RESEARCH TRIANGLE INSTITUTE

System Reliability Technology

Technical Report No. 1

ON CERTAIN FUNCTIONALS OF NORMAL PROCESSES

by

M. R. Leadbetter and J. D. Cryer

September 1964

Prepared for

Office of Reliability and Quality Assurance
National Aeronautics and Space Administration
Washington, D. C. 20546

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FOREWORD

This report is one of a series of technical reports issued by RTI (Research Triangle Institute), Durham, North Carolina, under NASA (National Aeronautics and Space Administration) Contract NASw-905, "Development of Reliability Methodology for Systems Engineering". The contract is administered under the direction of J. E. Condon, Director, ORQA (Office of Reliability and Quality Assurance) NASA Headquarters, Washington, D. C.

The major objective of this contract is to apply probabilistic modeling techniques developed at RTI under a previous NASA contract to a NASA in-house R and D system and to conduct research necessary to further develop techniques appropriate to probabilistic modeling. A complex 250 volt-ampere static inverter under design and development by the Astrionics Laboratory of MSFC (Marshall Space Flight Center) was selected as a representative NASA system for application and demonstration of the techniques, and the majority of the technical reports in this series are devoted to documenting results of this effort. The additional research, both basic and applied in nature, on general probabilistic modeling is documented in several specific reports also included in this series.

The effort under this contract began in April, 1964 and will continue for a period of approximately two years. The studies are being performed jointly in the Institute's Solid State Laboratory and Statistics Research Division under the general direction of Dr. R. M. Burger with W. S. Thompson serving as project leader.

PREFACE

This report is the first technical report of the series to be issued under Contract NASw-905 and is devoted specifically to documenting some theoretical results on probabilistic modeling obtained in the basic research effort under the contract. The basic research studies consist mainly of investigations in curve, level and zero crossings by certain normal stochastic processes (both stationary and non-stationary). Such investigations provide measures of the quality of performance and the reliability of certain complex systems. In this report, a number of such measures are discussed. These include the time which the process of interest spends outside given (undesirable) levels and related random variables.

CONTENTS

On Certain Functionals of Normal Processes

1. System Performance Indices

When considering the performance quality or reliability of complex systems, it is sometimes convenient to define certain performance "indices" or "measures" based on the characteristics of a stochastic process associated with the system. In some cases, the indices and their statistical properties are very simple, for example, when we are concerned with a particular property of the system at <u>one</u> instant of time. On the other extreme, we may be interested in properties of the system for all values of time during some operational period. For example, in a missile system, the angular error in rocket attitude can be considered as a stochastic process which, for good performance, should be kept small during the entire mission period.

Suppose then that we have such a stochastic process $x(t)$ of interest and that for good performance $x(t)$ should never become "too large". Specifically, suppose that there is a known function $u(t)$ such that for good performance $x(t)$ should always be kept less than $u(t)$. Let h be a function which is zero for negative arguments and strictly positive for positive arguments. Define

$$Z = \frac{1}{T} \int_0^T h[x(t) - u(t)]dt \; . \tag{1.1}$$

(We assume that the behavior of h is such that, with probability one, the integral will exist.)

Now the event $\{Z=0\}$ is equivalent to the event $\{x(t) \leq u(t),\ t \in [0,T]\}$ (if $x(t)$ is continuous), and hence

$$P\{Z=0\} = P\{x(t) \leq u(t),\ t \in [0,T]\}$$

and Chebyshev bounds on $P\{Z=0\}$ (using the mean and variance of Z) give bounds on $P\{x(t) \leq u(t),\ t \in [0,T]\}$. This latter quantity represents, of course, the <u>reliability</u>, or probability of a successful mission from this point of view.

We wish to investigate forms of the function h which lead to tractable Chebyshev bounds. One particular, amenable choice for h is

$$(1.2) \qquad h(x) = h_n(x) = \begin{cases} x^n & , \text{ if } x \geq 0 \\ 0 & , \text{ if } x < 0 . \end{cases}$$

We then write Z_n for Z with the corresponding h_n as integrand. Note that Z_0 is the proportion of <u>time</u> which the process spends above the curve u(t) on the interval [0,T] and TZ_1 is the <u>area</u> which the process cuts off above the curve in the same interval.

