grade zero, we see that for a solution for which $P \equiv 0$, $\sigma = \gamma$ for all time which clearly violates the kinematic requirement (4.3b) for accelerationless motion and we arrive at conclusions similar to those of the grade zero material. Ignoring the kinematic requirement on $\sigma(t)$, we obtain as an approximate solution

$$P \equiv 0$$

$$\frac{dT}{d\Gamma} = [2\mu(1 + \gamma) - \frac{1}{2}(1 - 2\gamma)T]\frac{1}{1 + \Gamma}$$

$$T_a = \frac{4\mu(1 + \gamma)}{(1 - 2\gamma)}\left[1 - (1 + \Gamma)^{-\frac{1}{2}(1 - 2\gamma)}\right] \qquad (4.30)$$

where we have taken $K(t)$ and Γ in accordance with the previous work and have set $T(0) = 0$. On the other hand, if we take $\sigma(t)$ and $k(t)$ in accordance with (4.3), then the exact solution for simple statical extension becomes

$$(1 + \Gamma)(1 - \gamma\Gamma)\frac{dP}{d\Gamma} = -2\mu\gamma\frac{(1 + \gamma)}{(1 - 2\gamma)}\Gamma - \frac{(1 - 2\gamma - 3\gamma\Gamma)}{2}P$$

$$P = \frac{-4\mu\gamma(1 + \gamma)}{3(1 - 2\gamma)(1 - \gamma\Gamma)}\left[\frac{2}{(1 + \Gamma)^{1/2}} + (\Gamma - 2)\right] \qquad (4.31)$$

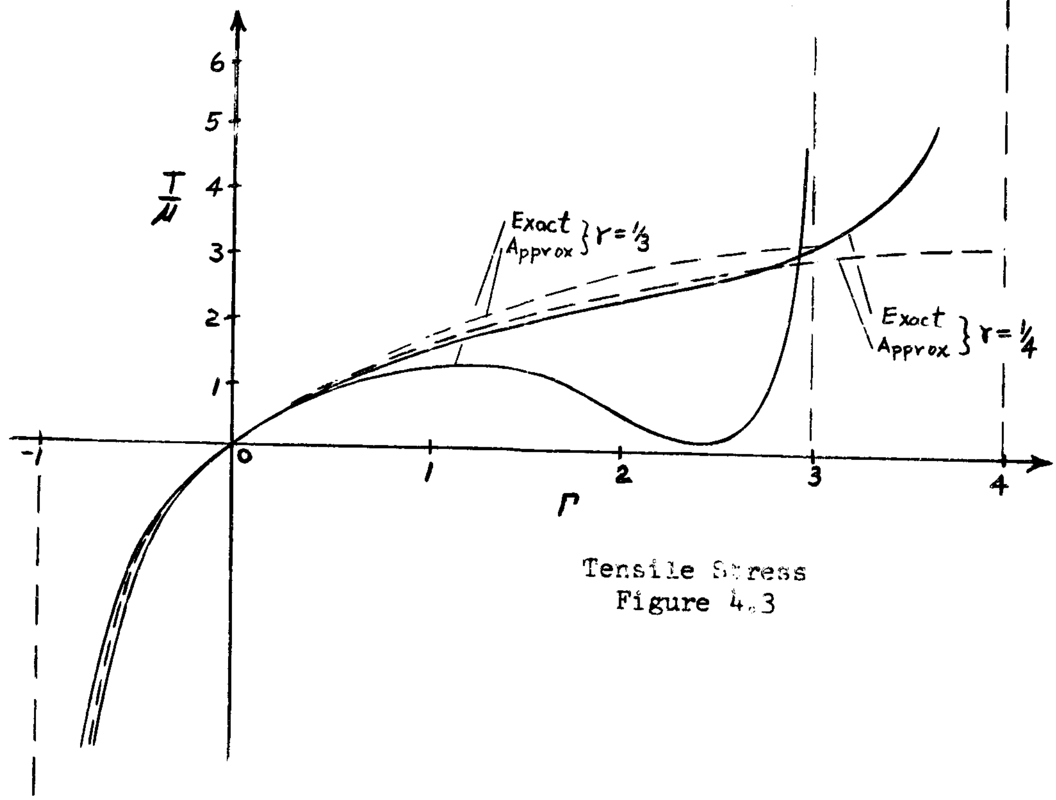
and

$$(1 + \Gamma)(1 - \gamma\Gamma)\frac{dT}{d\Gamma} = 2\mu\frac{(1 + \gamma)}{(1 - 2\gamma)}(1 - 2\gamma - \gamma\Gamma) - \frac{(1 - 2\gamma - 3\gamma\Gamma)}{2}T$$

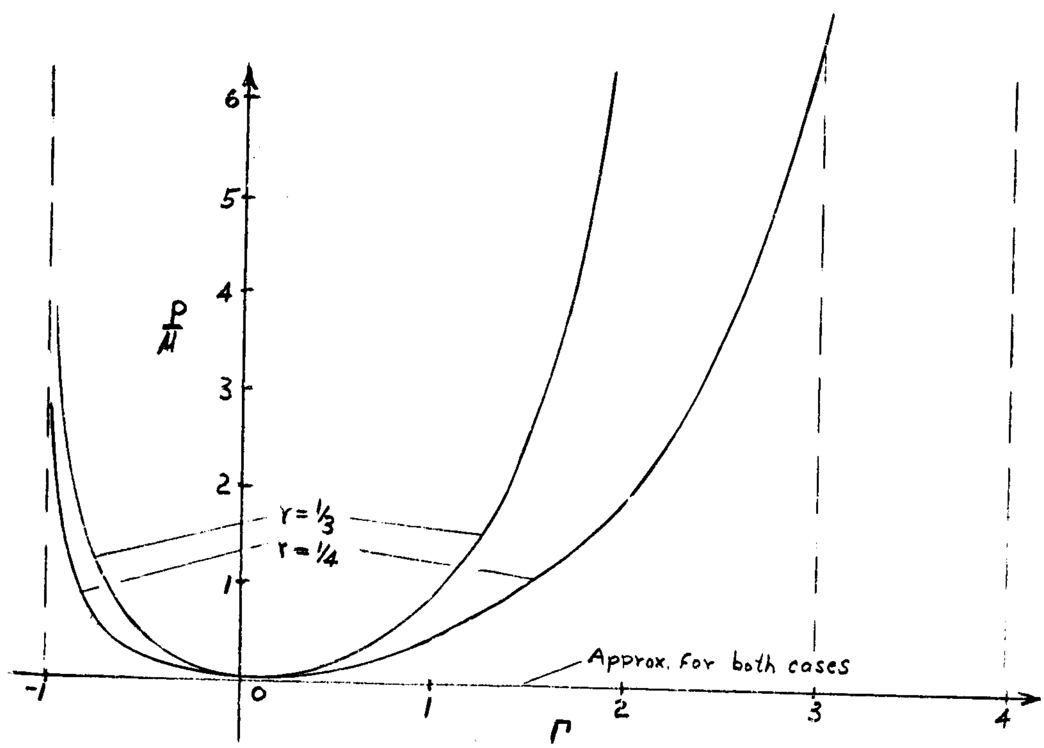
$$T = \frac{4\mu(1 + \gamma)}{3(1 - 2\gamma)(1 - \gamma\Gamma)}\left[-\frac{(3 - 4\gamma)}{(1 + \Gamma)^{1/2}} + (3 - 4\gamma - \gamma\Gamma)\right] \qquad (4.32)$$

where we have taken $T(0) = P(0) = 0$. The correlation between the approximate and exact solutions is illustrated by the figures on the next page for the same range of Γ as indicated previously.

