AN APPROXIMATE METHOD FOR SOLVING THE SINGLE-DEGREE-OF-FREEDOM ROLL EQUATION WITH TIME-DEPENDENT COEFFICIENTS

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SUMMARY

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An approximate analytical method (with three alternate integration procedures) is developed for solving the single-degree-of-freedom roll equation with time-dependent coefficients. The method is applied with each integration procedure to two sample problems and shows good agreement with more exact numerical solutions. Information governing the approximate error of each integration procedure is also presented.

Author

INTRODUCTION

One method often used to improve the accuracy of predicting the flight trajectories of rocket vehicle systems has been that of spinning the vehicle slowly very soon after launching. This tends to average out the effects of any asymmetrical vehicle loads on the trajectory. In other instances a vehicle may be designed to spin at higher rates for the purpose of spin stabilization. In either case the spin is usually provided in flight by means of deflected control surfaces, canted or twisted fins, or with small spinner rockets or gas jets.

During the preliminary design of these vehicles, the fin cant (if this is the method chosen to produce the rolling moments) must be determined for each stage to produce the desired spin rates. This computation involves solutions of a first-order differential equation having time-variable coefficients. These solutions are carried out with a step-by-step numerical integration procedure usually with the aid of a high-speed digital computer.

The present paper presents an approximate analytical method (with three alternate integration procedures) for solving the single-degree-of-freedom roll equation with typically varying coefficients. The use of this method avoids the tedious step-by-step method of solution and allows quick and accurate hand computation of the results. The method is applied to two actual flight vehicle situations and the results compared with more exact solution determined numerically.[1]

In this paper, the approximate method is applied only to the roll equation. However, the method can be (and has been) applied successfully to other physical problems.

SYMBOLS

The units used for the physical quantities defined in this paper are given in both the U.S. Customary Units and the International System of Units (SI). Factors relating the two systems are given in reference 1.

A,a,D — constants for simulating roll damping terms, second^{-1} (see eq. (4))

$A_1 = Ae^{at} + D$

$\bar{A}_1 = Ae^{a\bar{t}} + D$

B,b,C,c — constants for simulating roll driving terms, second^{-1} (see eq. (4))

C_1,C_2 — proportionality factors, seconds per radian and radians, respectively

E_1,E_2 — complex Fresnel integrals defined by equations (9a)

I_X — inertia of vehicle about spin axis, slug-foot^2 (kilogram-meter^2)

k_1,k_2,k_3 — constants defined by equations (15a) to (15c)

$M_{X,p}$ — roll damping moment per unit p, foot-pound-seconds per radian (joule-seconds per radian)

$M_{X,\delta}$ — aerodynamic roll driving moment per unit δ, foot-pounds per degree (joules per degree)

M_j — jet damping moment coefficient, foot-pound-seconds per radian (joule-seconds per radian)

M_X — time-dependent roll input moment, foot-pounds (joules)

p — vehicle spin rate, radians per second

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[1]The more exact numerical method uses a fourth-order Runge-Kutta procedure with extrapolation to zero interval size as a correction factor and with tabular look-ups for the coefficients.

Δp	approximate error in spin rate, radians per second
$\bar{p}$	spin rate at midinterval time, radians per second
p_o	spin rate at zero interval time, radians per second (In multistep problems, this quantity is final spin rate of previous interval.)
p_{ss}	steady-state spin rate, radians per second
t	interval time, seconds ($t = 0$ at beginning of each interval)
$\bar{t}$	midinterval time, seconds
u	independent variable used in complex Fresnel integral
y	transformation variable defined by equation (5)
δ	differential deflection of control surfaces or fins producing rolling moment, degrees
Δe^{at}	error in e^{at} (see eq. (13))

Subscript:

o	zero interval time

A dot over a symbol indicates the first derivative of the quantity with respect to time.

ANALYSIS

The single degree-of-freedom rolling-moment equation can be written as follows:

$$I_X\dot{p} + M_{X,p}p = M_{X,\delta}\delta + M_X + M_jp \qquad (1)$$

$I_X\dot{p}$	$M_{X,p}p$	$M_{X,\delta}\delta$	M_X	M_jp
Inertial acceleration moment	Aerodynamic damping moment	Aerodynamic driving moment	Auxiliary driving moment (time dependent)	Jet damping moment

The aerodynamic driving moment can be generated by deflected control surfaces, canted fins, or any asymmetric rolling-moment condition dependent upon aerodynamics. The M_X term was included for time-dependent rolling moments such as would be obtained with small spinner rockets or gas jets. The jet damping moment, included for completeness, is rather hard to predetermine and is highly dependent upon the configuration of the rocket motor propellant. In the following analysis, the jet damping term is considered absorbed within the aerodynamic damping term. For conditions where the coefficients in equation (1) are constants, the solution can be written as

$$p = \left(p_o - p_{ss}\right)e^{\frac{-M_{X,p}t}{I_X}} + p_{ss} \tag{2}$$

where p_o is the initial spin rate and the steady-state spin rate is $p_{ss} = \dfrac{M_{X,\delta}\delta + M_X}{M_{X,p}}$.

Of primary interest in this paper is the condition in which the coefficients of equation (1), normalized with respect to I_X, are not constant. The normalized equation to be solved is

$$\dot{p} + \frac{M_{X,p}}{I_X} p = \frac{M_{X,\delta}\delta}{I_X} + \frac{M_X}{I_X} \tag{3}$$

In this equation, I_X can change continuously during a thrust period. The terms $M_{X,p}$ and $M_{X,\delta}\delta$ are both functions of air density, vehicle velocity, and Mach number which change throughout a flight; also, M_X/I_X must be given the capability of varying with time. The general form of equation (3), chosen from practical considerations to simulate these possible variations, is

$$\dot{p} + p\left(D + Ae^{at}\right) = Be^{bt} + Ce^{ct} \tag{4}$$

The method for solving equation (3), then, consists of fitting actual problem histories of the normalized driving and damping moments to the constant and/or exponential coefficients of equation (4) which can be integrated for spin rate.

