TECHNICAL NOTE R-159

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HYDROSTATICS OF A FLUID BETWEEN PARALLEL PLATES AT LOW BOND NUMBERS

by F.W. Geiger

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RESEARCH LABORATORIES

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Prepared By

F. W. Geiger

ABSTRACT

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The hydrostatics of a fluid between parallel plates at low but positive Bond numbers is re-examined as a preliminary to dynamic calculations. The results of this study differ from those of a previous study by Reynolds. It is believed that the results of Reynolds are in error.

Author

Approved

E. J. Rodgers, Manager
Mechanics and Thermodynamics
Department

Approved

R. C. Watson, Jr.
Director of Research

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LIST OF SYMBOLS

A	A constant in the pressure equation, lbm/ft-sec^2
B	Bond number in Reynold's notation
B_o	Bond number in present notation
c	An integration constant
$E(k)$	Complete elliptic integral of the second kind
$E(k, \alpha)$	Elliptic integral of the second kind
$F(B, \theta)$	A function used in comparing results of present paper with those of Reynold's paper
$F(k, \alpha)$	Elliptic integral of the first kind
g	The effective acceleration of gravity, ft/sec^2
h	The y-coordinate of the surface of the liquid (or fluid), ft
$\hbar$	Modified y-coordinate of the surface $(h - \tau)$, ft
$K(k)$	Complete elliptic integral of the first kind
L	Characteristic length used by Reynolds $(w/2)$, ft
P	Pressure of vapor and gas above liquid, lbm/ft-sec^2
p	Static pressure in the liquid, lbm/ft-sec^2
R	Radius of curvature of the surface, ft
s	Dimensionless parameter proportional to vertical displacement of surface
T	Surface tension of liquid, lbm/sec^2
w	Width of tank, ft
x	Distance across tank, ft
Y	Reynolds' vertical displacement $(h - h_m)$, ft

y	Distance perpendicular to x-direction, ft
θ	Contact angle between liquid and wall, rad
ρ	Density of the liquid (fluid), lbm/ft^3
τ	Distance related to A $(A/\rho g)$, ft

Subscript

m	Indicates mean or average value
o	At point where $dh/dx = 0$ (at middle of tank)
s	Indicates surface condition
u	Upper value or value at the wall

INTRODUCTION

When the valves in the propellant lines of a missile in flight are suddenly closed, there are oscillations of fluid flow at the propellant tank outlets. When, in addition, ullage rockets are operating, the effective acceleration of gravity (for the fluid in the tanks) is very low but positive and is directed along the axes of symmetry of the tanks. It is the purpose of this project to investigate the behavior of the liquid-vapor (plus gas) interfaces in the propellant tanks under these conditions.

For the propellants under consideration (specifically, liquid hydrogen and liquid oxygen), the contact angles are small. Under the conditions of low acceleration (small Bond number) and small contact angle, the deflections from any constant height and the slopes of the liquid-gas interface will be moderate to large over a fair portion of the tank. Then those assumptions of the usual small perturbation theory which involve small deflections of this surface from a constant height and small slopes of that surface are invalid.

For the present problem it is appropriate to assume perturbations about a static equilibrium surface. It is then necessary that (1) the static equilibrium surface be known with considerable accuracy and that (2) dynamic variations from that surface be amenable to analysis.

Propellant tanks are usually circular cylinders, and a final aim of analysis must be to solve the problem in which the static case is axially symmetric. However, for the moment, the dynamic two-dimensional (three-dimensional, including time) problem seems difficult enough to handle; and efforts have been confined to solving that problem. The static solution required is therefore the two-dimensional one.

Treatment of the two-dimensional static problem is not new. The problem is reported by Otto[1] to have been treated by both Reynolds[2] and Benedikt[3] for the case of vertical walls. Some justification of the present paper, which deals in detail with the same problem, is therefore required. The justifications are these:

- Efforts to obtain the original papers of Reynolds and Benedikt were unsuccessful.

- What is needed here are detailed calculations for particular Bond numbers and for particular low contact angles. It could not be expected that either the needed accuracy was attained or the particular contact angles were treated in the original papers.

- It is shown that the results of Reynolds, as reported by Otto, are at variance with those to be obtained through the present analysis. It is believed that Reynolds' results are in error.

This paper starts with the governing equations, develops the differential equation for the vertical displacement of the surface, integrates that equation for positive Bond numbers and for small contact angles, treats the special case of Bond number zero, calculates results for a particular Bond number and contact angle, and, finally, questions the results of Reynolds.

STATEMENT OF THE PROBLEM

Consider two plane parallel walls a distance w apart, as in Figure 1, which extend to infinity (or to a very long distance compared to w) both out of and into the plane of the figure. In the plane of the figure choose a horizontal or x-axis perpendicular to the walls at an arbitrary vertical location and a y-axis perpendicular to the x-axis and half-way between the walls*. Let the effective acceleration of gravity, g, act in the minus y-direction. Let a fluid fill the lower part of the region to a mean depth, h_m. Consider the density of the fluid, ρ, its surface tension, T, the pressure of the gas (plus vapor) above the liquid, P, the effective acceleration of gravity, and the contact angle of the liquid at the wall, θ, to be constant. Find the location of the surface of the liquid as a function of x.

*The latter choice is made only because of the symmetry of the problem but is not essential in obtaining the solution.

