NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Report No. 32-774

# Stiffness Matrix Structural Analysis

B. Wada



JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

October 31, 1965

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

*Technical Report No. 32-774*

# *Stiffness Matrix Structural Analysis*

*B. Wada*

M. E. Alper

M. E. Alper, Manager
*Applied Mechanics Section*

JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

October 31, 1965

BLANK PAGE

# CONTENTS



# FIGURES

## FIGURES (Cont'd)

## FOREWORD

This document supersedes Technical Memorandum No. 33-75, dated February 12, 1962, titled *Stiffness Matrix Structural Analysis.* The computer program has been modified to meet the needs of the engineers that have used the program. The major modifications are:

1. Input data is part of output format
2. Evaluation of mass properties
3. Thermal analysis
4. Jacobi's method for eigenvalue and eigenvector evaluation
5. Orthogonality check
6. Addition of a non-circular rigid-jointed member.

The original program was modified by Lincoln Laboratory of MIT to increase the degrees of freedom that can be handled. The identification given for the program by Lincoln Laboratory is STEIGR.

BLANK PAGE

## ABSTRACT

14376

A computer program is described that solves structural problems having lumped masses connected by weightless members. The program is capable of handling 130 degrees of freedom with the option of using any one of five different member types.

Using the stiffness formulation, static deflections and loads, thermal deflections and loads, eigenvalues and eigenvectors can be evaluated.

# I. INTRODUCTION

### A. General Description

A program has been developed at the Jet Propulsion Laboratory (JPL) for the analysis of structural frameworks. Since the program is intended for use as a design tool, particular attention has been given to simplicity and flexibility of input and output. It may thus be used by personnel who have had little training in computer utilization, and input may easily be revised to reflect changes in a design.

The program is coded in FORTRAN-II version-3 language, operating under IBSYS, and may be run at any IBM 7090 installation whose system is compatible with that of the Jet Propulsion Laboratory and whose machine has a 32K memory.

The program has been written for the analysis of five types of structure:

1. Three-dimensional structure, pinned joints
2. Three-dimensional structure, rigid joints, equal member cross-section moment of inertia
3. Planar structure, rigid joints, loaded in-plane
4. Planar grid structure, rigid joints, loaded normal-to-plane
5. Three-dimensional structure, rigid joints, doubly symmetric cross-sections

### B. Function of Program

A structural framework will be defined as a stable system of uniform, weightless members, and joints at which loads are applied and weights are lumped. Such a framework and its environment may be described by the following quantities:

1. Coordinates of joints
2. Geometric and elastic properties of members

3. Locations of restraints
4. Weights at joints
5. Static loads at joints
6. Temperature changes of members
7. Acceleration of a joint during free vibration in a normal mode

Given these as input, the program will perform the computations to provide the following as output:

1. Center of weight and weight moments of inertia of the structure
2. Deflections and member loads for static loadings
3. Reactions and equilibrium checks at each joint for static loadings
4. Deflections and member loads for thermal loadings
5. Frequencies, mode shapes, and member loads during free vibration in normal modes
6. Reactions and equilibrium checks at each joint for dynamic loadings
7. Orthogonality check of normal modes

## C. *Method of Analysis*

The program generates the stiffness matrix **K** for a particular type of structure from geometrical data, and performs static and normal-mode analyses by solving the equations

$$\mathbf{U} = \mathbf{K}^{-1}\,\mathbf{F} \text{ and } \frac{1}{\omega^2}\,\mathbf{U} = \mathbf{K}^{-1}\,\mathbf{M}\,\mathbf{U}$$

where **F** is a matrix of static loads, **M** is a matrix of inertia terms, **U** is a matrix of static deflections or a normal-mode shape, and $\omega$ is the circular frequency of a normal mode. Member loads are computed from a set of deflections **U** and geometrical properties of the members.

The thermal loads are computed by first calculating member loads with all degrees of freedom fixed and forces at each joint required to prevent joint motion caused by temperature increase. The thermal deflections of joints and thermal loads in members are obtained by superimposing the member loads evaluated above to the member loads and joint defections evaluated by applying forces equal, but opposite in sign, to the joint-restraining forces to the structure.

The stiffness matrix method of analysis was chosen over possible techniques (e.g., flexibility matrix, force relaxation) because it most fully satisfies the following criteria:

1. That it provide a complete analysis (deflections, loads, normal modes)
2. That input be in a simple form
3. That it analyze statically indeterminate structures with no extra effort on the part of the user
4. That it be adaptable to any type of framework
5. That a useful program be easy to write
6. That the computer be utilized efficiently with respect to storage capacity and running time
7. That the accuracy of the solution be sufficient for engineering use and be predictable

## D. *Operating Experience*

The program has been used extensively during design of various spacecraft vehicles. In the few cases where prototype experimental data are available, correlation with predicted results is good. Analyses of structures of 130 degrees of freedom have been performed with no accuracy problems, as indicated by a check on static equilibrium of the structure and orthogonality of the modes.

Machine time on the 7094 for complete analyses (static and normal mode) varies from 1 min for 20 degrees of freedom to 20 min for 130 degrees of freedom.

The input for the original *Mariner-A* basic structure, as an example, could be written in about 2 hr after appropriate idealization. (The structure was of 90 degrees of freedom, statically indeterminate to the 48th degree.) Key-punching the data cards required 20 min; machine time was about 10 min. More than 25 revisions to the original data have been run during the design process.

Some experimentation has been done with very poorly conditioned matrices (in particular, Hilbert matrices) to determine the effect of conditioning on accuracy. Empirical results of these tests are presented in Section II-K.

# II. MATHEMATICS

## A. Notation

**A** square matrix

$A$ member area

$\mathbf{A}^{(k)}$ $k^{th}$ matrix

$A_i$ input member section property; thermal strains of members

$a_i$ acceleration of structure in $\mathbf{x}_i$ direction

$a_{ij}$ element of matrix **A**

$a_{ij}^{(k)}$ element of matrix $\mathbf{A}^{(k)}$

**B** square matrix

**C** square matrix

$c_{ij}$ element of matrix **C**

$D$ outside diameter of circular member cross section

$E$ elastic modulus

**F** matrix of static loadings

$f_i$ load in $i^{th}$ generalized component direction; natural frequency of $i^{th}$ mode, cps

$f_{pi}$ load at point $p$ in the $\mathbf{x}_i$ direction

$\mathbf{f}_p$ vector load applied to joint $p$ in $\mathbf{x}_i$ coordinate system

$g$ gravity acceleration, 386.4 in./sec$^2$

$h$ depth of member cross-section

**I** unit matrix

$I$ moment of inertia of member cross-section

$I_i$ moment of inertia of member cross-section about $\xi_i$ axis

$I_{jj}$ weight moment of inertia about the $x_j$ axis through the center of weight

$I_{jk}$ cross product weight moment of inertia about center of weight with respect to $x_j$ and $x_k$ axis

$j$ input joint number

**K** square stiffness matrix

$K$ member section torsional stiffness parameter

$\mathbf{K}_{pq}$ square stiffness matrix relating joint $p$ to joint $q$

$k_{ij}$ element of matrix **K**

$\bar{k}_{ij}$ the $i^{th}$ row, $j^{th}$ column component of generalized spring matrix

$k^*_{ij}$ the $i^{th}$ row, $j^{th}$ column component of normalized generalized spring matrix

**L** diagonal matrix of eigenvalues; load matrix

**M** diagonal matrix of inertia terms

$m$ number of joints in structure, normal mode

$m_i$ inertia in $i^{th}$ generalized component direction

$m_{pi}$ inertia at point $p$ in $\mathbf{x}_i$ direction

$\bar{m}$ the $i^{th}$ row, $j^{th}$ column component of generalized weight matrix

$m^*_{ij}$ the $i^{th}$ row, $j^{th}$ column component of normalized generalized spring matrix

$N$ input control parameter

$n$ degree of freedom of structure, control parameter

$p$ first joint specified to describe member in input data

$P$ condition number

**pq** vector from joint $p$ to joint $q$

**pr** vector from joint $p$ to joint $r$

$p_k$ constant

$q$ second joint specified to describe member in input data

$q_m$ acceleration in $m^{th}$ mode, in/sec$^2$

$r$ symbol for a joint

$r_i$ input restraint parameter

$S$ member length

$t$ time variable

$T$ wall thickness of circular member cross-section

$\delta T$ thermal gradient across member cross-section

$\Delta T$ thermal increase of member

**U** matrix of static deflection

$\mathbf{U}_m$ vector mode shape, $m^{th}$ normal mode

$u_i$ deflection in $i^{th}$ generalized component direction

$u_{pi}$ deflection of joint $p$ in $\mathbf{x}_i$ direction; amplitude of $u_p$ in the $i^{th}$ normal mode

$\mathbf{u}_p$ vector deflection of joint $p$ in $x_i$ coordinate system

**V** matrix of eigenvectors

$\mathbf{V}_m$ eigenvector, $m^{th}$ mode

**W** diagonal matrix of weights

$W_i$ component of weight (or weight moment of inertia) in $\mathbf{x}_i$ direction

$W_{ji}$ $i^{th}$ weight component of $j^{th}$ joint

$\mathbf{X}_m^{(i)}$ trial vector, $m^{th}$ mode, $i^{th}$ iteration

$\mathbf{X}$ trial vector

$x_{jm}^{(i)}$ $j^{th}$ component of $\mathbf{X}_m^{(i)}$

$\mathbf{X}^{(k)}$ $k^{th}$ trial vector

$x_{pi}$ coordinate of joint $p$ in $\mathbf{x}_i$ direction

$x_i$ reference coordinate system

$\mathbf{x}_i$ unit vector in $x_i$ coordinate direction

$\bar{x}_j$ location of center of weight from $x_j$ axis

$\mathbf{Z}$ transformation matrix

$\mathbf{Z}^{(k)}$ $k^{th}$ transformation matrix

$\alpha$ coefficient of thermal expansion

$\alpha_i$ constant

$\gamma_i$ cosine of angle between member axis and $x_i$ axis

$\epsilon$ constant

$\lambda_m$ eigenvalue, $m^{th}$ mode $= 1/\omega_m^2$

$\nu$ Poisson's ratio

$\nu_I$ initial off-diagonal norm

$\nu_F$ final off-diagonal norm

$\nu_i$ $i^{th}$ off-diagonal norm

$\sigma$ constant

$\rho$ accuracy requirement

$\boldsymbol{\xi}_i$ unit vector in $i^{th}$ coordinate direction of member-oriented coordinate system

$\xi_i$ member coordinate system

$\omega_m$ circular frequency, rad/sec

Sign Convention:

1. Right-handed coordinate systems
2. Forces and displacements positive in positive coordinate directions
3. Moments and rotations positive by right-hand rule about positive coordinate axes

## B. Derivation of Matrix Equations

At any joint in a structure, a component of load $f_i$ applied to the joint must be in equilibrium with member loads reacting on the joint in the same direction. Since member loads in a linear structure are proportional to deflections $u_j$, the expression of force equilibrium in an $n$ degree-of-freedom system may be written

$$f_i = \sum_{j=1}^{n} k_{ij} u_j \qquad i = 1, n \tag{1}$$

where the $k_{ij}$ are constants of proportionality. In matrix notation, the same equation is

$$\mathbf{F} = \mathbf{KU}$$

When a joint undergoes free vibration in a normal mode $m$, its component deflections must be of the form

$$u_i = u_{im} \sin \omega_m t$$

The inertia load acting in the same direction is

$$f_i = -m_i \ddot{u}_i = m_i u_{im} \omega_m^2 \sin \omega_m t \tag{2}$$

Substituting this load into the expression for force equilibrium,

$$m_i u_{im} \omega_m^2 = \sum_{j=1}^{n} k_{ij} u_{jm} \qquad i = 1, n$$

or, in matrix notation

$$\mathbf{MU}_m \omega_m^2 = \mathbf{KU}_m$$

The analysis thus involves solution of two matrix equations: knowing a set of loads $\mathbf{F}$ to compute static displacements $\mathbf{U}$ from

$$\mathbf{F} = \mathbf{KU}$$

and knowing the inertia of the structure $\mathbf{M}$ to compute normal-mode shapes and frequencies (eigenvectors and eigenvalues) $\mathbf{U}_m$ and $\omega_m^2$ from

$$\mathbf{MU}_m \omega_m^2 = \mathbf{KU}_m$$

Member loads may be computed from static displacements $\mathbf{U}$ or properly normalized mode shapes $\mathbf{U}_m$.

## C. Generation of the Stiffness Matrix

In Eq. 1, if all displacements $u_k = 0, k \neq j$, the resulting equations are

$$f_i = k_{ij} u_j \qquad i = 1, n$$

The coefficient $k_{ij}$ is thus the force component in the $i^{th}$ direction per unit deflection in the $j^{th}$ direction, all other deflections being zero.

1. Matrices for the members meeting at joint 1 are computed as

$$\begin{Bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{Bmatrix} = 10^4 \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix} \{\mathbf{u}_1\}$$

for member (1–2) and

$$\begin{Bmatrix} \mathbf{f}_1 \\ \mathbf{f}_3 \end{Bmatrix} = 10^4 \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \\ -0.5 & -0.5 \\ -0.5 & -0.5 \end{bmatrix} \{\mathbf{u}_1\}$$

for member (1–3).

2. The stiffness matrix of the structure will be set up in an 8×8 array with forces (and deflections) in the order

$$\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4.$$

3. Due to unit component deflections of joint 1, forces are produced which are the elements of the first two columns of the stiffness matrix. These forces, being reacted by loads in the members at joint 1, are determined by adding the matrices of members 1–2 and 1–3 as follows:

$$\begin{Bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \\ \mathbf{f}_4 \end{Bmatrix} = 10^4 \left[\begin{array}{cc} 1.5 & 0.5 \\ 0.5 & 0.5 \\ \hdashline -1 & 0 \\ 0 & 0 \\ \hdashline -0.5 & -0.5 \\ -0.5 & -0.5 \\ \hdashline 0 & 0 \\ 0 & 0 \end{array}\right] \{\mathbf{u}_1\}$$

4. The complete matrix is formed by similar superpositions:

$$\begin{Bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \\ \mathbf{f}_4 \end{Bmatrix} = 10^4 \left[\begin{array}{cc:cc:cc:cc} 1.5 & 0.5 & -1 & 0 & -0.5 & -0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & -0.5 & -0.5 & 0 & 0 \\ \hdashline -1 & 0 & 2 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ \hdashline -0.5 & -0.5 & 0 & 0 & 1 & 0 & -0.5 & 0.5 \\ -0.5 & -0.5 & 0 & -1 & 0 & 2 & 0.5 & -0.5 \\ \hdashline 0 & 0 & -1 & 0 & -0.5 & 0.5 & 1.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & -0.5 & 0.5 & 0.5 \end{array}\right] \begin{Bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \mathbf{u}_4 \end{Bmatrix}$$

5. Restraints are introduced in the form $u_{11} = u_{12} = u_{42} = 0$. Multiplying these known deflections through the matrix, the first, second, and last columns make no contribution to the product and may be omitted from the operation. Also, the forces $f_{11}, f_{12}, f_{42}$ are unknown reactions to be determined from the unrestrained deflection components.

6. Analysis for the unknown deflections thus reduces to solution of the equation

$$\begin{Bmatrix} f_{21} \\ f_{22} \\ f_{31} \\ f_{32} \\ f_{41} \end{Bmatrix} = 10^4 \begin{bmatrix} 2 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -0.5 \\ 0 & -1 & 0 & 2 & 0.5 \\ -1 & 0 & -0.5 & 0.5 & 1.5 \end{bmatrix} \begin{Bmatrix} u_{21} \\ u_{22} \\ u_{31} \\ u_{32} \\ u_{41} \end{Bmatrix}$$

In summary, the program generates the stiffness matrix as follows:

1. Step through the joints consecutively.
2. For joint $p$, search list of member numbers for $p$.
3. For each member $pq$, generate and store (temporarily) the matrix columns corresponding to deflections $\mathbf{u}_p$. The submatrices $\mathbf{K}_{pp}$, $\mathbf{K}_{qp}$ have the following locations:

$$\begin{Bmatrix} \mathbf{f}_1 \\ \cdot \\ \cdot \\ \mathbf{f}_p \\ \cdot \\ \cdot \\ \mathbf{f}_q \\ \cdot \\ \cdot \\ \mathbf{f}_n \end{Bmatrix} \begin{bmatrix} \mathbf{K}_{1p} \\ \cdot \\ \cdot \\ \mathbf{K}_{pp} \\ \cdot \\ \cdot \\ \mathbf{K}_{qp} \\ \cdot \\ \cdot \\ \mathbf{K}_{np} \end{bmatrix} \{\mathbf{u}_p\}$$

4. Search list of component restraints; delete rows and contract stiffness matrix columns vertically.
5. Store contracted columns into main stiffness matrix array, except where a column corresponds to a zero deflection component as determined by checking the list of component restraints.

A few properties of the stiffness matrix are evident from its derivation:

1. It is symmetric, a consequence of Maxwell's reciprocity theorem.
2. Each diagonal term is positive and is large compared with all other elements in its row, since diagonal blocks are formed by superposition of off-diagonal blocks.
3. Stability of the structure is reflected in the linear independence of rows, after rows have been deleted to account for restraints.
4. It is generally sparse (many elements are zero), since the position of matrix elements reflects the presence of members.

## D. Generation of Weight and Load Matrices

The matrices $\mathbf{M}$ and $\mathbf{F}$ are generated by appropriate storage of input quantities and contracted to account for restraints in the same manner as for stiffness matrix columns. The diagonal matrix $\mathbf{M}$ is stored as a vector.

Loads may be specified either as concentrated forces or moments on the joint, or as linear and/or rotary accelerations of the structure as a rigid body. Loads at joint $p$ corresponding to acceleration in the $i^{th}$ coordinate direction $a_i$ are computed as

$$f_{pi} = m_{pi}\, a_i$$

## E. Weights, Center of Weight, and Weight Moments of Inertia About Center of Weight

The weights are obtained by adding the values in each $i^{th}$ coordinate direction separately as

$$\sum_{p=1}^{m} W_{pj} \qquad j = 1, 2 \cdots 6$$

where $m$ is the number of joints and $W_{pj}$ is the $j^{th}$ weight component of the $p^{th}$ joint.