It is not implied that $M_{X,\delta}\delta/I_X$ is simulated by Be^{bt} and M_X/I_X by Ce^{ct}. Rather, it is suggested for reasons of flexibility that the sum of the input terms be simulated by the sum of the two exponentials. Since the two input terms on the right of equation (4) are of identical form, their particular solutions will be of the same form. Thus, for purposes of solving equation (4), only the B term will be considered. The response of the C term will be added to this solution by the principle of superposition.

Three procedures are presented for integrating equation (4). These procedures referred to as the asymptotic method, the tabular method, and the mid-damping method are discussed in the following sections. With all three methods, it is sometimes necessary to divide total problem time into two or more steps or time intervals as explained in the section entitled "Limitations and Applications." The method to be used is then applied once in each time interval, the final value of spin rate in an interval being used as the initial value of spin rate for the next interval.

Asymptotic Method

In the asymptotic method, the dependent variable is first transformed as follows. If

$$y = pe^{\frac{A}{a}e^{at}+Dt} \tag{5}$$

equation (4) (neglecting the C term) becomes

$$\dot{y} = Be^{bt+\frac{A}{a}e^{at}+Dt} \tag{6}$$

This equation can be integrated repeatedly by parts in two different ways. Each result is in the form of an asymptotic series which transforms through equation (5) to a separate spin rate solution. The more useful solution is the following convergent series:

$$\begin{aligned} p = \frac{Be^{bt}}{b + D}\left[1 - \frac{Ae^{at}}{a + B + D} + \frac{(Ae^{at})^2}{(a + b + D)(2a + B + D)} - \frac{(Ae^{at})^3}{(a + b + D)(2a + b + D)(3a + b + D)} + \ldots\right] \\ + e^{-\frac{A}{a}(e^{at}-1)-Dt}\left\{p_o - \frac{B}{b + D}\left[1 - \frac{A}{a + b + D} + \frac{A^2}{(a + b + D)(2a + b + D)}\right.\right. \\ \left.\left. - \frac{A^3}{(a + b + D)(2a + b + D)(3a + b + D)} + \ldots\right]\right\} + \text{similar C terms} \end{aligned} \tag{7a}$$

In application, each series should be extended until the first term not used is "negligible."

The second solution, an asymptotic series useful for very small values of $\left|\frac{b + D - a}{Ae^{at}}\right|$, is

$$p \approx \frac{Be^{bt}}{Ae^{at}}\left[1 - \frac{b + D - a}{Ae^{at}} + \frac{(b + D - a)(b + D - 2a)}{(Ae^{at})^2} - \frac{(b + D - a)(b + D - 2a)(b + D - 3a)}{(Ae^{at})^3} + \ldots\right]$$

$$+ e^{-\frac{A}{a}(e^{at}-1)-Dt}\left\{p_o - \frac{B}{A}\left[1 - \frac{b + D - a}{A} + \frac{(b + D - a)(b + D - 2a)}{A^2}\right.\right.$$

$$\left.\left. - \frac{(b + D - a)(b + D - 2a)(b + D - 3a)}{A^3} + \ldots\right]\right\} + \text{similar } C \text{ terms} \tag{7b}$$

As before, the series are extended until the first terms not used are "negligible."

Normally, equation (7a) is a fast, efficient, and accurate means of approximating the solution of equation (3). However, if convergence is too slow, another of the integration methods can be used.

Tabular Method

The tabular method picks up the integration of equation (4) at equation (6) which is

$$\dot{y} = Be^{bt+\frac{A}{a}e^{at}+Dt}$$

Approximating e^{at} by $1 + at + \frac{a^2t^2}{2}$, rearranging terms, and integrating give

$$y = p_o e^{\frac{A}{a}} + Be^{\frac{A}{a} - \frac{(A+b+D)^2}{2Aa}} \int_0^t e^{\frac{Aa}{2}\left(t+\frac{A+b+D}{Aa}\right)^2} dt$$

The independent variable is transformed by letting

$$\frac{Aa}{2}\left(t + \frac{A + b + D}{Aa}\right)^2 = i\,\frac{\pi}{2}\,u^2$$

Then

$$y = p_o e^{\frac{A}{a}} + Be^{\frac{A}{a} - \frac{(A+b+D)^2}{2Aa}} \sqrt{\frac{i\pi}{Aa}} \int_{\sqrt{\frac{Aa}{i\pi}}\left(\frac{A+b+D}{Aa}\right)}^{\sqrt{\frac{Aa}{i\pi}}\left(t+\frac{A+b+D}{Aa}\right)} e^{i\frac{\pi}{2}u^2}\, du$$

The integral is now in the form of the complex Fresnel integral tabulated in reference 2.