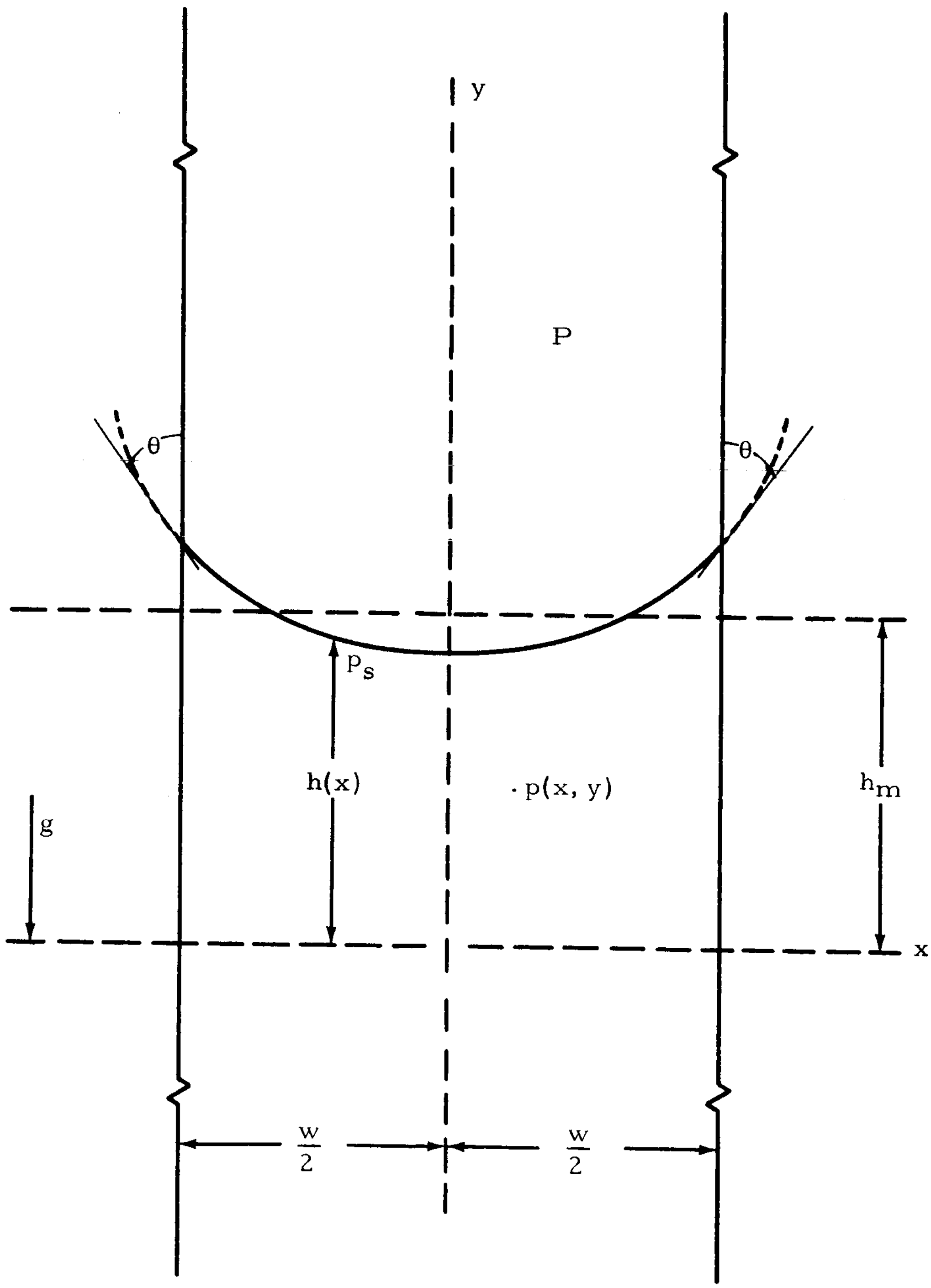

Figure 1. Fluid Between Two Parallel Plane Walls Which are Also Parallel to the Effective Acceleration of Gravity

ANALYSIS

PRESSURE IN THE LIQUID AS A FUNCTION OF DEPTH

The pressure, p, at any point (x, y) in the fluid is given by

$$p + \rho g y = P + A \tag{1}$$

where A is a constant to be determined. Let the equation of the surface be $y = h(x)$ and the pressure at the surface be p_s. Then

$$p_s + \rho g h = P + A \quad . \tag{2}$$

The pressure at the surface is related to the surface tension, T, by

$$p_s = P - \frac{T}{R} \tag{3a}$$

where R is the radius of curvature of the surface, or by

$$p_s = P - T \frac{d^2h/dx^2}{[1 + (dh/dx)^2]^{3/2}} \tag{3b}$$

The elimination of p_s from Equations 2 and 3b yields

$$\frac{T\,(d^2h/dx^2)}{[1 + (dh/dx)^2]^{3/2}} = \rho g h - A = \rho g\,(h - \tau) \tag{4}$$

where

$$A = \rho g \tau \quad . \tag{5}$$

τ is a mathematical (not physical) value of h for which $d^2h/dx^2 = 0$. ($h = \tau$ will generally be below the surface of the liquid.) Before Equation 4 is integrated, τ will be obtained. The pressure at any point in the liquid will then be known.

The terms of Equation 4 are integrated with respect to x from wall to wall. The result is

$$\int_{-w/2}^{w/2} \frac{(d^2h/dx^2)\,dx}{[1 + (dh/dx)^2]^{3/2}} = \left\{ \frac{dh/dx}{[1 + (dh/dx)^2]^{1/2}} \right\}_{-w/2}^{w/2} = \frac{\rho g}{T} \int_{-w/2}^{w/2} (h - \tau)\,dx$$

$$= \frac{\rho g}{T} \left[\int_{-w/2}^{w/2} h(x)\,dx - \tau w \right]$$

so that

$$\tau = \frac{1}{w} \left(\int_{-w/2}^{w/2} h(x)\,dx - \frac{T}{\rho g} \left\{ \frac{dh/dx}{[1 + (dh/dx)^2]^{1/2}} \right\}_{-w/2}^{w/2} \right) \tag{6}$$

Let θ be the contact angle of the liquid at the wall. Then

$$\frac{dh}{dx} = \cot\theta \quad \text{for } x = \frac{w}{2}$$

$$\frac{dh}{dx} = -\cot\theta \quad \text{for } x = -\frac{w}{2}$$

so that

$$\left\{ \frac{dh/dx}{[1 + (dh/dx)^2]^{1/2}} \right\}_{-w/2}^{w/2} = 2\cos\theta$$

Now

$$\int_{-w/2}^{w/2} h(x)\,dx$$

is the cross-sectional area of the fluid as measured from the x-axis. It is a function of the amount of fluid in the tank. Let h_m be the mean value of h. Then

$$h_m = \frac{1}{w} \int_{-w/2}^{w/2} h(x)\,dx \tag{7}$$

and Equation 5 can be written

$$\tau = h_m - \frac{2\,T}{\rho\, g\, w} \cos\,\theta \quad . \tag{8}$$

Then

$$A = \rho\, g\, \tau = \rho\, g\; h_m - \frac{2\,T}{w} \cos\,\theta$$

and Equation 1 becomes

$$p + \rho\, g\, y = P + \rho\, g\; h_m - \frac{2\,T}{w} \cos\,\theta \tag{9}$$

so that the pressure is calculable for any point in the fluid. Observe that this equation holds no matter where the origin of coordinates is located since its location affects y and h_m similarly.

INTEGRATION OF THE DIFFERENTIAL EQUATION FOR BOND NUMBERS GREATER THAN ZERO

Equation 4 will now be integrated. Let $\hbar = h - \tau$. Then Equation 4 becomes

$$\frac{d^2\hbar/dx^2}{[1 + (d\hbar/dx)^2]^{3/2}} = \frac{\rho\, g}{T}\, \hbar \quad .$$

This can be integrated once after multiplying both sides by $d\hbar/dx$. The result is

$$\frac{-1}{[1 + (d\hbar/dx)^2]^{1/2}} = \frac{\rho\, g}{2\,T}\, (\hbar^2 - c)$$

where c is the constant of integration. Let $\hbar_o$ be the value of $\hbar$ where $d\hbar/dx = 0$ (at the middle of the tank). (The value of $\hbar_o$ is at present unknown.) Then

$$c = \hbar_o^2 + \frac{2\,T}{\rho\, g}$$

and

$$\frac{1}{[1 + (d\hbar/dx)^2]^{1/2}} = 1 - \frac{\rho g}{2T} (\hbar^2 - \hbar_o^2) \tag{10}$$

or

$$\frac{d\hbar}{\left\{\frac{1}{[1 - (\rho g/2T)(\hbar^2 - \hbar_o^2)]^2} - 1\right\}^{\frac{1}{2}}} = \pm\, dx$$

so that

$$\int_{\hbar_o}^{\hbar} \frac{d\hbar}{\left\{\frac{1}{[1 - (\rho g/2T)(\hbar^2 - \hbar_o^2)]^2} - 1\right\}^{\frac{1}{2}}} = \pm \int_0^x dx = \pm\, x$$

