

Ionospheric Research

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on

"An Application of Sampling Theory to the Reduction of Ionograms"

by

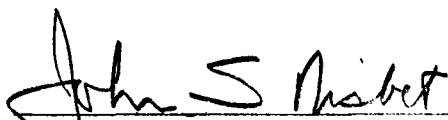
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ABSTRACT

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When ionograms are sampled to provide input data for a true height-electron density reduction program, errors, which are dependent on the sampling accuracy and on the quality of the ionograms, are introduced. The errors, introduced in the sampled values, are transformed by the reduction program to give errors in the estimation of the layer shape.

The relationship between the errors in the virtual height profile and in the resulting true height profile has been examined by consideration of a parabolic layer model. This allows the layer shape to be characterized by three basic parameters: a layer thickness, a maximum layer density and a height at which this maximum occurs.

Relationships between the distribution of the sampling points with respect to one another and to the layer maximum critical frequency are examined. Conclusions are drawn about the effect of incomplete ionograms in the region of the layer maximum on the estimate of  $h_m F_2$ . A suggested distribution of sampling points is given for fitting a parabola to the layer maximum for the estimate of  $h_m F_2$  and "SCAT".

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## I. INTRODUCTION

### A. General Statement of the Problem

Estimates of the maximum electron density, the height at which this maximum occurs and the curvature in the electron density-height profile at the region of the peak are of considerable importance to the study of the physics of the F region of the ionosphere.

Rishbeth and Barron (1960), Nisbet (1963), Tornatore (1964) and others have shown that the electron density in the region of the peak is proportional to the ratio of the production coefficient to the recombination coefficient of the ions, and that the height of the maximum electron density is related to the ratio of the diffusion coefficient to the recombination coefficient of the ions under equilibrium conditions. The curvature of the profile in the region of the peak was shown by Dungey (1956), Stubbe (1964) and others to be related to the temperature of the neutral atmosphere.

The oldest and most widely used method for measuring the distribution of electrons in the ionosphere is by means of an ionosonde. While the ionospheric electron distribution can now be measured by other methods, such as direct rocket and satellite probes, rocket propagation methods, and incoherent scatter soundings, these methods can only be used at isolated times or locations, and they are relatively new, thus data from them is not presently available for an entire solar cycle. Ionosonde records are available from locations all over the world for several solar cycles and are thus of considerable interest. This paper is concerned only with ionosonde measurements.

Ionosonde measurements consist of ground-based vertical incidence soundings made by transmitting vertically upwards pulses of radio waves whose carrier frequency is slowly varied, and recording the time delay of the pulses returned from the ionosphere as a function of the logarithm of the carrier frequency. The time delay is recorded as a "virtual height", which is the height from which the pulses would have been reflected if they had traveled at the free space velocity for the same time interval. This record of virtual height versus the logarithm of the carrier frequency, is known as an ionogram.

When ionograms are sampled to provide input data for a true height-electron density reduction program, errors are introduced dependent on the accuracy of the samples, the number of samples, the distribution of the samples, and the general quality of the ionograms. These errors, introduced in the sampled values, are transformed by the reduction program to give errors in the estimate of the shape of the true height-electron density profile. It is of interest to study how these errors in the ionogram are related to errors in the profile for the case neglecting the effects of the magnetic field. Even though the magnetic field causes great changes in an ionogram, as compared to an ionogram from the same electron distribution with no magnetic field, the mapping of the ordinary and extraordinary rays with a magnetic field is similar to the mapping of the rays with no magnetic field. Since the errors in virtual height are transformed into errors in true height in a similar manner in either case, the case

where the magnetic field is neglected is studied, for the expressions in this case are analytic and simpler to use.

Past efforts to reduce these errors have consisted mainly of increases in the number of samples taken, usually along equal frequency increments; and in the refinement of the reduction technique. Kelso (1952), Budden (1955), Jackson (1956), Thomas (1959), Titheridge (1961), Doupnik (1963), Brown (1964) and Doupnik and Schmerling (1965) give some excellent examples of the refinements of the reduction technique. Some work has also been done to improve the quality of the ionograms (Barry and Fenwick, 1965). Little, however, has been done to find an optimum distribution of the sample points at the region of the peak.

In present practice, ionograms are usually plotted on a logarithmic frequency scale, and since the frequency at which reflection occurs is proportional to the square root of the electron density at the height of reflection, the top 20% of the true height-electron density profile corresponds to only the top 5% of the ionogram. This distortion of the scale occurs in a very important region of the ionogram-the region where the virtual height is changing most rapidly and where the errors are greatest. Figure 1. illustrates this distortion. The three points on the true height profile correspond respectively to the three points on the ionogram.

In the lower regions of an ionogram, the slope of the virtual height profile is increasing at a nearly constant rate. It is shown in Appendix A, that under these conditions it is sufficient to sample the ionogram at equal

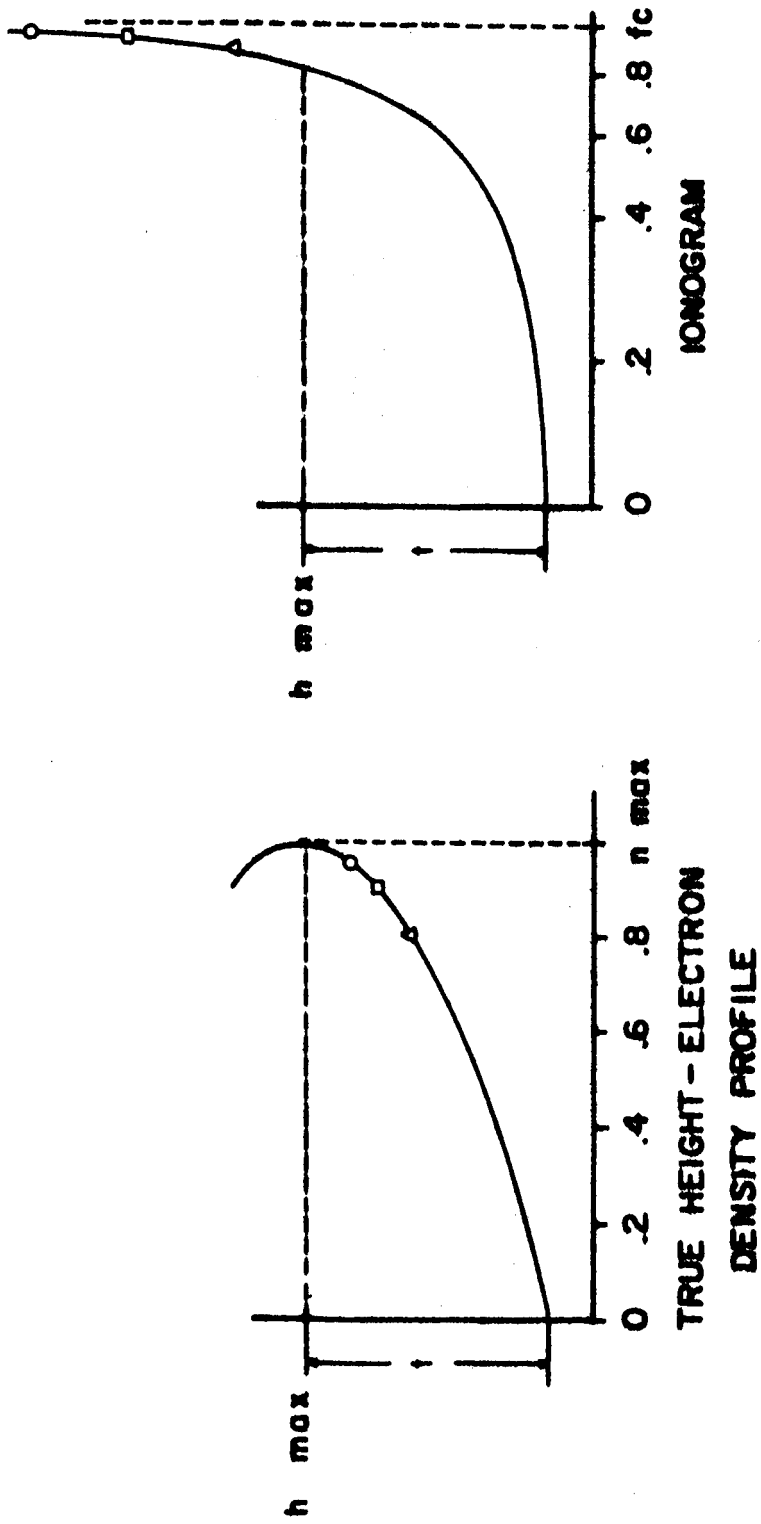


FIGURE 1



frequency intervals.

In the region of the peak, however, the virtual height and all its derivatives rapidly approach infinity. It is thus apparent that the considerations affecting the choice of sampling points when reducing ionograms to electron density profiles are very different in this region from those relevant to the lower regions of the ionogram. The main topic of this paper is the study of the effect of the choice of sampling points upon the errors produced in the reduction of ionograms, with the object of minimizing these errors by a judicious choice of sampling frequencies.

#### B. Sources of Error

The pulses sent out by an ionosonde have a duration of the order of 50 microseconds, corresponding to a virtual height resolution of about 15 kilometers. The pulse width will be increased slightly by the narrow bandwidth of the receivers and the average height of the interval will be modulated by small random fluctuations of recorder gating time, irregularities in the resolution of the film on which the ionograms are recorded, the noise at the receiver input, and irregularities in the ionosphere close to the peak. Thus, the ionogram will consist of a virtual height "band" modulated by "noise".

When an ionogram is sampled and one value of virtual height is chosen from this noisy band, errors result which are generally independent of sampling frequency.

When the pulses used in obtaining ionograms are represented by a Fourier integral, it is found that they contain a spectrum of different frequency components in the vicinity of the carrier frequency. Since the slope of the virtual height profile is high in the region of the peak, the lower frequency components of the pulse will be reflected from a substantially lower virtual height than the higher frequency components. Rydbeck (1942) shows that the pulse shape is modified to that of its own Fresnel diffraction pattern, and the pulse is dispersed over a larger virtual height interval than would have been otherwise expected. Budden (1961) has shown that attenuation in the atmosphere is proportional to the difference between the virtual height and the phase height. The phase height at a given frequency is the mean value of virtual height up to that frequency, and the virtual height is an increasing function which rapidly approaches infinity at the region of the peak. Thus, the absorption also approaches infinity in the region of the peak.