The p solution with the C term included by superposition is

$$p = e^{\frac{A}{a}(1-e^{at})-Dt}\left[p_o + Be^{-\frac{(A+b+D)^2}{2Aa}}\sqrt{\frac{i\pi}{Aa}}\left(E_1 - E_{1,o}\right) + Ce^{-\frac{(A+c+D)^2}{2Aa}}\sqrt{\frac{i\pi}{Aa}}\left(E_2 - E_{2,o}\right)\right] \tag{8}$$

where

$$\left.\begin{aligned} E_1 &= E\left[\sqrt{\frac{Aa}{i\pi}}\left(t + \frac{A + b + D}{Aa}\right)\right] \\ E_2 &= E\left[\sqrt{\frac{Aa}{i\pi}}\left(t + \frac{A + c + D}{Aa}\right)\right] \end{aligned}\right\} \text{Complex Fresnel integrals tabulated in reference 2} \tag{9a}$$

$$\left.\begin{aligned} \sqrt{\frac{i\pi}{Aa}} &= \sqrt{\frac{2\pi}{-Aa}}\left(\frac{1 - i}{2}\right) = \sqrt{\frac{2\pi}{Aa}}\left(\frac{1 + i}{2}\right) \\ \sqrt{\frac{Aa}{i\pi}} &= \sqrt{\frac{-Aa}{2\pi}}(1 + i) = \sqrt{\frac{Aa}{2\pi}}(1 - i) \end{aligned}\right\} \text{Convenient relationships} \tag{9b}$$

The spin acceleration $\dot{p}$ can be determined from equation (4). Note that this solution does not degenerate for zero damping ($A = D = 0$) or constant damping ($a = 0$). However, for $A = D = 0$ a straightforward integration of equation (4) yields the following solution:

$$\left.\begin{aligned} p &= p_o + \frac{B}{b}\left(e^{bt} - 1\right) + \frac{C}{c}\left(e^{ct} - 1\right) \\ \dot{p} &= Be^{bt} + Ce^{ct} \end{aligned}\right\} \text{Zero damping}$$

and, for $a = D = 0$,

$$\left.\begin{aligned} p &= p_o e^{-At} + \frac{B}{A + b}\left(e^{bt} - e^{-At}\right) + \frac{C}{A + c}\left(e^{ct} - e^{-At}\right) \\ \dot{p} &= -Ap + Be^{bt} + Ce^{ct} \end{aligned}\right\} \begin{array}{l}\text{Constant}\\ \text{damping}\\ \text{coefficient}\end{array} \qquad (10)$$

Mid-Damping Method

The use of a constant damping coefficient in equations (10) introduces the third method for integrating equation (4) - that is, the mid-damping method. In this method the value of the damping coefficient at the midinterval time is used throughout the interval and the B and C terms are exponential as before. Therefore, the terms $D + Ae^{at}$ are replaced with their midinterval value $\bar{A}_1$ and the constant damping solution becomes

$$p = p_o e^{-\bar{A}_1 t} + \frac{B}{\bar{A}_1 + b}\left(e^{bt} - e^{-\bar{A}_1 t}\right) + \frac{C}{\bar{A}_1 + c}\left(e^{ct} - e^{-\bar{A}_1 t}\right) \qquad (11)$$

where

$$\bar{A}_1 = Ae^{a\bar{t}} + D$$

LIMITATIONS AND APPLICATION

The method for solving the roll equation consists of fitting actual histories of the driving and damping moments, normalized with respect to I_X, to the constant and/or exponential coefficients of equation (4). The equation is then integrated in one of three ways. This curve fitting is a limitation on the accuracy of the method. However, the errors generated depend upon the skill of the curve fitter and in any case can be kept as small as desired with the use of more intervals. When fitting the coefficients of the damping term, first plot the values of $M_{X,p}/I_X$ on semilog paper as a function of time. If the plot is essentially a straight line over a given time interval, the constant term D is zero and the values of A and a are easily determined from the zero intercept and slope of the faired curve. If the plot is curved, determine by trial and error the value of D required to straighten the plot over the required time interval and compute the values of A and a from the straightened and faired curve.

Think now about fitting the normalized input moments of equation (3) by considering a rolling-moment input which increases linearly with time from 2 ft-lb (2.72 J) to 6 ft-lb (8.16 J) over a 10-sec period. One exponential term cannot adequately simulate this curve. However, by thinking of fitting an exponential curve having a displaced origin, one can visualize a flatter curve which will more closely simulate the requirements as shown in figure 1.

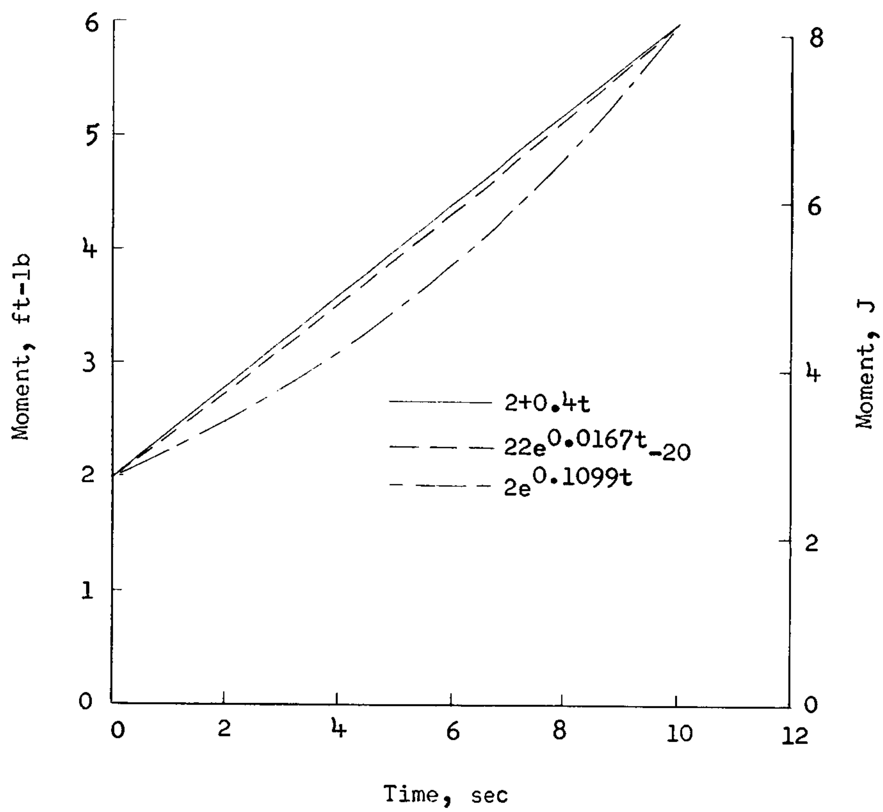

Figure 1.- Exponential simulation of a linear moment variation.