Now let $s = (\rho g/2T)^{\frac{1}{2}}\, \hbar$, $s_o = (\rho g/2T)^{\frac{1}{2}}\, \hbar_o$, then

$$\left(\frac{2T}{\rho g}\right)^{\frac{1}{2}} \int_{s_o}^{s} \frac{ds}{\left\{\frac{1}{[1 - (s^2 - s_o^2)]^2} - 1\right\}^{\frac{1}{2}}} = \pm\, x$$

or

$$\left(\frac{2}{B_o}\right)^{\frac{1}{2}} \int_{s_o}^{s} \frac{(1 + s_o^2 - s^2)\, ds}{[(s^2 - s_o^2)(2 - s^2 + s_o^2)]^{\frac{1}{2}}} = \pm \frac{x}{w} \tag{11}$$

where B_o is the Bond number, defined here by

$$B_o = \frac{\rho g w^2}{T} \quad . \tag{12}$$

On making the successive substitutions,

$$t = 1 + s_o^2 - s^2$$

$$\sin\alpha = t$$

$$\phi - \frac{\pi}{2} = \alpha$$

$$2\psi = \phi$$

while retaining s in the upper limit of integration and expressing constants in terms of s_o, this becomes

$$\pm\frac{x}{w} = \frac{1}{(B_o)^{\frac{1}{2}}}\left[2\left(1+\frac{s_o^2}{2}\right)^{\frac{1}{2}}\int_{\pi/2}^{(\pi/4)+(1/2)\sin^{-1}(1+s_o^2-s^2)}\left(1-\frac{1}{1+s_o^2/2}\sin^2\psi\right)^{\frac{1}{2}}d\psi\right.$$

$$\left.-\frac{1+s_o^2}{\left(1+\frac{s_o^2}{2}\right)^{\frac{1}{2}}}\int_{\pi/2}^{(\pi/4)+(1/2)\sin^{-1}(1+s_o^2-s^2)}\frac{d\psi}{\left(1-\frac{1}{1+s_o^2/2}\sin^2\psi\right)^{\frac{1}{2}}}\right]$$

$$= \frac{1}{(B_o)^{\frac{1}{2}}}\left(\frac{1+s_o^2}{(1+s_o^2/2)^{\frac{1}{2}}}\left\{K\left[\frac{1}{(1+s_o^2/2)^{\frac{1}{2}}}\right] - F\left[\frac{1}{(1+s_o^2/2)^{\frac{1}{2}}}, \frac{\pi}{4}+\frac{1}{2}\sin^{-1}(1+s_o^2-s^2)\right]\right\}\right.$$

$$\left.-2\left(1+\frac{s_o^2}{2}\right)^{\frac{1}{2}}\left\{E\left[\frac{1}{(1+s_o^2/2)^{\frac{1}{2}}}\right] - E\left[\frac{1}{(1+s_o^2/2)^{\frac{1}{2}}}, \frac{\pi}{4}+\frac{1}{2}\sin^{-1}(1+s_o^2-s^2)\right]\right\}\right) \tag{13}$$

where

$$K\left[\frac{1}{(1+s_o^2/2)^{\frac{1}{2}}}\right]$$

is the complete elliptic integral of the first kind,

$$F\left[\frac{1}{(1+s_o^2/2)^{\frac{1}{2}}}, \frac{\pi}{4}+\frac{1}{2}\sin^{-1}(1+s_o^2-s^2)\right]$$

is the elliptic integral of the first kind,

$$E\left[\frac{1}{(1+s_o^2/2)^{\frac{1}{2}}}\right]$$

is the complete elliptic integral of the second kind, and

$$E\left[\frac{1}{(1+s_o^2/2)^{\frac{1}{2}}}, \frac{\pi}{4}+\frac{1}{2}\sin^{-1}(1+s_o^2-s^2)\right]$$

is the elliptic integral of the second kind.

THE RANGE OF VALUES OF s

The dimensionless variable s is related to the physical variable h through

$$s = \left(\frac{\rho g}{2T}\right)^{\frac{1}{2}} \hbar = \left(\frac{B_o}{2}\right)^{\frac{1}{2}} \frac{\hbar}{w} = \left(\frac{B_o}{2}\right)^{\frac{1}{2}} \left(\frac{h-\tau}{w}\right)$$

$$= \left(\frac{B_o}{2}\right)^{\frac{1}{2}} \left[\frac{h-h_m}{w} + \frac{2}{B_o}\cos\theta\right]$$

$$= \left(\frac{2}{B_o}\right)^{\frac{1}{2}} \cos\theta + \left(\frac{B_o}{2}\right)^{\frac{1}{2}} \left(\frac{h-h_m}{w}\right) \tag{14}$$

or

$$\frac{h-h_m}{w} = \left(\frac{2}{B_o}\right)^{\frac{1}{2}} s - \frac{2}{B_o}\cos\theta \quad . \tag{15}$$

It follows that the range of values of s also determines the range of values of h (except at zero Bond number).

An equation previously obtained was

$$\frac{1}{[1 + (d\hbar/dx)^2]^{\frac{1}{2}}} = 1 - \frac{\rho g}{2T} (\hbar^2 - \hbar_o^2) \tag{10}$$

which can be written

$$\frac{1}{[1 + (d\hbar/dx)^2]^{\frac{1}{2}}} = 1 + s_o^2 - s^2 \quad . \tag{16}$$

The maximum values of $|d\hbar/dx| = |dh/dx|$ and of s occur at the wall $[x = \pm (1/2)\, w]$, where $|dh/dx| = \cot\theta$. Call the corresponding value of s, s_u (for s upper). Then at the wall

$$\frac{1}{[1 + (d\hbar/dx)^2]^{\frac{1}{2}}} = \sin\theta$$

and

$$s_u^2 = 1 + s_o^2 - \sin\theta \tag{17}$$

Substitution of $s^2 = s_u^2$ at $\frac{x}{w} = \pm\frac{1}{2}$ into Equation 13 yields

$$+\frac{1}{2} = \frac{1}{(B_o)^{\frac{1}{2}}} \left[2\left(1 + \frac{s_o^2}{2}\right)^{\frac{1}{2}} \int_{\pi/2}^{(\pi/4)+(\theta/2)} \left(1 - \frac{1}{1+s_o^2/2}\sin^2\psi\right)^{\frac{1}{2}} d\psi \right.$$