Extension of Grade One



Tensile Stress
Figure 4.3



(-) Cross Stress
Figure 4.4

(d) Discussion of simple extensional deformation: The two materials considered in the preceeding work produce strikingly different results for simple statical extensional deformation. This difference was not completely unexpected because of the inherent character of the two forms of the constitutive equations. More specifically, we observe from equations (4.6) and (4.9) that for the grade zero material the stress rate will tend to zero as the limiting values of Γ are approached. Hence, one would expect that at these limits the stress approaches a finite value as indicated in Figures 4.1 and 4.2. On the other hand, equation (4.28) indicates that as the limiting values of Γ are approached the stress rates become infinite and one would expect asymptotic behavior as indicated by the vertical dashed lines in Figures 4.3 and 4.4. Thus, these two materials characterize two distinctly different types of material behavior and yet both are linear materials according to our usage of the term linear. We can now appreciate the generality of the rate form of the constitutive equations for materials although, until suitable experimental data are available, it is impossible to say what form of constitutive equation best characterizes any particular real material.

For the grade zero material, Figure 4.1 shows that the approximate quasi-static solution makes the material appear to be more stiff than it actually is. Also, for small deformations (up to approximately $\Gamma = 1/2$) the exact and approximate solutions agree rather well, but for larger Γ the differences become rather large. H ever, it should be remembered that when we speak of a value of Γ, say $\Gamma=1$ for example, that this is comparable to an extension ratio of 100 percent. Thus while we may speak of small values of Γ we are in effect talking about considerably significant finite deformations. Notice that the approximate solution indicates a monotonic increase in stress with increasing deformation while the exact solution apparently indicates the presence of a yield point at which the stress begins to decrease with increasing deformation, a phenomenon which is not uncommon in solids. We also see from a comparison of Figures 4.1 and 4.2 that the cross stress becomes dominant over the tensile stress for larger Γ but that for small Γ it is far less effective, an observation in complete agreement with our previous analysis.

For the grade one material, the exact solution appears to be much more sensitive to Poisson's ratio, γ, tha. for he grade zero material. With $\gamma = 1/3$, the material indicates a yield point while for $\gamma = 1/4$ no yield point is indicated. Also, for the latter value of γ, the exact and approximate solutions correspond very well up to deformations of $\Gamma = 3$, while for $\gamma = 1/3$, this correspondence is good only up to about $\Gamma = 1$. As with the grade zero material, the cross stress eventually becomes dominant over the tensile stress. For this material, however, both of these stresses tend to infinity as the limiting values of Γ are approached whereas these stresses remained finite for the grade zero material.

Perhaps the most significant feature of these results, aside from the demonstration of the great generality of this form of formulation of constitutive equations, is that the stress-strain relationship, as thought of by many engineers, is the <u>outcome</u> of the problem and is therefore predicted by the theory itself. Furthermore, such terminology as small rate of loading or long-time experiment is meaningless since time has been explicitly removed from the final result which is stated in terms of a deformation measure.

V. Uniqueness and Extremum Principles.

Using d'Alembert's Principle [12] let us write the equations of motion in the form

$$\tau_{ij,j} + \rho(f_i - \overset{\circ}{v}_i) = 0 \tag{5.1}$$

which represents the resultant effective force on a representative volume element of the media. Consider now a small, kinematically admissible, virtual velocity $\delta\underline{v}$ (not a variation) and form its scalar product with the resultant force (5.1). Then the rate of virtual work of the media for the admissible virtual velocity becomes

$$\int_V \{\tau_{ij,j} + \rho(f_i - \overset{\circ}{v}_i)\}\, \delta v_i dV = 0 \tag{5.2}$$

where the integral is taken over the instantaneous volume V characterized by the coordinates x_i and time t. By application of the divergence theorem this

$$\int_V (\tau_{ij,j}\delta v_i + \rho f_i \delta v_i) dV = \int_V \rho \overset{\circ}{v}_i \delta v_i dV$$

becomes

$$\int_S \tau_{ij} n_j \delta v_i dS + \int_V \rho f_i \delta v_i dV = \int_V \tau_{ij} \delta v_{i,j} dV + \int_V \rho \overset{\circ}{v}_i \delta v_i dV$$

or

$$\int_S \underline{T} \circ \delta\underline{v} dS + \int_V \rho\underline{f} \circ \delta\underline{v} dV =$$

$$\int_V \tau_{ij}\delta(d_{ij}) dV + \int_V \rho\overset{\circ}{\underline{v}} \circ \delta\underline{v} dV \tag{5.3}$$

The first integral on the left of (5.3) can be interpreted as the rate of virtual work done by the surface tractions acting on the bounding surface S, and the second integral is similarly the rate of virtual work of the body forces. As yet, no simple interpretation can be made of either of the integrals appearing on the right side of (5.3) except to say that the second one results from the presence of the inertia term in equation (5.1) while the first one would appear regardless of whether the inertia term were present or not.

(a) <u>A variational principle</u>: Consider the functional

$$J = \int_V \rho \underline{f} \cdot \underline{v} dV + \int_S \underline{T} \cdot \underline{v} dS - \int_V \tau_{ij} d_{ij} dV - \int_V \rho \overset{\circ}{\underline{v}} \cdot \underline{v} dV$$

or

$$J = \int_V \rho \underline{f} \cdot \underline{v} dV + \int_S \underline{T} \cdot \underline{v} dS - \int_V \tau_{ji} v_{i,j} dV - \int_V \rho \dot{\underline{v}} \cdot \underline{v} dV \quad (5.4)$$

where we have utilized the symmetry of τ_{ij} and the definition of d_{ij}. We wish to perform a variation of this functional and to avoid confusion as to what is implied by the variational operations let us rewrite (5.4) in terms of some arbitrary initial reference configuration V_o and S_o characterized by the coordinates X_i and the time t_o. Then by use of the principle of conservation of mass we have

$$J = \int_{V_o} \rho_o \underline{f} \cdot \underline{v} dV_o + \int_{S_o} \underline{T}_o \cdot \underline{v} dS_o - \int_{V_o} \pi_{Ki} \frac{\partial v_i}{\partial X_K} dV_o - \int_{V_o} \rho_o \dot{\underline{v}} \cdot \underline{v} dV_o \quad (5.5)$$

where $\underset{\sim}{T}_o$ is the unit surface traction measured per unit of the reference surface area S_o and

$$\Pi_{Ki}(X,t) = \frac{\partial X_K}{\partial x_k} \frac{\rho_o}{\rho} \tau_{ki}$$

is the non-symmetric Piola pseudostress [2] measured per unit area of the reference configuration. In this form the variation δJ becomes

$$\begin{aligned} \delta J = & \int_{V_o} \rho_o \underline{f} \cdot \delta \underline{v} dV_o + \int_{S_o} \underline{T}_o \cdot \delta \underline{v} dS_o - \int_{V_o} \Pi_{Ki} \delta(v_{i,K}) dV_o \\ & - \int_{V_o} \rho_o \overset{\circ}{\underline{v}} \, \delta \underline{v} dV_o + \int_{V_o} \rho_o \delta \underline{f} \cdot \underline{v} dV_o + \int_{S_o} \delta \underline{T}_o \cdot \underline{v} dS_o \\ & - \int_{V_o} \delta \Pi_{Ki} \frac{\partial v_i}{\partial X_K} dV_o - \int_{V_o} \rho_o \delta \overset{\circ}{\underline{v}} \cdot \underline{v} dV_o \end{aligned} \tag{5.6}$$

where the comma denotes partial differentiation with respect to the X_K coordinates. Rewriting in an obvious notation and using the divergence theorem, we have

$$\begin{aligned} \delta J = & \int_{V_o} (\rho_o f_i + \Pi_{Ki,K} - \rho_o \overset{\circ}{v}_i) \delta v_i dV_o \\ & + \int_{S_o} (T_{o_i} - \Pi_{Ki} n_{oK}) \, \delta v_i dS_o \\ & + \int_{V_o} (\rho_o \delta f_i - \rho \delta \overset{\circ}{v}_i + \delta \Pi_{Ki,K}) v_i dV_o \\ & + \int_{S_o} (\delta T_{o_i} - \delta \Pi_{Ki} n_{oK}) v_i dS_o \end{aligned} \tag{5.7}$$