The Z_n's describe excursions of x(t) above u(t) in various ways - for example, Z_0 takes no account of the size of such an excursion whereas Z_1, Z_2, ... do. Z_0 and Z_1 have been considered previously in Leadbetter [1963] and Cryer [1963] for a normal, stationary process x(t). The generalization of these results to the Z_n statistics is the purpose of this report.

2. Mean and Variance of the Z_n- statistics

Let x(t) be a normal stationary stochastic process with zero mean and covariance function $r(\tau)$, assumed such that, with probability one, the sample functions are everywhere continuous. Sufficient conditions for this are given, for example, in Belaev [1961].

For convenience, we use the following notation:

$$\sigma^2 = r(0)$$
$$v_t = u(t)/\sigma$$

$$\rho(\tau) = r(\tau)/\sigma^2$$

$$\phi(x) = \frac{1}{(2\pi)^{1/2}} e^{-x^2/2}, \text{ the normal density function,}$$

$$\Phi(x) = \int_{-\infty}^{x} \phi(t)dt, \text{ the normal distribution function,}$$

$$y_n(t) = h_n[x(t)-u(t)] .$$

We have

$$\mathcal{E}[Z_n] = \mathcal{E}[\frac{1}{T} \int_0^T y_n(t)dt]$$

$$= \frac{1}{T} \int_0^T \mathcal{E}[y_n(t)]dt .$$

By the definition of h_n we find

$$\mathcal{E}[y_n(t)] = \frac{1}{\sigma} \int_{u(t)}^{\infty} [x-u(t)]^n \phi(x/\sigma)dx$$

$$= \sigma^n \int_{v_t}^{\infty} [x-v_t]^n \phi(x)dx$$

and hence

(2.1) $$\mathcal{E}[Z_n] = \frac{\sigma^n}{T} \int_0^T \int_{v_t}^{\infty} [x-v_t]^n \phi(x)dx\, dt .$$

Using the binomial expansion for $[x-v(t)]^n$ the integral of the form $\int_c^{\infty} [x-c]^n \phi(x)dx$ may be evaluated as a finite sum of incomplete gamma functions. This would give a useful form for computing purposes.

For the variance of Z_n, we note that

$$(2.2)\qquad \operatorname{Var} Z_n = \mathcal{E}[\frac{1}{T^2}\int_0^T\int_0^T y_n(t)y_n(s)ds\,dt] - \frac{1}{T^2}\int_0^T\int_0^T \mathcal{E}[y_n(t)]\mathcal{E}[y_n(s)]ds\,dt$$

$$= \frac{1}{T^2}\int_0^T\int_0^T \operatorname{cov}[y_n(t),y_n(s)]ds\,dt\ .$$

Now

$$(2.3)\qquad \mathcal{E}[y_n(t)y_n(s)] = \frac{1}{\sigma^2}\int_{u(t)}^{\infty}\int_{u(s)}^{\infty} [x-u(t)]^n[y-u(s)]^n\,\phi(\tfrac{x}{\sigma},\tfrac{y}{\sigma};\ \rho)dx\,dy$$

$$= \sigma^{2n}\int_{v_t}^{\infty}\int_{v_s}^{\infty} (x-v_t)^n(y-v_s)^n\,\phi(x,y;\ \rho)dx\,dy\ ,$$

where $\phi(x,y;\ \rho)$ is the standardized bivariate normal density with correlation coefficient $\rho=\rho(t-s)$. From Cramér [1946] p. 290, we have the expansion

$$\phi(x,y;\ \rho) = \sum_{j=0}^{\infty} \frac{\rho^j}{j!}\,\Phi^{(j+1)}(x)\,\Phi^{(j+1)}(y)\ ,$$

where

$$\Phi^{(k)}(x) = \frac{d^k}{dx^k}\Phi(x)\ .$$

Using this in (2.3) we obtain

$$(2.4)\qquad \mathcal{E}[y_n(t)y_n(s)] = \sigma^{2n}\sum_{j=0}^{\infty}\frac{\rho^j}{j!}\int_{v_t}^{\infty}(x-v_t)^n\,\Phi^{(j+1)}(x)dx\int_{v_s}^{\infty}(y-v_s)^n\,\Phi^{(j+1)}(y)dy\ .$$