This, of course, is the same as using two exponentials - one with a zero exponent - to meet the need. However, this improved simulation requires calculations of increased quality and quantity since the additional term involves the difference of two large numbers.

In addition to curve-fitting errors, the accuracy of the method is limited by approximations which were used in the integration procedures. Consider first the tabular method in which the approximation

$$e^{at} \approx 1 + at + \frac{a^2t^2}{2}$$

was used. As shown in the appendix, this approximation causes errors in spin rate over an interval as given approximately by

$$\Delta p = \frac{Be^{bt} + Ce^{ct} - p(Ae^{at} + D)}{ae^{at}} \Delta e^{at} \tag{12}$$

where

$$\Delta e^{at} = e^{at} - \left(1 + at + \frac{a^2t^2}{2}\right) \tag{13}$$

The absolute value of the spin-rate error becomes larger as the value of at is increased with the result that solutions usually must be divided into two or more steps or time intervals. Equation (12) can be used to determine these step lengths for a given accuracy requirement. However, a much faster and more desirable method is to use step lengths consistent with a factor-of-2 change in the damping term $M_{X,p}/I_X$. This "factor-of-2 criterion" is consistent with good accuracy (about 2 percent error). If more or less accuracy is desired, a smaller or larger factor can be used and the relative error will be governed by equation (13).

Now consider the mid-damping method. The use of a constant (midinterval) damping coefficient over the entire interval causes a loss in accuracy toward the midpoint of the interval. The approximate value of the spin-rate error, developed in the appendix, is given by

$$\Delta p \approx k_1 t + \frac{k_2}{\bar{A}_1 - a}\left[e^{\left(a - \bar{A}_1\right)t} - 1\right] - \frac{k_3}{a + b}\left[e^{(a+b)t} - 1\right] - D\int_0^t (p - \bar{p})\, dt \tag{14}$$

where

$$k_1 = k_2 e^{\left(a - \bar{A}_1\right)\bar{t}} + k_3 e^{(a+b)\bar{t}} \tag{15a}$$

$$k_2 = A\left(p_o - \frac{B}{\bar{A}_1 + b}\right) \tag{15b}$$

$$k_3 = \frac{AB}{\bar{A}_1 + b} \tag{15c}$$

$$\int_0^t \left(p - \bar{p}\right) dt = -\bar{p}t + \left(\frac{B/\bar{A}_1}{\bar{A}_1 + b} - \frac{p_o}{\bar{A}_1}\right)\left(e^{-\bar{A}_1 t} - 1\right) + \frac{B/b}{\bar{A}_1 + b}\left(e^{bt} - 1\right) \tag{15d}$$

Equation (14) shows that the absolute value of the error builds up near the center of the intervals while remaining lower near the end points. Also, the use of longer intervals does not necessarily decrease the accuracy of the method near the end points. Thus, longer intervals can be used accurately if only the final spin rate of each interval is required.

Finally, with the asymptotic method of integration, no additional errors are involved since approximations are not used with this method.

In some problems, the semilog plots of $M_{X,p}/I_X$, $M_{X,\delta}\delta/I_X$, or M_X/I_X have discontinuities where the rocket motor cases drop off or torque motors start or stop, and so forth. These discontinuities are the usual cause for additional time intervals, and are illustrated in figures 2 and 3, the data plots for the sample problem.

Sample Problem

The approximate method was used separately with each integration procedure to simulate the flight spin history of the second stage of the Trailblazer II vehicle described in reference 3. These simulations start with the separation of a first-stage rocket from the second-stage configuration during an exiting trajectory. Ignition of the second-stage rocket motor occurs at this separation and the motor thrusts about 6 sec followed by a 20-sec coasting period. During this 26-sec postseparation period, the second-stage configuration is exiting the sensible atmosphere, and its spin rate must be increased from about 0 to 65 rad/sec by means of precanted booster fins. The problem is to predetermine the fin cant required to produce this increase in spin rate and in the process to generate the spin history of the vehicle over the 26 sec.

It is assumed that a particle trajectory has been computed to furnish time histories of Mach number, velocity, and dynamic pressure for converting the aerodynamic driving and damping coefficients into the time histories of $M_{X,p}/I_X$ and $M_{X,\delta}/I_X$ shown in the semilog plots of figures 2 and 3, respectively. These plots were faired with straight-line segments for reasons previously discussed. For these straight-line variations, the C and D terms of equation (4) are not needed and are set equal to zero. Both plots have natural breaks at the end of thrusting (41.2 sec), and the break at 48 sec was chosen to best fit the curves.