In terms of the pseudostress Π_{Ki} and the variables X_i, the equations of motion are

$$\Pi_{Ki,K} + \rho_o f_i - \rho_o \overset{\circ}{v}_i = 0 \tag{5.8}$$

and on the boundary

$$T_{o_i} = \Pi_{Ki} n_{o_K} \tag{5.9}$$

where $\underset{\sim}{n}$ is the outward unit normal to the surface S_o. Hence if we impose the admissibility requirement that the <u>variations satisfy the equations of motion</u> then clearly

$$\delta J = \int_{V_o} (\rho_o f_i + \Pi_{Ki,K} - \rho_o \dot{v}_i)\delta v_i dV_o$$

$$+ \int_{S_o} (T_{oi} - \Pi_{Ki} n_{oK}) dS_o$$

Thus, if the state of stress and velocity is such that the equations of motion are satisfied, then

$$\delta J = 0$$

Hence, if we let t_o be any arbitrary time, we can formulate a variational theorem as follows: among all possible velocities $\underline{v}$ compatible with the geometrical constraints, among all the stress fields τ_{ij} which satisfy the equations of motion in the interior and on the boundary, and among all the strain rate components d_{ij}, the actual ones render the functional

$$J = \int_V \rho \underline{f} \cdot \underline{v} dV + \int_S \underline{T} \cdot \underline{v} dS - \int_V \tau_{ji} d_{ij} dV - \int_V \rho \dot{\underline{v}} \cdot \underline{v} dV \tag{5.10}$$

stationary. With this theorem we are now prepared to develop some minimum theories for quasi-static deformations.

(b) <u>A uniqueness theorem for quasi-static deformations</u>: Consider a generic stage in some process of <u>quasi-static</u> deformation for which the internal distribution of stress, the material parameters, the shape of the body, and the body forces are regarded as known. We enquire to know under what conditions the velocity field is uniquely determined when the traction rates are prescribed over part of the bounding surface S_t, geometrical constants are prescribed over the remaining surface S_v, and the body force rate is prescribed throughout V. It is assumed that these prescribed rates are functions of position and time but independent of the velocity; i.e., the changes of the load vectors are pre-ordained regardless of the ensuing change of shape or position.

Suppose that τ, d and $\underline{v}$ denote the actual stress, strain rate, and velocity field for the body so that

$$\tau_{ij,j} + \rho f_i = 0, \quad \tau_{ij} = \tau_{ji} \quad \text{in } V \qquad \text{(a)}$$

$$d_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) \quad \text{in } V \qquad \text{(b)}$$

$$\tau_{ij}n_j = T_i^* \quad \text{on } S_t \qquad \text{(c)}$$

$$v_i = v_i^* \quad \text{on } S_v \qquad \text{(d)} \qquad (5.11)$$

For the material we postulate that there exists a single-valued tensor relationship of the type

$$\text{stress rate} = f(\text{strain rate})$$

where f is homogeneous of degree one in the strain rate and may also depend explicitly on the current stress and density.

In terms of the non-symmetric Piola pseudostress, equations (5.11a) and (5.11c) become

$$\begin{aligned} \Pi_{Ki,K} + \rho_o f_i &= 0 \qquad \text{in } V_o \qquad &(a)\\ \Pi_{Ki} n_{oK} &= T^*_{oi} \qquad \text{on } S_{ot} \qquad &(b) \end{aligned} \tag{5.12}$$

where $\Pi_{Ki} = \Pi_{Ki}(X, t)$. Hence, differentiating (5.12) with respect to time and employing an obvious notation we have

$$\begin{aligned} \overset{\circ}{\Pi}_{Ki,K} + \rho_o \overset{\circ}{f}_i &= 0 \qquad \text{in } V_o \qquad &(a)\\ \overset{\circ}{\Pi}_{Kj} n_{oK} &= \overset{\circ}{T}{}^*_{oi} \qquad \text{on } S_{ot} \qquad &(b) \end{aligned} \tag{5.13}$$

for continuing equilibrium.

Suppose that there are two solutions S and S' to the stated problem. If

$$\Delta S = S - S'$$

denotes the difference of any corresponding quantities, then clearly

$$(\Delta\overset{\circ}{\Pi}_{Ki})_{,K} = 0 \quad \text{in } V_o$$

$$\Delta\overset{\circ}{T}{}^{*}_{oi} = \Delta\overset{\circ}{\Pi}_{Ki}\, n_{oK} = 0 \quad \text{on } S_{ot}$$

$$\Delta v_i^{*} = \Delta v_i = 0 \quad \text{on } S_{ov} \tag{5.14}$$

Hence, recalling that for a quasi-static process

$$\int_{S_o} \underline{T}_o \cdot \underline{v}dS_o + \int_{V_o} \rho_o \underline{f} \cdot \underline{v}dV_o = \int_{V_o} \Pi_{Ki} v_{i,K} dV_o$$

we see, by use of (5.13) and (5.14) that for the field ΔS we must have

$$\int_{V_o} \Delta\overset{\circ}{\Pi}_{Ki}(\Delta v_i)_{,K} dV_o = \int_{S_o} \Delta\overset{\circ}{\Pi}_{Ki}\Delta v_i n_{oK} dS_o = 0$$

Therefore, a sufficient condition for uniqueness is

$$\int_{V_o} \Delta\overset{\circ}{\Pi}_{Ki}(\Delta v_i)_{,K}\, dV_o > 0 \tag{5.15}$$

for all pairs of continuous differentiable velocity fields taking prescribed values on S_v; i.e., the field Δv_i vanishes on S_v. The $\Delta\overset{\circ}{\Pi}_{Ki}$ are functions of the respective velocity gradients and may not be in equilibrium nor make the traction rate vanish on S_t. The reverse inequality would also give a sufficient uniqueness condition but will not be considered here.

This sufficient, but not necessary, uniqueness criterion is similar to that obtained by Hill [14, 15] except that Hill's formulation is in terms of a nominal stress using convected coordinates (see appendix for a brief discussion of this). While convected coordinates have several manipulatory advantages, the writer feels that their use tends to obscure the physical interpretations of the quantities involved and has contributed to erroneous conclusions sometimes drawn from Hill's work.