The center of weight and weight moment of inertia about center of weight are calculated assuming the first weight component represents the weights in all three directions. The center of weight and weight moment of inertia are calculated as

$$\bar{x}_j = \frac{\sum_{p=1}^{m} W_{p1}\, x_{pj}}{\sum_{p=1}^{m} W_{p1}}$$

$$I_{jj} = \sum_{p=1}^{m} W_{p1}\left(x_{pk}^2 + x_{pl}^2\right) - \sum_{p=1}^{m} W_{p1}\left(\bar{x}_k^2 + \bar{x}_l^2\right) \qquad (j \neq k \neq l)$$

and

$$I_{jk} = \sum_{p=1}^{m} W_{p1} x_{pj} x_{pk} - \bar{x}_j \bar{x}_k \sum_{p=1}^{m} W_{p1} \quad (j \neq k)$$

The terms $\bar{x}_j$, $x_{pj}$, $I_{jj}$, and $I_{jk}$ are defined as location of center of weight along $x_j$ direction, coordinate of joint $p$ in $x_j$ direction, weight moment of inertia about the $x_j$ axis through the center of weight, and cross-product weight moment of inertia about center of weight with respect to $x_j$ and $x_k$ axes, respectively.

## F. Static Analysis

Given the matrices **K** and **F**, the deflections of **U** are computed from

$$\mathbf{U} = \mathbf{K}^{-1}\mathbf{F}$$

by gaussian elimination. No row interchanges or pivot tests are performed, since the diagonal of the stiffness matrix is always strong; i.e., the diagonal element is the largest number in its row. Overflow or underflow during an arithmetic operation is not sensed, so the elimination process continues with whatever remains in the accumulator.

Experience with this basic procedure has been good. It has provided results to highly ill-conditioned problems which compare favorably with those computed by more sophisticated techniques.

Static member loads are computed from the deflections **U** and the geometry of the structure. Appropriate equations for each member type are given in Appendix B.

Equilibrium check at each joint is made by summing the various loads (member loads plus external loads) in the $x_j$ directions. The unbalanced loads at the restrained joints are the reaction loads on the structure.

## G. Thermal Analysis

The thermal analysis is performed as follows:

1. The load in each member of the structure induced by temperature changes, with all joints restrained, are calculated and stored.

2. Equilibrium checks at all originally unrestrained joints of the structure are made to determine the loads imposed by the temporary restraints. The restraint forces on the joints are stored.

3. Forces equal and opposite to the restraint forces (determined in step 2) are applied as static loads to the structure and member loads and joint deflections are calculated (Section II-F).

4. The addition of member loads calculated in steps 1 and 3 give the thermal loads of the structure, and the deflections calculated in step 3 are the thermal displacements of the structure.

## H. Normal-Mode Analysis

An iterative procedure for computing solutions $\mathbf{U}_m$ and $\omega_m$ of the equation

$$\mathbf{K}\mathbf{U}_m = \omega_m^2 \mathbf{M}\mathbf{U}_m \tag{3}$$

is developed in this Section.

First, the above equation will be transformed into

$$\mathbf{A}\mathbf{V}_m = \omega_m^2 \mathbf{V}_m$$

or

$$(\mathbf{A} - \omega_m^2 \mathbf{I})\,\mathbf{V}_m = 0 \tag{4}$$

where **A** is real and symmetric. Solutions to this equation have the following properties (Refs. 1–6):

1. There are $n$ solutions $\omega_m^2$, $\mathbf{V}_m$, where **A** is of order $n \times n$.

2. The eigenvalues $\omega_m^2$ are all real and positive, and the eigenvectors $\mathbf{V}_m$ are real.

3. The eigenvectors are orthogonal with respect to the unit matrix

$$\mathbf{V}_k^T \mathbf{V}_m = 0 \qquad (k \neq m) \tag{5}$$

4. The length of an eigenvector is indeterminate; i.e., if $\mathbf{V}_m$ is a solution, $\alpha_m \mathbf{V}_m$ is also a solution, where $\alpha_m$ is a constant.

5. Any vector **X** of order $n$ may be represented by a linear combination of eigenvectors

$$\mathbf{X} = \sum_{m=1}^{n} \alpha_m \mathbf{V}_m \tag{6}$$

There are two principal reasons for performing the transformation:

1. Equation 3, representing an undamped structure, can only have real, positive eigenvalues. It is possible, however, for roundoff during operations on **K** and **M** to produce an equation of similar form with

imaginary components in its solution. The convergent process, using real arithmetic, will not converge on such solutions. This problem is avoided if the matrix **A** in Eq. 4 is kept symmetric.

2. Use of the orthogonality condition is simpler if eigenvectors are orthogonal with respect to the unit matrix rather than to another matrix

$$\mathbf{V}_k^T \mathbf{B} \mathbf{V}_m = 0 \qquad (k \neq m)$$

The transformation is effected by defining

$$\mathbf{M} = \mathbf{M}^{1/2}\mathbf{M}^{1/2} \qquad (7)$$

where, since **M** is a diagonal matrix of positive elements, $\mathbf{M}^{1/2}$ is also a diagonal matrix whose element in the $i^{th}$ row is $(m_i)^{1/2}$, and the corresponding element in $\mathbf{M}^{-1/2}$ is $1/(m_i)^{1/2}$. Also, let

$$\mathbf{V}_m = \mathbf{M}^{1/2}\mathbf{U}_m \qquad (8)$$

Substituting Eq. 7 into Eq. 3,

$$\mathbf{K}\mathbf{U}_m = \omega_m^2 \mathbf{M}^{1/2}\mathbf{M}^{1/2}\mathbf{U}_m \qquad (9)$$

Substituting Eq. 8 into Eq. 9,

$$\mathbf{M}^{-1/2}\mathbf{K}\mathbf{U}_m = \omega_m^2 \mathbf{V}_m$$

or

$$\mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}\mathbf{V}_m = \omega_m^2 \mathbf{V}_m$$

Since **K** is symmetric, the product

$$\mathbf{A} = \mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}$$

is symmetric and the desired formulation

$$\mathbf{A}\mathbf{V}_m = \omega_m^2 \mathbf{V}_m$$

is achieved, where

$$\mathbf{U}_m = \mathbf{M}^{-1/2}\mathbf{V}_m$$

and $\omega_m$ are the desired solutions.

The solutions of Eq. 3 corresponding to smallest values of $\omega_m$ are of primary importance in structural applications, since larger deflections and loads occur during vibration at lower frequencies. The iterative process to be described converges most readily on the eigenvalue of largest magnitude, so a transformation of Eq. 4 is performed:

$$\mathbf{C}\mathbf{V}_m = \lambda_m \mathbf{V}_m \qquad (10)$$

where $\mathbf{C} = \mathbf{A}^{-1}$ and $\lambda_m = 1/\omega_m^2$

The inverse is computed by straightforward gaussian elimination on the upper triangular half. No row interchanges or checks for division by zero pivot elements are performed.

Solutions of Eq. 10 for the largest value of $\lambda_m$ and the corresponding value of $\mathbf{V}_m$ will now be found. From Eq. 6 any vector

$$\mathbf{X} = \sum_{m=1}^{n} a_m \mathbf{V}_m$$

so

$$\mathbf{C}\mathbf{X} = \mathbf{C}\left(\sum_{m=1}^{n} a_m \mathbf{V}_m\right)$$

$$= \sum_{m=1}^{n} a_m \mathbf{C}\mathbf{V}_m$$

$$= \sum_{m=1}^{n} a_m \lambda_m \mathbf{V}_m$$

and similarly,

$$\mathbf{C}(\mathbf{C}\mathbf{X}) = \mathbf{C}^2\mathbf{X} = \mathbf{C}\left(\sum_{m=1}^{n} a_m \lambda_m \mathbf{V}_m\right)$$

$$= \sum_{m=1}^{n} a_m \lambda_m^2 \mathbf{V}_m$$

or, in general,

$$\mathbf{C}^k\mathbf{X} = \sum_{m=1}^{n} a_m \lambda_m^k \mathbf{V}_m \qquad (11)$$

If the multiplication process is continued, the right side of Eq. 11 will eventually be dominated by powers of the largest eigenvalue $\lambda_1$:

$$\mathbf{C}^k\mathbf{X} \rightarrow a_1 \lambda_1^k \mathbf{V}_1 \qquad k \rightarrow \infty$$

In practice, to keep the components of $\mathbf{X}^{(i+1)} = \mathbf{C}\mathbf{X}^{(i)}$ from becoming too large, $\mathbf{X}^{(i+1)}$ is normalized after each multiplication so that its largest component is 1. (This is permitted since the lengths of the **V**'s are arbitrary.) Normalized versions of $\mathbf{X}^{(i)}$ are multiplied through $\mathbf{C}$ until $\mathbf{X}^{(k)}$ converges to $\mathbf{V}_1$ and the normalization factor to $\lambda_1$. Multiplications continue until the maximum difference between components of $\mathbf{X}^{(i)}$ and $\mathbf{X}^{(i+1)}$ is within a given tolerance, or until a maximum number of cycles has been performed.

For obvious reasons, the foregoing procedure is called the "power method." It is a generalization of Stodola's method, where successive guesses at a mode shape $\mathbf{X}_m^{(i)}$ are used to compute better guesses:

$$\mathbf{C}\mathbf{X}_m^{(i)} = \lambda_m \mathbf{X}_m^{(i+1)}$$

etc.

At any stage in the convergent process, an approximate eigenvalue of better accuracy than the current eigenvector is given (Ref. 1) by Rayleigh's Quotient, defined as

$$\lambda_r = \frac{\mathbf{X}^T\mathbf{C}\mathbf{X}}{\mathbf{X}^T\mathbf{X}}$$

If, in Eq. 6, $\alpha_1 = 0$

$$\mathbf{X} = \sum_{m=2}^{n} \alpha_m \mathbf{V}_m$$

and

$$\mathbf{C}^k \mathbf{X} = \sum_{m=2}^{n} \alpha_m \lambda_m^k \mathbf{V}_m$$

then convergence will be to the next largest eigenvalue $\lambda_2$ and eigenvector $\mathbf{V}_2$. This condition may be obtained by application of the orthogonality condition of Eq. 5 to keep an arbitrary vector $\mathbf{X}$ orthogonal to $\mathbf{V}_1$ (or any known eigenvectors). Thus if $\mathbf{V}_1$ is known, the transformation of an arbitrary vector $\mathbf{X}$ to a vector $\mathbf{X}_1$ orthogonal to $\mathbf{V}_1$ is as follows:

$$\mathbf{X} = \sum_{m=1}^{n} \alpha_m \mathbf{V}_m$$

$$\mathbf{V}_1^T \mathbf{X} = \sum_{m=1}^{n} \alpha_m \mathbf{V}_1^T \mathbf{V}_m = \alpha_1 \mathbf{V}_1^T \mathbf{V}_1$$

$$\mathbf{V}_1^T \mathbf{X}_1 = 0 = \mathbf{V}_1^T \mathbf{X} - \alpha_1 \mathbf{V}_1^T \mathbf{V}_1$$

$$\mathbf{X}_1 = \mathbf{X} - \alpha_1 \mathbf{V}_1$$

$$= \mathbf{X} - \frac{\mathbf{V}_1^T \mathbf{X}}{\mathbf{V}_1^T \mathbf{V}_1} \mathbf{V}_1$$

Similar transformations orthogonalize $\mathbf{X}$ to other eigenvectors $\mathbf{V}_2$, $\mathbf{V}_3$, etc.

When eigenvalues are close, say

$$\lambda_1 \approx \lambda_2$$

the trial vector becomes

$$\mathbf{X}^{(k)} = \alpha_1 \lambda_1^k \mathbf{V}_1 + \alpha_2 \lambda_2^k \mathbf{V}_2$$

in which powers of $\lambda_1$ cannot dominate those of $\lambda_2$ for any reasonable $k$. The process described above will be modified to speed convergence to the larger of close eigenvalues. As before,

$$\mathbf{X} = \sum_{m=1}^{n} \alpha_m \mathbf{V}_m$$

Given an arbitrary number $p$,

$$(\mathbf{C} - p\mathbf{I})\,\mathbf{X} = \sum_{m=1}^{n} \alpha_m (\mathbf{C} - p\mathbf{I})\mathbf{V}_m$$

$$= \sum_{m=1}^{n} \alpha_m (\mathbf{C}\mathbf{V}_m - p\mathbf{V}_m)$$

$$= \sum_{m=1}^{n} \alpha_m (\lambda_m - p)\mathbf{V}_m$$

Powers of both sides are

$$(\mathbf{C} - p\mathbf{I})^k \mathbf{X} = \sum_{m=1}^{n} \alpha_m (\lambda_m - p)^k \mathbf{V}_m$$

which converges to

$$(\mathbf{C} - p\mathbf{I})^k \mathbf{X} \to \alpha_m(\lambda_m - p)_M^k \mathbf{V}_m, \qquad k \to \infty$$

where $(\lambda_m - p)_M$ is the largest value of the difference. The problem here is to choose values of $p$ which

1. will hasten convergence by increasing the ratio

$$\frac{\lambda_1 - p}{\lambda_2 - p} > \frac{\lambda_1}{\lambda_2}$$

2. will not force convergence to a mode other than the first by causing $(\lambda_m - p)$ to be greater than $(\lambda_1 - p)$.

The effect of the procedure is to orthogonalize the trial vector $\mathbf{X}^{(k)}$ to an eigenvector $\mathbf{V}_m$ if $p = \lambda_m$ is chosen, since

$$\mathbf{X}^{(k)} = (\mathbf{C} - \lambda_m \mathbf{I})\mathbf{X}^{(k-1)} = \alpha_1 \lambda_1^{k-1} (\lambda_1 - \lambda_m) \mathbf{V}_1 + \cdots (0)\,\mathbf{V}_m + \cdots$$

has no component of $\mathbf{V}_m$. In this context, a "troublesome" eigenvector is one whose eigenvalue is close to that being sought. Convergence is hastened if components of the troublesome eigenvector in the trial vector $\mathbf{X}$ are "suppressed." Components of troublesome vectors are never completely suppressed, since even if $p = \lambda_m$, roundoff will soon replace troublesome components in $\mathbf{X}$ as $p$ takes on values far from $\lambda_m$.

When $\lambda_1$ is close to $\lambda_2$, the trial vector approximates $\mathbf{V}_1$ after many cycles, although convergence to the true eigenvector is slow. If the approximate first eigenvector is $\mathbf{X}_1$, a trial vector $\mathbf{X}_2$ orthogonalized to $\mathbf{X}_1$ will converge to an approximation of $\mathbf{V}_2$. The Rayleigh's Quotient computed from $\mathbf{X}_2$ is a better estimate of the true $\lambda_2$, and is an effective value of $p$ to accelerate convergence on $\mathbf{V}_1$. But, since

$$|\lambda_n - \lambda_2| \gg |\lambda_1 - \lambda_2|$$

use of $p = \lambda_2$ will also strongly increase components of the lowest eigenvectors $\mathbf{V}_m, \cdots, \mathbf{V}_n$ in $\mathbf{X}_1$. The solution to this quandry is to alternate values of $p$ between an estimated upper eigenvalue and zero, thereby suppressing components in $\mathbf{X}$ of eigenvectors at each end of the range of eigenvalues. Eigenvector components near the middle of the range are then suppressed by varying $p$ between zero and $0.6125\ \lambda_m$, where $\lambda_m$ is the eigenvalue currently being sought.

Variations of $p$ with the power method may be concisely described by the continued product notation

$$\prod_{k=1}^{q} a_k = a_1 a_2 \cdots a_q$$

or

$$\prod_{k=1}^{q} (\mathbf{C} - p_k \mathbf{I})\mathbf{X} = \sum_{m=1}^{n} \alpha_m \prod_{k=1}^{q} (\lambda_m - p_k)\, \mathbf{V}_m \qquad (12)$$

which denotes products of $(\mathbf{C} - p_k\mathbf{I})$ and $(\lambda_m - p_k)$ with $p_k$ varying from $p_1$ to $p_q$.

The procedure for automatic selection of $p_k$ in Eq. 12 may be summarized as follows:

1. Set $p = 0$. Obtain estimates of the highest six eigenvalues by five iterations on each.
2. Alternate $p_k = \lambda_{m+1}, 0, \cdots, \lambda_6, 0, 0.9\ \lambda_{m-1}, 0, 0.81\ \lambda_{m-1}, 0$ to force convergence on $\lambda_m$. If $\lambda_{m+1} > 0.999\ \lambda_m$, use $p_1 = 0.99\ \lambda_m$ to prevent undue suppression of the desired eigenvector.
3. Vary $p_k$ in the range $0 \leq p_k \leq 0.6125\ \lambda_m$ by a quadratic formula emphasizing values of $p_k$ near zero. Repeat a maximum of 20 times for each mode, checking convergence at each cycle.
4. Repeat steps 2 and 3 five times.

When two (or more) eigenvalues are equal,

$$\lambda_1 = \lambda_2$$

then, after many iterations the trial vector

$$\mathbf{X} = \alpha_1 \mathbf{V}_1 + \alpha_2 \mathbf{V}_2$$

where the $\alpha$'s are arbitrary; thus, there can be no convergence to a "first" eigenvector although the eigenvalue $\lambda_1 = \lambda_2$ is well defined. Consider, for example, a mass at the end of a weightless cantilever that is rigid axially and of circular cross-section. The position of the mass is defined by two coordinate components so the system has two degrees of freedom, two mode shapes, and two equal frequencies Using the power method, there would be no convergence to a first mode shape, since this could be deflection in any direction. When iterations are stopped, however, convergence on a second mode orthogonal to the first will be obtained. The same is true in the case of many degrees of freedom and several identical frequencies.

There is still the possibility that convergence on the first eigenvalue and

$$\mathbf{X} = \alpha_1 \mathbf{V}_1 + \alpha_2 \mathbf{V}_2$$

will not be refined enough to eliminate components of lower vectors in $\mathbf{X}$. No test on this error is available. In practice, enough iterations have been made that physically reasonable mode shapes have been obtained in several problems with multiple eigenvalues.

Convergence is tested by searching for the maximum difference between elements in successive trial eigenvectors. If all

$$\left| x_{jm}^{(i)} - x_{jm}^{(i-1)} \right| \leq \epsilon, \quad j = 1, n$$

iterations are stopped on that mode and begun on the next. The criterion $\epsilon$ varies from $4 \times 10^{-4}$ when coarse estimates of the eigenvalues are required to $4 \times 10^{-7}$ for the final cycles.