This combination of absorption and dispersion usually causes some of the upper portion of the ionogram to be lost. Estimates of the peak of the ionogram must then be made from information derived entirely from the lower regions. It is of interest to determine what type and what magnitude errors result when this occurs, and how such errors are to be minimized.

## II. RELATION BETWEEN ERRORS IN VIRTUAL HEIGHT AND TRUE HEIGHT

In the case of a monotonic electron density versus height profile, for the simple case neglecting the effects of the magnetic field, the relation between the true and virtual heights is given, according to Appleton (1930) and de Groot (1930), by the transform pair, in normalized coordinates:

$$H'(F) = F \int_0^{H_R} \frac{dH}{\sqrt{F^2 - F_N^2}} \quad (2.1)$$

$$H(N) = \frac{2}{\pi} \int_0^{F_N} \frac{H'(F) dF}{\sqrt{F_N^2 - F^2}} \quad (2.2)$$

where

$$N = F_N^2 \quad (2.3)$$

$$H_R = H \text{ where } F = F_N \quad (2.4)$$

The terms are defined in Appendix B.

If  $H'$  is assumed to be in error such that  $H'_\epsilon = X + YH'(ZF)$  (2.5)

where  $X$  represents an error in the zero height marker,  $Y$  represents an error in vertical scaling and  $Z$  represents an error in horizontal scaling, and  $X$ ,  $Y$ ,  $Z$  are independent of  $F$ ; then we find the corresponding value of  $H'_\epsilon$  from equation (2.2)

$$H_{\epsilon} = \frac{2}{\pi} \int_0^{F_N} \frac{X + YH^1(XF) dF}{\sqrt{F_N^2 - F^2}} \quad (2.6)$$

$$= \frac{2}{\pi} \int_0^{F_N} \frac{X dF}{\sqrt{F_N^2 - F^2}} + \frac{2}{\pi} \int_0^{F_N} \frac{YH^1(XF) dF}{\sqrt{F_N^2 - F^2}} \quad (2.7)$$

Solving the first integral of equation (1.7) and multiplying the numerator and denominator of the second integral by Z yields:

$$H_{\epsilon} = X + \frac{2}{\pi} Y \int_0^{ZF_N} \frac{H^1(ZF) d(ZF)}{\sqrt{Z^2 F_N^2 - Z^2 F^2}} \quad (2.8)$$

Except for the dummy variable and the limits, the integral in (1.8) is seen to be the same as in (1.2); therefore

$$H_{\epsilon} = X + Y H_{\epsilon}(Z^2 N) \quad (2.9)$$

Thus comparing (2.5) to (2.9) it seems apparent that an error in the zero height marker and vertical scale of an ionogram transform into identical errors in the true height profile. An error in horizontal scaling changes to an identical error in the scaling of  $F = N^{1/2}$ .

Since these errors transform linearly from one profile to the other, in order to minimize the errors in the electron density profile it is sufficient to minimize the errors in the ionogram samples, provided the

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reduction program is perfect .

### III. APPLICATION OF SAMPLING THEORY TO THE MINIMIZATION OF ERRORS

#### A. General Theory

In order to reduce the errors appearing in the electron density - height profile, it is necessary to reduce them at the source. There are but two sources of error - the errors may be introduced by the ionogram reduction program, or they may be found on the sampled values taken from the ionogram for the reduction program. Much work has been done recently by Titheridge (1961), Doupnik (1963), Brown (1964), Grebowsky (1965) and many others, to improve the ionogram reduction techniques to the point where the errors due to reduction are very small. For the purpose of this paper the reduction technique is assumed to be perfect so that all errors are assumed to come from errors in the ionogram samples.

Present reduction techniques, such as those of Thomas (1959), Titheridge (1961) and Doupnik (1963) do not use the full ionogram curve, but break the curve up into  $M$  sample points. These  $M$  points are then reduced, either one by one or in groups, to give points in the true height profile, which are then joined together by some assumed relation between these points. The shape of the electron density profile, and of the ionogram, is therefore assumed to be completely determined by a curve which contains only  $n$  independent parameters, where the value of  $n$  depends on the reduction technique, but is never more than  $M$ . Let the virtual height on an ionogram be expressible by a function  $H'(F)$ , where  $H'(F)$  is a curve with  $n$  independent para-

meters. This curve has the value  $H'_i$  at frequency  $F_i$ . When the ionogram is sampled at the frequency  $F_i$ , an error is introduced and the sampled value is given by

$$H'_{ri} = H'_i + \epsilon_i \quad (3.1)$$

where  $\epsilon_i$  is the error introduced at frequency  $F_i$ .

The ionogram reduction technique takes  $M$  of these  $H'_{ri}$ , where  $i = 1, 2, 3, 4, \dots, M$ , and fits them into a curve  $H'_\epsilon$ , which is derived from  $H'(F)$  by adjusting the  $n$  parameters of  $H'(F)$ . This adjustment may be represented by multiplying the  $k^{\text{th}}$  parameter of  $H'(F)$  by  $(1 + a_k)$ , where  $k = 1, 2, 3, 4, \dots, n$ .

If a least square error curve fitting technique is used, the  $a_k$  are adjusted to minimize the function.

$$E = \sum_{i=1}^M \left[ H'_\epsilon(F_i, a_k) - H'_{ri} \right]^2 \quad (3.2)$$

This is done by setting  $\frac{\delta E}{\delta a_k} = 0$  for all  $a_k$ , and solving the  $n$  resulting simultaneous equations in  $a_k$ ,  $\epsilon_i$  and  $F_i$  to give the values for the  $a_k$  as a function of  $\epsilon_i$  and  $F_i$ ,  $i = 1, 2, 3, 4, \dots, M$ .

The  $a_k$  represent the deviation of  $H'_\epsilon$  from the actual ionogram,  $H'(F)$ ; thus they can be represented as error terms, giving the errors in the  $n$  parameters of  $H'(F)$ . For  $H'_\epsilon$  to be close to  $H'(F)$ , all the  $a_k$  must be made small.

This may be done by choosing the sampling frequencies so that the  $(a_k)^2$  will be minimized.

$$\text{Therefore, } a_k \frac{\partial a_k}{\partial F_i} = 0 \quad (3.3) \quad \text{for all } a_k \text{ and for all } F_i.$$

This unfortunately gives  $n \times M$  equations in  $M$  unknowns. Clearly all these equations do not in general have a simultaneous solution, therefore, all  $a_k$  can not be minimized by the same set of sampling frequencies. In general, only one of the  $a_k$ 's can be minimized in this way.

#### B. The $a_k$ Function

Since all the  $a_k$  can not be minimized simultaneously, a function must be chosen to relate the  $a_k$  according to their importance. This function must be such that as all  $a_k$  approach zero, the function approaches zero. One such function may be

$$P(a_k) = \sum_{k=1}^M C_k a_k^2 \quad (3.4)$$

Where the  $C_k$  weight the  $a_k^2$  according to their importance in the final electron density profile. The function  $P(a_k)$  may be minimized by setting.

$$\frac{\partial P(a_k)}{\partial F_i} = 0 \quad (3.5)$$

and solving the resulting  $M$  equations for the  $F_i$ .



### C. Eliminating the $\epsilon_i$

The  $a_k$ , and therefore the  $F_i$ , are functions of  $\epsilon_i$ . Since the  $\epsilon_i$  are unknown error values, the solutions for the  $F_i$  will contain unknown values and thus  $F_i$  can not be found directly.

The  $\epsilon_i$  can, however, be represented statistically by their variances  $\sigma^2(\epsilon_i)$ , which may be given as

$$\sigma^2(\epsilon_i) = G_i^2 \sigma_\epsilon^2 \quad (3.6)$$

where  $\sigma_\epsilon^2$  represents the general estimated variance of the error terms, and  $G_i^2$  gives the frequency dependence of  $\sigma^2(\epsilon_i)$ , as determined by an examination of the sources of error.

With the new representation for  $\epsilon_i$ ,  $a_k$  is replaced by  $\sigma^2(a_k)$ , the variance in  $a_k$ , which has the form

$$\sigma^2(a_k) = Q(F_i, G_i^2) \sigma_\epsilon^2 \quad (3.7)$$

A function  $P(\sigma^2 a_k)$  may be formed corresponding to the  $P(a_k)$  of equation (3.4). Then if

$$\frac{\delta P(\sigma^2 a_k)}{\delta F_i} = 0 \quad (3.8)$$

is solved for the  $F_i$ ; the  $F_i$  are found to be independent of  $\sigma_\epsilon^2$  and, since  $G_i^2$  is a known function of  $F_i$ , the  $F_i$  may be determined exactly.

### D. Linear Function Case

The minimization problem can be simplified, if it is possible to

represent the difference between  $H^1_{\epsilon}$  and  $H^1_{(F)}$  as a linear function

$$H^1_{\epsilon} - H^1_{(F)} = \sum_{j=1}^n b_j B_j \quad (3.9)$$

where  $b_j$  is some function of  $a_k$  such that as  $b_j$  approaches zero,  $a_k$  approaches zero also. Equation (3.2) can then be written as

$$E = \sum_{i=1}^M \left[ \left( \sum_{j=1}^n b_j B_j \right)_i + \epsilon_i \right]^2 \quad (3.10)$$

Since  $E$  is to be minimized by adjusting the  $b_j$ ,

$$\frac{\delta E}{\delta b_j} = 2 \sum_{i=1}^M \left[ \left( \sum_{k=1}^n b_k B_k \right)_i + \epsilon_i \right] B_{ji} = 0 \quad (3.11)$$

This is a system of  $n$  simultaneous linear equations which has solutions of the form

$$b_j = \frac{N_j(\epsilon_i, F_i)}{D(F_i)} \quad (3.12)$$

Here the denominator is independent of  $\epsilon_i$  and is common to all  $b_j$ , therefore all the  $b_j$  can be decreased simultaneously by maximizing  $D(F_i)$ , if the  $N_j$  do not change much with the  $F_i$ .