(c) <u>A minimum principle for quasi-static deformations</u>: Our formulation will now be developed further for a class of materials for which there exists a stress rate potential such that

$$\dot{\Pi}_{Ki} = \frac{\partial \chi}{\partial\left(\frac{\partial v_i}{\partial X_K}\right)} \qquad \chi = \chi(v_{i,K}) \tag{5.16}$$

where χ is a homogeneous function of degree two in the velocity gradients, depending also on the current stress and density. χ is assumed to have continuous first derivatives and at least sectionally continuous second derivatives. Also when equations (5.16) have a unique inverse, then by means of the Legendre dual transformation there exists a function χ_c such that

$$\frac{\partial v_i}{\partial X_K} = \frac{\partial \chi_c}{\partial \dot{\Pi}_{Ki}} \qquad \text{(a)}$$

where

$$\chi_c = \dot{\Pi}_{Ki} v_{i,K} - \chi[v_{i,K}(\dot{\Pi}_{Ki})] \qquad \text{(b)}$$

or by Euler's theorem

$$\chi_c = 2\chi - \chi = \chi(\dot{\Pi}_{Ki}) \qquad \text{(c)} \tag{5.17}$$

Consider the functional

$$G\{\underline{v}\} = \int_{V_o} \chi(v_{i,K})dV_o - \int_{V_o} \rho_o \underline{\mathring{f}} \cdot \underline{v}dV_o - \int_{S_{ot}} \underline{\mathring{T}}_o^* \cdot \underline{v}dS_{ot} \quad (5.18)$$

along with equations (5.13)

Theorem 1. For any solution, unique or not, the class of continuous differentiable velocity fields satisfying (5.16) and taking prescribed values on S_v make the functional (5.18) stationary.

To prove this we see

$$\begin{aligned} \delta G &= \int_{V_o} \delta\chi dV_o - \int_{V_o} \rho_o \underline{\mathring{f}} \cdot \delta\underline{v}dV_o - \int_{S_{ot}} \underline{\mathring{T}}_o^* \cdot \delta\underline{v}dS_{ot} \\ &= \int_{V_o} \mathring{\Pi}_{Ki}\delta(v_{i,K})dV_o - \int_{V_o} \rho_o \mathring{f}_i \delta v_i dV_o - \int_{S_{ot}} \mathring{T}_{oi}^* \delta v_i dS_{ot} \\ &= \int_{S_o} \mathring{T}_{oi}\delta v_i dS_o - \int_{V_o} \mathring{\Pi}_{Ki,K}\delta v_i dV_o - \int_{V_o} \rho_o \mathring{f}_i \delta v_i dV_o - \\ &\quad \int_{S_{ot}} \mathring{T}_{oi}^* \delta v_i dS_{ot} \\ &= 0 \end{aligned}$$

since $\delta v_i = 0$ on S_{ov} and by virtue of (5.13). Hence, for this class of velocity fields we see that the first variation of $G\{\underline{v}\}$ vanishes which is necessary and sufficient for G to be stationary.

Furthermore, under the hypothesis stated for the existence of χ_c, by use of (5.17b) we can obtain a second functional

$$H\{\mathring{\Pi}_{Ki}\} = \int_{S_{ov}} \underline{\mathring{T}}_o \underline{v}^* \, dS_{ov} - \int_{V_o} \chi_c(\mathring{\Pi}_{Ki})dV_o \quad (5.19)$$

Theorem 2. The class of continuous differentiable pseudostress fields satisfying (5.13) and (5.17) make the functional (5.19) stationary.

To prove this we see

$$\begin{aligned}
\delta H &= \int_{S_{ov}} \delta \overset{\circ}{T}_{oi} v_i^* dS_{ov} - \int_{V_o} \delta\chi_c dV_o \\
&= \int_{S_{ov}} \delta T_{oi} v_i^* dS_{ov} - \int_{V_o} v_{i,K} \delta \overset{\circ}{\Pi}_{Ki} dV_o \\
&= \int_{S_{ov}} \delta T_{oi} v_i^* dS_{ov} - \int_{S_o} \delta T_{oi} v_i dS_o + \int_{V_o} \delta \Pi_{Ki,K} v_i dV_o \\
&= 0
\end{aligned}$$

by virtue of (5.14) Hence for this class of pseudostress fields the first variation of $H\{\overset{\circ}{\Pi}_{Ki}\}$ vanishes and H is therefore stationary for these fields.

Notice that the uniqueness condition (5.15) can now be written as

$$\int_{V_o} \Delta \frac{\partial\chi}{\partial(v_{i,K})} \quad (\Delta v_i)_{,K} \quad dV_o > 0 \qquad \text{(a)}$$

or

$$\int_{V_o} \Delta \frac{\partial\chi^c}{\partial\dot{\Pi}_{Ki}} \quad \Delta\dot{\Pi}_{Ki} \quad dV_o > 0 \qquad \text{(b)} \qquad (5.20)$$

In what follows, let $\overset{\circ}{\Pi}_\alpha$ and e_α, $\alpha = 1, 2, \ldots 9$, denote the nine unsymmetrical components of $\overset{\circ}{\Pi}_{Ki}$ and $v_{i,K}$, respectively. Then, recalling the assumed continuity conditions on χ, we have by the mean value theorem

$$\frac{\partial\chi(e')}{\partial e_\alpha} - \frac{\partial\chi(e)}{\partial e_\alpha} = \frac{\partial^2\chi(\bar{e})}{\partial e_\alpha \partial e_\beta}(e'_\beta - e_\beta) \tag{5.21}$$

where e'_α and e_α belong to any two admissible velocity fields and $\bar{e}_\alpha$ is on the 'join' of those two fields. Now let us assume that the uniqueness criterion (5.20a) is fulfilled and let e_α correspond to the unique solution S of the problem stated in the first paragraph of (V-b). Then obviously

$$\int_{V_o} \frac{\partial^2\chi(\bar{e})}{\partial e_\alpha \partial e_\beta} (e'_\alpha - e_\alpha)(e'_\beta - e_\beta)dV_o > 0$$

from which, by a re-application of the mean value theorem, we have

$$\int_{V_o} \left[\chi(e') - \chi(e) - \frac{\partial\chi(e)}{\partial e_\alpha}(e'_\alpha - e_\alpha)\right] dV_o > 0 \tag{5.22}$$

for permissible e'_α. Thus we have for any kinematically admissible velocity field $\underline{v}'$,

$$\begin{aligned} G\{\underline{v}'\} - G\{\underline{v}\} &= \int_{V_o} \chi(e')dV_o - \int_{V_o} \rho_o \underline{\dot{f}} \cdot \underline{v}'dV_o - \int_{S_{ot}} \underline{\overset{\circ}{T}}{}_o^* \cdot \underline{v}'dS_{ot} \\ &\quad - \int_{V_o} \chi(e)dV_o + \int_{V_o} \rho_o \underline{\dot{f}} \cdot \underline{v}dV_o - \int_{V_o} \underline{\overset{\circ}{T}}{}_o^* \cdot \underline{v}dS_{ot} \\ &= \int_{V_o} \left[\chi(e') - (e)\right] dV_o - \int_{V_o} \rho_o \underline{\dot{f}} \cdot (\underline{v}' \cdot \underline{v})dV_o \\ &\quad - \int_{S_{ot}} \overset{\circ}{T}{}_o^* \cdot (\underline{v}' - \underline{v})\, dS_{ot} \end{aligned}$$

But, recalling that the stress field associated with S satisfies (5.13) and (5.16), we have

$$G\{\underline{v}'\} - G\{\underline{v}\} = \int_{V_o}\left[\chi(e') - \chi(e) - \dot{\Pi}_{Ki}(v_i' - v_i)_{,K}\right] dV_o$$
$$+ \int_{S_o} \overset{\circ}{T}_{oi}\,(v_i' - v_i)dS_o - \int_{S_{ot}} \overset{\circ}{T}{}^*_{oi}\,(v_i' - v_i)dS_{ot}$$
$$> 0$$

which follows from (5.22) and the fact that $(v_i' - v_i) = 0$ on S_{ov}. Thus we arrive at the following theorem.