Initial guesses at the trial vectors $\mathbf{X}_m^{(1)}$ are required to start convergence on each of the six modes computed. These are taken as successive normalized products of the diagonal of $\mathbf{C}\ X\ \mathbf{C}$, in reverse order:

$$x_{i6}^{(1)} = c_{ii}$$

$$\mathbf{X}_m^{(1)} = \mathbf{C}^{6-m+1}\, \mathbf{X}_6^{(1)}$$

The vector $\mathbf{X}_6$ should be a fair guess at the first mode shape, since its components are largest where mass and flexibility are largest. This guess improves with successive iterations, so $\mathbf{X}_1$ may be close to $\mathbf{V}_1$ before convergence is tested. Higher mode guesses $\mathbf{X}_m$ are similarly affected by mass and flexibility, so when they are orthogonalized to lower modes they may be expected to converge relatively rapidly as well. Experience with this procedure has been satisfactory.

In theory, it is possible to compute lower frequencies from the original Eq. 4 of the problem

$$\mathbf{A}\mathbf{V}_m = \omega_m^2\, \mathbf{V}_m$$

by choosing $p > \omega_1^2$ so that $(\omega_n^2 - p)$ has a larger magnitude than any other $(\omega_m^2 - p)$. This has the advantage that computation of $\mathbf{C} = \mathbf{A}^{-1}$, with attendant errors, is eliminated. In practice, however, the lower eigenvalues are so close in comparison with the upper ones that convergence is prohibitively slow. Also, there is evidence that eigenvectors computed by this process are more in error than those obtained from even a poor inverse. Similarly, although the accuracy of eigenvectors computed from

$$\mathbf{CV}_m = \lambda_m \mathbf{V}_m$$

is dependent on the accuracy of the inverse $\mathbf{C} = \mathbf{A}^{-1}$, the eigenvectors will not be improved by iteration through

$$\mathbf{AV}_m = \omega_m^2 \mathbf{V}_m$$

If the acceleration in a component direction at a joint is known when a structure is undergoing vibration in a normal mode, the absolute amplitude of the mode shape is determined and loads may be computed. An acceleration may be known from previous dynamic testing, or analysis of an idealized damped version of the structure. Deflections are of the periodic form

$$u_i = u_{im} \sin \omega_m t$$

so the amplitude of acceleration in the $i^{th}$ generalized direction is

$$\ddot{u}_{im} = -u_{im}\,\omega_m^2$$

when $q_{im}$ is the acceleration amplitude input, the eigenvector will be renormalized by the factor

$$\frac{q_{im}\,g}{u_{im}\,\omega_m^2}$$

In summary, the program operates on the given matrices **K** and **M** as follows:

1. Compute $\mathbf{A} = \mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}$.
2. Compute $\mathbf{C} = \mathbf{A}^{-1}$.
3. Compute first-guess vectors $\mathbf{X}_m$.
4. Attempt to converge on the six eigenvalues and eigenvectors corresponding to modes of lowest frequency.
5. Compute $\mathbf{U}_m = \mathbf{M}^{-1/2}\mathbf{V}_m$, $\omega_m = 1/(\lambda_m)^{1/2}$.
6. Renormalize mode shape to input acceleration levels; compute loads from equations given in Appendix B.
7. Equilibrium checks at each joint.
8. Output mode shapes, frequencies, dynamic loads.

## I. Jacobi's Method

The Jacobi method (Ref. 23) of determining eigenvalues and eigenvectors exists in the program. The advantages of the method is that all eigenvalues and eigenvectors are evaluated with equal accuracy, multiple roots can be evaluated, and zero frequencies systems can be handled. The modified power method (described in Section H) is retained because it has been successfully used for the past 3 yr.

Equation 4, $\mathbf{AV}_m = \omega_m^2 \mathbf{V}_m$, is used for the evaluation of the eigenvalues and eigenvectors by Jacobi's Method. For an $n$ degree-of-freedom system, $n$ equations ($m = 1, 2, \cdots n$) can be written as

$$\mathbf{AV} = \mathbf{VL} \tag{13}$$

where **V** is the matrix of eigenvectors, and **L** is a diagonal matrix of eigenvalues. Since **V** is an orthogonal matrix, Eq. 13 can be written as

$$\mathbf{V}^T\mathbf{AV} = \mathbf{L} \tag{14}$$

Jacobi's Method is to start with a given matrix **A** and transform it by a number of pre- and post-multiplications $\mathbf{Z}^{(k)T}$ and $\mathbf{Z}^{(k)}$ where

$$\prod_{k=1}^{\ell} \mathbf{Z}^{(k)} = \mathbf{Z}$$

and

$$\prod_{k=1}^{\ell} \mathbf{Z}^{(k)T} = \mathbf{Z}^T$$

($\ell$ = number of transformations), such that

$$\mathbf{Z}^T\mathbf{AZ} = \text{diagonal matrix} = \mathbf{L}^{\prime} \tag{15}$$

The $\mathbf{Z}^T$ and $\mathbf{Z}$ must satisfy the relation $\mathbf{ZZ}^T = \mathbf{I}$. If the satisfactory matrix condition can be obtained, then comparing Eq. 15 with Eq. 13, **Z** is the desired eigenvector matrix and $\mathbf{L}^{\prime}$ is the diagonal eigenvalue matrix.

The Jacobi process of obtaining the orthogonal **Z** matrix is to annihilate, in turn, selected off-diagonal elements of **A** by orthogonal transformations. To eliminate an element $a_{ij}^{(k)}$ $(i < j)$ of $\mathbf{A}^{(k)}$ the elements of the transformation matrix $\mathbf{Z}^{(k)}$ would be

$$z_{ii} = \cos\theta, \qquad z_{ij} = \sin\theta$$

$$z_{ji} = -\sin\theta, \qquad z_{jj} = \cos\theta$$

$$z_{kk} = 1, \text{ and } z_{il} = z_{kj} = z_{kl} = 0$$

where

$$\begin{aligned} k &\neq i, j \\ l &\neq i, j \end{aligned} \tag{16}$$

Represent the $k^{th}$ transformation matrix product by

$$\mathbf{Z}^{(k)T}\mathbf{Z}^{(k-1)T}\cdots\mathbf{Z}^{(1)T}\mathbf{A}\mathbf{Z}^{(1)}\mathbf{Z}^{(2)}\cdots\mathbf{Z}^{(k)} = \mathbf{A}^{(k)} \quad (17)$$

The elements of $\mathbf{A}^{(k)}$ are

$$\left.\begin{aligned} a_{il}^{(k)} &= a_{il}^{(k-1)}\cos\theta - a_{jl}^{(k-1)}\sin\theta \\ a_{jl}^{(k)} &= a_{il}^{(k-1)}\sin\theta + a_{jl}^{(k-1)}\cos\theta \\ a_{ki}^{(k)} &= a_{ki}^{(k-1)}\cos\theta - a_{kj}^{(k-1)}\sin\theta \\ a_{kj}^{(k)} &= a_{ki}^{(k-1)}\sin\theta + a_{kj}^{(k-1)}\cos\theta \\ a_{kl}^{(k)} &= a_{kl}^{(k-1)} \end{aligned}\right\} k, l \neq i, j \quad (18)$$

$$\begin{aligned} a_{ii}^{(k)} &= a_{ii}^{(k-1)}\cos^2\theta + a_{jj}^{(k-1)}\sin^2\theta - 2a_{ij}^{(k-1)}\sin\theta\cos\theta \\ a_{jj}^{(k)} &= a_{ii}^{(k-1)}\sin^2\theta + a_{jj}^{(k-1)}\cos^2\theta + 2a_{ij}^{(k-1)}\sin\theta\cos\theta \\ a_{ij}^{(k)} &= \frac{1}{2}\left(a_{ii}^{(k-1)} - a_{jj}^{(k-1)}\right)\sin 2\theta + a_{ij}\cos 2\theta \end{aligned} \quad (19)$$

In order to eliminate $a_{ij}^{(k)}$, the equation

$$\frac{1}{2}\left(a_{ii}^{(k-1)} - a_{jj}^{(k-1)}\right)\sin 2\theta + a_{ij}\cos 2\theta = 0$$

or

$$\tan 2\theta = -\frac{a_{ij}^{(k-1)}}{\frac{1}{2}\left(a_{ii}^{k-1} - a_{jj}^{k-1}\right)} \quad (20)$$

is the angle $\theta$ required for annihilation of $a_{ij}^{(k)}$.

The orthogonal transformation designed to annihilate an off-diagonal term may undo the previously annihilated off-diagonal terms. For this reason, the Jacobi's method is an iterative process, rather than a finite one, that is carried on indefinitely until a predetermined accuracy requirement is satisfied. The success of the method depends on each transformation reducing the sum of the squares of the off-diagonal terms; the proof is outlined below. Stability of the convergence process against round-off error is mentioned (Ref. 23).

The Pope-Tompkins scheme for convergence of Jacobi's method when subjected to the transformations shall be proved.

From Eq. 18

$$a_{il}^{(k)2} + a_{jl}^{(k)2} = a_{il}^{(k-1)2} + a_{jl}^{(k-1)2}$$

and

$$a_{li}^{(k)2} + a_{lj}^{(k)2} = a_{li}^{(k-1)2} + a_{lj}^{(k-1)2} \quad (21)$$

and since other elements $a_{kl}(l,k \neq i,j)$ are unaffected by the transformation, with the exception of $a_{ij}^{(k)}$ and $a_{ii}^{(k)}$, the sum of the squares of the off-diagonal element is invariant.

Also

$$a_{ii}^{(k)2} + a_{ij}^{(k)2} + a_{ji}^{(k)2} + a_{jj}^{(k)2}$$
$$= a_{ii}^{(k-1)2} + a_{ij}^{(k-1)2} + a_{ji}^{(k-1)2} + a_{jj}^{(k-1)2} \quad (22)$$

Since $a_{ij}^{(k)} = a_{ji}^{(k)} = 0$, from Eq. 19

$$a_{ii}^{(k)2} + a_{jj}^{(k)2} = a_{ii}^{(k-1)2} + a_{jj}^{(k-1)2} + 2a_{ij}^{(k-1)2} \quad (23)$$

Equation 23 shows that the quantity $2a_{ij}^{(k-1)2}$ has been lost from the sum of squares of the off-diagonal terms.

$$\text{Define } \nu_I = \left\{\sum_{\substack{l,k=1 \\ l \neq k}}^{n} a_{lk}^{(1)2}\right\}^{1/2} \quad (24)$$

where $\nu_I$ is the initial off-diagonal norm. A threshold $\nu_1$ is established by dividing $\nu_I$ by a fixed constant $\sigma \geq n$; the threshold value is used to determine the terms to be annihilated first. The off-diagonal elements for which

$$\left|a_{kl}^{(k)}\right| \geq \nu_1 = \frac{\nu_I}{\sigma}, k \neq l \quad (25)$$

is annihilated.

Since $\sigma \geq n$, there exists at least one off-diagonal element $\geq \nu_1$, since if all were $\leq$

$$\sum_{l \neq k} a_{kl}^2 \leq \sum \nu_1^2 = n(n-1)\,\nu_1^2 \leq n^2\nu_1^2 \leq \sigma^2\nu_1^2 = \nu_I^2$$

which contradicts Eq. 24. Note that, because of symmetry, only one half of diagonal elements are used. For any element whose magnitude is not smaller than $\nu_1$, the appropriate transformation is performed. Thus, from Eq. 23 the off-diagonal squared norm is decreased by at least $2\nu_1^2$. If all off-diagonal terms $< \nu_1$, the off-diagonal norm is bounded as follows:

$$\nu_{od}^2 = \nu_I^2 - \sum_{|a_{kl}| \geq \nu_1} 2a_{kl}^2 < \nu_I^2 - 2\nu_1^2 = \left(1 - \frac{2}{\sigma^2}\right)\nu_I^2 \quad (26)$$

Lower the threshold by $\nu_2 = \frac{\nu_1}{\sigma}$ and proceed as before. Continue until $\nu_r \leq \frac{\nu_{od}}{\sigma}$; at this point let $\nu_{od}$ play the role of $\nu_I$ and as shown before, there exists an off-diagonal element $\geq \frac{\nu_{od}}{\sigma}$ and, thus, $> \nu_r$. Performing the appropriate transformations on all elements $> \nu_r$, a bound on the new off-diagonal norm follows

$$(\nu_{od}^{(2)})^2 = \nu_{od}^2 - \sum_{|a_{kl}| \geq \nu_r} 2a_{kl}^2 < \nu_{od}^2 - 2\nu_{od}^2/\sigma^2$$

$$= \left(1 - \frac{2}{\sigma^2}\right)\nu_{od}^2 < \left(1 - \frac{2}{\sigma^2}\right)^2 \nu_I^2 \quad (27)$$

By induction, if $\nu_{od}^{(m)}$ is the off-diagonal norm after $m$ stages in which at least one transformation has been performed, then at worst

$$(\nu_{od}^{(m)})^2 \leq \left(1 - \frac{2}{\sigma^2}\right)^m \nu_I^2 \quad (28)$$

For convergence a final threshold $\nu_F$ must be established such that

$$\nu_{od}^2 = \sum_{k \neq l} a_{kl}^2 \leq n(n-1)\,\nu_F^2 < n^2\,\nu_F^2 \quad (29)$$

or the accuracy requirement may be specified as

$$\nu_{od}^2 \leq \rho^2\,\nu_I^2, \quad \text{where}$$

$$\nu_F = \left(\frac{\rho}{n}\right)\nu_I. \quad (30)$$

For this problem $\frac{\rho}{n} = 2^{-27}$ has been selected; thus

$$\nu_{od}^2 \leq n^2\,(2^{-27})^2\,\nu_I^2 = n^2\,2^{-54}\,\nu_I^2 \quad (31)$$

is the convergence criteria of this program.

### J. Orthogonality Check

If the structure has discrete nonequal eigenvalues, Eq. 3 can be written

$$\mathbf{K}\mathbf{U}_m = \omega_m^2\,\mathbf{M}\mathbf{U}_m$$

and

$$\mathbf{K}\mathbf{U}_n = \omega_n^2\,\mathbf{M}\mathbf{U}_n$$

where $m \neq n$.

Premultiply the first equation by $\mathbf{U}_n^T$ and the second equation by $\mathbf{U}_m^T$. Subtracting the transpose of the second equation from the first results in

$$(\omega_m^2 - \omega_n^2)\,\mathbf{U}_n^T\,\mathbf{M}\mathbf{U}_m = 0$$

Since $n \neq m$ was assumed, $\omega_m^2 \neq \omega_n^2$; thus, $\mathbf{U}_n^T\,\mathbf{M}\mathbf{U}_m = 0$ for all $m$ and $n$ not equal to each other. By a similar argument

$$\mathbf{U}_n^T\,\mathbf{K}\,\mathbf{U}_m = 0,\ n \neq m \quad (32)$$

can be shown. A $6 \times 6$ generalized weight and spring matrix representing the orthogonality check is outputed.

To obtain a better comparison of the magnitude of the off-diagonal terms, the generalized weight and spring matrix are normalized as

$$m_{ij}^{\star} = \frac{\overline{m}_{ij}}{(\overline{m}_{ii})^{1/2}\,(\overline{m}_{jj})^{1/2}} \quad (33)$$

and

$$k_{ij}^{\star} = \frac{\overline{k}_{ij}}{(\overline{k}_{ii})^{1/2}\,(\overline{k}_{jj})^{1/2}}. \quad (34)$$

The values $\overline{m}_{ij}$ and $\overline{k}_{ij}$ are the elements of the $i^{th}$ row, $j^{th}$ column of the generalized weight and spring matrix, and $\overline{m}_{ij}^{\star}$ and $\overline{k}_{ij}^{\star}$ are the elements of the $i^{th}$ row, $j^{th}$ column of the normalized generalized weight and spring matrices.

### K. Accuracy

Although no analytic studies have been made of error inherent in the numerical process described herein, enough has been learned from production runs and experiment with abnormal cases to permit some general comments on accuracy of the program. (The discussion is for Section H and not Section I.) Several tests, available to the user when results are in doubt, are discussed in this Section.

Accuracy of a structural analysis performed by the program is affected adversely by the following factors:

1. Errors in idealization of a structure. All structures must be idealized by one of the standard-structure types before analysis; a basic discussion of this procedure is presented in Section II-K.
2. Gross errors in input. These may be indicated by obvious errors in output, but all inputs should be carefully hand-checked.
3. Characteristics of the stiffness matrix **K** that lead the static deflections to be in error.
4. Characteristics of the matrix $\mathbf{C} = \mathbf{M}^{1/2}\,\mathbf{K}^{-1}\,\mathbf{M}^{1/2}$ that lead normal-mode shapes and frequencies to be in error.
5. Failure to properly test convergence of the normal-mode analysis.
6. Gross program or machine errors. The programs have been tested on check problems and on nearly 200 production runs. Correlation with test results has been good where tests have been run.

Single-precision arithmetic is used throughout; this provides storage of approximately eight decimal digits plus an exponent for all quantities. Required accuracy for the proposed engineering is two or more significant figures for the largest quantities in a set of deflections.

Input will usually be provided with three or more significant digits, with zeros filling out the stored number of eight digits. Computation during matrix generation introduces roundoff in the last one or two places of the elements of the stiffness matrix.

Accuracy of the static analysis

$$\mathbf{U} = \mathbf{K}^{-1}\mathbf{F}$$

is impaired if the stiffness matrix is singular or ill-conditioned. Singularity is caused by structural instability which, in turn, causes division by zero; since no overflow checks are made to detect division by zero, the only indicators are those cited below for ill-conditioning. This latter is a qualitative description of the loss of accuracy during computation of an inverse matrix. Generally, significant figures are lost during subtraction operations when digits ($r$) subject to roundoff are drawn into the significant places of a number:

$$\begin{array}{r} 0.1234567r \times 10^{0} \\ -0.1234566r \times 10^{0} \\ \hline 0.0000001r \times 10^{0} = 0.1r000000 \times 10^{-6} \end{array}$$

It has been observed in structural usage that ill-conditioning becomes a problem when the stiffness of members are greatly different. Thus, when the ratios of diagonal elements of $K$ were

$$\frac{k_{ii}}{k_{jj}} < 10$$

systems of 130 degrees of freedom were successfully analyzed, while smaller systems with

$$\frac{k_{ii}}{k_{jj}} > 100$$

gave obviously false results. A second indication of ill-conditioning is the ratio of maximum to minimum eigenvalues of $\mathbf{K}$ or condition number

$$P = \frac{\lambda_{\max}}{\lambda_{\min}}$$

This number is computed for

$$\mathbf{A} = \mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}$$

and only indicates the condition of $\mathbf{K}$ when the elements of $\mathbf{M}$ are nearly equal. Condition numbers $P > 10^{n}$ may indicate loss of all significance from computed deflections.