If $1 \leq j \leq n$, repeated integration by parts gives

$$\int_v^{\infty}(x-v)^n\,\Phi^{(j+1)}(x)dx = (-1)^j\frac{n!}{(n-j)!}\int_v^{\infty}(x-v)^{n-j}\phi(x)dx\ ,$$

and if $j > n$,

$$\int_v^{\infty}(x-v)^n\,\Phi^{(j+1)}(x)dx = (-1)^n\, n![-\Phi^{(j-n)}(v)\ ]\ .$$

Hence (2.4) may be written

$$(2.5)\quad \mathcal{E}[y_n(t)y_n(s)] = \sigma^{2n}\Big\{ \sum_{j=0}^{n} \frac{\rho^j}{j!}\Big[\frac{n!}{(n-j)!}\Big]^2 \int_{v_t}^{\infty} (x-v_t)^{n-j}\phi(x)dx \int_{v_s}^{\infty} (y-v_s)^{n-j}\phi(y)dy$$

$$+ \sum_{j=n+1}^{\infty} \frac{\rho^j}{j!} (n!)^2\, \Phi^{(j-n)}(v_t)\ \Phi^{(j-n)}(v_s) \Big\} .$$

Note that the first term $(j=0)$ is

$$\sigma^{2n} \int_{v_t}^{\infty} (x-v_t)^n\phi(x)dx \int_{v_s}^{\infty} (y-v_s)^n\phi(y)dy = \mathcal{E}[y_n(t)]\ \mathcal{E}[y_n(s)]$$

and therefore

$$\mathrm{Cov}[y_n(t),y_n(s)] = \sigma^{2n}\Big\{ \sum_{j=1}^{n} \frac{\rho^j}{j!}\Big[\frac{n!}{(n-j)!}\Big]^2 \int_{v_t}^{\infty} (x-v_t)^{n-j}\phi(x)dx \int_{v_s}^{\infty} (y-v_s)^{n-j}\phi(y)dy$$

$$+ \sum_{j=n+1}^{\infty} \frac{\rho^j}{j!} (n!)^2\, \Phi^{(j-n)}(v_t)\ \Phi^{(j-n)}(v_s) .$$

Substituting this into equation (2.2) gives the final result

(2.6)

$$\mathrm{Var}\ Z_n = \sigma^{2n}\Big(\frac{n!}{T}\Big)^2\Big\{ \sum_{j=1}^{n} \frac{1}{j![(n-j)!]^2} \int_0^T\!\!\int_0^T \Big[\rho^j(t-s) \int_{v_t}^{\infty} (x-u_t)^{n-j}\phi(x)dx \int_{v_s}^{\infty} (y-v_s)^{n-j}\phi(y)dy\Big]ds\ dt$$

$$+ \sum_{j=n+1}^{\infty} \frac{1}{j!} \int_0^T\!\!\int_0^T \rho^j(t-s)\, \Phi^{(j-n)}(v_t)\ \Phi^{(j-n)}(v_s)ds\ dt\Big\} ,$$

where it is understood that the first summation does not appear if $n = 0$.