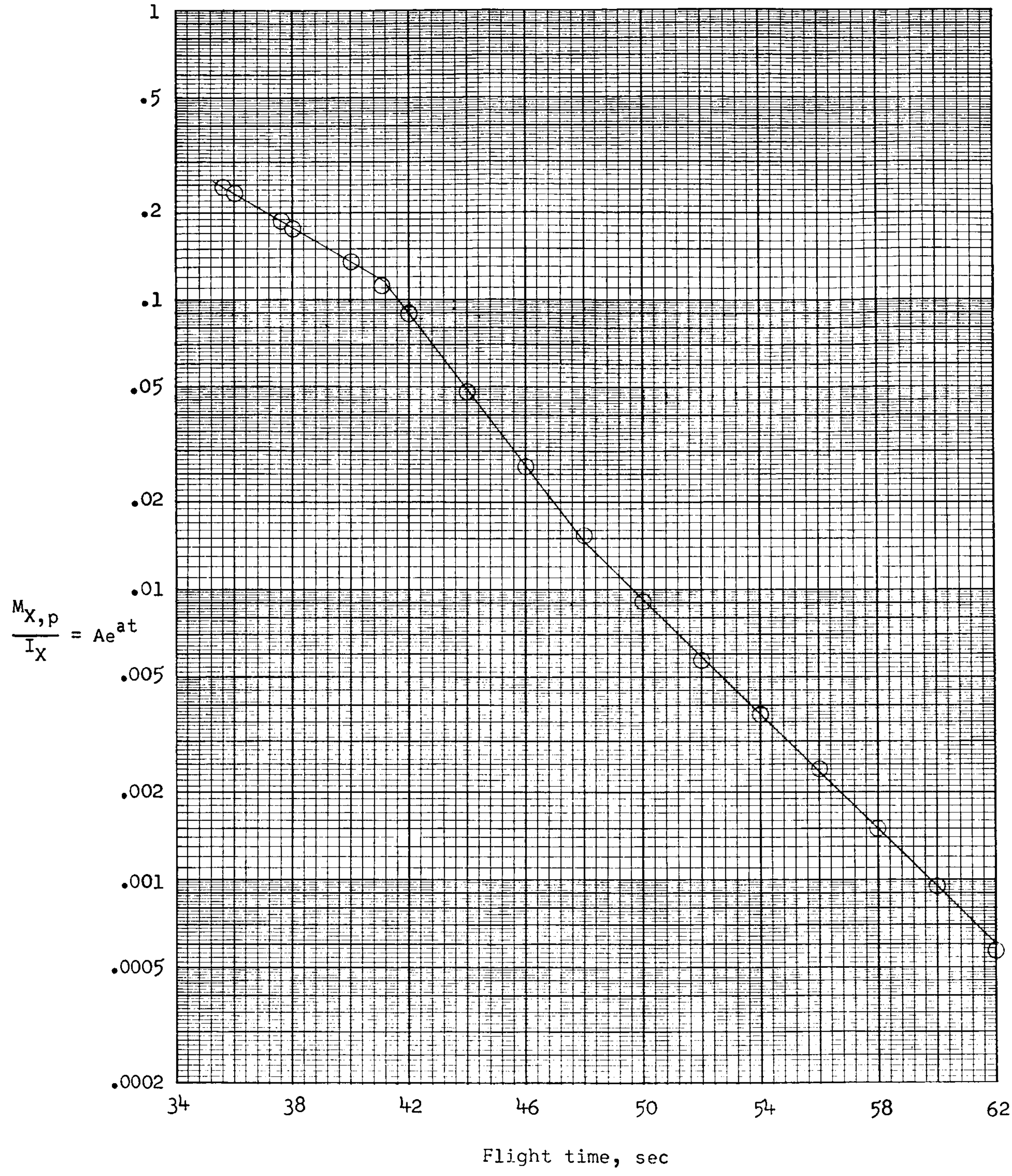

Figure 2.- History of the roll damping term for Trailblazer II second-stage flight.