Theorem 3. Of all the kinematically admissible velocity fields, the unique field which satisfies (5.13) and (5.16) makes $G\{\underline{v}\}$ an absolute minimum.

Similarly, suppose that the uniqueness criterion (5.21b) is fulfilled. Then one has

$$\int_{V_o}\left[\chi_c(\overset{\circ}{\Pi}') - \chi_c(\overset{\circ}{\Pi}) - \frac{\partial\chi_c(\overset{\circ}{\Pi})}{\partial\dot{\Pi}_\alpha}(\overset{\circ}{\Pi}'_\alpha - \dot{\Pi}_\alpha)\right] dV_o > 0 \qquad (5.23)$$

where $\overset{\circ}{\Pi}_\alpha$ corresponds to the unique solution of the problem. Then if $\overset{\circ}{\Pi}'_\alpha$ is any other statically admissible stress field satisfying (5.13) we have as a consequence of (5.23) and (5.17)

$$H\{\dot{\Pi}'_{Ki}\} - H\{\dot{\Pi}_{Ki}\} = \int_{S_{ov}} \underline{\overset{\circ}{T}}_o{}' \cdot \underline{v}^*\, dS_{ov} - \int_{V_o} \chi_c(\overset{\circ}{\Pi}')\, dV_o$$
$$- \int_{S_{ov}} \underline{\overset{\circ}{T}}_o \cdot \underline{v}^*\, dS_{ov} + \int_{V_o} \chi_c(\dot{\Pi})dV_o$$
$$= \int_{S_{ov}} (\underline{\overset{\circ}{T}}_o{}' - \underline{\overset{\circ}{T}}_o) \cdot \underline{v}^*\, dS_{ov} - \int_{V_o}\left[\chi_c(\dot{\Pi}') - \chi_c(\dot{\Pi})\right] dV_o$$

$$< \int_{S_{ov}} (\overset{\circ}{\underline{T}}_o{}' - \underline{T}_o) \cdot \underline{v}^* \, dS_{ov} - \int_{V_0} v_{i,K} (\overset{\circ}{\Pi}'_{Ki} - \overset{\circ}{\Pi}_{Ki}) dV_o$$

$$< \quad 0$$

where we have used the fact that $(\Pi'_{Ki} - \Pi_{Ki})$ satisfies (5.14) and $\underline{v} = \underline{v}^*$ on S_{ov}. Thus we arrive at the following theorem.

Theorem 4. Of all the statically admissible pseudostress fields, the unique one which satisfies (5.17) and the kinematic conditions makes $H\{\overset{\circ}{\Pi}_{Ki}\}$ an absolute maximum.

Notice that theorems 3 and 4 are analogous to the minimum theories of elastostatics (see Chapter I). Also, in view of the fact that if both χ and χ_c exist and both (5.20a) and (5.20b) are satisfied simultaneously, then H and G are identical and we have succeeded in bounding this quantity from above and below.

(d) Application to linear preferred media: Of particular interest to us is the form which the uniqueness and minimum theories assume when the material has a linear homogeneous constitutive equation of the form

$$\overset{\circ}{\Pi}_{Ki} = C_{KiLj} v_{j,L} \tag{5.24}$$

where the coefficients may depend on the current stress and density. With this constitutve equation the stress potential is

$$\chi = \frac{1}{2} C_{KiLj} v_{i,K} v_{j,L} \tag{5.25}$$

with the restriction that

$$C_{KiLj} = C_{LjKi} \tag{5.26}$$

Also, if (5.24) is required to have an inverse, then we have

$$v_{i,K} = D_{KiLj}\dot{\Pi}_{Lj} \tag{5.27}$$

and

$$\chi_c = \frac{1}{2} D_{KiLj}\overset{\circ}{\Pi}_{Ki}\overset{\circ}{\Pi}_{Lj} \tag{5.28}$$

with the restrictions

$$D_{KiLj} = D_{LjKi}$$

$$D_{LjKi}C_{KiMr} = C_{LjKi}D_{KiMr} = \delta_{LM}\delta_{jr} \tag{5.29}$$

In view of (5.24) and (5.27), the differences between any two stress rate and strain rate fields are related by

$$\Delta\overset{\circ}{\Pi}_{Ki} = C_{KiLj}\Delta v_{j,L}$$

$$\Delta v_{i,K} = D_{KiLj}\Delta\overset{\circ}{\Pi}_{Lj}$$

Hence, if there are two or more distinct solutions to the general boundary value problem, we must have

$$\int_{V_o} \chi(\Delta v_{i,K})dV_o = 0$$

or

$$\int_{V_o} \chi_c(\Delta\overset{\circ}{\Pi}_{Ki})dV_o = 0$$

according to the work preceeding equation (5.15). Consequently by virtue of (5.20a) and (5.20b), a sufficient condition for uniqueness is either

$$\int_{V_o} \chi(v_{i,K})dV_o > 0 \tag{5.30a}$$

for <u>all</u> velocity fields with the analytic properties of the actual field and vanishing on S_{oV} but not identically zero, or

$$\int_{V_o} \chi_c(\overset{\circ}{\Pi}_{K,i})dV_o > 0 \tag{5.30b}$$

for all continuous stress-rate fields in equilibrium with zero body force in V_o and zero traction-rate on S_{ot}, but not identically zero.

It is important to note that unless χ is strictly convex [15], the uniqueness criteria (15.20a) and (15.20b) differ and need not be satisfied, simultaneously. By strictly convex we mean

$$\chi(e') - \chi(e) - \frac{\partial\chi(e)}{\partial e_\alpha}(e'_\alpha - e_\alpha) > 0 \tag{5.31}$$

for all admissible e'_α not identical with e_α. Then, assuming (5.31) to hold, we have from (5.17b)

$$\chi_c(\dot{\Pi}') - \chi_c(\dot{\Pi}) = \dot{\Pi}'_c e'_\alpha - \chi(e') - \dot{\Pi}_\alpha e_\alpha + \chi(e)$$

$$> \overset{\circ}{\Pi}'_\alpha e'_\alpha - \overset{\circ}{\Pi}_\alpha e_\alpha + \overset{\circ}{\Pi}'_\alpha (e_\alpha - e'_\alpha)$$

or

$$\chi_c(\mathring{\Pi}') - \chi_c(\dot{\Pi}) - \frac{\partial \chi\ (\mathring{\Pi})}{\partial \mathring{\Pi}_\alpha} (\mathring{\Pi}'_\alpha - \dot{\Pi}_\alpha) > 0 \tag{5.32}$$

Hence, (5.31) implies (5.32) which, in turn, imply (5.22) and (5.23) respectively. Thus under this condition the uniqueness criteria (5.20a) and (5.20b) hold simultaneously.

In particular, for linear materials the requirement of convexity is tantamount to the positive definiteness of the coefficient matrix (C_{KiLj}) and the inverse coefficient matrix (D_{KiLj}).