3. Special Cases

a. If $u(t)$ is in fact a constant level $u(t) = u$, for all t, then putting $v = u/\sigma$ we have

$$\mathcal{E}[Z_n] = \sigma^n \int_v^\infty (x-v)^n \phi(x)dx, \text{ and} \tag{3.1}$$

(3.2)

$$\operatorname{Var} Z_n = 2\sigma^{2n}\left(\frac{n!}{T}\right)^2 \Big\{ \sum_{j=1}^{n} \frac{1}{j![(n-j)!]^2} \left[\int_v^\infty (x-v)^{n-j} \phi(x)dx\right]^2 \int_0^T (T-\tau)\rho^j(\tau)d\tau$$

$$+ \sum_{j=n+1}^{\infty} \frac{1}{j!} \left[\int_0^T (T-\tau)\rho^j(\tau)d\tau\right] \left[\Phi^{(j-n)}(v)\right]^2 \Big\} ,$$

where use is made of the fact that for an even function $f(x)$

$$\int_0^T\int_0^T f(t-s)ds\, dt = 2\int_0^T (T-\tau)f(\tau)d\tau .$$

b. The cases $n = 0$ and $n = 1$ ($u(t)$ not constant) were obtained in Cryer [1963].

c. The results for $n = 1$, $u(t) = u$ constant (in addition to other versions of the problem) were reported in Leadbetter [1963] .

d. The case $n = 0$ and $u(t) = u$ constant is well known; see, e.g., Siddiqui [1961].

4. Computational Aspects and Asymptotic Formulae

Calculation of the mean, $\mathcal{E}[Z_n]$, can be carried out, as already noted, using tables of the incomplete gamma function. Presumably numerical methods would be needed to obtain the final integration over t, although in the special cases where $u(t) = u$, a constant, the final integration over t is trivial.

The formula for the variance is always convergent (if $u(t)$ is at least bounded on $[0,T]$), but the number of terms which must be computed for a good approximation varies with the form of the covariance function $r(\tau)$.

A typical correlation function is $r(\tau) = \exp(-\alpha|t|)$, i.e., the covariance for a normal Markov process. If v, i.e. u(t), is constant and in the range 1 to 5. Leadbetter [1963] has shown that for n = 1 and T not too small, five terms of the infinite series for Var Z_1 are a good approximation. Also, for large T we have the approximation

$$\mathrm{Var}\ Z_1 \simeq \frac{2}{\alpha} \sum_{j=1}^{\infty} \frac{\tau_{j-1}^2}{j^2} / T$$

where $\tau_j = \tau_j(v)$ is the j-th tetrachoric function (see, e.g., Pearson [1931]).

Similar limiting results hold for other covariance functions which are integrable over $(0,\infty)$. For non-integrable covariance functions the results can be simpler. For example, if

$$\frac{r(\tau)}{r(0)} \sim \frac{A}{|\tau|} \quad \text{as } |\tau| \longrightarrow \infty$$

then

$$\mathrm{Var}\ Z_n \sim 2\ \sigma^{2n}\ n[\int_v^{\infty} (x-v)^{n-1}\ \phi(x)dx]^2\ A \log T/T\ .$$

5. Conclusion

The above investigation has been concerned with the first two moments of the Z_n-statistics, for n = 0, 1, 2 ... . As noted, for each fixed n we may obtain an upper bound to the probability that the process will never exceed a given curve u(t) in the time period $0 \leq t \leq T$. Specifically we have

$$\mathrm{Prob}\{x(t) \leq u(t),\ 0 \leq t \leq T\} = \mathrm{Prob}\{Z_n=0\} \tag{5.1}$$

$$\leq (1 + \frac{[\mathcal{E}\{Z_n\}]^2}{\mathrm{var}\ Z_n})^{-1}\ ,$$

using a one-sided Chebyshev inequality.

If it is very desirable that the process never exceed $u(t)$ in the time period T, or more specifically, if exceedance of the curve by the process induces system failure, then the left-hand side of Equation (5.1) is just the probability of correct performance. That is, Equation (5.1) provides an upper bound for the reliability of the system when this particular failure mode is considered. Equations (2.1) and (2.6) provide the appropriate quantities to be inserted on the right of (5.1). For each fixed n, we obtain such a bound. It is not known at present whether some of these bounds are sharper than others - and this is an interesting question. Finally, we note that these techniques lead to *upper* bounds for system reliability. The calculation of *lower* bounds may be accomplished by different methods, as described, for example, in Leadbetter and Lewis [1962].

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