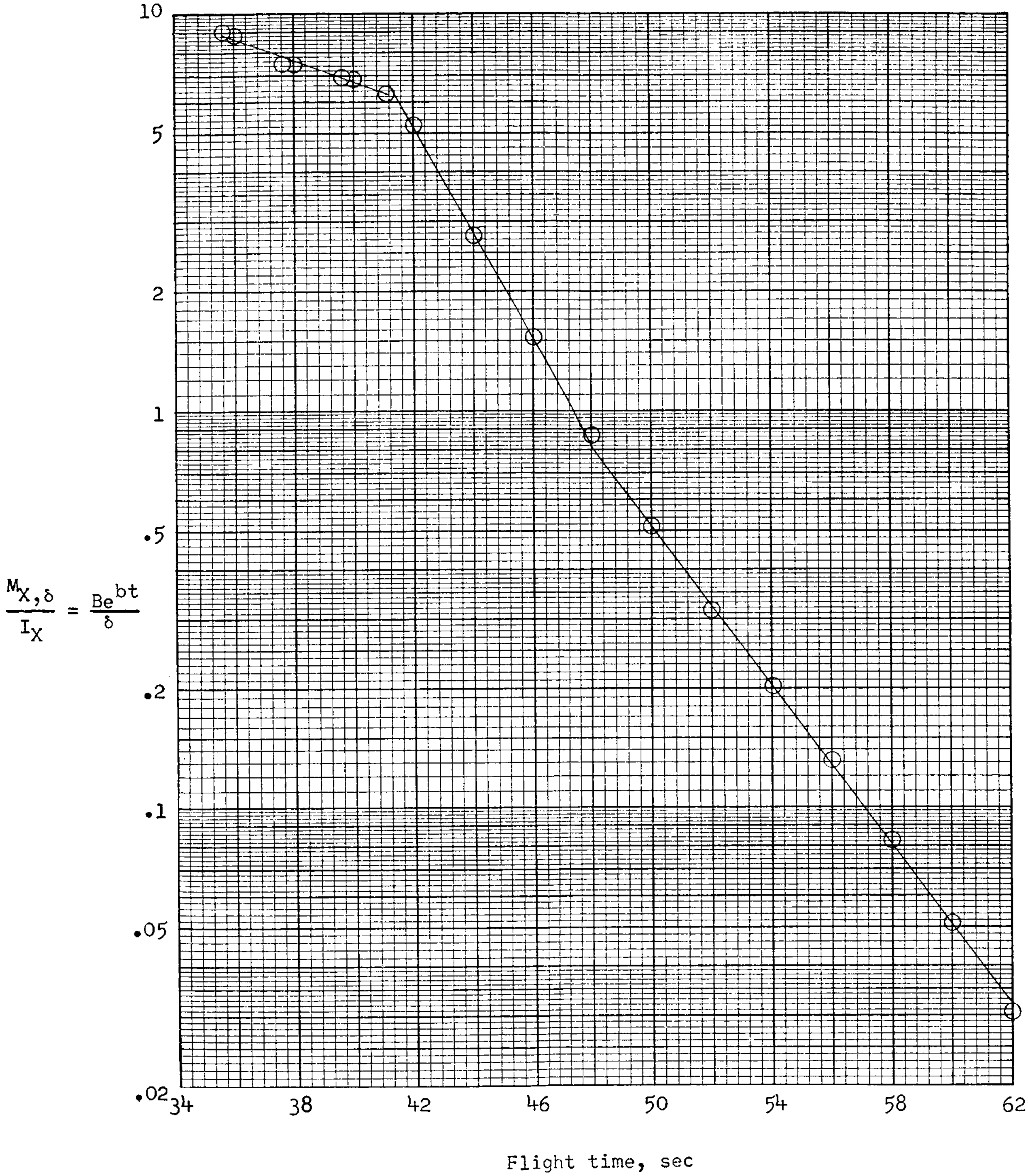

Figure 3.- Variation of $\frac{M_{X,\delta}}{I_X}$ during Trailblazer II second-stage flight.

In application of the tabular method to the problem, intervals were determined by the factor-of-2 criterion and modified (at no loss in accuracy) with equation (12) to the intervals given in table I.

TABLE I

CONSTANTS FOR PROBLEMS 1 AND 2

Problem 1:

Constants	Step				
	1	2	3	4	5
A, sec^{-1}	0.244	0.114	0.0433	0.0146	0.00296
B/δ, sec^{-1}/deg . . .	8.85	6.65	2.48	0.82	0.162
C, sec^{-1}	0	0	0	0	0
D, sec^{-1}	0	0	0	0	0
a, sec^{-1}	-0.1359	-0.3022	-0.3022	-0.2280	-0.2280
b, sec^{-1}	-0.0621	-0.3078	-0.3078	-0.2317	-0.2317
c, sec^{-1}	0	0	0	0	0
$\bar{t}$, sec	2.8	1.6	1.8	3.5	3.5
t_o, sec	0	5.6	8.8	12.4	19.4
Initial p_o = 0 rad/sec					

Problem 2:

Constants for problem 2 are the same as for problem 1 except initial p_o = 200 rad/sec.

Although these time intervals are tailored to the tabular method, they were used also with the mid-damping method for purposes of comparison. Equations (8) and (11), with constants determined from figures 2 and 3 and given in table I, were used to generate the spin rate histories shown in figure 4 for these two approximation methods and compared with the more accurate results of the numerical integration method. Continuous curves are illustrated for these two approximate methods. In practice, however, only end points of the intervals would be computed with the mid-damping method. These end points would then be faired for the final spin-rate history. Also shown in figure 4 are several values computed by the asymptotic method (eq. (7)). The use of the asymptotic method reduced the number of problem intervals to three - each corresponding to one of the straight-line segments of the semilog plots in figures 2 and 3 and to steps 1, 2, and 4 of table I. All results were computed for $\delta = 1.75^{\circ}$, a value that was determined by trial and error to produce the required spin rate. Since spin rate at any given time is proportional to δ