Finally, while we have a formulation of χ, χ_c and the uniqueness criteria for a linear material with the form (5.24), it is important that we interpret these results in terms of the true stress τ and a true stress-rate form of the constitutive equation. To do this, recall

$$\Pi_{Kj}(X, t) = \frac{\partial X_K}{\partial x_i} \frac{\rho_o}{\rho} \tau_{ij}(x, t)$$

Thus, differentiating both sides with respect to time

$$\mathring{\Pi}_{Kj} = (\frac{\partial \dot{X}_K}{\partial x_i}) \frac{\rho_o}{\rho} \tau_{ij} + \frac{\partial X_K}{\partial x_i} (\frac{\mathring{\rho}_o}{\rho}) \tau_{ij} + \frac{\partial X_K}{\partial x_i} \frac{\rho_o}{\rho} \mathring{\tau}_{ij}$$

$$= -\frac{\partial X_K}{\partial x_\ell} v_{\ell,i} \frac{\rho_o}{\rho}\tau_{ij} + \frac{\partial X_K}{\partial x_i} \frac{\rho_o}{\rho} v_{m,m}\tau_{ij} + \frac{\partial X_K}{\partial x_i} \frac{\rho_o}{\rho} \mathring{\tau}_{ij} \tag{5.33}$$

Now, if we let the arbitrary referenct time t_o correspond to the instantaneous time t we have

$$\dot{\Pi}_{Ki} = -\delta_{K\ell} v_{\ell,i} \tau_{ij} + \delta_{Ki} v_{m,m} \tau_{ij} + \delta_{Ki} \dot{\tau}_{ij} \qquad (5.34)$$

Because of the appearance of the term containing $v_{m,m}$, let us choose Truesdell's form for the stress-rate equations.

$$\breve{\tau}_{ij} = \mathring{\tau}_{ij} + \tau_{ij} v_{m,m} - \tau_{ip} v_{j,p} - \tau_{pj} v_{i,p}$$

$$= A_{ijk\ell}(\rho, \tau) d_{k\ell}$$

$$= A_{ijk\ell} v_{\ell,k}$$

$$A_{ijk\ell} = A_{jik\ell} = A_{ij\ell k}$$

or

$$\mathring{\tau}_{ij} = B_{ijk\ell} v_{\ell,k} \qquad (5.35)$$

where

$$B_{ijk\ell} = \tau_{ik}\delta_{j\ell} + \tau_{jk}\delta_{i\ell} - \tau_{ij}\delta_{\ell k} + A_{ijk\ell} \qquad (5.36)$$

Substituting (5.35) and (5.36) into (5.34) yields

$$\dot{\Pi}_{Kj} = \delta_{Ki} \tau_{i\ell} \gamma_{j,\ell} + A_{ijk\ell} \delta_{Ki} v_{\ell,k} \qquad (5.37)$$

But, from (5.24), at $t = t_o$

$$\mathring{\Pi}_{Kj} = C_{KjLi} v_{i,L} = C_{KjLi} v_{i,\ell} \delta_{\ell L} \qquad (5.38)$$

Thus from (5.37) and (5.38) we have

$$C_{KjLi}\delta_{L\ell} = A_{kj\ell i}\delta_{Kk} + \tau_{k\ell}\delta_{ij}\delta_{Kk}$$

or, dropping the unnecessary distinction in the subscripts

$$C_{kj\ell i} = A_{kj\ell i} + \tau_{k\ell}\delta_{ij} \tag{5.39}$$

which clearly satisfies the required symmetry in kj, ℓi provided $A_{kj\ell i}$ does. Hence, for any instant t, we have now formulated the form of χ, and hence χ_c, in terms of the coefficients $A_{ijk\ell}$. As a result, we have also arrived at an instantaneous formulation of the uniqueness and extremum principles. Thus we have succeeded in bounding the functionals G and H of theorems 3 and 4, for any particular instant of the motion. In effect we have bounded the integral of the form

$$\int_V C_{ijk\ell}v_{j,i}v_{\ell,k}\,dV \tag{5.40}$$

where $C_{ijk\ell}$ are functions of the instantaneous stress and density.

VI. Comments and Conclusions

As stated in Chapter I of this paper, the object of this investigation was to identify and resolve some of the individual problems related to the overall problem of multi-phase media with particular attention devoted to linear preferred media. We will now review the results of this investigation in relation to the stated objectives.

First of all, let it be understood that the formulation of constitutive equations in terms of stress rates and strain rates is an attempt to explain or predict the dynamical response of deformable media to various loading conditions and kinematic restraints; i.e., resolution of the first and second fundamental boundary value problems. The formulation is exact and dynamically admissible and is valid for all types of media whether they be solid, fluid, plastic, etc., and, in fact, such special classification of behavior can be predicted from the solution of the equations governing the deformation process. Furthermore, the reference configuration can be selected arbitrarily so long as the compatibility and boundary conditions are satisfied. Of course, many types of material behavior are not included in this formulation, most notably the viscoelastic time-dependent type of material response.

Unfortunately, the general form of the constitutive equations,

$$\text{Stress rate} = f(\text{Strain rate})$$

where f is linear and homogeneous in the strain rate and may also depend on the current stress and density, is much too general to indicate clearly the multitude of possible mechanical and thermodynamic responses that can be exhibited by materials with this characterization. Hence it was felt that a more definite characterization was necessary to provide a better understanding of such materials. We have attempted to do this in two ways: by consideration in Chapter III of the existence of a deformation energy density function; by consideration in Chapter IV of the response to simple extensional deformations. In both cases, two relatively simple forms of the constitutive equations were used as simple illustrations of what kind of results can be expected from these types of materials by either explicitly including or excluding the density in the constitutive equations.

Under the assumption of the existence of an energy of deformation density function it was found that for the two special materials considered, the energy density could be expressed as a scalar function of the stress invariants in the one case and as a scalar function of the density and stress invariants in the other. That these functions were remarkably similar in form to the elastic energy density for isotropic materials leads one to conjecture that linear preferred materials are elastic and vice versa. A little reflection would soon indicate that this surmise is clearly incorrect. Indeed, Bernstein [15] has shown that isotropic Cauchy elastic materials are hypoelastic but, very definitely, that all hypoelastic materials are not elastic, nor are all elastic materials hypoelastic.

It must be remembered that our derivation of the deformation energy density function placed no restrictions on the amount of deformation or stress. Large deformation of real materials generally is accompanied by, or results from, the irreversible process of material flow. Hence it is highly unlikely that all the work energy supplied during the deformation process is available for full recovery or is independent of the path as would be the case for materials which possess an energy of deformation function such as those discussed in Chapter 3. However, it is possible for materials to possess such an energy density function for values of stresses or deformations within the neighborhood of a given configuration pair. Or, one might consider that just the <u>reversible</u> part of the deformation energy is expressible in terms of an energy density function. In any case, the existence of an energy of deformation function should not be assumed <u>a priori</u> for real materials undergoing large deformations, and it is highly unlikely that the constitutive equations of media undergoing large flow-like deformations can be properly formulated without thermodynamic considerations. Our results here appear to indicate that real media undergoing <u>small but finite</u> deformations might well be characterized by linear rate-type constitutive equations along with the possible existence of an energy of deformation function.

Of course, the tried and true way of classifying or identifying types of real material behavior is by laboratory experiment. With this in mind the author investigated the simple extensional deformation process for the two simple representative materials chosen. These results were enlightening for several reasons.

First of all, the results clearly indicated that for such linear materials tensile (compressive) stresses alone may not be sufficient to produce homogeneous statical simple extension (compression). Or, conversely, the application of tensile (compressive) stresses alone will generally not produce a uniform homogeneous deformation. This conclusion is not too startling when one considers such readily observed deformation processes as necking. Secondly, the results definitely indicated the existence of yield points and multiple-valued stress=strain relations, both of which can be readily observed in real materials.