Under these conditions the minima of  $b_j$  lie somewhere near the maximum of  $D(F_i)$ , and the solution to the equation  $\frac{\delta D(F_i)}{\delta F_i} = 0$

$$(3.13)$$

gives the sampling frequencies.

#### E. Conclusions

A general method of approach for minimizing the errors in an ionogram was outlined, since it is not practical to study all possible variations of ionogram sampling schemes in detail. It is of interest to apply this method of approach to a particular type of profile in order to study the means for minimizing the errors appearing in that profile. The profile which is studied here is a monotonically increasing function expressible by three parameters.

#### IV. THREE PARAMETER MODEL STUDIES

##### A. General Study

In some regions of the ionogram, such as the region which corresponds to the peak of the electron density profile, the ionograms may be represented by a monotonically increasing function which can be completely specified by three parameters. In such a region the virtual height may be written as

$$h' = h_m - t + t \frac{f}{f_c} S \left( \frac{f}{f_c} \right) \quad (4.1)$$

where  $S$  is a function which goes monotonically from zero to infinity as  $f$  goes from zero to  $f_c$ . The three variable parameters are  $h_m$ ,  $t$  and  $f_c$ .

The value of the virtual height read from the ionogram by the experimenter will be in error, and can be expressed as

$$h = h' + \epsilon h_m \quad (4.2)$$

where the error  $\epsilon$  is expressed as a fraction of  $h_m$ . The read values,  $h_r$  are fitted by a virtual height curve in which the three parameters have been varied

$$h'_\epsilon = (1 + \alpha) h_m - (1 + \beta) t + (1 + \beta) t \frac{(1 + \gamma) f}{f_c} S \left[ \frac{(1 + \gamma) f}{f_c} \right] \quad (4.3)$$

Here  $\alpha$ ,  $\beta$  and  $\gamma$  represent the fractional changes in  $h_m$ ,  $t$  and  $f_c$  respectively.

Equations (4.1), (4.2) and (4.3) can be normalized to:

$$H' = 1 - T + TF S(F) \quad (4.4)$$

$$H' = H^* + \epsilon \quad (4.5)$$

$$H' = (1 + \alpha) - (1 + \beta)T + (1 + \beta)(1 + \gamma)TF S \left[ (1 + \gamma)F \right] \quad (4.6)$$

A square difference function corresponding to equation (3.2) can be formed

$$E = \sum_{i=1}^M \left( H'_{\epsilon i} - H'_{r i} \right)^2 \quad (3.2)$$

Substituting (4.4), (4.5) and (4.6) in (3.2) gives

$$E = \sum_{i=1}^M \left[ (\alpha - \beta T) + (1 + \beta)(1 + \gamma)TF_i S \left[ (1 + \gamma)F_i \right] - TF_i S(F_i) - \epsilon_i \right]^2 \quad (4.7)$$

This equation may be simplified by substituting

$$R_i = TF_i S(F_i) \quad (4.8)$$

$$\text{and } A_i = \frac{TF_i}{\gamma} \left[ S \left( (1 + \gamma)F_i \right) - S(F_i) \right] \quad (4.9)$$

into the equation.

$A_i$  is assumed to be independent of  $\gamma$ . This is true to a first order approximation if  $\gamma$  is very small, since

$$\lim_{\gamma \rightarrow 0} A_i = T F_i^2 \frac{d S_i}{d F_i} \quad (4.10)$$

which is independent of  $\gamma$ .

Substituting (4.8) and (4.9) in (4.7)

$$E = \sum_{i=1}^M \left[ (\alpha - \beta T) + (\beta + \gamma + \beta \gamma) R_i + \gamma(1 + \gamma)(1 + \beta) A_i - \epsilon_i \right]^2 \quad (4.11)$$

This can be linearized by substituting

$$\eta = \alpha - \beta T; \quad \lambda = \beta + \gamma + \beta \gamma; \quad \mu = \gamma(1 + \gamma)(1 + \beta) \quad (4.12)$$

These meet the criterion mentioned in Chapter III, that as  $\eta$ ,  $\lambda$  and  $\mu$  approach zero;  $\alpha$ ,  $\beta$  and  $\gamma$  must approach zero also. Equation (4.11)

then becomes

$$E = \sum_{i=1}^M \left[ \eta + \lambda R_i + \mu A_i - \epsilon_i \right]^2 \quad (4.13)$$

The least square difference formulation requires that E be minimized. Then

$$\frac{\delta E}{\delta \eta} = \frac{\delta E}{\delta \lambda} = \frac{\delta E}{\delta \mu} = 0 \quad (4.14)$$

These are three simultaneous equations:

$$1/2 \frac{\delta E}{\delta \eta} = \sum_{i=1}^M \left[ \eta + \lambda R_i + \mu A_i - \epsilon_i \right] = 0 \quad (4.15)$$

which becomes

$$\eta M + \lambda \sum_{i=1}^M R_i + \mu \sum_{i=1}^M A_i - \sum_{i=1}^M \epsilon_i = 0 \quad (4.16)$$

$$1/2 \frac{\delta H}{\delta \lambda} = \sum_{i=1}^M \left[ \eta + \lambda R_i + \mu A_i - \epsilon_i \right] R_i = 0 \quad (4.17)$$

which becomes

$$\eta \sum_{i=1}^M R_i + \lambda \sum_{i=1}^M R_i^2 + \mu \sum_{i=1}^M R_i A_i - \sum_{i=1}^M \epsilon_i R_i = 0 \quad (4.18)$$

$$1/2 \frac{\delta H}{\delta \mu} = \sum_{i=1}^M \left[ \eta + \lambda R_i + \mu A_i - \epsilon_i \right] A_i = 0 \quad (4.19)$$

which becomes

$$\eta \sum_{i=1}^M A_i + \lambda \sum_{i=1}^M R_i A_i + \mu \sum_{i=1}^M A_i^2 - \sum_{i=1}^M \epsilon_i A_i = 0 \quad (4.20)$$

These equations (4.16), (4.18) and (4.20) are linear equations in

$\eta$ ,  $\lambda$  and  $\mu$ , and their solutions are therefore of the form:

$$\eta = \frac{N}{D} \eta \quad \lambda = \frac{N}{D} \lambda \quad \mu = \frac{N}{D} \mu \quad (4.21)$$

where the denominator is common to all three error terms. Therefore, all three error terms can be decreased by increasing D.

$D^2$  is the determinant of the coefficients of the error terms; since the solutions (4.21) actually are of the form:

$$\eta = \frac{N D}{D^2} \quad (4.22)$$

Then

$$\begin{aligned} D^2 = & 2 \sum_{i=1}^M R_i \sum_{i=1}^M A_i \sum_{i=1}^M R_i A_i + M \sum_{i=1}^M A_i^2 \sum_{i=1}^M R_i^2 \\ & - M \left( \sum_{i=1}^M R_i A_i \right)^2 - \sum_{i=1}^M A_i^2 \left( \sum_{i=1}^M R_i \right)^2 - \sum_{i=1}^M R_i^2 \left( \sum_{i=1}^M A_i \right)^2 \end{aligned} \quad (4.23)$$

It is shown in Appendix C that this equation reduces to:

$$\begin{aligned} D^2 = & 1/2 \left[ \sum_{i=1}^M \sum_{k=1}^M \sum_{e=1}^M \left( A_i - A_k \right)^2 \left( R_i - R_{e'} \right)^2 \right. \\ & \left. - \frac{M-2}{2} \sum_{i=1}^M \sum_{k=1}^M \left( A_i - A_k \right)^2 \left( R_i - R_k \right)^2 \right] \end{aligned}$$

$$\text{where } i \neq k \neq e \quad (4.24)$$

Equation (4.24) may be further reduced to give:



$$D^2 = \sum_{i=3}^M \sum_{k=2}^{M-1} \sum_{e=1}^{M-2} \left[ \left( A_i - A_k \right) \left( R_k - R_e \right) - \left( A_k - A_e \right) \left( R_i - R_k \right) \right]^2 \quad (4.25)$$

where  $i > k > e$

Comparing equations (4.4) and (4.8) gives the relation

$$R_i - R_k = H'_i - H'_k \quad (4.26)$$

Substituting this in (4.25) yields

$$D^2 = \sum_{k=3}^M \sum_{i=2}^{M-1} \sum_{e=1}^{M-2} \left[ \left( A_i - A_k \right) \left( H'_k - H'_e \right) - \left( A_k - A_e \right) \left( H'_i - H'_k \right) \right]^2 \quad (4.27)$$

This is the equation which is to be maximized by adjusting the  $F_i$ .

If the numerators of equation (4.27) can be shown to remain nearly constant as the sampling frequencies are varied, then, the error functions  $\eta$ ,  $\lambda$  and  $\mu$  will have a minimum very close to the place where  $D^2$  has a maximum, and the above values of  $F_i$  are the optimum sampling frequencies.

The behavior of the functions in this section are hard to visualize in the general case. In order to see if the numerators remain approximately constant and how the optimum sampling frequencies depend on  $H'$  and  $A$ , some

elementary cases were studied.

### B. Three Point Case, the Numerators

The simplest solution to the sampling problem occurs in the degenerate case where there are only three sample points. The function  $H'_\epsilon$  in this case goes through all three of these points, and the square difference function has a minimum value of zero.