Normal mode analysis is subject to the same problems of singularity and ill-conditioning as static analysis, plus problems caused by the nature of $\mathbf{M}$, and the convergent means of solution. If $\mathbf{M}$ contains zero diagonal elements, then $\mathbf{M}^{-1/2}$ will have elements produced by division by zero and

$$\mathbf{A} = \mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}$$

proves to be singular. If the ratios of elements of $\mathbf{M}$ are large, so that ratios of diagonal elements of $\mathbf{A}$ are large,

$$\frac{a_{ii}}{a_{jj}} > 100$$

then $\mathbf{A}$ may be ill-conditioned and $\mathbf{C} = \mathbf{A}^{-1}$ subject to large error.

Convergence is tested by comparing elements of normalized trial vectors at successive iterations

$$\mathbf{C}\mathbf{X}_{m}^{(i)} = \alpha_m \mathbf{X}_{m}^{(i+1)}$$

when

$$\left|x_{jm}^{(i)} - x_{jm}^{(i+1)}\right| \leq \epsilon$$

iterations are stopped. Alternate tests are

$$\left(\mathbf{C} - \lambda_{m}^{(i)}\mathbf{I}\right)\mathbf{V}_{m}^{(i)} = \mathbf{X}_{m}^{(i)}$$

$$\left|x_{jm}^{(i)}\right| \leq \epsilon$$

which may never stop iterations, and

$$\left|\lambda_{m}^{(i)} - \lambda_{m}^{(i+1)}\right| \leq \epsilon$$

which proves to stop the convergent process too soon. There is always a risk that convergence will stop too soon when it is very slow, since the change in any parameter then becomes small even when the parameter is far from its true value. Checks against this possibility include model testing and computation of normal modes by Jacobi's method. The maximum change in a vector element is output for checking; this value should be $\epsilon \leq 1 \times 10^{-7}$ at the final iteration of each mode.

The ratio of maximum to minimum eigenvalues of a matrix or condition number is a measure of the degree of ill-conditioning of the matrix. The maximum eigenvalue of

$$\mathbf{A}\mathbf{V}_m = \omega_m^2 \mathbf{V}_m.$$

$\omega_n^2$ may usually be found easily and accurately by the power method. The minimum eigenvalue, $\omega_1^2$, is found from

$$\mathbf{C}\mathbf{V}_n = \lambda_n \mathbf{V}_n$$

where

$$\mathbf{C} = \mathbf{A}^{-1}$$

As noted before, condition numbers

$$P = \frac{\omega_n^2}{\omega_1^2} < 10^6$$

usually indicate that engineering accuracy can be obtained.

A consequence of the power method is that eigenvalues are computed in descending order. If such is not the case, there probably has been no reliable convergence to one or more eigenvectors. Since frequencies are proportional to reciprocals of eigenvalues of $\mathbf{C} = \mathbf{A}^{-1}$, the output frequencies must be in ascending order. (When frequencies are very close, differing in the third place, this rule may be violated without prejudicing the results.)

The following eigenvalues are defined as:

$\lambda_A$ = computed lower eigenvalue of $\mathbf{A}$

$\lambda_C$ = corresponding computed upper eigenvalue of $\mathbf{C} = \mathbf{A}^{-1}$

$\lambda_T$ = true magnitude of lower eigenvalue of $\mathbf{A}$

It has been observed in tests with Hilbert matrices that the difference

$$\left| \lambda_A - \frac{1}{\lambda_C} \right| >> \left| \lambda_T - \frac{1}{\lambda_C} \right|$$

Also, in all reliable production runs

$$\lambda_A \lambda_C \approx 1$$

Thus, if

$$| \lambda_A \lambda_C - 1 | > 10^{-2}$$

the validity of $\lambda_C$ and its eigenvector should be doubted; and, if not, then the error in the computed eigenvalue is

$$\left| \lambda_T - \frac{1}{\lambda_C} \right| < \left| \lambda_A - \frac{1}{\lambda_C} \right|$$

Experience with the program has indicated that the accuracy rules mentioned above are to be used as a guide and are not absolute in ensuring accurate data. The best methods of evaluating the results have been the equilibrium checks at the joints and the orthogonality of the mode shapes. An estimate of the accuracy of the results can be determined by the equilibrium check from the non-zero terms at the unrestrained joints of the structure. Usually the off-diagonal terms of the generalized weight or spring matrix are orders of magnitude less than the diagonal terms if the mode shapes are correct.

In summary, the following checks are available to the user in program output:

1. Normal-mode convergence test, $\epsilon < 4 \times 10^{-7}$
2. Condition number, $P < 10^6$
3. Frequencies output in ascending order unless nearly equal
4. Equilibrium check at the joints
5. Eigenvalues of $\mathbf{C}$ equal eigenvalues of $\mathbf{A}$ within

$$| \lambda_A \lambda_C - 1 | < 10^{-2}$$

6. Orthogonality check of mode shapes

The following tests may be applied as the need arises:

1. Hand-check of program input
2. Reasonableness of program output
3. Ratios of stiffness matrix diagonal elements $k_{ii}/k_{jj} < 100$
4. Solution by independent numerical methods
5. Comparison with different idealizations of the same configuration
6. Model testing

### L. Structural Idealization

Matrix representations of five distinct types of structure have been programmed. Any structure to be analyzed must be idealized by a structure composed entirely of members of one of these types:

1. Three-dimensional, pin-jointed
2. Three-dimensional, rigid-jointed, with circular member cross-sections
3. Planar, rigid-jointed, loaded in-plane
4. Planar grid, rigid-jointed, loaded normal-to-plane
5. Three-dimensional, rigid-jointed doubly symmetric cross-section

Some types of idealization are commonly used in structural analysis; for example, trusses are usually assumed to be pin-jointed, and continuous slabs are often analyzed as grids. The following remarks will be concerned with typical approximations that extend the power of the program:

1. A continuous structure may be approximated by a "lumped-mass" system. Natural frequencies of a lumped-mass system will always be lower than those of the represented system, with the degree of approximation dependent on the quantity of mass points and connecting members in the idealization.

2. The stiffness (and thus normal modes) of a structure as stable as a truss will usually be well-represented by a pin-jointed truss. Secondary loads will not be found directly, but may be estimated from deflections.

3. The stiffness of a shear panel may be represented by a lattice of pin-connected members (Ref. 14). In most cases, if a negative area is required as specified by the lattice analogy, negative frequencies (obviously erroneous) results.

4. Flexible supports may be represented by inserting members with appropriate stiffness at points of support.

5. A beam of varying section properties may be approximated by several beams of constant section. Care should be taken that the problem does not become ill-conditioned by making the stiffness of the small beam segments very large in comparison to other elements of the structure.

6. Members normal to the plane of a grid may be included by adding appropriate stiffnesses to the matrix of the grid in the normal direction.

7. Pin-ended members in a rigid-jointed frame may be input with zero moment of inertia.

8. Loads applied at the interior of a bending member may be approximated by shears and fixed-end moments at its ends.

9. The validity of an idealization may be checked by comparison with a continuous structure idealization, comparison with another lumped-mass idealization, or by model testing.

10. Increments of elements of the stiffness matrix may be input to the program; thus, the stiffness of structural components that are not conveniently idealized by a standard member may be included in an analysis.

11. Various end conditions can be approximated by a linkage system of several members; additional degrees of freedom will be required.

# III. PROGRAMMING

## A. *Input Format*

Input to the program is provided in the following blocks. An example of the input format is given in the sample problem, Appendix D.

1. Comment
2. Control
3. Joint coordinates
4. Member properties
5. Restraints
6. Stiffness matrix elements (optional)
7. Static loadings (optional)
8. Accelerations (optional)

The most convenient sheet on which to enter data for punching contains nine or more columns. Each line is punched on one card, with a maximum of nine words per card. Where a word is not required as input for a problem, it may be left blank; blanks are always read as 0. Most of the input is written with fixed-point numbers (integers) in the first two columns and floating-point numbers (mixed numbers) in succeeding columns. Floating-point numbers must be written with a decimal point regardless of whether the fractional part is present.

### 1. Comment

Comments up to 72 characters
(I2A6)

The first column must not be used

### 2. Control

| $N_1$ | $N_2$ | $N_3$ |
|---|---|---|

(3I8)

$N_1$ Structure type
- 1 Pin-jointed member, three dimensions
- 2 Rigid-jointed member, equal member cross-section moment of inertia, three dimensions
- 3 Planar member, rigid joints, loaded in-plane
- 4 Planar member, rigid joints loaded normal-to-plane (grid)
- 5 Rigid-jointed member, doubly symmetric cross-section, three dimensions.

$N_2$ Mode shape card output control (I3,6F8.5)
- 0 No output desired
- 1 Output desired
- *n* Number of modes desired for Jacobi Method (use only if $N_{10} < 0$)

$N_3$ Eigenvalue control
- 1 No output desired
- 0 Output desired
- *n* Numbers of rigid body modes to be eliminated, if any. ($N_{10} < 0$)

| A | $N_4$ | $N_5$ | $N_6$ | $N_7$ | $N_8$ | $N_9$ | $N_{10}$ | $N_{11}$ |
|---|---|---|---|---|---|---|---|---|

(9I8)

A Problem number (six characters or less)

$N_4$ Quantity of joints in structure

$N_5$ Quantity of members in structure

$N_6$ Quantity of static loadings

$N_7$ Weight code
- 0 No weight input
- 1 Weight input included

$N_8$ Quantity of joints having one or more components of restraint

$N_9$ Degrees of freedom per joint

$N_{10}$ Normal-mode code
- 0 Compute no normal modes
- 1 Compute lowest six mode shapes, normalized to input accelerations, and compute dynamic loads
- 2 Compute lowest six mode shapes only, normalized to the largest component ($u_{max} = 1.0$)
- −1 Jacobi's method for evaluating eigenvectors and eigenvalues; the lowest six non-rigid body eigenvectors normalized to input accelerations, and compute dynamic loads
- −2 Jacobi's method for evaluating eigenvectors and eigenvalues; the lowest six non-rigid body eigenvectors normalized to the largest component ($u_{max} = 1.0$)

$N_{11}$ Output code
- 0 No output of **K**, **L**, **W** matrices
- 1 Output **K**, **L**, **W** matrices

| $N_{12}$ | $N_{13}$ | $E$ | $\nu$ | $\gamma$ |
|---|---|---|---|---|

(2I8,3E8.0)

$N_{12}$ Quantity of stiffness matrix elements to be altered

$N_{13}$ Temperature code

0 Temperature problem not to be solved

1 Temperature problem to be solved

$E$ Elastic modulus, $10^6$ lb/in.$^2$

$\nu$ Poisson's ratio

$\gamma$ Specific weight lb/in.$^3$

**3. Joint Coordinates**

| $j$ | Blank | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|---|

(2I8,3F8.0)

$j$ Joint number (must be listed consecutively starting with 1)

$\left.\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}\right\}$ Joint coordinates, in.

In two-dimensional problems $x_3$ must be normal to the plane

**4. Member Properties**

Properties are entered on one line (one card per member); when temperature code, $N_{13} = 1$, then $A_5$, $A_6$, or $A_7$ must be included. Values for $A_i$ are not required unless specifically indicated.

| $p$ | $q$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ |
|---|---|---|---|---|---|---|---|---|

(2I8, 7E8.0)

$\left.\begin{matrix} p \\ q \end{matrix}\right\}$ Member ends (enter in any order; enter each member once only)

Member properties and temperature inputs are defined for each structure type as follows (all quantities to be input in inch units):

*a. Structure type 1, three-dimensional, pin-jointed members*

$A_1 = A$, section area

$A_3$ = non-zero term

or

$A_1 = D$, outside diameter of circular tube

$A_2 = T$, wall thickness of circular tube

$A_3 = 0$

and

$A_5 = \alpha\Delta T$, coefficient of thermal expansion times change in temperature of member. Positive $\Delta T$ indicates increase in temperature.

*b. Structure type 2, three-dimensional, rigid-jointed, equal member cross-section moment of inertia*

$A_1 = A$, section area

$A_2 = I$, section moment of inertia

$A_3 = K$, section torsional stiffness

or

$A_1 = D$, outside diameter of circular tube

$A_2 = T$, wall thickness of circular tube

$A_3 = 0$

and

$A_5 = \alpha\Delta T$, coefficient of thermal expansion times change in temperature of member. Positive $\Delta T$ indicates increase in temperature.

*c. Structure type 3, two-dimensional, rigid-jointed members, loaded in-plane*

$A_1 = A$, section area

$A_2 = I$, section moment of inertia

$A_3$ = non-zero term

or

$A_1 = D$, outside diameter of circular tube

$A_2 = T$, wall thickness of circular tube

$A_3 = 0$

and

$A_5 = \alpha\Delta T$, coefficient of thermal expansion times change in temperature of member. Positive $\Delta T$ indicates increase in temperature.

$A_6 = \alpha\delta T/h$, coefficient of thermal-expansion times change in temperature across member cross-section divided by height of rectangular cross-section. $\delta T$ is positive if a change in temperature will tend to rotate joint $p$ of the member in positive $\mathbf{x}_3$ direction. Joint $p$ of member is the first joint listed in III A-4 to describe the member.

*d. Structure type 4, two-dimensional rigid-jointed, loaded normal-to-plane (grid)*

$A_2 = I$, section moment of inertia

$A_3 = K$, section torsional constant

or

$A_1 = D$, outside diameter of circular tubes

$A_2 = T$, wall thickness of circular tubes

$A_3 = 0$

and

$A_6 = \alpha\delta T/h$, coefficient of thermal-expansion times change in temperature across member cross-section divided by height of rectangular cross-section. $\delta T$ is positive if the increase in temperature across member cross-section is in positive $\mathbf{x}_3$ direction.

*e. Structure, type 5, three-dimensional rigid-jointed member, doubly symmetric cross-section*

$A_1 = A$, section area

$A_2 = I_1$, section torsional constant

$A_3 = I_2$, section moment of inertia about $\xi_2$ axis

$A_4 = I_3$ section moment of inertia about $\xi_3$ axis

$A_5 = \alpha\Delta T$, coefficient of thermal expansion times change in temperature of member. Positive $\Delta T$ indicates increase in temperature.

$A_7 =$ Joint number in *input list* (III A-3) not along member axis. The section moment of inertia $I_2$ is about $\xi_2$, which is perpendicular to the plane formed by the member $\mathbf{pq}$ ($p$ represents first joint listed to describe member in member properties input, and $q$ the second joint) and $\overline{pA_7}$; $\xi_3$ is positive in the direction $\boldsymbol{pq} \times \overline{\boldsymbol{pA_7}}$.

**5. Restraints**

The restraints must be followed by a zero card if no stiffness matrix element cards are incorporated in a temperature problem.

| $j$ | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ | $r_6$ |
|---|---|---|---|---|---|---|

(7I8)

$j$ Joint number (may be listed in any order)

$r_i$ Restraint code (integer)

0 No restraint

1 $i$th component of deflection at joint $j$ is 0. The order of deflection components at a joint in each structure type is as follows:

*a. Structure type 1, three-dimensional, pin-jointed members*

$u_{j1}$ = displacement in $\mathbf{x}_1$ direction

$u_{j2}$ = displacement in $\mathbf{x}_2$ direction

$u_{j3}$ = displacement in $\mathbf{x}_3$ direction

*b. Structure type 2, three-dimensional, rigid-jointed members, equal member cross-section moment of inertia*

$u_{j1}$ = displacement in $\mathbf{x}_1$ direction

$u_{j2}$ = displacement in $\mathbf{x}_2$ direction

$u_{j3}$ = displacement in $\mathbf{x}_3$ direction

$u_{j4}$ = rotation about $x_1$ axis

$u_{j5}$ = rotation about $x_2$ axis

$u_{j6}$ = rotation about $x_3$ axis

*c. Structure type 3, two-dimensional, rigid-jointed members, loaded in-plane*

$u_{j1}$ = displacement in $\mathbf{x}_1$ direction

$u_{j2}$ = displacement in $\mathbf{x}_2$ direction

$u_{j3}$ = rotation about $x_3$ axis

*d. Structure type 4, two-dimensional, rigid-jointed, loaded normal-to-plane (grid)*

$u_{j1}$ = displacement in $\mathbf{x}_3$ direction

$u_{j2}$ = rotation about $x_1$ axis

$u_{j3}$ = rotation about $x_2$ axis

*e. Structure type 5, three-dimensional, rigid-jointed member, doubly symmetric cross-section*

$u_{j1}$ = displacement in $x_1$ direction

$u_{j2}$ = displacement in $x_2$ direction

$u_{j3}$ = displacement in $x_3$ direction

$u_{j4}$ = rotation about $x_1$ axis

$u_{j5}$ = rotation about $x_2$ axis

$u_{j6}$ = rotation about $x_3$ axis

**6. Stiffness Matrix Elements**

To account for the effect of structural elements that cannot be idealized by members of the type with which an analysis is being performed, increments to elements of the matrix may be inputed. This block may be inputed only if the control parameter $N_{12} \neq 0$. The stiffness matrix elements must be followed by a zero card for a temperature problem.

| i | j | $\Delta k_{ij}$ |
|---|---|---|

(2I8,E8.2)

i, j — Row and column, respectively, of revised element in non-contracted stiffness matrix (insert as if rows and columns have not been deleted to account for restraints).

$\Delta k_{ij}$ Incremental change to element $k_{ij}$ of original stiffness matrix. The new element, $k_{ij} = k_{ij} + \Delta k_{ij}$.

**7. Weights**

| $j$ | Blank | $W_1$ | $W_2$ | $W_3$ | $W_4$ | $W_5$ | $W_6$ |
|---|---|---|---|---|---|---|---|

(2I8,6E8.2)

$j$ Joint number (may be listed in any order)

$W_i$ $i^{th}$ component of inertia at joint $j$. The order of translational inertia (lb) and rotary inertia (lb-in.$^2$) components is as specified for deflections in III A-5.