Thus we see that this formulation of material response is at least in qualitative agreement with observed phenomenological behavior. The significant feature, however, is that these phenomena are predicted by the theory and do not have to be assumed a priori, as is usually done in most elasticity and plasticity theories. The question of whether or not this formulation is in quantitative agreement with real material behavior remains open because of an acute lack of quantitative experimental evidence. In this regard it may be said that one of the most sparsely inhabited areas of experimental mechanics is that of field analysis of large deformations. It is hoped that as the experimental needs become more clearly defined this lack of experimental endeavor will disappear. In this respect, the author would like to suggest that the problems of simple shear and hydrostatic loading receive their share of experimental and analytical attention.

Insofar as extremum and uniqueness theorems are concerned, several points are worth mentioning. Our uniqueness theorem has been formulated in terms of stress rates and velocity fields. The question might naturally arise as to the uniqueness of the stress and displacement fields. In this regard, one has recourse to the lack of a general uniqueness theorem for partial differential equations. Thus, the uniqueness theorems presented in this paper represent a major contribution to the proper formulation of well set boundary value problems for this theory of material response. The question of how severe are the restrictions which must be imposed on the constitutive equations in order for the theorems to be valid can only be answered by application of these theorems to specific porblems. The same statement applies to the extremum principles and the author hopes to answer some of these questions in a subsequent application of these theorems to the multi-phase material problem.

APPENDIX

(a) Time derivatives of tensor fields: In order to fix our ideas in relation to the terminology which has been used in this paper, consider the following definitions.[4]

X . . . fixed spatial reference frame

x^i . . deformed spatial curvilinear coordinates for any arbitrary time t, referred to the X frame; i.e., Eulerian coordinates.

X^i . . spatial curvilinear coordinates referred to X frame for some initial reference time t_o and may thus be thought of as material or Lagrangian coordinates.

The motion of the media is specified by the three functions $x^i = x^i(X^i, t)$, everything being measured relative to the X frame.

Now, define a second frame

ξ. . . . . . An arbitrary curvilinear moving frame of reference

$\xi^\alpha(x^i,t)$ . . an arbitrary curvilinear coordinate

---

4. Throughout this appendix generalized tensor notation is employed as distinguished from Cartesian tensor notation used in the body of this paper. The transformation from the former to the latter is immediate.

system moving and possibly deforming in some continuous manner with velocity $V^i(x^i, t)$, rotation $\Omega_{ij}(x^i, t)$, and the rate of deformation $D_{ij}(x^i, t)$. In effect we are considering the ξ frame to be moving and deforming like a continuous media.

For any time t, the components of any <u>absolute</u> tensor field T with reference to the ξ frame may be determined from the components of T with reference to the X frame by the usual laws of tensor transformation, the transformation from fixed to moving system being different at different times. Then Oldroyd [16, 17] has shown that the time derivative of this tensor field in the ξ frame holding the ξ^α coordinates constant has the components in the X frame given by:

$$\frac{dT^{i\,\cdots}_{j\,\cdots}(x^i, t)}{dt} = \frac{\partial T^{i\,\cdots}_{j\,\cdots}}{\partial t} + V^m T^{i\,\cdots}_{j\,\cdots,m}$$

$$- \Sigma\Omega^{\cdot m}_{j}\, T^{i\,\cdots}_{m\,\cdots} - \Sigma\Omega^{i}_{\cdot m}\, T^{m\,\cdots}_{j\,\cdots}$$

$$+ \Sigma D^{m}_{j}\, T^{i\,\cdots}_{m\,\cdots} - \Sigma D^{i}_{m}\, T^{m\,\cdots}_{j\,\cdots} \qquad \text{(A-1)}$$

where the usual summation convention applies, a comma followed by a latin index denotes covariant differentiation with respect to the x^i coordinate, and the summation signs denote the summation of all similar terms, one for each covariant (contravariant) index.

In particular, let us look at a few special choices of the ξ^i coordinates.

(i) Let $\xi^{\alpha}(x^i, t)$ be rigid and stationary. Then (A-1) becomes

$$\frac{\partial T^{i\cdots}_{j\cdots}(x^i, t)}{\partial t}$$

Time rate of change at a fixed spatial position (a fixed observer).

(ii) Let $\xi^{\alpha}(x^i, t)$ be a rigid system moving with no rotation, but with a velocity at x^i and time t equal to the velocity v^i of the material at that point. Then (A-1) becomes

$$\frac{\partial T^{i\cdots}_{j\cdots}}{\partial t} + v^m T^{i\cdots}_{j\cdots,m} = \frac{DT^{i\cdots}_{j\cdots}}{Dt} = \dot{T}^{i\cdots}_{j\cdots}$$

Intrinsic or material time derivative (an observer moves translationally with the material.)

(iii) Let $\xi^{\alpha}(x^i, t)$ be a rigid system moving with the rotation ω_{ij} and translational velocity v^i at x^i and time t equal to that of the material. Then (A-1) becomes

$$\frac{\partial T^{i\cdots}_{j\cdots}}{\partial t} + v^m T^{i\cdots}_{j\cdots,m} - \Sigma\omega_{j}^{\,m} T^{i\cdots}_{m\cdots}$$

$$- \Sigma\omega^{i}_{\,m} T^{m\cdots}_{j\cdots} = \hat{T}^{i\cdots}_{j\cdots}$$

Absolute time derivative (an

observer translates and rotates with the material).

(iv) Let $\xi^{\alpha}(x^i, t)$ be imbedded in the media and thus it moves and deforms with the material with a rate of deformation tensor of d_{ij}. Then (A-1) becomes

$$\frac{\partial T^{i\,\cdots}_{j\,\cdots}}{\partial t} + v^m T^{i\,\cdots}_{j\,\cdots,m} - \Sigma\omega^{\cdot m}_{j} T^{i\,\cdots}_{m\,\cdots}$$

$$- \Sigma\omega^{i}_{\cdot m} T^{m\,\cdots}_{j\,\cdots} + \Sigma d^{m}_{j} T^{i\,\cdots}_{m\,\cdots} - \Sigma d^{i}_{m} T^{m\,\cdots}_{j\,\cdots} = \tilde{T}^{i\,\cdots}_{j\,\cdots}$$

Convected time derivative (the observer is the material itself).

Let us note for future use that the convected derivative can also be written in the form

$$\tilde{T}^{i\,\cdots}_{j\,\cdots} = \overset{\circ}{T}{}^{i\,\cdots}_{j\,\cdots} + \Sigma v^{m}_{\ ,j} T^{i\,\cdots}_{m\,\cdots} - \Sigma v^{i}_{\ ,m} T^{r\,\cdots}_{j\,\cdots} \qquad \text{(A-2)}$$

which is easily obtained by recalling the definitions of ω_{ij} and d_{ij}.