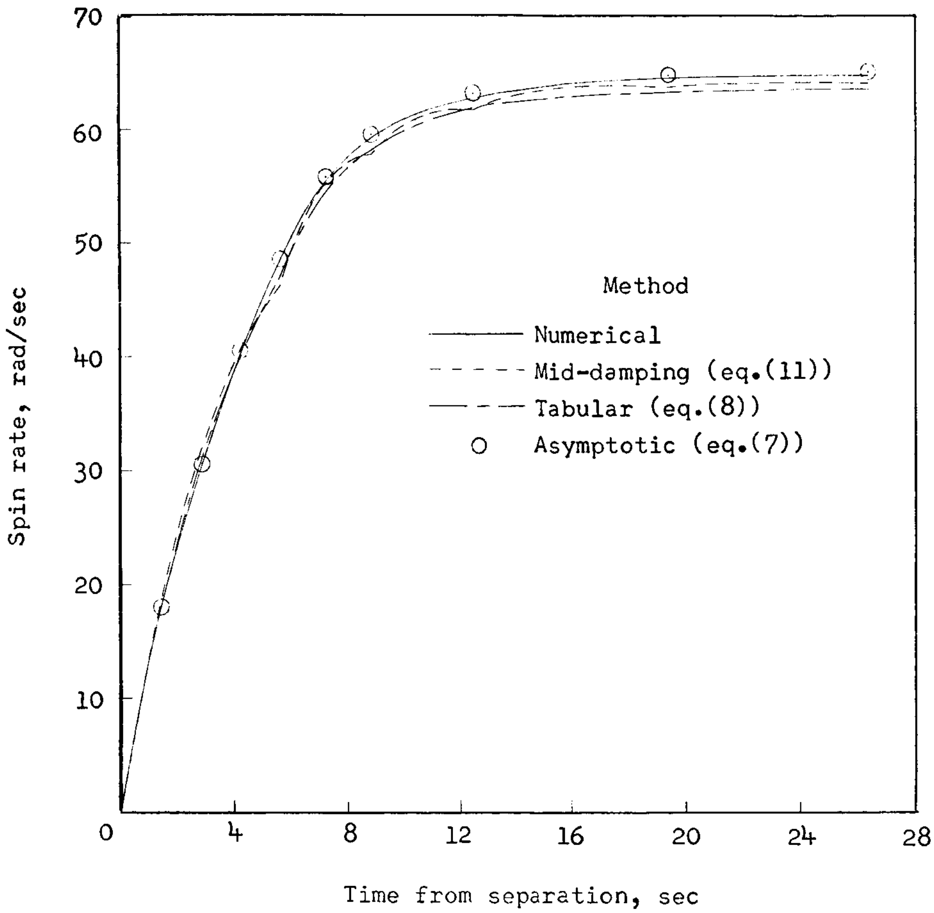

Figure 4.- Comparison of Trailblazer II second-stage spin histories computed by tabular, mid-damping, and asymptotic methods with more accurate numerical integration results for initial spin rate of zero. $\delta = 1.75^{\circ}$.

and to initial spin rate $\left(\text{i.e.,}\quad p = C_1\left(p_o\right) + C_2(\delta)\right)$, the trial-and-error process is fast and simple after once computing the proportionality factors.

As might be expected from consideration of the errors involved, the asymptotic method appears to be slightly more accurate than either of the other two approximation methods; however, all approximations are within about 2 percent of the numerical solution.

In an effort to exploit the weakness of the mid-damping method, problem 1 was rerun with an initial spin rate of 200 rad/sec, a value considerably above steady-state roll. These results, presented in figure 5, show that all approximation methods again compare quite well with the numerical solution. As in problem 1, the spin-rate error generated by assuming exponential dependence of the driving and damping terms is illustrated by the difference between the asymptotic and numerical solutions.

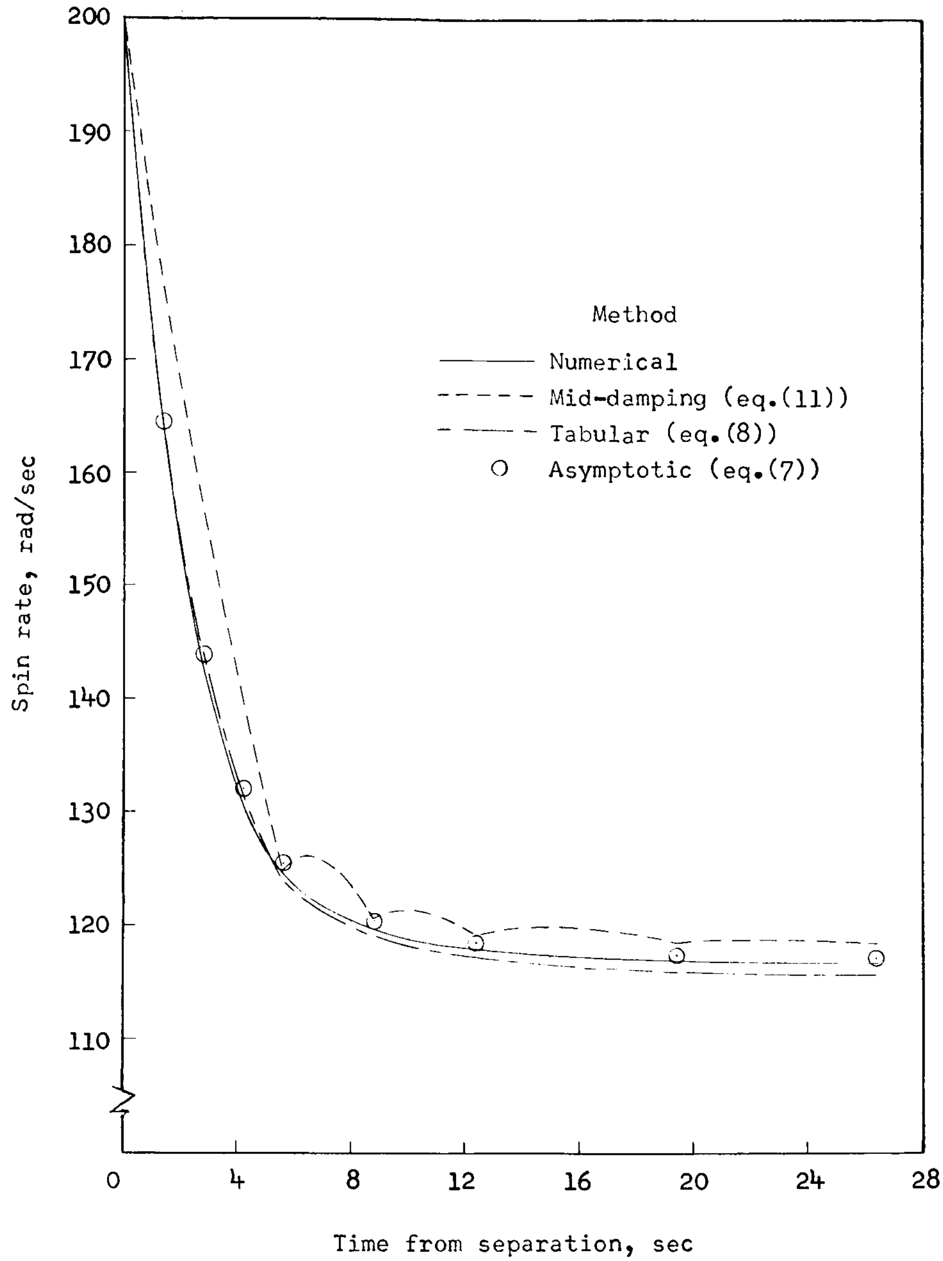

Figure 5.- Comparison of Trailblazer II second-stage spin histories computed by tabular, mid-damping, and asymptotic methods with more accurate numerical integration results for initial spin rate of 200 rad/sec. $\delta = 1.75^{\circ}$.