For the theory of section IV A. to be applicable, the numerators of equation (4.21) must remain nearly constant as the frequency of the center point is varied. These numerators can be found by solving equations (4.16), (4.18), (4.20); for the three point case they are:

$$N_\lambda = \epsilon_1 (A_3 - A_2) - \epsilon_2 (A_3 - A_1) + \epsilon_3 (A_2 - A_1) \quad (4.28)$$

$$N_\mu = -\epsilon_1 (H'_3 - H'_2) + \epsilon_2 (H'_3 - H'_1) - \epsilon_3 (H'_2 - H'_1) \quad (4.29)$$

$$N_\eta = -\epsilon_1 (A_3 H'_2 - H'_3 A_2) + \epsilon_2 (A_3 H'_1 - H'_3 A_1) - \epsilon_3 (A_2 H'_1 - H'_2 A_1) \quad (4.30)$$

These are functions of  $\epsilon_i$  and since  $\epsilon_i$  are unknowns, can not be calculated directly. The  $\epsilon_i$  can, however, be given in terms of their variances  $\sigma^2_{\epsilon_i}$ , since  $\sigma_{\epsilon_i}$  represents the probable range of  $\epsilon_i$ . Then, adding the vari-

ances by the laws of statistics, and denoting the variance of  $N_\eta$  as  $\bar{N}_\eta^{-2}$ , the equations become

$$\bar{N}_\eta^{-2} = \left[ \left( A_3 - A_2 \right)^2 + \left( A_3 - A_1 \right)^2 + \left( A_2 - A_1 \right)^2 \right] \sigma_\epsilon^2 \quad (4.31)$$

$$\bar{N}_\mu^{-2} = \left[ \left( H'_3 - H'_2 \right)^2 + \left( H'_3 - H'_1 \right)^2 + \left( H'_2 - H'_1 \right)^2 \right] \sigma_\epsilon^2 \quad (4.32)$$

$$\bar{N}_\eta^{-2} = \left[ \left( A_3 H'_2 - H'_3 A_2 \right)^2 + \left( A_3 H'_1 - H'_3 A_1 \right)^2 + \left( A_2 H'_1 - H'_2 A_1 \right)^2 \right] \sigma_\epsilon^2 \quad (4.33)$$

if the error is assumed to be independent of frequency.

To see how the variances of the numerators vary with  $F_2$ , equation (4.31) is examined as a typical case. Equation (4.31) is minimum when:

$$1/2 \frac{\delta \bar{N}_\eta^{-2}}{\delta F_2} = \left( 2A_2 - A_3 - A_1 \right) \frac{\delta A_2}{\delta F_2} = 0 \quad (4.34)$$

$\frac{\delta A_2}{\delta F_2}$  is not zero, since A is a monotonically increasing function

of F. Then the first term on the right hand side must be zero, and the value of  $A_2$  at minimum is :

$$A_{2m} = \frac{A_3 + A_1}{2} \quad (4.35)$$

Substituting this into equation ( 4.31 ) gives

$$\bar{N}_{\lambda \min}^2 = \frac{3}{2} \left( A_3 - A_1 \right) \sigma^2 \epsilon \quad (4.36)$$

The maximum value of  $\bar{N}_{\lambda}^2$  occurs at the endpoints, where  $F_2 = F_1$  or  $F_2 = F_3$ , and where

$$\bar{N}_{\lambda \max}^2 = 2 \left( A_3 - A_1 \right) \sigma^2 \epsilon \quad (4.37)$$

The ratio of the minimum to the maximum values of  $\bar{N}_{\lambda}^2$  is only three to four, thus the numerator of  $\lambda$  remains nearly constant, and  $\lambda$  has a minimum very close to where  $D^2$  has a maximum.

Since the equations for  $\bar{N}_{\lambda}^2$  and  $\bar{N}_{\mu}^2$  are identical except for the variable,  $\bar{N}_{\mu}^2$  is also expected to remain nearly constant as  $F_2$  is varied. In the same manner  $\bar{N}_{\eta}^2$  can also be shown to vary only slightly with  $F_2$ . Thus all the numerators remain nearly constant as the center frequency is varied between  $F_1$  and  $F_3$ ; and in order to minimize the error terms it is sufficient to maximize the denominator of these terms.

### C. Three Point Case, the Denominator

The denominator of the error terms for the three point case is :

$$D^2 = \left[ \left( A_2 - A_1 \right) \left( H_3' - H_2' \right) - \left( A_3 - A_2 \right) \left( H_2' - H_1' \right) \right]^2 \quad (4.38)$$

which can also be written as

$$D^2 = \left[ (A_2 - A_1)(H_3' - H_1') - (A_3 - A_1)(H_2' - H_1') \right]^2 \quad (4.39)$$

$A$  and  $H'$  are both monotonic functions of  $F$ , and it is assumed that  $A$  does not equal  $H'$ , and that  $F_3 > F_2 > F_1$ .  $D^2$  is a function of  $(H_3 - H_1)$  and  $(A_3 - A_1)$ , both of which increase monotonically with  $(F_3 - F_1)$ ; therefore the larger  $(F_3 - F_1)$  becomes, the larger will be  $D^2$ . Therefore, in order to decrease the size of the error terms,  $F_3$  should be as large, and  $F_1$  as small as is practical.

Equation (4.38) shows that  $D^2 = 0$  when  $F_2 = F_1$  and when  $F_2 = F_3$ . As  $F_2$  goes from  $F_1$  to  $F_3$ ,  $D^2$  must either remain zero throughout the interval or have at least one maximum in the interval.  $D^2$  is zero throughout the interval only in the trivial case where  $H' = 0$  or  $A = 0$ . In the general case there must then be a maximum of  $D^2$  for some value of  $F_2$ . This maximum is given by the relation:

$$\frac{\delta D^2}{\delta F_2} = 2D \left( (H_3' - H_1') \frac{\delta A_2}{\delta F_2} - (A_3 - A_1) \frac{\delta H_2'}{\delta F_2} \right) = 0 \quad (4.40)$$

In the non-trivial case, in order to maximize  $D^2$ ,  $F_2$  must satisfy the relation:

$$\frac{\delta A_2}{\delta F_2} - \frac{(A_3 - A_1)}{(H_3 - H_1)} \frac{\delta H_2'}{\delta F_2} = 0 \quad (4.41)$$

This is recognized as the well known law of the mean of calculus when it is written as:

$$\frac{\delta A_2}{\delta H_2} = \frac{(A_3 - A_1)}{(H_3 - H_1)} \quad (4.42)$$

The value of  $F_2$  found as the solution to this equation is the optimum value for the center sampling frequency.

The equations which have been developed determine the optimum values for the sampling frequencies in the case where there are only three samples taken. If the sampling is done on these frequencies, the errors introduced into the final electron density profile in the region of the peak will be minimized.

#### D. Solutions for $M > 3$

The first non-degenerate solution occurs when there are four sample points — when  $M = 4$ . Since there are more sample points than variable parameters in the matching curve, the curve does not generally go through all of the sample points and a true "least square error" situation is established.

The denominator of the error terms is shown in equation (4.26), and in the four point case is given by:

$$D^2 = \left[ (A_3 - A_2)(H'_2 - H'_1) - (A_2 - A_1)(H'_3 - H'_2) \right]^2 + \left[ (A_4 - A_2)(H'_2 - H'_1) - (A_2 - A_1)(H'_4 - H'_2) \right]^2 + \left[ (A_4 - A_3)(H'_3 - H'_1) - (A_3 - A_1)(H'_4 - H'_3) \right]^2 + \left[ (A_4 - A_3)(H'_3 - H'_2) - (A_3 - A_2)(H'_4 - H'_3) \right]^2 \quad (4.43)$$

This equation has four terms of the type found in the three sample point case. In general, if there are  $M$  sample points and the  $H$  curve has  $n$  independent variable parameters, the equation for  $D^2$  has

$$\frac{M!}{n! (M-n)!} \quad (4.44)$$

such terms.

For the three parameter curve  $n = 3$ , and this function has the progressive values of 1, 4, 10, 20, 35, 56, 84, and 120 as  $M$  goes from 3 to 10. The solutions for the optimum sampling frequencies become increasingly difficult to find as the number of sampling points is increased.

The denominator of the error terms for the four point case is maximized in the same way as it was for the three point case. Using the same argument that was used for the three point case, it can be shown that  $F_4$  should be made as large, and  $F_1$  as small, as is practical. The optimum values of  $F_2$  and  $F_3$  are found by setting

$$\frac{\delta (D^2)}{\delta F_2} = \frac{\delta (D^2)}{\delta F_3} = 0 \quad (4.45)$$

To simplify the notation somewhat, the terms of  $D^2$  in equation (4.43) are written as the sum of  $J_i^2$ , where  $J_i^2$  indicates the term which is independent of  $F_i$ . Equation (4.43) may then be written as :

$$D^2 = J_4^2 + J_3^2 + J_2^2 + J_1^2 \quad (4.46)$$

Then equations ( 4.45 ) become:

$$\begin{aligned} \frac{1}{2} \frac{\delta D^2}{\delta F_2} &= \left[ (A_3 - A_1) J_4 + (A_4 - A_1) J_3 + (H_4' - H_1') J_2 \right] \frac{\delta H_2'}{\delta F_2} \\ &- \left[ (H_3' - H_1') J_4 + (H_4' - H_1') J_3 + (H_4' - H_1') J_2 \right] \frac{\delta A_2}{\delta F_2} = 0 \quad (4.47) \end{aligned}$$

$$\begin{aligned} \frac{1}{2} \frac{\delta D_2}{\delta F_3} &= \left[ (A_2 - A_1) J_4 + (A_4 - A_1) J_2 + (A_4 - A_2) J_1 \right] \frac{\delta H_3'}{\delta F_3} \\ &- \left[ (H_2' - H_1') J_4 + (H_4' - H_1') J_2 + (H_4' - H_2') J_1 \right] \frac{\delta A_3}{\delta F_3} = 0 \quad (4.48) \end{aligned}$$

These are two simultaneous equations which are soluble for  $F_2$  and  $F_3$ . The three solutions to these equations are:

$$\begin{aligned} \text{(a.)} \quad F_3 &= F_4 & J_1 &= J_2 = 0 & J_3 &= J_4 \\ \text{(b.)} \quad F_2 &= F_3 & J_1 &= J_4 = 0 & J_2 &= J_3 & (4.49) \\ \text{(c.)} \quad F_2 &= F_1 & J_3 &= J_4 = 0 & J_2 &= J_1 \end{aligned}$$

The independent center sampling point corresponds in each case to the center sampling point for the three sample solutions, and it can therefore be found by use of equation ( 4.42 ).

The value of the denominator of the error terms is the same for all three solutions, and to find which of the three solutions gives the smallest



values for the error terms  $\eta$ ,  $\lambda$  and  $\mu$ ; the numerators of these terms must be examined.