$$\left. - \frac{1+s_o^2}{\left(1 + \frac{s_o^2}{2}\right)^{\frac{1}{2}}} \int_{\pi/2}^{(\pi/4)+(\theta/2)} \frac{d\psi}{\left(1 - \frac{1}{1+s_o^2/2}\sin^2\psi\right)^{\frac{1}{2}}} \right]$$

$$= \frac{1}{(B_o)^{\frac{1}{2}}} \left(\frac{1+s_o^2}{(1+s_o^2/2)^{\frac{1}{2}}} \left\{ K\left[\frac{1}{(1+s_o^2/2)^{\frac{1}{2}}}\right] - F\left[\frac{1}{(1+s_o^2/2)^{\frac{1}{2}}}, \frac{\pi}{4} + \frac{\theta}{2}\right] \right\} \right.$$

$$\left. - 2\left(1 + \frac{s_o^2}{2}\right)^{\frac{1}{2}} \left\{ E\left[\frac{1}{(1+s_o^2/2)^{\frac{1}{2}}}\right] - E\left[\frac{1}{(1+s_o^2/2)^{\frac{1}{2}}}, \frac{\pi}{4} + \frac{\theta}{2}\right] \right\} \right) \tag{18}$$

This equation determines s_o^2, and Equation 17 then determines s_u^2.

In order to be able to convert s back to a physical coordinate (h), it remains to determine the signs of s_o and s. Equation 13 has significance only for $s^2 \geqq s_o^2$ or for

$$s^2 - s_o^2 = (s - s_o)(s + s_o) \geqq 0 \quad . \tag{19}$$

Now from Equation 14 one obtains

$$s_o = \left(\frac{2}{B_o}\right)^{\frac{1}{2}} \cos\theta + \left(\frac{B_o}{2}\right)^{\frac{1}{2}} \frac{h_o - h_m}{w} \tag{20}$$

$$s + s_o = 2\left(\frac{2}{B_o}\right)^{\frac{1}{2}} \cos\theta + \left(\frac{B_o}{2}\right)^{\frac{1}{2}} \left(\frac{h_o + h - 2\,h_m}{w}\right)$$

$$s - s_o = \left(\frac{B_o}{2}\right)^{\frac{1}{2}} \left(\frac{h - h_o}{w}\right)$$

For $\theta < \frac{\pi}{2}$ it follows that physically $\frac{h - h_o}{w} \geqq 0$ so that

$$s - s_o \geqq 0 \quad .$$

Then, from Equation 19,

$$s + s_o \geqq 0$$

and from the above two equations

$$s \geqq 0$$

$$s_o \geqq 0$$

and

$$s \geqq s_o \geqq 0 \quad .$$

For $\theta > \frac{\pi}{2}$ it follows that

$$\frac{h - h_o}{w} \leqq 0$$

$$s - s_o \leqq 0$$

$$s + s_o \leqq 0$$

$$s \leqq 0$$

$$s_o \leqq 0$$

and

$$s \leqq s_o \leqq 0 \quad .$$

CALCULATION PROCEDURE FOR POSITIVE BOND NUMBERS AND FOR SMALL CONTACT ANGLES

A procedure for determining the shape of the interface can now be given. The steps in this procedure are:

(1) Calculate the Bond number from

$$B_o = \frac{\rho g w^2}{T}$$

(2) Determine s_o, using B_o and the contact angle, θ, from Equation 18.

(3) Calculate s_u from Equation 17.

(4) Select a number of values of s such that $s_o \leqq s \leqq s_u$. Calculate the corresponding values of $\frac{h - h_m}{w}$ using Equation 15 and of $\frac{x}{w}$ using Equation 13.

(5) Plot $\frac{h - h_m}{w}$ versus $\frac{x}{w}$.

The second of the above steps requires special consideration. The value of s_o (or of s_o^2) must be found by iteration; and, in order to minimize the labor involved, it is desirable to specify both maximum and minimum values of s_o.

The proper minimum value of s_o appears to be $s_o = 0$. However, at $s_o = 0$ the integral sum on the right side of Equation 18 becomes infinite (or that equation cannot possibly be satisfied unless the Bond number is also infinite). For finite Bond numbers, then, the substitution $s_o = 0$ yields an infinite value for the right side of the equation.

For the problems of present interest θ is small, and an upper limit for s_o will be found primarily for such problems. For $\theta < \frac{\pi}{2}$, $\frac{h_o - h_m}{w} < 0$, and it follows from Equation 20 that for $\theta < \frac{\pi}{2}$,

$$0 \leqq s_o < \left(\frac{2}{B_o}\right)^{\frac{1}{2}} \cos \theta \quad .$$

For small θ one may as well use

$$0 \leqq s_o \leqq \left(\frac{2}{B_o}\right)^{\frac{1}{2}} \quad .$$

$\left(\frac{2}{B_o}\right)^{\frac{1}{2}}$ is then the required upper bound. One may use it as a first guess in the iteration process for calculating s_o.

A FORTRAN IV computer program for calculating $\frac{h - h_m}{w}$ and $\frac{x}{w}$ for small contact angles has been written by Allen G. Collier of the Scientific Programming Section of Brown Engineering Company, Inc. His main program is given in Appendix A while his subroutine for calculating the elliptic integrals is given in Appendix B. His elliptic integral subroutine utilized the method of Fettis and Cashin[4].

A SPECIAL CASE: BOND NUMBER ZERO

In this section it is demonstrated that at Bond number zero the surface of the fluid is that of a right circular cylinder. The equation of the surface is obtained.

From Equations 4 and 8 it follows that

$$\frac{T}{R} = \rho g (h - \tau)$$

$$= \rho g \left(h - h_m + \frac{2T}{\rho g w} \cos \theta\right) \quad ,$$

where R is the local radius of curvature of the surface, or that

$$\frac{w}{R} = 2 \cos \theta + B_o \left(\frac{h - h_m}{w}\right) \quad .$$

At Bond number zero, then, it follows that

$$R = \frac{w}{2} \sec \theta \quad , \tag{21}$$

that the radius of curvature of the surface is a constant, or that in the h-x plane the cross-section of the surface is circular with the radius as given above. (This equation is expected since at Bond number zero, say $g = 0$, the pressure must be the same at all points in the fluid, which means that the radius of curvature of the surface must be constant, and since the fluid must reach the wall at the contact angle. This same type of result is found by Li[5], who determined that the interface was spherical for a cylindrical tank, using the principle of minimum energy.)