If normal modes are to be computed, finite (nonzero) inertia components should be specified for all degrees of freedom of the structure; the effect of a zero inertia is to produce accumulator overflow. (This is a peculiarity of the numerical procedure.) This block of input may be written only if the weight code, $N_7 = 1$. If no loadings follow ($N_6 = 0$), the last card of weights must be followed by a card with 0 as its first word. If temperature problem is to be solved ($N_{13} = 1$), the weight cards must not be incorporated.

**8. Static Loadings**

Each loading is initiated by a card with 0 or $-1$ as the first word, and the final loading must be followed by a zero card (blank card).

The initial card has the format:

| $j$ | Blank | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ |
|---|---|---|---|---|---|---|---|

(2I8,6E8.2)

$j = 0$, $A_i$ ($i = 1, 2, 3$) are the components of translational acceleration on the structure as a rigid body in the $i^{th}$ coordinate direction (III A-5) and $A_i$ ($i = 4, 5, 6$) are the components of rotational acceleration of the structure as a rigid body about the $i^{th}$ coordinate direction (III A-5) with respect to the origin of the coordinates. The effect of specifying $A_i$ ($g$) is to multiply component $W_i$ ($i = 1, 2, 3$) at each joint by $A_i$ ($i = 1, 2, 3$), or $\epsilon_{i(k-3)l}\, x_j A_k$ where ($k = 4, 5, 6$).

$j = -1$, same as for $j = 0$ except the rigid body rotational acceleration is with respect to the center of weight.

$j$ = joint number. $A_i$ are components of concentrated load on joint $j$ in the $i^{th}$ direction. Order of load component is as specified in III A-5.

**9. Accelerations**

If the normal-mode code, $N_{10} = 0$ or 2, this block must be omitted. If $N_{10} = 1$ or $-1$ deflections and dynamic loads will be computed. In this case, six cards must be given in the following format (one for each mode $m$ in order):

| $j$ | $i$ | $q_m$ |
|---|---|---|

(2I8,E8.2)

$j$ Joint number

$i$ Translational component direction number as specified for deflections (see III A-5)

$q_m$ Acceleration ($g$) of joint $j$ in direction $\mathbf{x}_i$. If $j = 0$, the acceleration $q_m$ applies to the maximum deflection component in the mode shape. The mode shape is renormalized with the factor $q_m g/\omega_m^2 u_{ji}$ before output and load calculation. Rotary accelerations have no meaning in this application. A zero card after accelerations is not required if $N_{10} = 1$.

The matrix of coefficients $k_{ij}$ for a member of any type connecting joints $p$ and $q$ is derived by introducing unit-component deflections of $p$ and $q$, and calculating the forces at $p$ and $q$ produced by each deflection. Matrices for several types of member are presented in Appendix A.

To illustrate the method by which such matrices are computed, and how they are used in the generation of a matrix for a structure, consider the pin-ended member in two dimensions as shown in Fig. 1:

1. Compute member length S and direction cosines $\gamma_1$ and $\gamma_2$ from joint coordinates.

2. Introduce $u_{p1} = 1$, holding $u_{p2} = u_{q1} = u_{q2} = 0$.

3. Axial load in member $= -(AE/S)\gamma_1$

4. Compute force components at $p$ and $q$, holding loaded member in equilibrium:

$$f_{p1} = \frac{AE}{S}\gamma_1^2$$

$$f_{p2} = \frac{AE}{S}\gamma_1\gamma_2$$

$$f_{q1} = -\frac{AE}{S}\gamma_1^2$$

$$f_{q2} = -\frac{AE}{S}\gamma_1\gamma_2$$

This set of forces constitutes the first column of the stiffness matrix in the following equation. Succeeding columns are formed similarly:

$$\begin{Bmatrix} f_{p1} \\ f_{p2} \\ f_{q1} \\ f_{q2} \end{Bmatrix} = \frac{AE}{S}\left[\begin{array}{cc|cc} \gamma_1^2 & \gamma_1\gamma_2 & -\gamma_1^2 & -\gamma_1\gamma_2 \\ \gamma_1\gamma_2 & \gamma_2^2 & -\gamma_1\gamma_2 & -\gamma_2^2 \\ \hline -\gamma_1^2 & -\gamma_1\gamma_2 & \gamma_1^2 & \gamma_1\gamma_2 \\ -\gamma_1\gamma_2 & -\gamma_2^2 & \gamma_1\gamma_2 & \gamma_2^2 \end{array}\right]\begin{Bmatrix} u_{p1} \\ u_{p2} \\ u_{q1} \\ u_{q2} \end{Bmatrix}$$

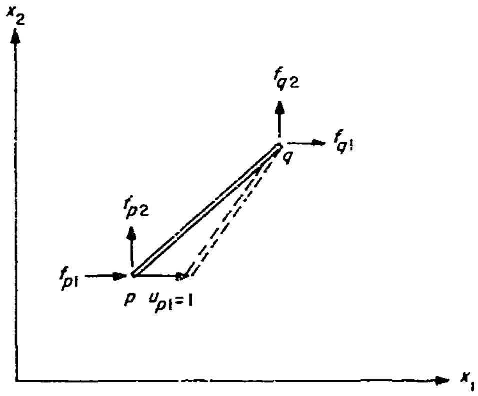



**Fig. 1. Illustrative member**

The notation of this equation may be further condensed by writing

$$\begin{Bmatrix} \mathbf{f}_p \\ \mathbf{f}_q \end{Bmatrix} = \begin{bmatrix} \mathbf{K}_{pp} & \mathbf{K}_{pq} \\ \mathbf{K}_{qp} & \mathbf{K}_{qq} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_p \\ \mathbf{u}_q \end{Bmatrix}$$

where the vectors have components

$$\mathbf{f}_p = \begin{Bmatrix} f_{p1} \\ f_{p2} \end{Bmatrix}, \quad \mathbf{u}_p = \begin{Bmatrix} u_{p1} \\ u_{p2} \end{Bmatrix}$$

etc., and the elements of $\mathbf{K}_{pq}$ are components of the force vector $\mathbf{f}_p$ for unit values of each component of $\mathbf{u}_q$.

The stiffness matrix for the simple truss illustrated in Fig. 2 will be generated by appropriate superposition of the matrices of its members.

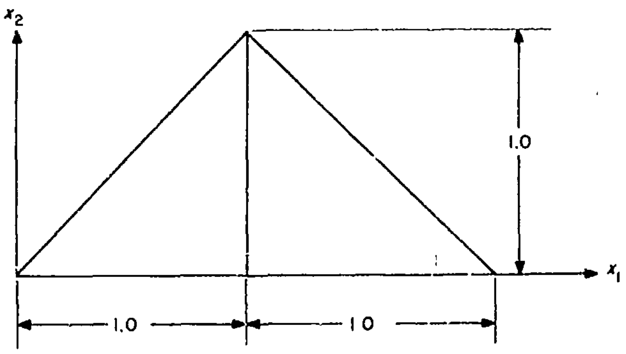



**Fig. 2. Illustrative problem**

*Size limitations*

| | |
|---|---|
| Degree of freedom of structure | 130 |
| Joints in structure (free or fixed) | 60 |
| Members in structure | 200 |
| Components of restraint | 100 |
| Loadings | 6 |
| Joints × degree of freedom per joint | 180 |

## B. Output Format

The output is printed in the following divisions. An example of the output format is given in the sample problem, Appendix D.

1. Input data
2. Stiffness matrix (lb/in.), weight (lb) matrix, load matrix (lb) printed columnwise, ten words per line. Each column is numbered.
3. Weight (lb), center of weight (in.), and weight moment-of-inertia matrix about center of weight (lb-in.$^2$).

   The $W_i$ in the directions $\mathbf{x}_i$ $(i = 1, 2, 3)$ are summed individually; the $\bar{x}_i$ (center of weight) is determined by using $W_i$'s in $\mathbf{x}_i$ directions; the weight inertia matrix with respect to center of weight is determined by using only weights in the $\mathbf{x}_1$ direction.
4. Static or thermal deflections. Each column corresponds to one loading; deflections at each joint follow the joint number in the order specified in III A-5.
5. Static or thermal member loads. The output values are defined in Appendix B.
6. Equilibrium check of static solution at each joint. The non-zero terms represent the reactions at the restraints; the reactions are positive if they act along the positive $\mathbf{x}_i$ directions. The equilibrium check is not made for the thermal loads. The unrestrained joints at which elements to stiffness matrix are added will not be 0 in equilibrium check; the non-zero term $f_i = \Delta k_{ij}\, u_j$.
7. Convergence data. The results of accuracy tests discussed in Section II-K are printed under appropriate headings.
8. Six frequencies, computed from the eigenvalues of the matrix C (see Section II-H), assuming input of weight in pound units and dimensions in inch units:

$$f_m = \frac{1}{2\pi}\left(\frac{g}{\lambda_m}\right)^{1/2} = 3.128518\left(\frac{1}{\lambda_m}\right)^{1/2}$$

9. Eigenvectors and eigenvalues using Jacobi's Method ($N_{1a} < 0$).
10. Dynamic member loads. The output values are defined in Appendix B.
11. Eigenvectors corresponding to the six eigenvalues of III-B-8.
12. Equilibrium check of dynamic solution at each joint. The non-zero terms represent the reactions at the restraints; the reactions are positive if they act along the positive $\mathbf{x}_i$ directions.
13. Generalized weight and spring matrix.
14. Normalized generalized weight and spring matrix.

To obtain an estimate of the time required to solve the various parts of the problem, the computer times are printed out after the following calculations:

1. Zero time
2. Reading input
3. Generating stiffness matrix
4. Generate load and weight matrices
5. Stiffness matrix inversion
6. Static displacement calculations
7. Static load calculations
8. Temperature calculations
9. Eigenvalue computation
10. Dynamic displacements
11. Dynamic loads

The first two numbers represent hours, the second two numbers represent minutes, and the fifth number represents tens of seconds.

Certain input errors will terminate the computation process and the cause will be part of the output format. The following errors will be detected:

1. ERROR READING JOINT COORDINATES. The joints coordinates are not in order.
2. PROBLEM EXCEEDS TOTAL DEGREE-OF-FREEDOM SIZE LIMITATION. The number of joints times the number of degrees of freedom at each joint exceeds 180.

3. NUMBER OF ALLOWABLE RESTRAINTS EXCEEDED. The number of restraint exceeds 100.

4. PROBLEM EXCEEDS NUMBER DEGREES-OF-FREEDOM SIZE LIMITATION. The number of joints times the number of degrees of freedom for each joint minus the number of restraints exceeds 130.

5. NEGATIVE EIGENVALUES.

6. STIFFNESS ELEMENT HAS BEEN PUT ON A A RESTRAINT. A change to the stiffness matrix corresponding to a restrained degree of freedom has been specified in the input.

7. NO ORTHOGONALITY CHECK, $pq$. For option 5, $\xi$, is not orthogonal to the member for member $pq$.

### C. Tape Requirements

| Logical | Channel-unit | Use |
|---|---|---|
| 4 | A4 | Intermediate input tape |
| 5 | A2 | BCD input |
| 6 | A3 | BCD output |
| 7 | B4 | BCD card output |
| 9 | A5 | Intermediate scratch |
| 14 | B7 | Intermedi te storage |
| 15 | A8 | Intermediate scratch |

## BIBLIOGRAPHY


1. Crandall, S. H., *Engineering Analysis,* McGraw-Hill Book Co., Inc., New York, N. Y., 1956.
2. von Karman, T. and Biot, M. A., *Mathematical Methods in Engineering,* McGraw-Hill Book Co., Inc., New York, N. Y., 1940.
3. Frazer, R. A., Duncan, W. J., and Collar, A. R., *Elementary Matrices, and Some Applications to Dynamics and Differential Equations,* Cambridge University Press, New York, N. Y., 1955.
4. Lanczos, C., *Applied Analysis,* Prentice Hall, Inc., Englewood Cliffs, N. J., 1956.
5. Bodewig, E., *Matrix Calculus,* Interscience Publishers, Inc., New York, N. Y., 1956.
6. Norris, C. H., et al., *Structural Design for Dynamic Load,* McGraw-Hill Book Co., Inc., New York, N. Y., 1959.
7. Hildebrand, F. B., *Introduction to Numerical Analysis,* McGraw-Hill Book Co., Inc., New York, N. Y., 1956.
8. Scanlan, R. H. and Rosenbaum, R., *Introduction to the Study of Aircraft Vibration and Flutter,* The Macmillan Co., New York, N. Y., 1951.
9. Hall, S. A. and Woodhead, R. W., *Frame Analysis,* John Wiley and Sons, Inc., New York, N. Y., 1961.
10. Timoshenko, S., *Vibration Problems in Engineering,* D. Van Nostrand Co., Inc., New York, N. Y., 1955.
11. Roark, R. J., *Formulas for Stress and Strain,* McGraw-Hill Book Co., Inc., New York, N. Y., 1954, p. 174.
12. Argyris, J. H., "Energy Theorems and Structural Analysis" *Aircraft Engineering,* October 1954, November 1954, February 1955, March 1955, April 1955, and May 1955.
13. Archer, J. S., "Digital Computation for Stiffness Matrix Analysis," *Proceedings of the American Society of Civil Engineers,* October 1958.
14. Archer, J. S. and Samson, C. H., Jr., "Structural Idealization for Digital Computer Analysis," *Conference Papers of the American Society of Civil Engineers,* Pittsburgh Conference on Electronic Computation, September 1960.
15. Martin, H. C., "Truss Analysis by Stiffness Consideration," *Proceedings of the American Society of Civil Engineers,* October 1956.

16. Clough, R. W., "Structural Analysis by Means of a Matrix Algebra Program," *Conference Papers of the American Society of Civil Engineers,* Kansas City Conference on Electronic Computation, November 1958.

17. Turner, M. J., Clough, R. W., Martin, H. C., and Topp, L. J., "Stiffness and Deflection Analysis of Complex Structures," *Journal of Aeronautical Sciences,* September 1956.

18. Klein, D., "A Simple Method of Matrix Structural Analysis," *Journal of Aeronautical Sciences,* January 1957, November 1957, and June 1958.

19. Samson, C. H., Jr. and Bergmann, H. W., "Analysis of Low-Aspect-Ratio Aircraft Structures," *Journal of Aerospace Sciences,* September 1960.

20. Melosh, R. J., "A Stiffness Matrix for the Analysis of Thin Plates in Bending," *Journal of Aerospace Sciences,* January 1961.

21. Martin, I. and Hernandez, J., "Orthogonal Gridworks Loaded Normally to Their Planes," *Journal of the American Society of Civil Engineers,* January 1960.

22. White, P. A., "The Computation of Eigenvalues and Eigenvectors of a Matrix," *Journal of the Society for Industrial and Applied Mathematics,* December 1958.

23. Ralston, A., and Wilf, H. S., *Mathematical Methods for Digital Computers,* John Wiley and Sons, Inc., New York, N. Y., 1962, pp. 84–91.

# APPENDIX A

## Matrices for Various Member Types

The following derivations are performed on typical members by introducing successive unit coordinate deflections of their ends and calculating forces reacting on the member. Coordinate deflections include both translations and rotatior loads are forces and moments. In each case, the first column of the required matr is derived in some detail to illustrate procedure.

Matrices relating forces and displacements in structure-oriented ($x_i$) coordinates are desired here; but intermediate use of member-oriented ($\xi_i$) coordinates is made in the more complicated derivations.

In the derivations below, the following quantities are input or computed for each member $p - q$:

1. Input coordinates $x_p$ , $x_{qi}$
2. Input member properties, $A_i$, $E$
3. Compute member length

$$S = \left[(x_{q1} - x_{p1})^2 + (x_{q2} - x_{p2})^2 + (x_{q3} - x_{p3})^2\right]^{1/2}$$

4. Compute direction cosines

$$\gamma_1 = \frac{(x_{q1} - x_{p1})}{S}$$

$$\gamma_2 = \frac{(x_{q2} - x_{p2})}{S}$$

and

$$\gamma_3 = \frac{(x_{q3} - x_{p3})}{S}$$

Matrices $\mathbf{K}_{pp}$, $\mathbf{K}_{qp}$ are written satisfying the expression

$$\begin{Bmatrix} \mathbf{f}_p \\ \mathbf{f}_q \end{Bmatrix} = \begin{bmatrix} \mathbf{K}_{pp} \\ \mathbf{K}_{qp} \end{bmatrix} \{\mathbf{u}_p\}$$

*1. Structure type 1, three-dimensional, pin-jointed members (Fig. A-1)*

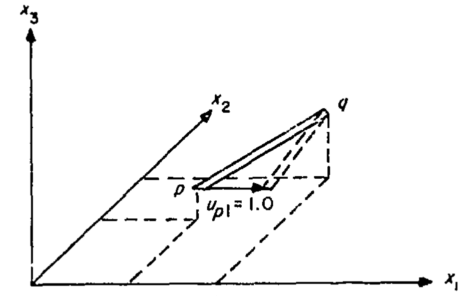



**Fig. A-1. Three-dimensional pin-jointed member**

Section property:

$$A = \mathrm{A}_1,$$

or if

$$\mathrm{A}_1 = 0$$

then

$$D = \mathrm{A}_1$$

and

$$T = \mathrm{A}_2$$

$$A = \pi T(D - T)$$

Introduce $u_{p1} = 1$

$$\text{Axial load} = \frac{AE}{S}\gamma_1$$

Force components at joints $p$ and $q$ are

$$f_{p1} = -f_{q1} = \frac{AE}{S}\gamma_1^2$$

$$f_{p2} = -f_{q2} = \frac{AE}{S}\gamma_1\gamma_2$$

and

$$f_{p3} = -f_{q3} = \frac{AE}{S}\gamma_1\gamma_3$$

The matrix relating displacements of joint $p$ to forces at joints $p$ and $q$ is

$$\frac{AE}{S}\left[\begin{array}{ccc} \gamma_1^2 & \gamma_1\gamma_2 & \gamma_1\gamma_3 \\ \gamma_1\gamma_2 & \gamma_2^2 & \gamma_2\gamma_3 \\ \gamma_1\gamma_3 & \gamma_2\gamma_3 & \gamma_3^2 \\ \hdashline -\gamma_1^2 & -\gamma_1\gamma_2 & -\gamma_1\gamma_3 \\ -\gamma_1\gamma_2 & -\gamma_2^2 & -\gamma_2\gamma_3 \\ -\gamma_1\gamma_3 & -\gamma_2\gamma_3 & -\gamma_3^2 \end{array}\right]$$

**2. *Structure type 2, three-dimensional, rigid-jointed members, equal member cross-section moment of inertia (Fig. A-2)***

Section properties:

$$A = \mathrm{A}_1$$

$$I = \mathrm{A}_2$$

$$K = \mathrm{A}_3$$

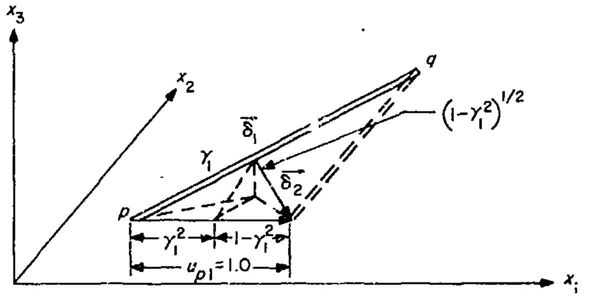



**Fig. A-2. Three-dimensional rigid-jointed member with equal member cross-section moment of inertia**

or if

$$A_3 = 0$$
$$D = A_1$$
$$T = A_2$$
$$A = T(D - T)\,\pi$$
$$I = \frac{\pi}{4}\left(\frac{1}{2}D^3T - \frac{3}{2}D^2T^2 + 2DT^3 - T^4\right)$$
$$K = 2I$$

Introduce $u_{p1} = 1$. Vector displacements in the axial and transverse directions at joint $p$ are

$$\boldsymbol{\delta}_1 = \gamma_1^2 \mathbf{x}_1 + \gamma_1\gamma_2 \mathbf{x}_2 + \gamma_1\gamma_3 \mathbf{x}_3$$
$$\boldsymbol{\delta}_2 = (1 - \gamma_1^2)\,\mathbf{x}_1 - \gamma_1\gamma_2 \mathbf{x}_2 - \gamma_1\gamma_3 \mathbf{x}_3$$

$\boldsymbol{\delta}_2$ is defined as a vector perpendicular to the plane defined by vectors $\boldsymbol{\delta}_1$ and $\mathbf{x}_1$.