CONCLUDING REMARKS

An approximate analytical method (with three alternate integration procedures) is developed for solving the single-degree-of-freedom roll equation with time-dependent coefficients. The method is applied with each integration procedure to two sample problems and found to compare closely with more exact numerical solutions. The closed-form solution avoids tedious step-by-step integration and allows rapid hand computation of results. Of the three integration procedures presented, the asymptotic method, if convergent, was judged superior for solving the roll equation. Information governing the approximate error of each integration procedure is also presented.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., August 13, 1965.

APPENDIX

ERRORS OF APPROXIMATION

Tabular Method

Errors considered herein are those resulting from the approximation

$$e^{at} \approx 1 + at + \frac{a^2t^2}{2}$$

The first-order approximation of the general error expression for p is

$$\Delta p = \frac{\partial p}{\partial e^{at}} \Delta e^{at} = \frac{dp}{de^{at}} \Delta e^{at} \tag{A1}$$

where

$$\Delta e^{at} = e^{at} - \left(1 + at + \frac{a^2t^2}{2}\right)$$

$$\frac{dp}{de^{at}} = \frac{\dot{p}}{ae^{at}} = \frac{Be^{bt} + Ce^{ct} - p\left(Ae^{at} + D\right)}{ae^{at}}$$

Mid-Damping Method

The use of a constant (midinterval) damping coefficient over an interval results in errors determined approximately as follows. No curve-fitting errors were considered. Equation (4) without the C term can be written as

$$\dot{p} = -A_1p + Be^{bt} \tag{A2}$$

where

$$A_1 = Ae^{at} + D$$

APPENDIX

The first-order approximation of the general error expression for $\dot{p}$ is

$$\Delta\dot{p} = \frac{\partial\dot{p}}{\partial A_1}\Delta A_1 + \frac{\partial\dot{p}}{\partial p}\Delta p + \frac{\partial\dot{p}}{\partial\left(Be^{bt}\right)}\Delta\left(Be^{bt}\right) \tag{A3}$$

and from equation (A2)

$$\frac{\partial\dot{p}}{\partial A_1} = -p$$

$$\frac{\partial\dot{p}}{\partial p} = -A_1$$

$$\frac{\partial\dot{p}}{\partial\left(Be^{bt}\right)} = 1$$

However, for best results, these slopes should be evaluated at their mean value over the associated error increment. Thus,

$$\frac{\partial\dot{p}}{\partial A_1}\Delta A_1 = -p_{mean}\Delta A_1 = -\Delta A_1\left(\frac{1}{\Delta A_1}\int_{\bar{A}_1}^{A_1} p\ dA_1\right) = -\int_{\bar{t}}^{t} paAe^{at}\ dt$$

Also,

$$\frac{\partial\dot{p}}{\partial p}\Delta p = -A_{1,mean}\Delta p = -\Delta p\left(\frac{1}{\Delta p}\int_{\bar{p}}^{p} A_1\ dp\right) = -\int_{\bar{t}}^{t} A_1\dot{p}\ dt$$

$$= -\int_{\bar{t}}^{t}\left(Ae^{at}\dot{p}\right) dt - D\left(p - \bar{p}\right)$$

Since in equation (A3), $\Delta\dot{p} \approx \frac{d(\Delta p)}{dt}$ and $\Delta\left(Be^{bt}\right) = 0$, the approximate error equation can be written as

$$\frac{d(\Delta p)}{dt} = -\int_{\bar{t}}^{t} paAe^{at}\ dt - \int_{\bar{t}}^{t} Ae^{at}\dot{p}\ dt - D\left(p - \bar{p}\right)$$

where values of p and $\dot{p}$ are obtained from equation (11). Finally, by integration,

$$\Delta p = \int_0^t \frac{d(\Delta p)}{dt}\, dt = k_1 t + \frac{k_2}{\bar{A}_1 - a}\left[e^{\left(a-\bar{A}_1\right)t} - 1\right]$$

$$- \frac{k_3}{a + b}\left[e^{(a+b)t} - 1\right] - D\int_0^t \left(p - \bar{p}\right) dt \qquad \text{(A4)}$$

where

$$k_1 = k_2 e^{\left(a-\bar{A}_1\right)\bar{t}} + k_3 e^{(a+b)\bar{t}}$$

$$k_2 = A\left(p_o - \frac{B}{\bar{A}_1 + b}\right)$$

$$k_3 = \frac{AB}{\bar{A}_1 + b}$$

$$\int_0^t \left(p - \bar{p}\right) dt = -\bar{p}t + \left(\frac{B/\bar{A}_1}{\bar{A}_1 + b} - \frac{p_o}{\bar{A}_1}\right)\left(e^{-\bar{A}_1 t} - 1\right) + \frac{B/b}{\bar{A}_1 + b}\left(e^{bt} - 1\right)$$

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