It remains only to obtain the equation of the surface. The equation of the surface (see Figure 2) is

$$[h - (R + h_o)]^2 + x^2 = R^2$$

from which

$$h = R + h_o - (R^2 - x^2)^{\frac{1}{2}} \quad . \tag{22}$$

h_m is assumed to be known

$$h_m = \frac{1}{w} \int_{-w/2}^{w/2} h(x)\, dx = \frac{1}{w} \int_{-w/2}^{w/2} \left[R + h_o - (R^2 - x^2)^{\frac{1}{2}}\right] dx$$

$$= R + h_o - \frac{1}{w} \int_{-w/2}^{w/2} (R^2 - x^2)^{\frac{1}{2}}\, dx \tag{23}$$

On solving the above two equations for $h - h_m$, substituting the value of R from Equation 21, and integrating, one obtains

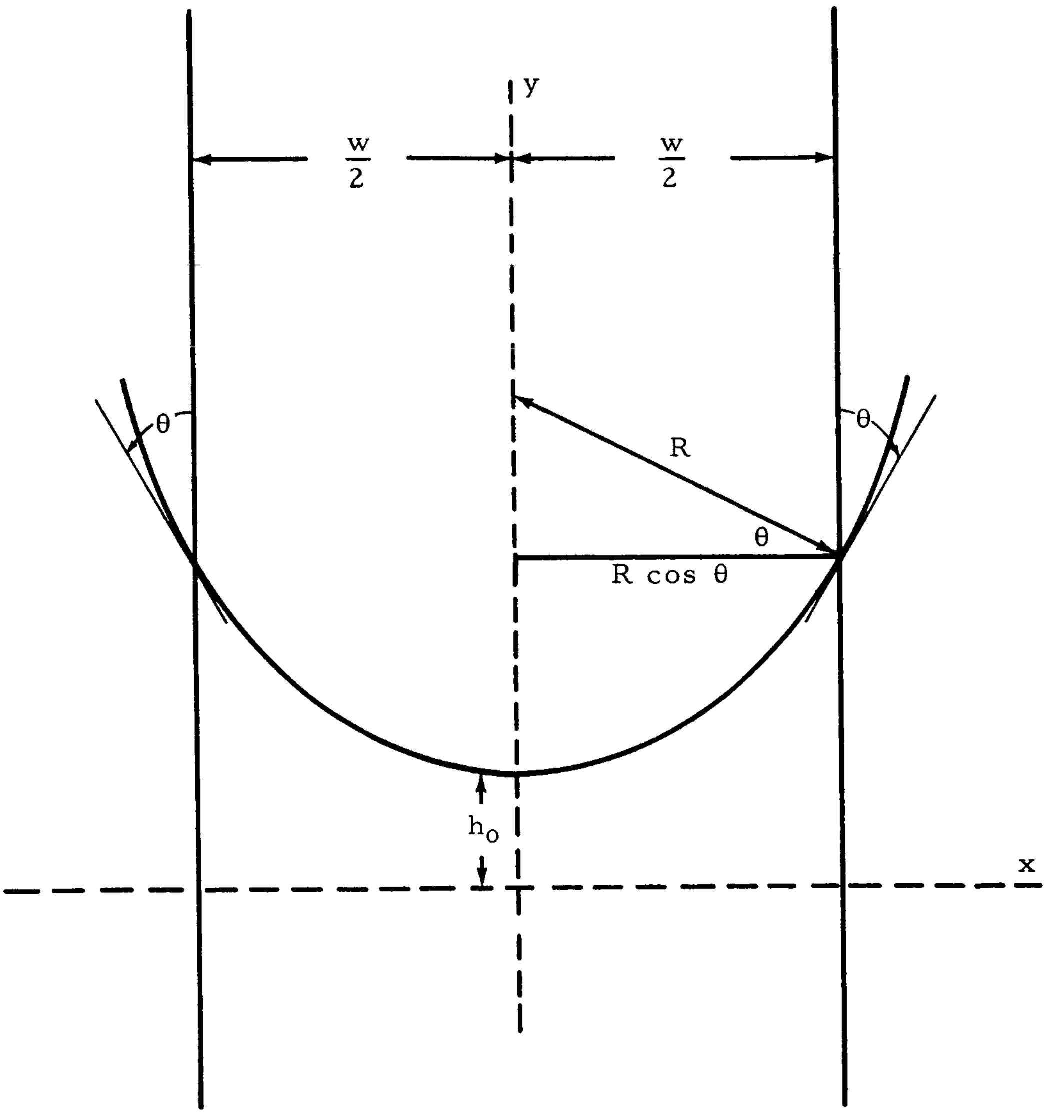

Figure 2. The Surface of the Liquid at Bond Number Zero

$$h - h_m = \frac{1}{w}\int_{-w/2}^{w/2} (R^2 - x^2)^{\frac{1}{2}}\, dx - (R^2 - x^2)^{\frac{1}{2}}$$

$$= \frac{1}{w}\int_{-w/2}^{w/2} \left(\frac{w^2}{4}\sec^2\theta - x^2\right)^{\frac{1}{2}} dx - \left(\frac{w^2}{4}\sec^2\theta - x^2\right)^{\frac{1}{2}}$$

$$= \frac{w}{2}\left[\frac{1}{2}\tan\theta + \frac{1}{2}\left(\frac{\pi}{2} - \theta\right)\sec^2\theta - \left(\sec^2\theta - \frac{4x^2}{w^2}\right)^{\frac{1}{2}}\right] , \qquad (24)$$

the equation of the surface.

DISCUSSION OF RESULTS

RESULTS FOR ONE BOND NUMBER AND CONTACT ANGLE

The missile of immediate concern is the Saturn V. The fuel (liquid hydrogen) tank of the S-IV B stage of that missile was selected for study under low acceleration (corresponding to the use of ullage rockets).

The input data for the calculations are given in Table 1. In the table the width, w, used was the diameter of the tank. (It is recognized that the real tank is axially symmetric.) The elliptic integrals were to be calculated to an accuracy of 10^{-8}.

The output data from the calculations are given in Table 2 and are plotted in Figure 3. Several observations are to be made concerning these data. These are:

(1) The Bond number calculated is approximately 120. The Bond number using the half-width of the tank (the radius of the real tank) as a characteristic length would have been about 30.

(2) The calculation is limited to a comparatively small number of values of s. Obviously as many more points could have been calculated as desired.

(3) The value of s_o is clearly the first value of s in the table.

(4) The difference between the final value of x/w and 1/2 indicates the overall accuracy of the program.

(5) Deflections and slopes of the surface near the wall are not small even for this rather high Bond number. Slopes of the surface are appreciable over about twenty percent of the width of the tank.