The numerators of the error terms, under the conditions given in ( 4.49 ), become the same as the numerators for the three point case, except the error for the double point is given as  $\frac{\epsilon_a + \epsilon_b}{2}$ . If the errors are independent of frequency, the variance of the error for the double point is halved. An examination of equations ( 4.31 ), ( 4.32 ), and ( 4.33 ) shows that the numerators will be the smallest if the term to be halved is the middle term of each equation, since the middle term is the largest in each case. The ratio of the maximum of the numerator to the minimum then becomes 3 to 2.

The middle term corresponds to the center sampling point, thus the condition for the optimum sampling point distribution corresponds to having  $F_4$  as large and  $F_1$  as small as practical, and having  $F_2 = F_3$  be the same frequency as the center frequency of the three point case.

If there are more than four sampling points, the equations corresponding to equations ( 4.45 ), ( 4.47 ) and ( 4.48 ) will still have solutions of the form of equation ( 4.49 ). Thus, for any number of sample points, the optimum distribution of these points is for them to coincide with the three optimum sampling frequencies found for the three point case.

It is not useful to take several samples at the same frequency unless all these samples are independent. One ionogram will not yield two independent samples at the sampling frequency, thus nothing will be gained

by sampling twice at this frequency. This difficulty can be overcome by taking both samples near the optimum frequency, one on either side, but keeping these samples far enough apart for them to be independent of one another. In the limit where there are many sample points, clusters of samples can be taken around each optimum sampling frequency keeping the samples far enough apart that the errors of each are independent of the errors of all the others.

## V. MODEL STUDIES OF THE PEAK REGION

A. Model of the Peak Region

In order to illustrate the techniques of Chapter IV, a simple model is adopted to represent the electron density profile at the region of the peak. Since the purpose of the model is to merely illustrate the techniques, the model must necessarily be kept simple. On the other hand, the model must have all of the characteristics of the profile that it represents, otherwise the results obtained from the model studies are meaningless. A good model of the electron density profile at the region of the peak can be derived by forming a Taylor series approximation of the curve at the peak.

$$N = 1 - (1-H) \frac{dH}{dN} + \frac{(1-H)^2}{2} \frac{d^2H}{dN^2} - \frac{(1-H)^3}{6} \frac{d^3H}{dN^3} + \dots \quad (5.1)$$

At the peak  $\frac{dH}{dN} = 0$ . In the region close to the peak  $\Delta H = (1-H)$  is small, thus all terms in  $(1-H)^n$  can be neglected for  $n$  greater than two.

This leaves the approximation

$$N \approx 1 + \frac{(1-H)^2}{2} \frac{d^2H}{dN^2} \quad (5.2)$$

The term  $\frac{d^2H}{dN^2}$  is, for simplicity, written as  $-\frac{2}{T^2}$ . The

approximation for  $N$  then becomes

$$N = 1 - \left( \frac{1-H}{T} \right)^2 \quad (5.3)$$

which is the equation for a parabola. The electron density versus height profile can therefore be represented by a parabola at the region sufficiently close to the peak.

The virtual height profile, if the magnetic field is neglected, is found by substituting equation ( 5.3 ) into equations ( 2.1 ) and ( 2.3 ), giving

$$H'(F) = F \int_0^{H_R} \frac{dH}{\sqrt{F^2 - 1 + \left(\frac{1-H}{T}\right)^2}} \quad (5.4)$$

Substituting

$$V = \frac{1-H}{T}, \quad \phi^2 = 1 - F^2 \quad \text{and} \quad V_R = \phi \quad (5.5)$$

into equation ( 5.4 ) yields

$$\begin{aligned} H'(F) &= 1 - T - FT \int_1^{V_R} \frac{dV}{\sqrt{V^2 - \phi^2}} \\ &= 1 - T - FT \log_e \left( V - \sqrt{V^2 - \phi^2} \right) \Big|_1^{\phi} \end{aligned}$$

which gives

$$H'(F) = 1 - T + FT \operatorname{arc} \tanh F \quad (5.6)$$

This equation corresponds to equation ( 4.4 ) of chapter V. when

$$S(F) = \operatorname{arc} \tanh F \quad (5.7)$$

Then R of equation ( 4.7 ) becomes

$$R = T F \operatorname{arc} \tanh F \quad (5.8)$$

### B. Calculation of A

The quantity A is important because it appears in the final equations giving the value of the optimum sampling frequencies.

A is defined by equation ( 4.8 ) and ( 5.7 ) as

$$\begin{aligned} A &\equiv \frac{TF}{\gamma} \left[ S(1+\gamma)F - S(F) \right] \\ &= \frac{TF}{\gamma} \left[ \operatorname{arc} \tanh (F + \gamma F) - \operatorname{arc} \tanh F \right] \end{aligned} \quad (5.9)$$

Making the substitution

$$\begin{aligned} \Theta &\equiv \operatorname{arc} \tanh F \\ \Theta + \Delta \Theta &\equiv \operatorname{arc} \tanh (F + \gamma F) \end{aligned} \quad (5.10)$$

in equation ( 5.9 ) yields

$$A \equiv \frac{TF}{\gamma} \Delta \Theta \quad (5.11)$$

From equation ( 5.10 )

$$F = \tanh \Theta$$

$$F + \gamma F = \tanh (\Theta + \Delta \Theta) = \frac{\tanh \Theta + \tanh \Delta \Theta}{1 + \tanh \Theta \tanh \Delta \Theta}$$

$$F + \gamma F = \frac{F + \tanh \Delta \Theta}{1 + F \tanh \Delta \Theta}$$

Solving this yields:

$$\tanh \Delta \theta = \frac{\gamma F}{1 - (1 + \gamma) F^2} \quad (5.12)$$

This is expanded in a Taylor series about  $\Delta \theta = 0$  to give

$$\tanh \Delta \theta = \Delta \theta - \frac{\Delta \theta^3}{3} + \dots \quad (5.13)$$

If  $\gamma$  is small,  $\Delta \theta$  must be small and  $\tanh \Delta \theta$  equals approximately  $\Delta \theta$ . Then

$$\Delta \theta = \frac{\gamma F}{1 - (1 + \gamma) F^2} = \frac{\gamma F}{1 - F^2} \left[ 1 + \frac{\gamma F^2}{(1 - F^2)} + \left( \frac{\gamma F}{1 - F^2} \right)^2 + \dots \right] \quad (5.14)$$

and if  $\gamma$  is much smaller than  $(1 - F^2)$ ,

$$A \cong \frac{T F^2}{1 - F^2} \quad (5.15)$$

This result agrees with equation (4.9). The optimum distribution of the sampling frequencies can now be calculated.

### C. Optimum Sampling Frequencies

The optimum sampling frequencies can be calculated by substituting

$$\frac{d A}{d F} = \frac{2 T F}{(1 - F^2)^2} \quad \frac{d H}{d F} = T \left[ \arctan F + \frac{F}{1 - F^2} \right] \quad (5.16)$$

into equation ( 4.40 ) or ( 4.41 ) to give

$$(1-F_2^2) \left[ 1 + \frac{1-F_2^2}{F_2^2} \operatorname{arc} \tanh F_2 \right] = 2 \frac{(H_3' - H_1')}{(A_3 - A_1)} \quad (5.17)$$

This may be written as

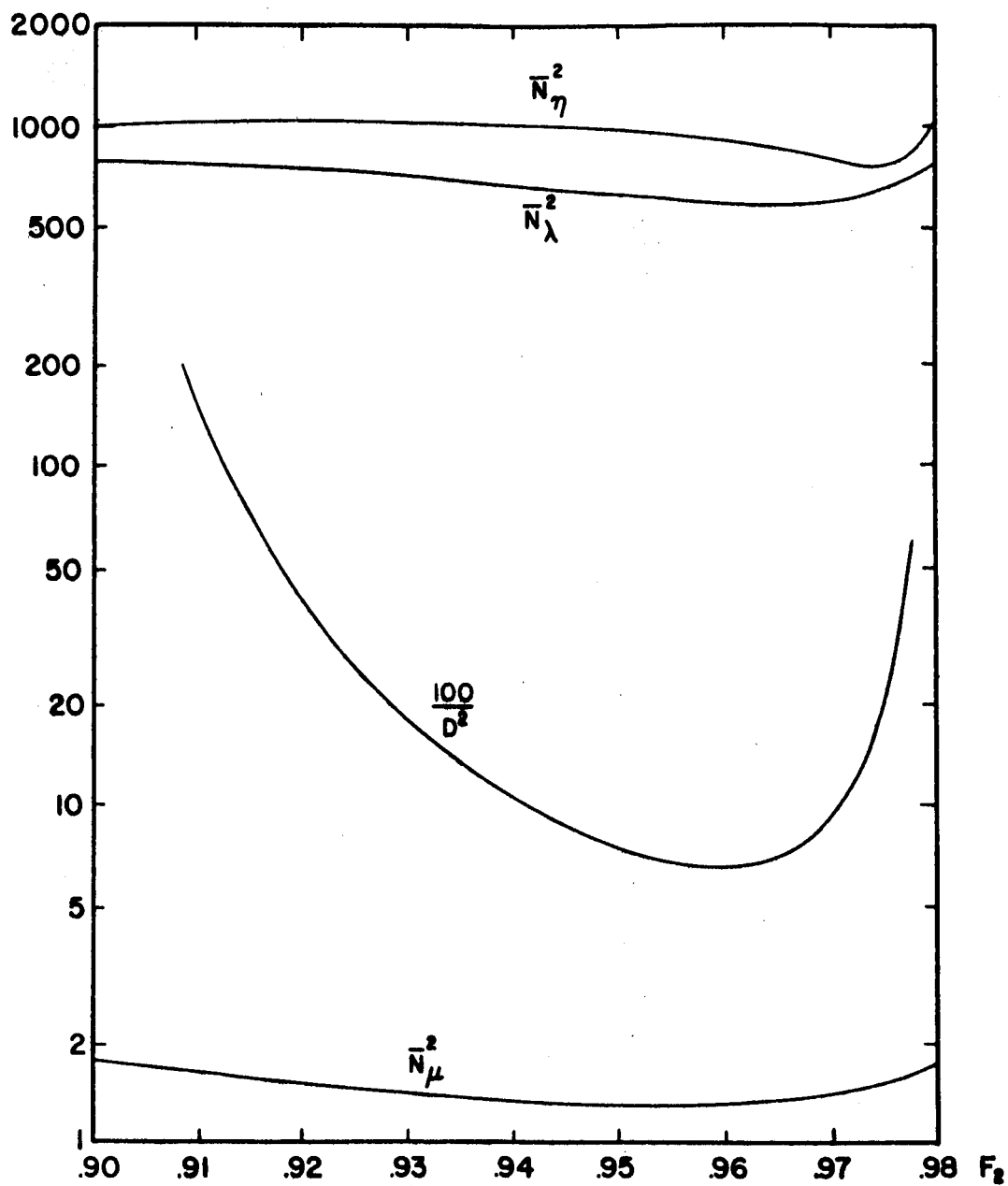
$$(1 - F_2^2) \left[ 1 + \frac{R_2}{A_2} \right] = 2 \left[ \frac{R_3 - R_1}{A_3 - A_1} \right] \quad (5.18)$$

and can be solved for  $F_2$  by use of an iterative technique.