(b) <u>Time derivative of the traction vector</u>: An important result which is apparently overlooked in much of the recent literature in continuum mechanics is the fact that the convected derivative and the intrinsic derivative of an absolute vector invariant are identical. For example, consider the traction vector

$$d\underline{F} = \underline{t}dS$$

where $\underline{t}$ is the stress vector. In convected coordinates this becomes

$$d\underline{F} = \sigma^{\alpha\beta} n_{\alpha} dS \underline{g}_{\beta} = d\underline{F}\,(\theta\ ,\ t) \qquad \text{(A-3)}$$

where the greek indices refer to tensor elements relative to the convected coordinates θ^{α}. Similarly, in fixed coordinates x^{i} we have

$$d\underline{F} = \sigma^{ij} n_{i} dS \underline{g}_{j} = d\underline{F}\,(x,\ t) \qquad \text{(A-4)}$$

In addition, we have the relations

$$x^{i} = x^{i}(\theta,\ t) \quad \text{or} \quad \theta^{\alpha} = \theta^{\alpha}(x,\ t)$$

$$\frac{\partial x^{i}}{\partial t} = v^{i}(\theta,\ t) \qquad \text{(A-5)}$$

Now, taking the partial derivative of $d\underline{F}$ with respect to time while holding the convected coordinates θ^{α} constant, we obtain

$$\frac{\partial}{\partial t}\,[d\underline{F}(\theta,\ t)] = \frac{\partial}{\partial t}\,\{d\underline{F}[x(\theta,\ t),t]\}$$

$$= \frac{\partial d\underline{F}}{\partial t} + \frac{\partial \underline{F}}{\partial x^{i}}\, v^{i}$$

Hence, if at time t the θ^{α} and x^{i} coordinates coincide, by their own definitions the left side is the convected time derivative while the right side is the intrinsic derivative and thus

$$(d\tilde{\underline{F}}) = (d\mathring{\underline{F}}) \qquad \text{(A-6)}$$

To see this more clearly as well as its implications, let us perform the corresponding operations on (A-3) and (A-4), respectively.

$$\frac{\partial[d\underline{F}(\theta, t)]}{\partial t} = \frac{\partial\sigma^{\alpha\beta}}{\partial t} n_\alpha dS \underline{g}_\beta + \sigma^{\alpha\beta}\frac{\partial}{\partial t}(n_\alpha dS)\underline{g}_\beta$$

$$+ \sigma^{\alpha\beta} n_\alpha dS \frac{\partial}{\partial t}(\underline{g}_\beta)$$

$$= \left\{\frac{\partial\sigma^{\alpha\beta}}{\partial t} + \sigma^{\alpha\beta} v^{\eta}{}_{,\eta} + \sigma^{\alpha\eta} v^{\beta}{}_{,\eta}\right\} n_\alpha dS \underline{g}_\beta \qquad \text{(A-7)}$$

and

$$[d\overset{\circ}{\underline{F}}(x, t)] = \dot{\sigma}^{ij} n_i dS \underline{g}_j + \tau^{ij}(n_i \dot{dS})\underline{g}_j$$

$$= \left\{\dot{\sigma}^{ij} + \sigma^{ij} v^{m}{}_{,m} - \sigma^{mj} v^{i}{}_{,m}\right\} n_i dS \underline{g}_j \qquad \text{(A-8)}$$

since $\overset{\circ}{\underline{g}}_j = 0$. Now, if at time t we choose the convected coordinates such that $\theta^\alpha = \delta^\alpha_i x^i$ everywhere, then $\underline{g}_\alpha = \delta^i_\alpha \underline{g}_i$ and $\sigma^{\alpha\beta} = \delta^\alpha_i \delta^\beta_j \sigma^{ij}$. Further

$$\frac{\partial(\sigma^{\alpha\beta})}{\partial t} \rightarrow (\tilde{\sigma}^{ij}) = \overset{\circ}{\sigma}^{ij} - v^{i}{}_{,m}\sigma^{mj} - v^{j}{}_{,m}\sigma^{im} \qquad \text{(A-9)}$$

by the definition of the convected derivative and equation (A-2).

Also

$$\sigma^{\alpha\beta} v^{\eta}{}_{,\eta} \rightarrow \sigma^{ij} v^{m}{}_{,m} \qquad \text{(A-10)}$$

$$\sigma^{\alpha\eta} v^{\beta}{}_{,\eta} \rightarrow \sigma^{im} v^{j}{}_{,m} \qquad \text{(A-11)}$$

Substitution of (A-9), (A-10), (A-11) into (A-7) results in an expression identical with (A-8) which once again proves the identity of the convected and intrinsic derivatives of the traction vector.

Similarly we may choose to formulate the traction vector in terms of the two-point [21] Kirchhoff stress tensor

$$
\begin{aligned}
d\underline{F} &= \tau^{\alpha\beta} n_{o\alpha} dS_o \underline{g}_\beta \\
&= \tau^{K\ell} n_{oK} dS_o \underline{g}_\ell
\end{aligned}
$$

where the subscript $_o$ denotes some initial time at which $x^k(\theta, t_o) = X^K = \delta^K_\alpha \theta_o^\alpha$. That is, at time t_o the X^K and θ_o^α coordinates coincide.

Performing the respective differentiation operations, we have

$$
\frac{\partial[d\underline{F}(\theta, t)]}{\partial t} = \frac{\partial \tau^{\alpha\beta}}{\partial t} n_{o\alpha} dS_o \underline{g}_\beta + \tau^{\alpha\beta} n_{o\alpha} dS_o \frac{\partial}{\partial t}(\underline{g}_\beta)
$$

$$
= \left\{ \frac{\partial \tau^{\alpha\beta}}{\partial t} + \tau^{\alpha\eta} v^\beta{}_{,\eta} \right\} n_{o\alpha} dS_o \underline{g}_\beta \qquad \text{(A-12)}
$$

$$
[d\dot{\underline{F}}(x, t)] = \overset{\circ}{\tau}{}^{K\ell} n_{oK} dS_o \underline{g}_\ell \qquad \text{(A-13)}
$$

Now, noting the two-point character of the Kirchhoff stress tensor

$$
\tau^{\alpha\beta} = \frac{\partial \theta_o^\alpha}{\partial X_o^K} \frac{\partial \theta^\beta}{\partial x^\ell} \tau^{K\ell} = \delta^\alpha_K \frac{\partial \theta^\beta}{\partial x^\ell} \tau^{K\ell}
$$

we see from equation (A-2) that at time t

$$
\bar{\tau}^{K\ell} = \overset{\circ}{\tau}{}^{K\ell} - v^\ell{}_{,m} \tau^{Km} \qquad \text{(A-14)}
$$

Then if we let the time t correspond to the arbitrary reference time t_o, (A-12) and (A-13) become

$$\frac{\partial[d\underline{F}(\theta, t)]}{\partial t} \rightarrow (d\tilde{\underline{F}}) = \left\{\tilde{\tau}^{K\ell} + v^{\ell}{}_{,m}\tau^{Km}\right\} n_K dS \underline{g}_{\ell} \qquad \text{(A-15)}$$

$$(d\overset{\circ}{\underline{F}}) = \overset{\circ}{\tau}{}^{K\ell} n_K dS \underline{g}_{\ell} \qquad \text{(A-16)}$$

which, by virtue of (A-14), are seen to be identical.

Finally, by a similar application of this same procedure, we can show that if $d\underline{F}$ is expressed in terms of a nominal stress

$$d\underline{F} = S^{\alpha\beta} n_{c\alpha} dS_o \underline{g}_{o\beta}$$

$$= S^{MN} n_{o\overset{\circ}{M}} dS_o \underline{\ell}_N$$

Then

$$(d\tilde{\underline{F}}) = (d\overset{\circ}{\underline{F}}) = \tilde{S}^{MN} n_M dS \underline{g}_N$$

$$= \overset{\circ}{S}{}^{MN} n_M dS \underline{g}_N \qquad \text{(A-17)}$$

for any particular instant $t = t_o$.

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