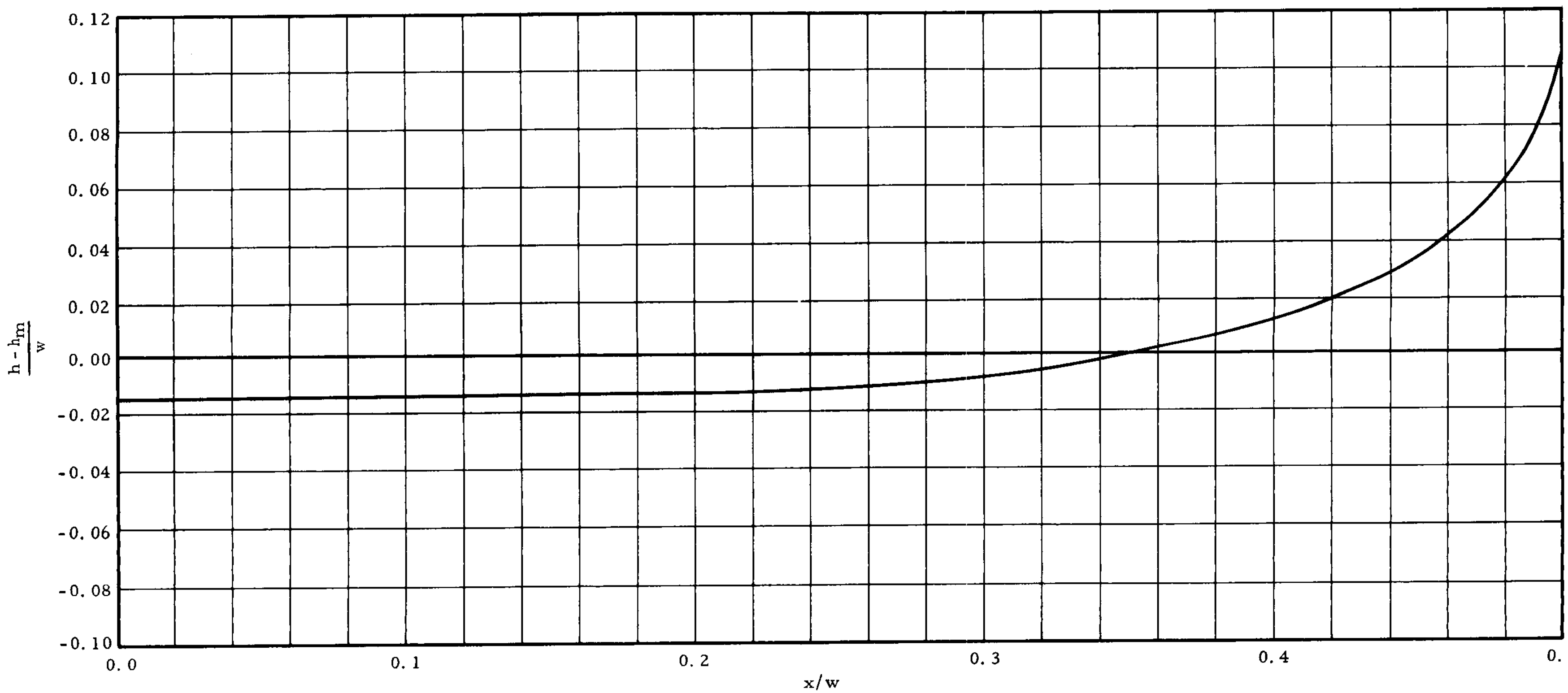

Figure 3. Displacement of the Surface, $\theta = 5°$, $B_o = 119.5$

TABLE 1

INPUT DATA FOR CALCULATIONS

$\rho = 4.44$ lb/ft^3 (liquid hydrogen)

$g = 32.3 \times 10^{-5}$ ft/sec^2

$T = 175 \times 10^{-6} \times 32.2$ pdl/ft (liquid hydrogen)

$w = 21.7$ ft

$\theta = 5° = \dfrac{5\pi}{180}$ rad

ACC = 0.00000001 (Accuracy of elliptic integral)

TABLE 2

OUTPUT DATA FROM CALCULATIONS

Bond Number s	0.11947152+03 x/w	$\theta = 5°$ $\frac{h - h_m}{w}$
0.55026502-02	0.00000000	-0.15964731-01
0.23094168-01	0.19329778-00	-0.13688658-01
0.40685687-01	0.24597565-00	-0.11412585-01
0.58277206-01	0.27900860-00	-0.91365126-02
0.75868724-01	0.30314618-00	-0.68604399-02
0.93460243-01	0.32216351-00	-0.45843671-02
0.11105176+00	0.33784146-00	-0.23082944-02
0.21660087-00	0.39780985-00	0.11348142-01
0.32214998-00	0.43215343-00	0.25004579-01
0.42769910-00	0.45528390-00	0.38661015-01
0.53324821-00	0.47180017-00	0.52317452-01
0.63879732-00	0.48372990-00	0.65973888-01
0.74434643-00	0.49210297-00	0.78630324-01
0.84989554-00	0.49746116-00	0.93286761-01
0.95544466-00	0.50003424-00	0.10694320+00

THE RESULTS OF REYNOLDS ARE QUESTIONED

It is reported by Otto[1] that Reynolds[2] has attacked the present problem, and Otto reports Reynolds' results as a plot of vertical displacement versus horizontal distance for various Bond numbers at fixed contact angle (Figure 5 of Reference 1). Unfortunately efforts to obtain Reynolds' paper were unsuccessful so that comments on his analysis cannot be made.

The result obtained in the preceding section for the displacement of the liquid at the walls appeared to be too large to fit Reynolds' results. (Reynolds' characteristic length is half of that used in this paper.) As a result, some method of checking the accuracy of his results was sought. The method used was to establish a lower limit for $h_u - h_m$ (the displacement at the wall) for small values of θ and to compare the results obtained at Reynolds' Bond numbers and at one of his two contact angles with his deflections (at the wall).

From Equations 14 and 17 it follows that for $\theta < \frac{\pi}{2}$

$$s_u = (1 + s_o^2 - \sin\theta)^{\frac{1}{2}} = \left(\frac{2}{B_o}\right)^{\frac{1}{2}} \cos\theta + \left(\frac{B_o}{2}\right)^{\frac{1}{2}} \frac{h_u - h_m}{w}$$

or that

$$\frac{h_u - h_m}{w} = \left(\frac{2}{B_o}\right)^{\frac{1}{2}} (1 + s_o^2 - \sin\theta)^{\frac{1}{2}} - \frac{2}{B_o} \cos\theta \quad .$$

s_o is a function of both Bond number and contact angle. For small contact angles it is small compared to unity for moderate Bond numbers, approaches zero as the Bond number increases, and approaches infinity as the Bond number decreases towards zero. The inequality

$$\frac{h_u - h_m}{w} > \left(\frac{2}{B_o}\right)^{\frac{1}{2}} (1 - \sin\theta)^{\frac{1}{2}} - \frac{2}{B_o} \cos\theta = F(B_o, \theta) \qquad (25)$$

can then be expected to be close to an equality except at low Bond numbers (for low contact angles).

It remains to express this inequality in Reynolds' notation and to compare the values calculated using it with the corresponding values of Reynolds. Reynolds uses the variables

$$Y\ (= h - h_m),\ L\left(= \frac{w}{2}\right),\ \text{and } B = \frac{\rho g L^2}{T}\ \left(= \frac{B_o}{4}\right) \quad .$$

Let $Y_u = h_u - h_m$, then Equation 25, when expressed in terms of these variables, is

$$\frac{Y_u}{L} > \left(\frac{2}{B}\right)^{\frac{1}{2}} (1 - \sin\theta)^{\frac{1}{2}} - \frac{\cos\theta}{B} = F\ (B, \theta) \quad . \tag{26}$$

Reynolds plots Y/L versus x/L for $\theta = 10°$ for B = 0, 0.58, 2.9, 5.6, and ∞. Of his curves, that for Bond number zero is clearly a circle; and that for Bond number infinity is clearly a horizontal line. Of the remaining three, Table 3, in which $F(B, \theta)$ and Reynolds' deflections at the walls are compared, indicates that for the two Bond numbers for which Equation 26 yields positive results, namely for Bond numbers 2.9 and 5.6, his results are too low and are therefore questionable unless there is an error in the present analysis.