#### D. The Three Point Case

In order to get numerical results it is necessary to examine the factors controlling the choice of sampling points. In general,  $F_1$  should be as small and  $F_2$  as large as possible. The parabolic approximation for the region of the peak of the electron density profile is valid only near the peak, say for  $N$  greater than .81. The corresponding region on the ionogram is the region where  $F$  is greater than .90. On the other end of the scale, some of the top of the ionogram is usually lost due to dispersion and absorption of the ionosonde signal; say the ionogram above  $F = .98$  is lost. Under these conditions  $F_1 = .90$  and  $F_3 = .98$ .

The values for the quantities  $\overline{N}_\eta^2$ ,  $\overline{N}_\lambda^2$ ,  $\overline{N}_\mu^2$ , and  $D^2$  can then be calculated for various values of  $F_2$  by use of equations ( 4.29 ), ( 4.30 ), ( 4.31 ) and ( 4.37 ). Figure 2 shows the variation of those quantities with  $F_2$  when  $T = 1$ . The numerators of the error terms are seen to vary only by



VARIATION OF  $\bar{N}_\eta^2, \bar{N}_\lambda^2, \bar{N}_\mu^2$ , AND  $\frac{100}{D^2}$  WITH  $F_2$  WHEN  
 $F_1 = .90$  AND  $F_3 = .98$

FIGURE 2



the ratio 4 : 3 from maximum to minimum. The value of  $\frac{100}{D^2}$ , however, has a minimum at  $F_2 = .96$  and approaches infinity as  $F_2$  approaches  $F_1$  or  $F_3$ . Since these curves are plotted on a logarithmic scale, the variance of an error term can be found by adding the curve of  $\frac{100}{D^2}$  to the curve of the corresponding numerator.

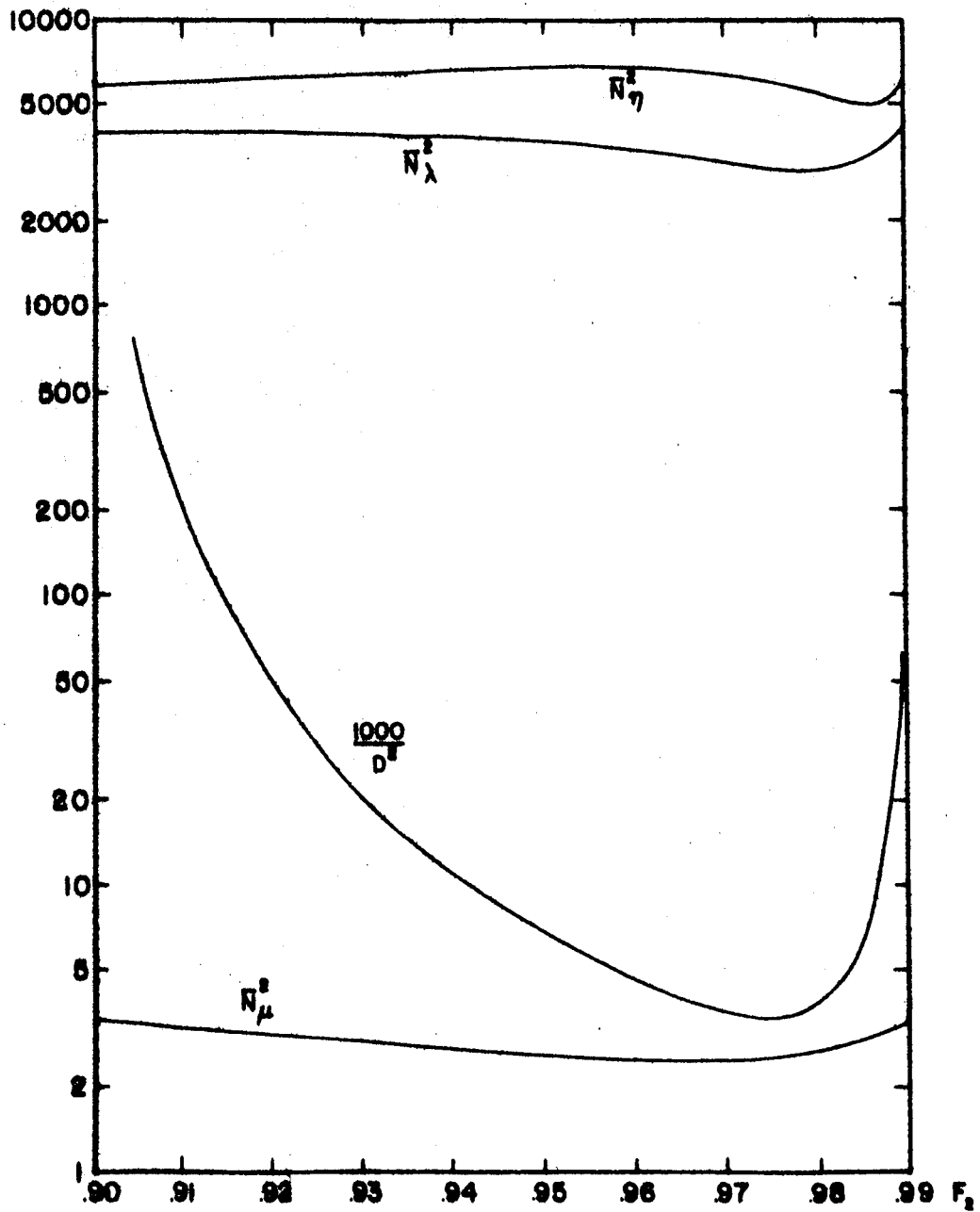
The same quantities are shown again on figure 3, except that the top of the ionogram is assumed to be at  $F_3 = .99$ . The numerators are again nearly constant. The minimum of  $\frac{1}{D^2}$  (plotted here as  $\frac{1000}{D^2}$ ) is sharper than it was when  $F_3$  was .98. The optimum value of  $F_2 = .974$ , when  $F_1 = .90$  and  $F_3 = .99$ .

The curves shown on figure 2 and figure 3 give the variances of the modified error terms  $\eta$ ,  $\lambda$  and  $\mu$ . The variances of the errors in the three parameters of the virtual height curve are found by use of equations ( 4.13 ), ( 4.28 ), ( 4.29 ), ( 4.30 ) and ( 4.38 ), to give the approximate relations:

$$\begin{aligned}\alpha &\cong \eta + T (\lambda - \mu) \\ \gamma &\cong \mu \\ \beta &\cong \lambda - \mu\end{aligned}\tag{ 5.19 }$$

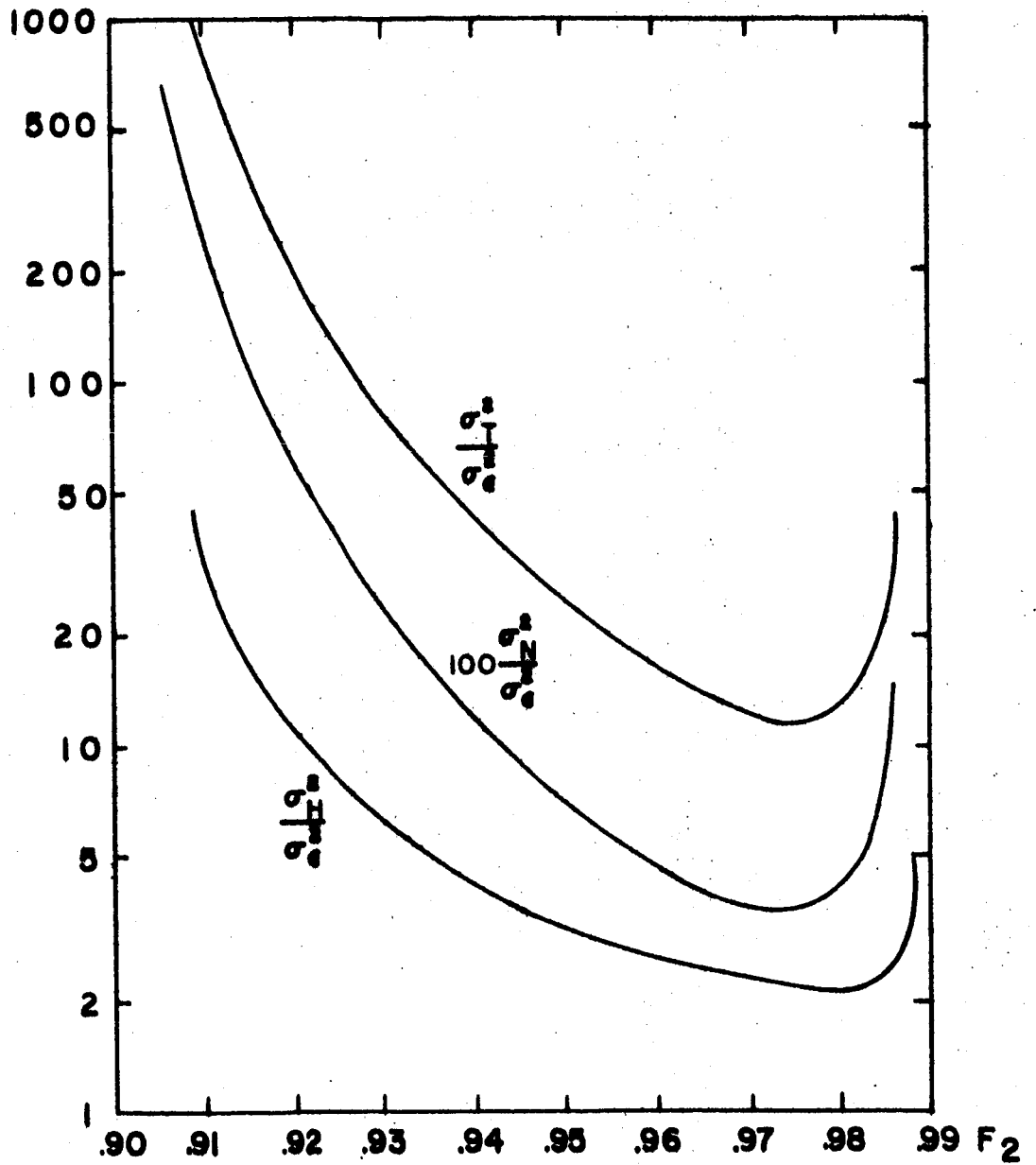
where  $\alpha$ ,  $\gamma$  and  $\beta$  are such that

$$\begin{aligned}h_{\epsilon} \text{ max} &= (1 + \alpha) h \text{ max} \\ n_{\epsilon} \text{ max} &= (1 + \gamma)^2 n_e \text{ max} \\ t_{\epsilon} &= (1 + \beta) t\end{aligned}\tag{ 5.20 }$$