A unit vector normal to $\boldsymbol{\delta}_1$ and $\boldsymbol{\delta}_2$ is

$$\boldsymbol{\delta}_3 = \frac{\boldsymbol{\delta}_1 \times \boldsymbol{\delta}_2}{|\boldsymbol{\delta}_1|\,|\boldsymbol{\delta}_2|}$$
$$= \frac{(\gamma_1\gamma_3 \mathbf{x}_2 - \gamma_1\gamma_2 \mathbf{x}_3)}{\gamma_1(1 - \gamma_1^2)^{1/2}}$$
$$= \frac{(\gamma_3 \mathbf{x}_2 - \gamma_2 \mathbf{x}_3)}{(1 - \gamma_1^2)^{1/2}}$$

The vector force exerted on joint $p$

$$= \frac{AE}{S}\,\boldsymbol{\delta}_1 + \frac{12EI}{S^3}\,\boldsymbol{\delta}_2$$

and the vector moment at joint $p$

$$= \frac{6EI}{S^2}\,(1 - \gamma_1^2)^{1/2}\,\boldsymbol{\delta}_3$$

Components of these load vectors are

$$f_{p1} = \text{force along } x_1 \text{ axis} = \frac{AE}{S}\gamma_1^2 + \frac{12EI}{S^3}(1-\gamma_1^2)$$

$$f_{p2} = \text{force along } x_2 \text{ axis} = \left(\frac{AE}{S} - \frac{12EI}{S^3}\right)\gamma_1\gamma_2$$

$$f_{p3} = \text{force along } x_3 \text{ axis} = \left(\frac{AE}{S} - \frac{12EI}{S^3}\right)\gamma_1\gamma_3$$

$$f_{p4} = \text{moment about } x_1 \text{ axis} = 0$$

$$f_{p5} = \text{moment about } x_2 \text{ axis} = \frac{6EI}{S^2}\gamma_3$$

$$f_{p6} = \text{moment about } x_3 \text{ axis} = -\frac{6EI}{S^2}\gamma_2$$

Similar load components at joint $q$ are

$$f_{q1} = -\frac{AE}{S}\gamma_1^2 - \frac{12EI}{S^3}(1-\gamma_1^2)$$

$$f_{q2} = \left(-\frac{AE}{S} + \frac{12EI}{S^3}\right)\gamma_1\gamma_2$$

$$f_{q3} = \left(-\frac{AE}{S} + \frac{12EI}{S^3}\right)\gamma_1\gamma_3$$

$$f_{q4} = 0$$

$$f_{q5} = +\frac{6EI}{S^2}\gamma_3$$

$$f_{q6} = -\frac{6EI}{S^2}\gamma_2$$

The required matrix will be written in terms of the quantities

$$C_0 = \frac{AE}{S}$$

$$C_1 = \frac{EK}{2S(1+\nu)}$$

$$C_{2A} = \frac{12EI}{S^3}$$

$$C_{2B} = \frac{6EI}{S^2}$$

$$C_{2C} = \frac{2EI}{S}$$

$$
\left[\begin{array}{cccccc}
C_0\gamma_1^2 + C_{2A}(1-\gamma_1^2) & (C_0-C_{2A})\gamma_1\gamma_2 & (C_0-C_{2A})\gamma_1\gamma_3 & 0 & C_{2B}\gamma_3 & -C_{2B}\gamma_2 \\
(C_0-C_{2A})\gamma_1\gamma_2 & C_0\gamma_2^2 + C_{2A}(1-\gamma_2^2) & (C_0-C_{2A})\gamma_2\gamma_3 & -C_{2B}\gamma_3 & 0 & C_{2B}\gamma_1 \\
(C_0-C_{2A})\gamma_1\gamma_3 & (C_0-C_{2A})\gamma_2\gamma_3 & C_0\gamma_3^2 + C_{2A}(1-\gamma_3^2) & C_{2B}\gamma_2 & -C_{2B}\gamma_1 & 0 \\
0 & -C_{2B}\gamma_3 & C_{2B}\gamma_2 & C_1\gamma_1^2 + 2C_{2C}(1-\gamma_1^2) & (C_1-2C_{2C})\gamma_1\gamma_2 & (C_1-2C_{2C})\gamma_1\gamma_3 \\
C_{2B}\gamma_3 & 0 & -C_{2B}\gamma_1 & (C_1-2C_{2C})\gamma_1\gamma_2 & C_1\gamma_2^2 + 2C_{2C}(1-\gamma_2^2) & (C_1-2C_{2C})\gamma_2\gamma_3 \\
-C_{2B}\gamma_2 & C_{2B}\gamma_1 & 0 & (C_1-2C_{2C})\gamma_1\gamma_3 & (C_1-2C_{2C})\gamma_2\gamma_3 & C_1\gamma_3^2 + 2C_{2C}(1-\gamma_3^2) \\
\hline
-C_0\gamma_1^2 - C_{2A}(1-\gamma_1^2) & (C_{2A}-C_0)\gamma_1\gamma_2 & (C_{2A}-C_0)\gamma_1\gamma_3 & 0 & -C_{2B}\gamma_3 & C_{2B}\gamma_2 \\
(C_{2A}-C_0)\gamma_1\gamma_2 & -C_0\gamma_2^2 - C_{2A}(1-\gamma_2^2) & (C_{2A}-C_0)\gamma_2\gamma_3 & C_{2B}\gamma_3 & 0 & -C_{2B}\gamma_1 \\
(C_{2A}-C_0)\gamma_1\gamma_3 & (C_{2A}-C_0)\gamma_2\gamma_3 & -C_0\gamma_3^2 - C_{2A}(1-\gamma_3^2) & -C_{2B}\gamma_2 & C_{2B}\gamma_1 & 0 \\
0 & -C_{2B}\gamma_3 & C_{2B}\gamma_2 & -C_1\gamma_1^2 + C_{2C}(1-\gamma_1^2) & (-C_1-C_{2C})\gamma_1\gamma_2 & (-C_1-C_{2C})\gamma_1\gamma_3 \\
C_{2B}\gamma_3 & 0 & -C_{2B}\gamma_1 & (-C_1-C_{2C})\gamma_1\gamma_2 & -C_1\gamma_2^2 + C_{2C}(1-\gamma_2^2) & (-C_1-C_{2C})\gamma_2\gamma_3 \\
-C_{2B}\gamma_2 & C_{2B}\gamma_1 & 0 & (-C_1-C_{2C})\gamma_1\gamma_3 & (-C_1-C_{2C})\gamma_2\gamma_3 & -C_1\gamma_3^2 + C_{2C}(1-\gamma_3^2)
\end{array}\right]
$$

*3. Structure type 3, two-dimensional, rigid-jointed members, loaded in-plane (Fig. A-3)*

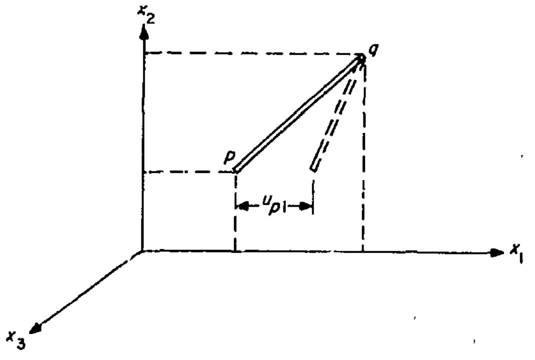



**Fig. A-3. Two-dimensional rigid-jointed loaded in-plane member**

Section properties:

$$A = A_1$$

$$I = A_2$$

or if

$$A_3 = 0$$

$$D = A_1$$

$$T = A_2$$

$$A = T(D - T)\pi$$

$$I = \frac{\pi}{4}\left(\frac{1}{2}D^3T - \frac{3}{2}D^2T^2 + 2DT^3 - T^4\right)$$

The derivation is similar to that preceding with $\gamma_3 = 0$.

The matrix is written in terms of

$$C_0 = \frac{AE}{S} \qquad C_{2B} = \frac{6EI}{S^2}$$

$$C_{2A} = \frac{12EI}{S^3} \qquad C_3 = \frac{2EI}{S}$$

Loads at joint $p$ are in the order

$f_{p1}$ = force along $x_1$ axis

$f_{p2}$ = force along $x_2$ axis

$f_{p3}$ = moment about $x_3$ axis

$$\left[\begin{array}{ccc}
C_0\gamma_1^2+C_{2A}(1-\gamma_1^2) & (C_0-C_{2A})\gamma_1\gamma_2 & -C_{2B}\gamma_2 \\
(C_0-C_{2A})\gamma_1\gamma_2 & C_0\gamma_2^2+C_{2A}(1-\gamma_2^2) & C_{2B}\gamma_1 \\
-C_{2B}\gamma_2 & C_{2B}\gamma_1 & 2C_3 \\
\hdashline
-C_0\gamma_1^2-C_{2A}(1-\gamma_1^2) & (C_{2A}-C_0)\gamma_1\gamma_2 & C_{2B}\gamma_2 \\
(C_{2A}-C_0)\gamma_1\gamma_2 & -C_0\gamma_2^2-C_{2A}(1-\gamma_2^2) & -C_{2B}\gamma_1 \\
-C_{2B}\gamma_2 & C_{2B}\gamma_1 & C_3
\end{array}\right]$$

***4. Structure type 4, two-dimensional, rigid-jointed, loaded normal-to-plane (grid) (Fig. A-4)***

Section properites:

$$I = A_2$$

$$K = A_3$$

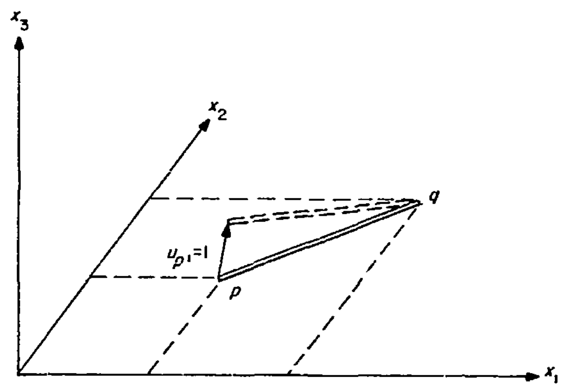



**Fig. A-4. Two-dimensional rigid-jointed loaded normal-to-plane member**

or if

$$A_3 = 0$$
$$D = A_1$$
$$T = A_2$$
$$I = \frac{\pi}{4}\left(\frac{1}{2}D^3T - \frac{3}{2}D^2T^2 + 2DT^3 - T^4\right)$$
$$K = 2I$$

Introduce $u_{p1} = 1$. Moment about an axis transverse to the member is of magnitude $6EI/S^2$. Components of load exerted on joints $p$ and $q$ are:

$$f_{p1} = -f_{q1} = \text{force in } \mathbf{x}_3 \text{ direction} = \frac{12EI}{S^3}$$

$$f_{p2} = f_{q2} = \text{moment about } x_1 \text{ axis} = \frac{6EI}{S^2}\gamma_2$$

$$f_{p3} = f_{q3} = \text{moment about } x_2 \text{ axis} = -\frac{6EI}{S^2}\gamma_1$$

As before, the matrix is written in terms of the parameters

$$C_1 = \frac{EK}{2S(1+\nu)} \qquad C_{2A} = \frac{12EI}{S^3}$$

$$C_{2B} = \frac{6EI}{S^2} \qquad C_3 = \frac{2EI}{S}$$

$$\left[\begin{array}{ccc}
C_{2A} & C_{2B}\gamma_2 & -C_{2B}\gamma_1 \\
C_{2B}\gamma_2 & C_1\gamma_1^2+2C_3\gamma_2^2 & (C_1-2C_3)\gamma_1\gamma_2 \\
-C_{2B}\gamma_1 & (C_1-2C_3)\gamma_1\gamma_2 & C_1\gamma_2^2+2C_3\gamma_1^2 \\
\hdashline
-C_{2A} & -C_{2B}\gamma_2 & C_{2B}\gamma_1 \\
C_{2B}\gamma_2 & -C_1\gamma_1^2+C_3\gamma_2^2 & -(C_1+C_3)\gamma_1\gamma_2 \\
-C_{2B}\gamma_1 & -(C_1+C_3)\gamma_1\gamma_2 & -C_1\gamma_2^2+C_3\gamma_1^2
\end{array}\right]$$

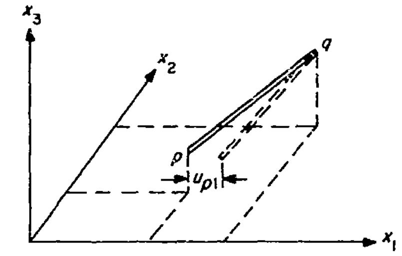



**Fig. A-5. Three-dimensional rigid-jointed member, doubly symmetric cross-section member**

***5. Structure type 5, three-dimensional, rigid-jointed member, doubly symmetrical cross-section (Fig. A-5)***

Section properties:

$$A = A_1$$

$$I_1 = A_2$$

$$I_2 = A_3$$

$$I_3 = A_4$$

$$\text{Joint } r = A_7$$

Calculate the direction cosine of the vector **pq** × **pr** and define the vector to be $\boldsymbol{\xi}_3 = \xi_{13}\,\mathbf{x}_1 + \xi_{23}\,\mathbf{x}_2 + \xi_{33}\,\mathbf{x}_3$ or the $I_3$ axis of the member. Using the right-handed coordinate system define the axis of $I_2$ to be

$$\boldsymbol{\xi}_2 = \frac{\boldsymbol{\xi}_3 \times \boldsymbol{\xi}_1}{|\boldsymbol{\xi}_3||\boldsymbol{\xi}_1|} = \mathbf{x}_1(\xi_{23}\gamma_3 - \xi_{33}\gamma_2) + \mathbf{x}_2(\xi_{33}\gamma_1 - \xi_{13}\gamma_3) + \mathbf{x}_3(\xi_{13}\gamma_2 - \xi_{23}\gamma_1)$$

$$= \mathbf{x}_1\beta_1 + \mathbf{x}_2\beta_2 + \mathbf{x}_3\beta_3$$

where $\boldsymbol{\xi}_1$ is a unit vector along the member.