TABLE 3

COMPARISON BETWEEN $F(B, \theta)$ AND REYNOLDS' RESULTS AT THE WALLS

$\theta = 10°$

B	$F(B, \theta)$	Y_u/L (Reynolds)
0.58	-0.0097	> 0.44
2.9	+0.4154	< 0.38
5.6	+0.3674	< 0.17

CONCLUSIONS

The present report results from a need for accuracy in static calculations (for low contact angles and Bond numbers) in order that the calculated static surfaces may serve as the unperturbed surfaces for dynamic calculations. Results are obtained for one particular Bond number and contact angle.

It is believed that the accuracy used in calculating the elliptic integrals involved is sufficient. It is shown that for one contact angle and for at least two Bond numbers the displacements calculated (at the walls) will be larger than those obtained by Reynolds. Reynolds' results are believed to be in error.

LIST OF REFERENCES

1. Otto, E. W., "Static and Dynamic Behavior of the Liquid-Vapor Interface During Weightlessness", Published by NASA in Proceedings of the Conference on Propellant Tank Pressurization and Stratification, Vol. II, pp. 281-354, January 20, 21, 1965

2. Reynolds, William C., "Hydrodynamic Considerations for the Design of Systems for Very Low Gravity Environments", Rep. LG-1, Stanford University, September 1961

3. Benedikt, E. T., "Epihydrostatics of Liquids in Vertical Tanks", Rep. ASG-TM-61-48, Norair Div., Northrop Corp., June 1961

4. Fettis, H. E. and J. C. Caslin, "FORTRAN Programs for Computing Elliptic Integrals and Functions", Applied Mathematics Research Laboratory, Aerospace Research Laboratories, Wright-Patterson Air Force Base, May 1964

5. Li, Ta, "Hydrostatics in Various Gravitational Fields", Journal of Chemical Physics, Vol. 36, No. 9, pp. 2369-2375, May 1, 1962

APPENDIX A

FORTRAN IV PROGRAM FOR $\theta < \frac{\pi}{2}$

DESCRIPTION OF PROGRAM

1. $B_o = \dfrac{\rho g w^2}{T}$

2. Solve the following equation for s_o

$$\frac{(B_o)^{\frac{1}{2}}}{2} = 2\left(1 + \frac{s_o^2}{2}\right)^{\frac{1}{2}} \left\{ \int_0^{(\pi/4)+(\theta/2)} \left[1 - \left(\frac{1}{1 + s_o^2/2}\right)\sin^2\psi\right]^{\frac{1}{2}} d\psi \right.$$

$$\left. - \int_0^{\pi/2} \left[1 - \left(\frac{1}{1 + s_o^2/2}\right)\sin^2\psi\right]^{\frac{1}{2}} d\psi \right\} - \frac{(1 + s_o^2)}{(1 + s_o^2/2)^{\frac{1}{2}}}$$

$$\left\{ \int_0^{(\pi/4)+(\theta/2)} \frac{d\psi}{\left[1 - \left(\frac{1}{1 + s_o^2/2}\right)\sin^2\psi\right]^{\frac{1}{2}}} - \int_0^{\pi/2} \frac{d\psi}{\left[1 - \left(\frac{1}{1 + s_o^2/2}\right)\sin^2\psi\right]^{\frac{1}{2}}} \right\}$$

3. Select values of s ranging from s_o to $(1 + s_o^2 - \sin\theta)^{\frac{1}{2}}$.

4. Solve the following equation for each value of s_i

$$\left(\frac{x}{w}\right)_i = \Bigg(2\left(1+\frac{s_o^2}{2}\right)^{\frac{1}{2}} \Bigg\{ \int_0^{(\pi/4)+(1/2)\sin^{-1}(1+s_o^2-s_i^2)} \left[1-\left(\frac{1}{1+s_o^2/2}\right)\sin^2\psi\right]^{\frac{1}{2}} d\psi$$

$$- \int_0^{\pi/2} \left[1-\left(\frac{1}{1+s_o^2/2}\right)\sin^2\psi\right]^{\frac{1}{2}} d\psi \Bigg\}$$

$$- (1+s_o^2)\,\frac{1}{\left(1+\frac{s_o^2}{2}\right)^{\frac{1}{2}}} \Bigg\{ \int_0^{(\pi/4)+(1/2)\sin^{-1}(1+s_o^2-s_i^2)} \frac{d\psi}{\left[1-\left(\frac{1}{1+s_o^2/2}\right)\sin^2\psi\right]^{\frac{1}{2}}}$$

$$- \int_0^{\pi/2} \frac{d\psi}{\left[1-\left(\frac{1}{1+s_o^2/2}\right)\sin^2\psi\right]^{\frac{1}{2}}} \Bigg\} \Bigg) (B_o)^{-\frac{1}{2}}$$

5. Solve the following equations for each value of s_i

$$\left(\frac{\eta}{w}\right)_i = s_i\left(\frac{2}{B_o}\right)^{\frac{1}{2}} - \frac{2}{B_o}\cos\theta$$

DATA SHEETS

Data sheets used contain the following information.

Columns	Item	Description
1-10	ρ	Density, lb/ft^3
11-20	g	Acceleration of gravity, ft/sec^2
21-30	T	Surface tension, pdl/ft or (lb/ft) $\times$ 32.3
31-40	w	Width of tank, ft
41-50	θ	Angle of contact of fuel with side of tank, degree
51-60	ACC	Accuracy of elliptic integral

EXPLANATION OF PARAMETERS AND CROSS-REFERENCE BETWEEN SYMBOLS (INPUT AND OUTPUT)

Algebraic Symbol	FORTRAN Symbol	Description
	A	Value of the integral $\int_0^{(\pi/4)+(\theta/2)} \left(1 - \frac{1}{1+s_o^2/2} \sin^2 \psi\right)^{\frac{1}{2}} d\psi$
	ACC	Accuracy of elliptic integral
	B	Value of the integral $\int_0^{\pi/2} \left(1 - \frac{1}{1+s_o^2/2} \sin^2 \psi\right)^{\frac{1}{2}} d\psi$
	BB	Value to be compared with Bond number in evaluating s_o
B_o	BZ	Bond number