VARIATION OF  $N_\gamma^2$ ,  $N_\lambda^2$ ,  $N_\mu^2$ , AND  $\frac{1000}{D^2}$  WITH  $F_2$  WHEN  $F_1 = .90$  AND  $F_3 = .99$

FIGURE 3



VARIATION OF  $\frac{9}{19}$ ,  $\frac{100}{97}$ ,  $100 \frac{9}{19}$  WITH

$F_2$  WHEN  $F_1 = .90$ ,  $F_3 = .99$  AND

$T = 1$

FIGURE 4

The statistical errors in  $h$  max,  $n$  max and  $t$  can be found by converting  $\alpha$ ,  $\gamma$ , and  $\beta$  into variances.

Figure 4 shows how the errors in the ionogram depend on  $F_2$ , when  $F_3 = .99$ . In this figure these errors are expressed directly in terms of the ratio of the variances of the errors in the three parameters of the ionogram to the variance of the errors in the ionogram samples.

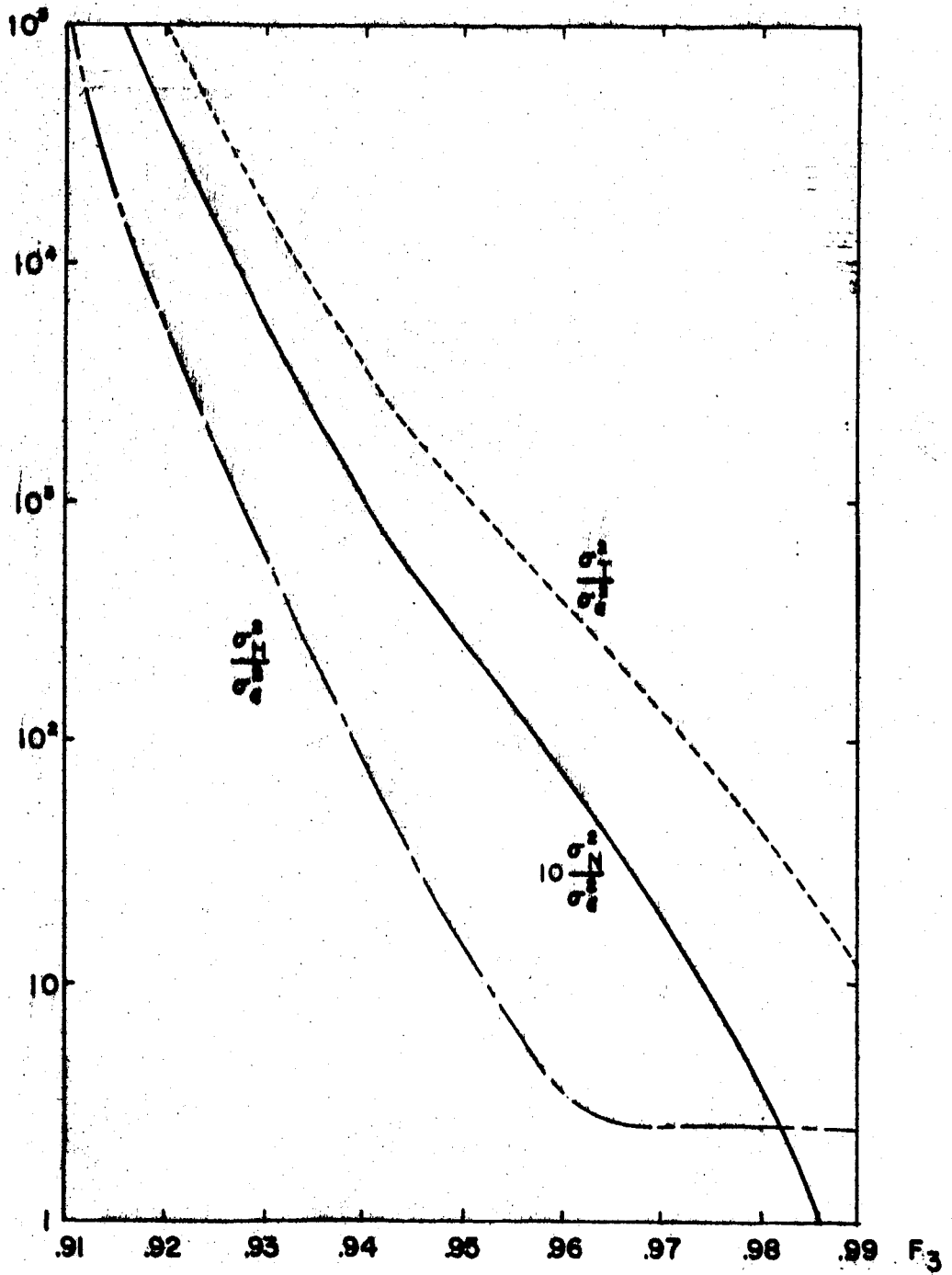
The notation of figure 4 is found from equation ( 5.20 ) to be

$$\sigma_T^2 = \sigma_\beta^2 ; \quad \sigma_H^2 = \sigma_\alpha^2 ; \quad \sigma_N^2 = 1/4 \sigma_\gamma^2 \quad ( 5.21 )$$

A comparison between figure 3 and figure 4 clearly shows that the dominant factor in determining the optimum sampling frequencies is the denominator of the error terms. The minima of the errors of the three parameters do not coincide exactly, but if the sampling is done where  $\frac{1}{D^2}$  is minimum the deviation of the errors of the parameters from minimum is negligible.

#### E. Loss of the Peak

If the peak of the ionogram is lost so that the position of the peak must be estimated from data which is taken at the lower regions, the errors in the parameters of the peak region are expected to increase. Figure 5 shows the errors produced in the ionogram when the peak is lost. The lower sampling point is kept constant, since below that point the approximations which were made for the peak region are not expected to be valid. The center sampling



VARIATION OF  $\frac{\sigma_H^2}{\sigma_e^2}$ ,  $\frac{\sigma_I^2}{\sigma_e^2}$ ,  $10 \frac{\sigma_N^2}{\sigma_e^2}$  WITH  $F_3$  WHEN

$F_1 = .90$  AND  $F_2$  IS THE OPTIMUM VALUE.

FIGURE 5

point was in each case taken at its optimum frequency. The graph clearly shows that the errors increase rapidly as the peak is lost - as  $F_3$  is forced to decrease. This stresses the fact that the top sample point must be as high as possible if a good estimate of the parameters of the peak region is to be made.

## VII. CONCLUSIONS

It was found that the errors in virtual height on an ionogram transform quite linearly into errors in the true height profile. Therefore, errors in the zero height marker and errors in the vertical scale on an ionogram are transformed into identical errors in true heights derived from this ionogram.

The distribution of sample points was found to be very important in the minimization of ionogram errors in the region of the peak. The common method of sampling at equal frequency intervals produced results which were far from optimum. Figure 6 shows the errors produced in a typical ionogram by four sampling schemes - sampling at equal frequency intervals, at equal distances along the virtual height curve, at equal virtual height intervals, and at the optimum sampling frequencies. In order to make good estimates of the height of the maximum electron density or the curvature at the region of the peak, it was found necessary to have a high density of sampling points close to the critical frequency.

When ionograms do not extend to the critical frequency, large errors can be introduced in the estimates of the height of the maximum electron density and the curvature of the true height profile in the region of that maximum. This is particularly important because indications are that this occurs at some stations quite regularly at certain times of day. Systematic errors can thus be introduced in the estimates of the parameters in the

SAMPLE	$F_2$	$\frac{\Delta t}{t}$	$\frac{\Delta n \max}{n \max}$	$\frac{\Delta h \max}{h \max}$
EQUAL F	.945	28.3%	1.50%	3.17%
EQUAL LENGTH	.958	21.2%	1.13%	2.78%
EQUAL H'	.967	18.2%	0.98%	2.56%
OPTIMUM	.974	16.9%	0.94%	2.53%

ERRORS IN  $t$ ,  $n \max$ , AND  $h \max$  WITH VARIOUS DISTRIBUTIONS OF  
THREE SAMPLE POINTS WHEN  $F_1 = .90$ ,  $F_3 = .99$ ,  $h \max = 300 \text{ km}$ ,

$t = 100 \text{ km}$ ,  $\sigma_t = 5 \text{ km}$ ,  $\epsilon = 1.67\% h \max$ .

FIGURE 6



region of the peak.

The examples presented here are extremely simplified. It is hoped that an extension of this method to more practical applications will enable a technique to be developed for the inclusion of a program to calculate realistic error estimates for the peak parameters into the practical true height reduction programs.

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## APPENDIX A.