Introduce $u_{p1} = 1$. Vector displacements of point $p$ in the member-oriented coordinate system ($\boldsymbol{\xi}_i$) are

$$\boldsymbol{\delta}_1 = \gamma_1^2\,\mathbf{x}_1 + \gamma_1\gamma_2\,\mathbf{x}_2 + \gamma_1\gamma_3\,\mathbf{x}_3$$

$$\boldsymbol{\delta}_2 = (\xi_{23}\gamma_3 - \xi_{33}\gamma_2)^2\,\mathbf{x}_1 + (\xi_{23}\gamma_3 - \xi_{33}\gamma_2)(\xi_{33}\gamma_1 - \xi_{13}\gamma_3)\,\mathbf{x}_2 + (\xi_{23}\gamma_3 - \xi_{33}\gamma_2)(\xi_{13}\gamma_2 - \xi_{23}\gamma_1)\,\mathbf{x}_3$$

$$= \beta_1^2\,\mathbf{x}_1 + \beta_1\beta_2\,\mathbf{x}_2 + \beta_1\beta_3\,\mathbf{x}_3$$

$$\boldsymbol{\delta}_3 = \xi_{13}^2\,\mathbf{x}_1 + \xi_{13}\xi_{23}\,\mathbf{x}_2 + \xi_{13}\xi_{33}\,\mathbf{x}_3$$

The vector force exerted on joints

$$-q \text{ and } p = \frac{AE}{S}\boldsymbol{\delta}_1 + \frac{12EI_3}{S^3}\boldsymbol{\delta}_2 + \frac{12EI_2}{S^3}\boldsymbol{\delta}_3$$

and the vector moment exerted on joints

$$p \text{ and } q = -\frac{6EI_2}{S^2}\,\delta_3\,\xi_2 - \frac{6EI_3}{S^2}\,\delta_2\,\xi_3$$

Components of these load vectors are:

$$f_{p1} = -f_{q1} = \text{force along } x_1 \text{ axis} = \frac{AE}{S}\gamma_1^2 + \frac{12EI_3}{S^3}(\xi_{23}\gamma_3 - \xi_{33}\gamma_2)^2$$

$$+ \frac{12EI_2}{S^3}\xi_{13}^2$$

$$f_{p2} = -f_{q2} = \text{force along } x_2 \text{ axis} = \frac{AE}{S}\gamma_1\gamma_2 + \frac{12EI_3}{S^3}(\xi_{23}\gamma_3 - \xi_{33}\gamma_2)$$

$$\times (\xi_{33}\gamma_1 - \xi_{13}\gamma_3) + \frac{12EI_2}{S^3}\xi_{13}\xi_{23}$$

$$f_{p3} = -f_{q3} = \text{force along } x_3 \text{ axis} = \frac{AE}{S}\gamma_1\gamma_3 + \frac{12EI_3}{S^3}(\xi_{23}\gamma_3 - \xi_{33}\gamma_2)$$

$$\times (\xi_{13}\gamma_2 - \xi_{23}\gamma_1) + \frac{12EI_2}{S^3}\xi_{13}\xi_{33}$$

$$f_{p4} = +f_{q4} = \text{moment about } x_1 \text{ axis} = \xi_{11}(\xi_{23}\gamma_3 - \xi_{33}\gamma_2)\left(\frac{6EI_3}{S^2} - \frac{6EI_2}{S^2}\right)$$

$$f_{p5} = f_{q5} = \text{moment about } x_2 \text{ axis} = -\frac{6EI_2}{S^2}\xi_{21}(\xi_{33}\gamma_1 - \xi_{13}\gamma_3)$$

$$+ \frac{6EI_3}{S^2}\xi_{21}(\xi_{23}\gamma_3 - \xi_{33}\gamma_2)$$

$$f_{p6} = f_{q6} = \text{moment about } x_3 \text{ axis} = -\frac{6EI_2}{S^2}\xi_{13}(\xi_{13}\gamma_2 - \xi_{23}\gamma_1)$$

$$+ \frac{6EI_3}{S^2}\xi_{33}(\xi_{23}\gamma_3 - \xi_{13}\gamma_2)$$

The stiffness matrix is written in terms of the parameters

$$C_1 = \frac{6EI_2}{S^2} \qquad K_1 = \frac{2EI_2}{S} = \frac{L_1}{2}$$

$$C_2 = \frac{6EI_3}{S^2} \qquad K_2 = \frac{2EI_3}{S} = \frac{L_2}{2}$$

$$C_3 = \frac{EI_1}{2S(1+\nu)} \qquad K_3 = \frac{12EI_2}{S^3}$$

$$C_4 = \frac{AE}{S} \qquad K_4 = \frac{12EI_3}{S^3}$$

$$
\left[\begin{array}{cccccc}
C_1\gamma_1^2+K_1\beta_1^2+K_3\xi_{13}^2 & K(2,1,p) & K(3,1,p) & K(4,1,p) & K(5,1,p) & K(6,1,p) \\
C_1\gamma_1\gamma_2+K_1\beta_1\beta_2+K_3\xi_{13}\xi_{23} & C_1\gamma_2^2+K_1\beta_2^2+K_3\xi_{23}^2 & K(3,2,p) & K(4,2,p) & K(5,2,p) & K(6,2,p) \\
C_1\gamma_1\gamma_3+K_1\beta_1\beta_3+K_3\xi_{13}\xi_{33} & C_1\gamma_2\gamma_3+K_1\beta_2\beta_3+K_3\xi_{23}\xi_{33} & C_1\gamma_3^2+K_1\beta_3^2+K_3\xi_{33}^2 & K(4,3,p) & K(5,3,p) & K(6,3,p) \\
\xi_{13}\beta_1(C_2-C_1) & -C_1\xi_{23}\beta_1+C_2\beta_2\xi_{13} & C_2\xi_{13}\beta_3-C_1\xi_{33}\beta_1 & C_3\gamma_1^2+L_2\xi_{13}^2+L_1\beta_1^2 & K(5,4,p) & K(6,4,p) \\
-C_1\xi_{13}\beta_2+C_2\xi_{23}\beta_1 & \xi_{23}\beta_2(C_2-C_1) & -C_1\xi_{33}\beta_2+C_2\xi_{23}\beta_3 & C_3\gamma_1\gamma_2+L_2\xi_{13}\xi_{23}+L_1\beta_1\beta_2 & C_3\gamma_2^2+L_2\xi_{23}^2+L_1\beta_2^2 & K(6,5,p) \\
-C_1\xi_{13}\beta_3+C_2\xi_{33}\beta_1 & -C_1\xi_{23}\beta_3+C_2\beta_2\xi_{33} & \xi_{33}\beta_3(C_2-C_1) & C_3\gamma_1\gamma_3+L_2\xi_{13}\xi_{33}+L_1\beta_1\beta_3 & C_3\gamma_2\gamma_3+L_2\xi_{23}\xi_{33}+L_1\beta_2\beta_3 & C_3\gamma_3^2+L_2\xi_{33}^2+L_1\beta_3^2 \\
\hdashline
-K(1,1,p) & -K(2,1,p) & -K(3,1,p) & -K(4,1,p) & -K(5,1,p) & -K(6,1,p) \\
-K(2,1,p) & -K(2,2,p) & -K(3,2,p) & -K(4,2,p) & -K(5,2,p) & -K(6,2,p) \\
-K(3,1,p) & -K(3,2,p) & -K(3,3,p) & -K(4,3,p) & -K(5,3,p) & -K(6,3,p) \\
K(4,1,p) & K(4,2,p) & K(4,3,p) & -C_3\gamma_1^2+K_2\xi_{13}^2+K_1\beta_1^2 & -C_3\gamma_1\gamma_2+K_2\xi_{13}\xi_{23}+K_1\beta_1\beta_2 & -C_3\gamma_1\gamma_3+K_2\xi_{13}\xi_{33}+K_1\beta_1\beta_3 \\
K(5,1,p) & K(5,2,p) & K(5,3,p) & -C_3\gamma_1\gamma_2+K_2\xi_{13}\xi_{23}+K_1\beta_1\beta_2 & -C_3\gamma_2^2+K_2\xi_{23}^2+K_1\beta_2^2 & -C_3\gamma_2\gamma_3+K_2\xi_{23}\xi_{33}+K_1\beta_2\beta_3 \\
K(6,1,p) & K(6,2,p) & K(6,3,p) & -C_3\gamma_1\gamma_3+K_2\xi_{13}\xi_{33}+K_1\beta_1\beta_3 & -C_3\gamma_2\gamma_3+K_2\xi_{23}\xi_{33}+K_1\beta_2\beta_3 & -C_3\gamma_3^2+K_2\xi_{33}^2+K_1\beta_3^2
\end{array}\right]
$$

# APPENDIX B

## Loads for Various Member Types

Expressions for member loads are developed, using the geometrical parameters of Section III and the joint deflections in the order specified in Section III A-5.

*1. Structure type 1, three-dimensional, pin-jointed members, axial extension of the member is*

$$\delta_1 = (u_{q1} - u_{p1})\gamma_1 + (u_{q2} - u_{p1})\gamma_2 + (u_{q3} - u_{p3})\gamma_3$$

The axial load is computed and output for each loading on each member in pound units, tension positive:

$$P = \frac{AE}{S}\,\delta_1$$

*2. Structure type 2, three-dimensional, rigid-jointed members, equal member cross-section moment of inertia*

A member-oriented coordinate system is defined as follows:

$\xi_1$ = unit vector along member axis

$$= \gamma_1 \mathbf{x}_1 + \gamma_2 \mathbf{x}_2 + \gamma_3 \mathbf{x}_3$$

$\xi_2$ = unit vector normal-to-plane of $\xi_1$ and $\mathbf{x}_1$ (or $\mathbf{x}_2$ if $\xi_1 = \mathbf{x}_1$)

$$= \frac{\xi_1 \times \mathbf{x}_1}{|\xi_1 \times \mathbf{x}_1|}$$

$$= \frac{(\gamma_3 \mathbf{x}_2 - \gamma_2 \mathbf{x}_3)}{(\gamma_3^2 + \gamma_2^2)^{1/2}}$$

$\xi_3$ = unit vector normal-to-plane of $\xi_1$ and $\xi_2$

$$= \xi_1 \times \xi_2$$

$$= \frac{-(\gamma_2^2 + \gamma_3^2)\mathbf{x}_1 + \gamma_2\gamma_1 \mathbf{x}_2 + \gamma_1\gamma_3 \mathbf{x}_3}{(\gamma_2^2 + \gamma_3^2)^{1/2}}$$

Net displacement components in the $\xi_i$ directions are

$$\delta_{10} = \left[(u_{q1} - u_{p1})\mathbf{x}_1 + (u_{q2} - u_{p2})\mathbf{x}_2 + (u_{q3} - u_{p3})\mathbf{x}_3\right] \cdot \xi_1$$

$$\delta_{20} = \left[(u_{q1} - u_{p1})\mathbf{x}_1 + (u_{q2} - u_{p2})\mathbf{x}_2 + (u_{q3} - u_{p3})\mathbf{x}_3\right] \cdot \xi_2$$

$$\delta_{30} = \left[(u_{q1} - u_{p1})\mathbf{x}_1 + (u_{q2} - u_{p2})\mathbf{x}_2 + (u_{q3} - u_{p3})\mathbf{x}_3\right] \cdot \xi_3$$

Net torsional rotation is

$$\delta_{10} = \left[(u_{q4} - u_{p4})\mathbf{x}_1 + (u_{q5} - u_{p5})\mathbf{x}_2 + (u_{q6} - u_{p6})\mathbf{x}_3\right] \cdot \xi_1$$

Transverse rotations of each end are

$$\delta_{p5} = (u_{p4}\,\mathbf{x}_1 + u_{p5}\,\mathbf{x}_2 + u_{p6}\,\mathbf{x}_3)\cdot\xi_2$$

$$\delta_{p6} = (u_{p4}\,\mathbf{x}_1 + u_{p5}\,\mathbf{x}_2 + u_{p6}\,\mathbf{x}_3)\cdot\xi_3$$

$$\delta_{q5} = (u_{q4}\,\mathbf{x}_1 + u_{q5}\,\mathbf{x}_2 + u_{q6}\,\mathbf{x}_3)\cdot\xi_2$$

$$\delta_{q6} = (u_{q4}\,\mathbf{x}_1 + u_{q5}\,\mathbf{x}_2 + u_{q6}\,\mathbf{x}_3)\cdot\xi_3$$

Moments about the transverse axes are

$$M_{p2} = \frac{-2EI}{S}\left(2\delta_{p5} + \delta_{q5} + \frac{3\delta_{30}}{S}\right)$$

$$M_{p3} = \frac{2EI}{S}\left(-2\delta_{p6} - \delta_{q6} + \frac{3\delta_{20}}{S}\right)$$

$$M_{q2} = \frac{-2EI}{S}\left(2\delta_{q5} + \delta_{p5} + \frac{3\delta_{30}}{S}\right)$$

$$M_{q3} = \frac{2EI}{S}\left(-2\delta_{q6} - \delta_{p6} + \frac{3\delta_{20}}{S}\right)$$

The following quantities are output, in order, for each member:

$P$ = axial load

$$= \frac{AE}{S}\,\delta_{10}$$

$M_p$ = resultant bending moment at $p$

$$= (M_{p2}^2 + M_{p3}^2)^{1/2}$$

$M_q$ = resultant bending moment at $q$

$$= (M_{q2}^2 + M_{q3}^2)^{1/2}$$

$M_t$ = twisting movement

$$= \frac{KE}{2S(1+\nu)}\,\delta_{10}$$

$V_p$ = resultant shear at $p$

$$= \frac{1}{S}\left[(M_{p2} + M_{q2})^2 + (M_{p3} + M_{q3})^2\right]^{1/2}$$

### 3. *Structure type 3, two-dimensional, rigid-jointed members, loaded in-plane*

Axial extension of the member is

$$\delta_1 = (u_{q1} - u_{p1})\,\gamma_1 + (u_{q2} - u_{p2})\,\gamma_2$$

Net transverse deflection of member is

$$\delta_2 = -(u_{q1} - u_{p1})\,\gamma_2 + (u_{q2} - u_{p2})\,\gamma_1$$

The following quantities are output, in order, for each member:

$$M_p = \text{bending moment at } p$$

$$= \frac{2EI}{S}\left(2u_{p3} + u_{q3} - \frac{3\delta_2}{S}\right)$$

$$M_q = \text{bending moment at } q$$

$$= \frac{2EI}{S}\left(2u_{q3} + u_{p3} - \frac{3\delta_2}{S}\right)$$

$$V_p = \text{shear at } p$$

$$= -\frac{1}{S}(M_p + M_q)$$

$$P = \text{axial load}$$

$$= \frac{AE}{S}\delta_1$$

*4. Structure type 4, two-dimensional, rigid-jointed, loaded normal-to-plane (grid)*

Net transverse displacement (normal-to-plane) is

$$\delta_3 = u_{q1} - u_{p1}$$

net axial rotation is

$$\delta_1 = (u_{q2} - u_{p2})\gamma_1 + (u_{q3} - u_{p3})\gamma_2$$

Transverse rotations of the ends are

$$\delta_{p2} = -u_{q2}\gamma_2 + u_{p3}\gamma_1$$

$$\delta_{q2} = -u_{q2}\gamma_2 + u_{q3}\gamma_1$$

The following quantities are output, in order, for each member:

$$M_p = \text{bending moment at } p$$

$$= \frac{2EI}{S}\left(2\delta_{p2} + \delta_{q2} - \frac{3\delta_3}{S}\right)$$

$$M_q = \text{bending moment at } q$$

$$= \frac{2EI}{S}\left(2\delta_{q2} + \delta_{p2} - \frac{3\delta_3}{S}\right)$$

$$M_t = \text{twisting moment}$$

$$= \frac{EK}{2S(1+\nu)}\delta_1$$

$$V_p = \text{shear at } p \text{ (normal-to-plane)}$$

$$= -\frac{1}{S}(M_p + M_q)$$

***5. Structure type 5, three-dimensional, rigid-jointed member, doubly symmetric, cross-section***

A member-oriented coordinate system is defined as follows:

$\boldsymbol{\xi}_1$ = unit vector along member axis $= \gamma_1 \mathbf{x}_1 + \gamma_2 \mathbf{x}_2 + \gamma_3 \mathbf{x}_3$

$\boldsymbol{\xi}_2$ = unit vector normal-to-plane of $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_3$

$$= \boldsymbol{\xi}_3 \times \boldsymbol{\xi}_1 = (\xi_{23}\gamma_3 - \xi_{33}\gamma_2)\,\mathbf{x}_1 + (\xi_{33}\gamma_1 - \xi_{13}\gamma_3)\,\mathbf{x}_2 + (\xi_{13}\gamma_2 - \xi_{23}\gamma_1)\,\mathbf{x}_3 = \beta_1 \mathbf{x}_1 + \beta_2 \mathbf{x}_2 + \beta_3 \mathbf{x}_3$$

$\boldsymbol{\xi}_3$ = unit vector normal-to-plane of $\boldsymbol{\xi}_1$ and **pr** ($p$ is first joint of member and $r$ is an inputed joint not on $\boldsymbol{\xi}_1$) $= \xi_{13}\mathbf{x}_1 + \xi_{23}\mathbf{x}_2 + \xi_{33}\mathbf{x}_3$

Net displacement components in the $\boldsymbol{\xi}_i$ directions are

$$\delta_{i0} = \left[(u_{q1} - u_{p1})\,\mathbf{x}_1 + (u_{q2} - u_{p2})\,\mathbf{x}_2 + (u_{q3} - u_{p3})\,\mathbf{x}_3\right] \cdot \boldsymbol{\xi}_i \qquad (i = 1, 2, 3)$$

Net torsion rotation is

$$\delta_{40} = \left[(u_{q4} - u_{p4})\,\mathbf{x}_1 + (u_{q5} - u_{p5})\,\mathbf{x}_2 + (u_{q6} - u_{p6})\,\mathbf{x}_3\right] \cdot \boldsymbol{\xi}_1$$

Transverse rotations of each end are

$$\delta_{pi} = (u_{p4}\,\mathbf{x}_1 + u_{p5}\,\mathbf{x}_2 + u_{p6}\,\mathbf{x}_3) \cdot \boldsymbol{\xi}_i \; (i = 2, 3)$$

$$\delta_{qi} = (u_{q4}\,\mathbf{x}_1 + u_{q5}\,\mathbf{x}_2 + u_{q6}\,\mathbf{x}_3) \cdot \boldsymbol{\xi}_i \; (i = 2, 3)$$

Moments about the transverse axes are

$$M_{p2} = \frac{2EI_2}{S}\left[2\delta_{p5} + \delta_{q5} + \frac{3\delta_{30}}{S}\right]$$

$$M_{p3} = \frac{2EI_3}{S}\left[2\delta_{p6} + \delta_{q6} - \frac{3\delta_{20}}{S}\right]$$

$$M_{q2} = \frac{2EI_2}{S}\left[2\delta_{q5} + \delta_{p5} + \frac{3\delta_{30}}{S}\right]$$

$$M_{q3} = \frac{2EI_3}{S}\left[2\delta_{q6} + \delta_{p6} - \frac{3\delta_{20}}{S}\right]$$

Shears along the transverse axes are:

$$V_{p2} = -V_{q2} = [M_{p3} + M_{q3}]/S$$

$$V_{p3} = -V_{q3} = -(M_{p2} + M_{q2})/S$$

The following quantities are output, in order, for each member:

$P$ = axial load $= \dfrac{AE}{S}\,\delta_{10}$

$M_{p2}, M_{p3}, M_{q2}, M_{q3}$ = moments at joints $p$ and $q$

$M_t$ = twisting moment $= \dfrac{I_t E}{2S(1+\nu)}\,\delta_{40}$

$V_{p2}, V_{q2}, V_{p3}, V_{q3}$ = shears at joints $p$ and $q$

# APPENDIX C

## Thermal Loads for Various Member Types

The analysis method is outlined in II-G. The equation used to calculate the loading in a member with the ends fixed in-space will be calculated.