Algebraic Symbol	FORTRAN Symbol	Description
	C	Value of the integral $\int_0^{(\pi/4)+(\theta/2)} \frac{d\psi}{\left(1 - \frac{1}{1+s_o^2/2} \sin^2 \psi\right)^{\frac{1}{2}}}$
	D	Value of the integral $\int_0^{\pi/2} \frac{d\psi}{\left(1 - \frac{1}{1+s_o^2/2} \sin^2 \psi\right)^{\frac{1}{2}}}$
	DB	Difference between BB and BZ for a particular value of s_o
	DBL	Difference between BB and BZ for a lower limit of s_o
	DBU	Difference between BB and BZ for an upper limit of s_o
	DS	Increments of s used for calculating (x/w) and (η/w)
$\frac{h - h_m}{w}$	ETAW	Ratio of the vertical displacement from the mean of a point to the width of the tank (η/w)
g	G	Acceleration of gravity
	I	Index item
	K	Control item
ρ	RHO	Density
	RK	K of $\int (1 - k^2 \sin^2 \psi)^{\frac{1}{2}} d\psi$
s	S	Dimensionless quantity

Algebraic Symbol	FORTRAN Symbol	Description
	SL	A lower limit of s_o
	SU	An upper limit of s_o
s_u	SU	The upper limit of s
s_o	SO	The lower limit of s
	S1	Used as a next guess in iterating for s_o
T	T	Surface tension
	THETA	Angle of contact of fluid with side of tank in degrees
θ	THETAR	THETA in radians
	UL	Upper limit of elliptic integral
w	W	Width of tank
(x/w)	XW	Ratio of the lateral displacement of a point from the middle of the tank to the width of the tank

FORMAT OF DECK AND OPERATING INSTRUCTIONS

The format of the operating decks is dependent upon the operating system of the computer being used. This program was written and checked out on the Univac 1107. However, any computer that accepts FORTRAN IV could be used. The program was written so that it could be run under a monitor operating system. The general format of the operating deck should be:

1. Control cards
2. Source or object cards for the main program
3. Source or object cards for the ELLIP subroutine
4. Card indicating that data follows

5. Data card(s)

6. Blank card

7. Card(s) returning control to monitor system

FORTRAN IV PROGRAM LISTING OF MAIN PROGRAM

```
      DIMENSION ETAW(101),S(101),XW(101)
10    FORMAT (6F10.0)
20    FORMAT (40H0        S                    X/W                  ETA/W/(3E16.8))
30    FORMAT (15H1  BOND NUMBER E16.8)
40    FORMAT (1H 3E16.8)
50    FORMAT (1H 7E16.8)
90    READ (5,10)RHO,G,T,W,THETA,ACC
      IF (W) 96,95,96
95    CALL EXIT
96    THETAR=.017453292*THETA
      BZ=RHO*G*W*W/T
      K=1
      SU=SQRT(2./BZ)
      UL=45.+THETA/2.
      S0=SU
100   RK=SQRT(2./(2.+S0*S0))
      CALL ELLIP (RK,UL,2,ACC,C,A)
      CALL ELLIP (RK,90.,2,ACC,D,B)
      BB=2./RK*(A-B)-(1.+S0*S0)*RK*(C-D)
      WRITE (6,50)RK,S0,A,B,C,D,BB
      DB=BZ-4.*BB*BB
      GO TO (110,120,130),K
110   DBU=DB
      S0=.5*S0
      K=2
```

FORTRAN IV PROGRAM LISTING OF MAIN PROGRAM

```
      GO TO 100
120   DBL=DB
      IF (DBL*DBU) 125,180,122
122   S0=.5*S0
      GO TO 100
125   SL=S0
      K=3
      GO TO 160
130   IF (DB*DBU) 140,180,150
140   DBL=DB
      SL=S0
      GO TO 160
150   DBU=DB
      SU=S0
160   S1=SL+DBL*(SL-SU)/(DBU-DBL)
      WRITE (6,60)S1,DBL,DBU
      IF (ABS((S1-S0)/S1)-ACC) 180,180,170
170   S0=S1
      GO TO 100
180   SU=SQRT(1.+S0*S0-SIN(THETAR))
      WRITE (6,30)DZ
      DS=(SU-S0)/100.
      S(1)=S0
      DO 190 I=2,101
185   S(I)=S(I-1)+DS
```

FORTRAN IV PROGRAM LISTING OF MAIN PROGRAM

```
190   CONTINUE
      DO 200 I=1,101
      UL=45.+ASIN(1.+S0*S0-S(I)*S(I))/.034906584
      CALL ELLIP (RK,UL,2,ACC,C,A)
      XW(I)=(2./RK*(A-B)-(1.+S0*S0)*RK*(C-D))/SQRT(PZ)
200   ETAW(I)=S(I)*SQRT(2./PZ)-2.*COS(THETAR)/BZ
      WRITE (6,20)(S(I),XW(I),ETAW(I),I=1,101)
      GO TO 90
      END
```

APPENDIX B

SUBROUTINE ELLIP
(Computes the Elliptic Integrals)

FORTRAN IV PROGRAM LISTING OF SUBROUTINE ELLIP

```
      SUBROUTINE ELLIP (SSK,TH,IND,E,BK,EK)
      P1=1.5707963
      TH1=P1*TH/90.
      IF (SSK-1.) 8,9,8
9     IF (TH-90.) 12,13,12
12    SN=SIN(TH1)
      A=(1.+SN)/(1.-SN)
      BK=.5*ALOG(A)
      EK=SN
      GO TO 30
13    GO TO (14,15),IND
15    BK=1.E10
      EK=1.
      GO TO 30
14    WRITE (6,100)SSK
      BK=1.E10
      GO TO 30
8     IF (TH-90.) 5,6,5
6     SN=1.
      CN=.0
      GO TO 7
5     SN=SIN(TH1)
      CN=COS(TH1)
7     SK=SSK
      R=1.
```

FORTRAN IV PROGRAM LISTING OF SUBROUTINE ELLIP

```
      T=1.
      Q=.0
      S=1.
      D=SQRT(1.-SK*SK*SN*SN)
3     SK2=SK*SK
      SKP=SQRT(1.-SK2)
      GO TO(11,10),IND
10    Q=Q+(T*SN*CN*SK2/(D+1.))
11    SK=(1.-SKP)/(1.+SKP)
      X=(1.+SKP)/(1.+D)
      SN=X*SN
      D=SQRT(1.-SK*SK*SN*SN)
      Z=(1.+SK*SN*SN)/D
      CN=Z*CN
      S=(S+S-SK2*T)/(1.+SKP)
      T=(1.+SKP)*T
      B=(1.+SK)*B
      IF (SK2-E) 4,4,3
4     IF (SN-CN) 22,22,24
22    P=ATAN(SN/CN)
      GO TO 23
24    P=P1-ATAN(CN/SN)
23    GO TO (25,26),IND
26    EK=P*S+Q
25    BK=P*B
```

FORTRAN IV PROGRAM LISTING OF SUBROUTINE ELLIP

```
50    RETURN
60    FORMAT (2H1 F4.1,16H IS NOT DEFINED)
      END
```

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13. ABSTRACT

The hydrostatics of a fluid between parallel plates at low but positive Bond numbers is re-examined as a preliminary to dynamic calculations. The results of this study differ from those of a previous study by Reynolds. It is believed that the results of Reynolds are in error.

14. KEY WORDS

sloshing
hydrostatics
fluid mechanics
free surface