## SAMPLING A FUNCTION WITH CONSTANTLY INCREASING SLOPE

If the slope of the virtual height profile is increasing at a constant rate, then

$$\frac{\delta^2 H'}{\delta F^2} = 2 C_2 \quad (\text{A.1})$$

Successive integrations of this yields

$$\frac{\delta H'}{\delta F} = C_1 + 2 C_2 F \quad (\text{A.2})$$

$$H' = C_0 + C_1 F + C_2 F^2 \quad (\text{A.3})$$

Comparing this with equations (4.4) and (4.10) gives

$$S(F) = C_1 + C_2 F \quad (\text{A.4})$$

and

$$A = C_2 F^2 \quad (\text{A.5})$$

differentiation of this gives

$$\frac{\delta A}{\delta F} = 2 C_2 F \quad (\text{A.6})$$

Equation ( 4.42 ) may be rewritten as

$$\frac{(H'_3 - H'_1)}{(A_3 - A_1)} = \frac{\delta H'_2}{\delta A_2} \quad (\text{A.7})$$

substituting equations ( A.2 ), ( A.3 ), ( A.5 ) and ( A.6 ) into ( A.7 ) gives

$$\frac{H'_3 - H'_1}{A_3 - A_1} = \frac{C_1 (F_3 - F_1) + C_2 (F_3^2 - F_1^2)}{C_2 (F_3^2 - F_1^2)} = \frac{C_1}{C_2 (F_3 + F_1)} + 1 \quad (\text{A.8})$$

$$\frac{\delta H'_2}{\delta A_2} = \frac{C_1 + 2 C_2 F_2}{2 C_2 F_2} = \frac{C_1}{2 C_2 F_2} + 1 \quad (\text{A.9})$$

Since equation ( A.8 ) equals equation ( A.9 )

$$2 F_2 = (F_3 + F_1) \quad (\text{A.10})$$

or

$$(F_3 - F_2) = (F_2 - F_1) \quad (\text{A.11})$$

Thus, if the slope of the virtual height profile is increasing at a constant rate, optimum sampling is done at equal frequency intervals.

## APPENDIX B.

## LIST OF SYMBOLS

$$A = \frac{TF}{\gamma} \left[ S [ (1 + \gamma) F ] - S(F) \right]$$

$a_k$  = error in the  $k^{\text{th}}$  parameter of the virtual height equation.

D = denominator of the error terms

E = square difference function

$f_c$  = critical frequency

$F = \frac{f}{f_c}$  normalized frequency

$F_N$  = normalized plasma frequency of ionosphere

$h_m$  = height of peak of electron density profile

$h$  = virtual height

$H = \frac{h}{h_m}$  normalized true height

$H = \frac{h}{h_m}$  normalized virtual height

$H_R$  = normalized height of reflection

$H_r$  = value of  $H$  read from ionogram.

$H_\epsilon$  = erroneous value of true height

$H'_\epsilon$  = erroneous value of virtual height

$i$  = subscript denoting the value of a function at a point where  $F = F_i$ .

M = total number of sample points

$n$  = total number of variable parameters in  $H(F)$  curve.



$n_m$  = maximum electron density

$N = \frac{n}{n_m}$  normalized electron density.

$N_j$  = numerator of the  $j^{\text{th}}$  error term

$\overline{N_j^2}$  = variance of  $N_j$ .

$R = T F S_{(F)}$

$S$  = function giving the frequency dependent part of  $H'$ .

$t$  = thickness parameter giving curvature in the region of the peak.

$T = \frac{t}{h_m}$  normalized thickness parameter.

$\alpha$  = term giving the error in  $h_m$

$\beta$  = term giving the error in  $t$

$\epsilon$  = normalized error in the ionogram

$\sigma_{\epsilon}^2$  = variance of  $\epsilon$

$\gamma$  = term giving the error in  $n_m$

$\eta = (\alpha - \beta T)$

$\lambda = (\beta + \gamma + \beta\gamma)$

$\mu = \gamma(1 + \gamma)(1 + \beta)$

## APPENDIX C

## DERIVATION OF EQUATIONS (4.24) AND (4.25)

The determinant of the coefficients of the error terms is given

by equation (4.23) as :

$$\begin{aligned}
 D^2 = & 2 \sum_{i=1}^M R_i \sum_{i=1}^M A_i \sum_{i=1}^M R_i A_i + M \sum_{i=1}^M A_i^2 \sum_{i=1}^M R_i^2 - M \left( \sum_{i=1}^M R_i A_i \right)^2 \\
 & - \sum_{i=1}^M A_i^2 \left( \sum_{i=1}^M R_i \right)^2 - \sum_{i=1}^M R_i^2 \left( \sum_{i=1}^M A_i \right)^2 \quad (4.23)
 \end{aligned}$$

Each of the products of the summations can be converted into summations of products by separating each product into its component parts. If  $i \neq k \neq e$ , then

$$\begin{aligned}
 \sum_{i=1}^M R_i \sum_{j=1}^M A_j \sum_{i=1}^M R_i A_i &= \sum_{i=1}^M R_i^2 A_i^2 + \sum_{i=1}^M \sum_{k=1}^M R_i A_i (R_i A_k + R_k A_i + R_k A_k) \\
 &+ \sum_{i=1}^M \sum_{k=1}^M \sum_{e=1}^M R_i R_k A_k A_e \quad (C.1)
 \end{aligned}$$

$$\sum_{i=1}^M A_i^2 \sum_{i=1}^M R_i^2 = \sum_{i=1}^M A_i^2 R_i^2 + \sum_{i=1}^M \sum_{k=1}^M A_i^2 R_k^2 \quad (C.2)$$

$$\left( \sum_{i=1}^M R_i A_i \right)^2 = \sum_{i=1}^M R_i^2 A_i^2 + \sum_{i=1}^M \sum_{k=1}^M R_i A_i R_k A_k \quad (C.3)$$

$$\begin{aligned} \sum_{i=1}^M A_i^2 \left( \sum_{i=1}^M R_i \right)^2 &= \sum_{i=1}^M A_i^2 R_i^2 + \sum_{i=1}^M \sum_{k=1}^M A_i^2 R_k (2R_i + R_k) \\ &+ \sum_{i=1}^M \sum_{k=1}^M \sum_{e=1}^M A_i^2 R_k R_e \end{aligned} \quad (C.4)$$

$$\begin{aligned} \sum_{i=1}^M R_i^2 \left( \sum_{i=1}^M A_i \right)^2 &= \sum_{i=1}^M A_i^2 R_i^2 + \sum_{i=1}^M \sum_{k=1}^M R_i^2 A_k (2A_i + A_k) \\ &+ \sum_{i=1}^M \sum_{k=1}^M \sum_{e=1}^M R_i^2 A_k A_e \end{aligned} \quad (C.5)$$

Equations (C.1) through (C.5) are substituted into equation (4.23) to give, after the terms are collected :

$$\begin{aligned} D^2 &= (M-2) \sum_{i=1}^M \sum_{k=1}^M A_i R_k (A_i R_k - R_i A_k) \\ &+ \sum_{i=1}^M \sum_{k=1}^M \sum_{e=1}^M A_i R_k (2R_i A_e - R_e A_i - R_k A_e) \end{aligned} \quad (C.6)$$

By a procedure similar to the one used above, the right side of equation (4.24) is separated to give

$$\sum_{i=1}^M \sum_{k=1}^M (A_i - A_k)^2 (R_i - R_k)^2 = 4(M-1) \sum_{i=1}^M A_i^2 R_i^2$$

$$+ 4 \sum_{i=1}^M \sum_{k=1}^M A_i R_k (A_i R_k - A_k R_k - A_i R_i - A_k R_i)$$
(C.7)

$$\sum_{i=1}^M \sum_{k=1}^M \sum_{e=1}^M (A_i - A_k)^2 (R_i - R_e)^2 = 2(M-1)(M-2) \sum_{i=1}^M A_i^2 R_i^2$$

$$+ 2(M-2) \sum_{i=1}^M \sum_{k=1}^M A_i R_k (2A_i R_k - A_i R_i - A_k R_k)$$
(C.8)

$$+ 2 \sum_{i=1}^M \sum_{k=1}^M \sum_{e=1}^M A_i R_k (2R_i A_e - A_i R_e - R_k A_e)$$

When equations (C.7) and (C.8) are substituted into equation (4.24) and the terms are collected, the result is identical to equation (C.6). Since equation (C.6) was shown to be equal to  $D^2$  in equation (4.23) and to the right side of equation (4.24), it then follows that

$$D^2 = \frac{1}{2} \sum_{i=1}^M \sum_{k=1}^M \sum_{e=1}^M (A_i - A_k)^2 (R_i - R_e)^2 - \frac{M-2}{4} \sum_{i=1}^M \sum_{k=1}^M (A_i - A_k)^2 (R_i - R_k)^2$$

This is equation (4.24), and the identity between equations (4.23) and (4.24) is established.

If the assumption is made that  $i > k > e$ , then equation (4.24)

can be written as

$$\begin{aligned}
 D^2 = \frac{1}{2} \sum_{i=3}^M \sum_{k=2}^{M-1} \sum_{e=1}^{M-2} \{ & (A_i - A_k)^2 [(R_i - R_e)^2 + (R_k - R_e)^2 - (R_i - R_k)^2] \\
 & + (A_i - A_e)^2 [(R_i - R_k)^2 + (R_k - R_e)^2 - (R_i - R_e)^2] \\
 & + (A_k - A_e)^2 [(R_i - R_e)^2 + (R_i - R_k)^2 - (R_k - R_e)^2] \} \quad (C.9)
 \end{aligned}$$

$(A_i - A_e)$  and  $(R_i - R_e)$  can be written as

$$(A_i - A_e) = (A_i - A_k) + (A_k - A_e) \quad (C.10)$$

$$(R_i - R_e) = (R_i - R_k) + (R_k - R_e) \quad (C.11)$$

Substituting equations (C.10) and (C.11) into equation (C.9), and collecting the terms yields equation (4.25):

$$D^2 = \sum_{i=3}^M \sum_{k=2}^{M-1} \sum_{e=1}^{M-2} \{ (A_i - A_k)(R_k - R_e) - (A_k - A_e)(R_i - R_k) \}^2$$

The identity between equations (4.23), (4.24) and (4.25) is established.