*1. Structure type 1, three-dimensional, pin-jointed members (Fig. C-1)*

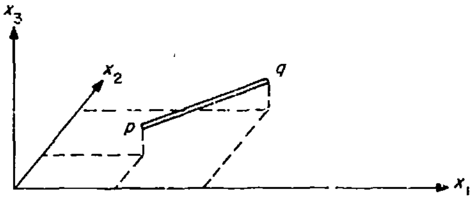



Fig. C-1. Three-dimensional pin-jointed member

Thermal input

$$\alpha\Delta T = A_5$$

where

$\alpha$ = coefficient of thermal expansion

$\Delta T$ = change of temperature of entire member; positive if increase in temperature

Force components at joints $p$ and $q$ are

$$f_{p1} = -f_{q1} = EA\,\gamma_1\,\alpha\Delta T$$

$$f_{p2} = -f_{q2} = EA\,\gamma_2\,\alpha\Delta T$$

$$f_{p3} = -f_{q3} = EA\,\gamma_3\,\alpha\Delta T$$

*2. Structure type 2, three-dimensional, rigid-jointed members, equal member cross-section moment of inertia*

The equations are identical to structure Type 1.

*3. Structure type 3, two-dimensional, rigid-jointed members, loaded in-plane (Fig. C-2)*

Thermal inputs:

$$\alpha\Delta T = A_5$$

$$\frac{\alpha\delta T}{h} = A_6$$

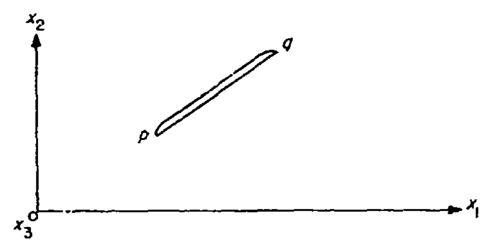



Fig. C-2. Two-dimensional rigid-jointed loaded in-plane member

where

$\alpha$ = coefficient of thermal expansion

$\Delta T$ = change of temperature of entire member; positive if increase in temperature

$\delta T$ = temperature gradient through member cross-section; positive if the unrestrained rotation of joint $p$ is in positive $x_3$ direction

$h$ = height of cross-section

Force components at joints $p$ and $q$ are

$$f_{p1} = -f_{q1} = EA\gamma_1 \alpha\Delta T$$

$$f_{p2} = -f_{q2} = EA\gamma_2 \alpha\Delta T$$

Moment component at $p$ and $q$ are

$$f_{p3} = -f_{q3} = \frac{EI\alpha\delta T}{h}$$

*4. Structure type 4, two-dimensional, rigid-jointed, loaded normal-to-plane (grid) (Fig. C-3)*

Thermal input

$$\frac{\alpha\delta T}{h} = A.$$

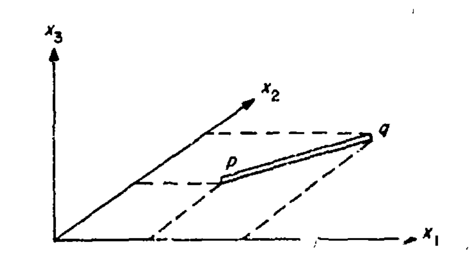



Fig. C-3. Two-dimensional rigid-jointed loaded normal-to-plane member where

$\alpha$ = coefficient of thermal expansion

$\delta T$ = thermal gradient through the member; positive if gradient increases in positive $\mathbf{x}_3$ direction

$h$ = height of cross-section

Moment component at joints $p$ and $q$ are

$$f_{p1} = -f_{q1} = -\gamma_2 \frac{EI\alpha\delta T}{h}$$

$$f_{p2} = -f_{q2} = \gamma_1 \frac{EI\alpha\delta T}{h}$$

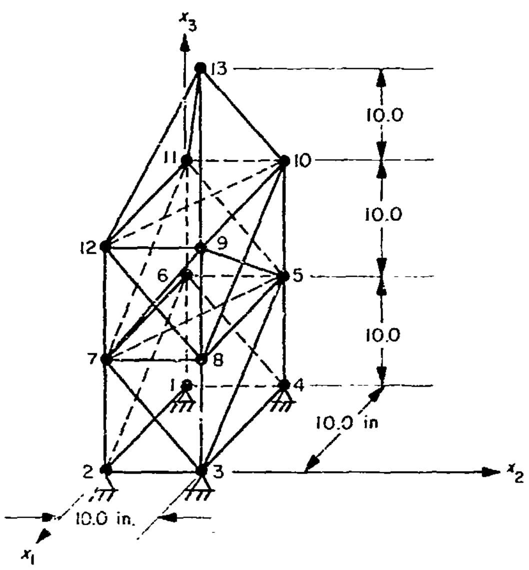
***5. Structure type 5, three-dimensional, rigid-jointed member, doubly symmetric cross-section***

The equations are identical to structure Type 1.

# APPENDIX D

## Example Problem

The option-1 sample problem chosen is shown in Fig. D-1.




INPUT
MEMBER AREAS – 0.01 in$^2$
WEIGHT AT EACH JOINT – 15.0 lb
MODULUS OF ELASTICITY – $10^7$ psi
SEE THE FIRST TWO PAGES OF THE SAMPLE PROBLEM FOR DETAILS OF THE INPUT

Fig. D-1. Example problem

```
        ZERO TIME = 005404
     OPTION ONE SAMPLE PROBLEM-STATIC AND DYNAMIC

  INPUT DATA
  STIFFNESS MATRIX ANALYSIS PROBLEM SAMPLE

     CONTROL CARD
        1          -0          -0          -0
     NO. JTS    NO. MEM.     NO. LDS    MASS NO.  NO. JT. RES. NO DEG F/JT  EIG CODE    OUT CODE
        13         35           3          1           4          3            2           1
     NO. ALT.   MOD. CODE       E          POISS     DENSITY
        4          0        0.10000E 05 0.30000E-00-0.
    JOINT COORDINATES
     _ JOINT    X1          X2          X3
         1     0.          0.          0.
         2     10.00000    0.          0.
         3     10.00000    10.00000    0.
         4     0.          10.00000    0.
         5     0.          10.00000    10.00000
         6     0.          0.          10.00000
         7     10.00000    0.          10.00000
         8     10.00000    10.00000    10.00000
         9     10.00000    10.00000    20.00000
        10     0.          10.00000    20.00000
        11     0.          0.          20.00000
        12     10.00000    0.          20.00000
        13     5.00000     5.00000     30.00000
    MEMBER PROPERTIES
     NO. MEM.  JT. A     JT. B   A1          A2         A3         A4       A5        A6        A7          VOLUME     WEIGHT
       1          1         2    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         1.000E-01-0.
       2          1         4    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         1.000E-01-0.
       3          1         6    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         1.000E-01-0.
       4          2         3    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         1.000E-01-0.
       5          2         6    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         0.141E-00-0.
       6          2         7    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         1.000E-01-0.
       7          3         4    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         1.000E-01-0.
       8          3         5    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         0.141E-00-0.
       9          3         7    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         0.141E-00-0.
      10          3         8    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         1.000E-01-0.
      11          4         5    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         1.000E-01-0.
      12          -         6    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         0.141E-00-0.
      13          5         6    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         1.000E-01-0.
      14          5         7    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         0.141E-00-0.
      15          5         8    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         1.000E-01-0.
      16          5        10    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         1.000E-01-0.
      17          5        11    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         0.141E-00-0.
      18          6         7    1.000E-07-0.           0.100E 01-0.        -0.       -0.       -0.         1.000E-01-0.
      19          6        11    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         1.000E-01-0.
      20          7         9    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         0.141E-00-0.
      21          7        12    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         1.000E-01-0.
      22          7        11    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         0.141E-00-0.
      23          8         9    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         1.000E-01-0.
      24          8        10    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         0.141E-00-0.
      25          8        12    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         0.141E-00-0.
      26          9        10    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         1.000E-01-0.
      27          9        12    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         1.000E-01-0.
      28          9        13    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         0.122E-00-0.
      29         10        11    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         1.000E-01-0.
      30         10        13    1.000E-07-..           0.100E 01-0.        -0.       -0.       -0.         0.122E-00-0.

      31         11        12    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         1.000E-01-0.
      32         11        13    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         0.122E-00-0.
      33         12        13    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         0.122E-00-0.
      34         10        12    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         0.141E-00-0.
      35          5         9    1.000E-02-0.           0.100E 01-0.        -0.       -0.       -0.         0.141E-00-0.

         SUM OF MEMBER VOLUMES =       4.080954          SUM OF MEMBER WEIGHTS =      0.

    RESTRAINTS
      NO. RES.     JT. NO.       R1          R2          R3          R4          R5          R6
         1            1           1           1           1
         2            2           1           1           1
         3            3           1           1           1
         4            4           1           1           1
    STIFFNESS MATRIX ELEMENTS
        NO.          ROW          COL.     ELE. CHANGS
         1           25          78 0.13300E 05
         2           28          25 0.13300E 05
         3           31          34 0.13300E 05
         4           34          31 0.13300E 05
    WEIGHTS
       JOINT          W1          W2          W3          W4          W5          W6
         1         0.15000E 02 0.15000E 02 0.15000E 02-0.          -0.         -0.
         2         0.15000E 02 0.15000E 02 0.15000E 02-0.          -0.         -0.
         3         0.15000E 02 0.15000E 02 0.15000E 02-0.          -0.         -0.
         4         0.15000E 02 0.15000E 02 0.15000E 02-0.          -0.         -0.
         5         0.15000E 02 0.15000E 02 0.15000E 02-0.          -0.         -0.
         6         0.15000E 02 0.15000E 02 0.15000E 02-0.          -0.         -0.
         7         0.15000E 02 0.15000E 02 0.15000E 02-0.          -0.         -0.
         8         0.15000E 02 0.15000E 02 0.15000E 02-0.          -0.         -0.
         9         0.15000E 02 0.15000E 02 0.15000E 02-0.          -0.         -0.
        10         0.15000E 02 0.15000E 02 0.15000E 02-0.          -0.         -0.
        11         0.15000E 02 0.15000E 02 0.15000E 02-0.          -0.         -0.
        12         0.15000E 02 0.15000E 02 0.15000E 02-0.          -0.         -0.
        13         0.15000E 02 0.15000E 02 0.15000E 02-0.          -0.         -0.
    LOADING
       JOINT        X1          X2          X3          X4          X5          X6
         0         0.          0.          0.10000E 02 0.          0.          0.
         0         0.          0.          0.          -0.10000E-00 0.         0.
        -1         0.          0.          0.          -0.10000E-00 0.         0.
         0        -0.         -0.         -0.          -0.          -0.         -0.

       TOTAL WEIGHTS=       195.000      195.000      195.000      0.          0.          0.

       CENTER OF WEIGHT   X=     5.000,    Y=     5.000,    Z=    11.538

       WEIGHT MOMENTS OF INERTIA ABOUT CENTER OF WEIGHT

            IXX= 22038.462,    IYY= 22038.462,    IZZ= 9000.000,    IXY=   -0.  ,    IXZ=    0.000,    IYZ=    0.000

            TIME AFTER READING INPUT = 005404

            TIME AFTER GENERATING STIFFNESS MATRIX = 005404
  STIFFNESS MATRIX
     ROW     1
  0.20607E 05 -0.35355E 04 -0.         -0.          -0.          -0.          -0.35355E 04  0.35355E 04  0.          -1.00000E 04
  0.           0.          -0.35355E 04  0.          -0.35355E 04 -0.          -0.           0.          -0.          -0.
  0.           0.           0.           0.           0.           0.           0.
     ROW     2
  -0.35355E 04  0.17071E 05 -0.35355E 04 -0.          -1.00000E 04 -0.          0.35355E 04 -0.35355E 04 -0.          0.
```

```
-0.          -0.          0.           -0.          0.           -0.          -0.          0.           -0.          -0.35355E 04
 0.35355E 04  0.          0.           0.           0.           0.           0.
    ROW    3
-0.          -0.35355E 04  0.30607E 05 -0.          -0.          -0.          0.           -0.          -0.          0.
-0.          -0.          -0.35355E 04  0.          -0.35355E 04  0.          0.           -1.00000E 04  0.          0.35355E 04
-0.35355E 04  0.          0.           0.           0.           0.           0.
    ROW    4
-0.          0.           -0.          0.13536E 05  0.          -0.35355E 04 -1.00000E 04  0.          0.           0.
 0.          0.           0.           0.           0.           0.           0.           0.           -0.          -0.
 0.          0.           0.           0.           0.           0.           0.
    ROW    5
 0.          -1.00000E 04  0.          0.           0.13536E 05 -0.35355E 04  0.          -0.          -0.          0.
 0.          0.           0.           0.           0.           0.           0.           0.           -0.          -0.
 0.          0.           0.           0.           0.           0.           0.
    ROW    6
-0.          0.           -0.          -0.35355E 04 -0.35355E 04  0.27071E 05  0.          -0.          -0.          0.
 0.          0.           0.           0.           0.           0.           0.           0.           0.           0.
-1.00000E 04  0.          0.           0.           0.           0.           0.
    ROW    7
-0.35355E 04  0.35355E 04 -0.          -1.00000E 04 -0.          -0.          0.17071E 05 -0.35355E 04 -0.35355E 04  0.
 0.          0.           -0.          0.           0.           0.           0.           0.           -0.35355E 04 -0.
 0.35355E 04 -0.          -0.          0.           0.           0.           0.
    ROW    8
 0.35355E 04 -0.35355E 04  0.          -0.          -0.          -0.          -0.35355E 04  0.10607E 05 -0.          0.
 0.          0.           0.           -0.35355E 04 -0.35355E 04  0.          0.           0.           -0.          -0.
 0.          -0.          -0.          0.           0.           0.           0.
    ROW
-0.          0.           -0.          -0.          -0.          -0.          -0.35355E 04 -0.          0.30607E 05  0.
 0.          0.           0.           -0.35355E 04 -0.35355E 04  0.          0.           0.           0.35355E 04  0.
-0.35355E 04  0.          0.           -1.00000E 04  0.          0.           0.
    ROW   10
-1.00000E 04 -0.          -0.          0.           0.           0.           0.           0.           0.           0.13536E 05
 0.          -0.35355E 04 -0.          -0.          0.           -0.35355E 04 -0.          0.35355E 04  0.          0.
 0.          -0.          -0.          0.           0.           0.           0.
    ROW   11
-0.          -0.          -0.          0.           0.           0.           0.           0.           0.           0.
 0.35355E 04 -0.35355E 04 -0.          -0.          0.           -0.          -0.          0.           0.           0.
 0.          -0.          -0.35355E 04  0.35355E 04  0.          0.           0.
    ROW   12
-0.          -0.          -0.          0.           0.           0.           0.           0.           0.           -0.35355E 04
-0.35355E 04  0.27071E 05  0.          0.           -1.00000E 04  0.35355E 04  0.          -0.35355E 04  0.          0.
 0.          0.           0.35355E 04 -0.35355E 04  .           0.           0.
    ROW   13
-0.35355E 04 -0.          -0.35355E 04  0.          0.           0.           -0.          -0.          -0.          -0.
-0.          -0.          0.14896E 05  0.13608E 04  0.81386E 03  0.33000E 04 -0.          -0.          0.           0.
 0.          -0.          -0.          -0.          -0.13608E 04 -0.13608E 04  0.27217E 04
    ROW   14
-0.          -0.          -0.          0.           0.           0.           -0.          -0.35355E 04 -0.35355E 04 -0.
-0.          -0.          0.13608E 04  0.14896E 05  0.81386E 03 -0.          -0.          -0.          0.           0.
 0.          -0.          -1.00000E 04 -0.          -0.13608E 04 -0.13608E 04  0.27217E 04
    ROW   15
-0.35355E 04 -0.          -0.35355E 04  0.          0.           0.           -0.          -0.35355E 04 -0.35355E 04 -0.
-0.          -1.00000E 04  0.81388E 03  0.81388E 03  0.22514E 05 -0.          -0.          -0.          0.           0.
 0.          -0.          -0.          -0.          0.27217E 04  0.27217E 04 -0.54433E 04
    ROW   16
-0.          -0.          -0.          0.           0.           0.           0.           0.           0.           -0.35355E 04
 0.          0.35355E 04  0.33000E 04  0.           0.           0.18432E 05 -0.48964E 04 -0.81388E 03 -0.          -0.
-0.          -0.35355E 04  0.35 [illegible] 04  0.   -0.13608E 04  0.13608E 04 -0.27217E 04
    ROW   17
-0.          -0.          -0.          0.           0.           0.           0.           0.           0.           0.

-0.          -0.          0.           -0.          -0.          -0.48964E 04  0.14896E 05 -0.27217E 04 -0.          -1.00000E 04
-0.          0.35355E 05 -0.35355E 04 -0.          0.13608E 04 -0.13608E 04  0.27217E 04
    ROW   18
-0.          -0.          -1.00000E 04  0.          0.           0.           0.           0.           0.           0.35355E 04
-0.          -0.35355E 04  0.          -0.          -0.          -0.81388E 03 -0.27217E 04  0.18979E 05 -0.          -0.
-0.          0.           -0.          -0.          -0.27217E 04  0.27217E 04 -0.54433E 04
    ROW   19
-0.          0.           -0.          -0.          -0.          -0.          -0.35355E 04  0.          0.35355E 04  0.
 0.          0.           0.           0.           0.           -0.          0.           -0.          0.14896E 05  0.13608E 04
-0.81388E 03  0.33000E 04  0.          0.           -0.13608E 04 -0.13608E 04 -0.27217E 04
    ROW   20
 0.          -0.35355E 04  0.35355E 04 -0.          -0.          -0.          0.           -0.          -0.          0.
 0.          0.           0.           0.           .            0.           -1.00000E 04  0.          0.13608E 04  0.14896E 05
-0.81388E 03  0.          -0.          -0.          -0.13608E 04 -0.13608E 04 -0.27217E 04
    ROW   21
-0.          0.35355E 04 -0.35355E 04 -0.          -0.          -1.00000E 04  0.35355E 04 -0.          -0.35355E 04  0.
 0.          0.           0.           0.           0.           -0.          0.           -0.          -0.81388E 03 -0.81388E 03
 0.22514E 05  0.          -0.          -0.          -0.27217E 04 -0.27217E 04 -0.54433E 04
    ROW   22
 0.          0.           0.           0.           0.           0.           -0.          -0.          -0.          -0.
 0.          -0.          -0.          0.           -0.          -0.35355E 04  0.35355E 04 -0.          0.33000E 04 -0.
-0.          0.14896E 05 -0.48964E 04 -0.27217E 04 -0.13608E 04  0.13608E 04  0.27217E 04
    ROW   23
 0.          0.           0.           0.           0.           0.           -0.          -0.          -0.          0.
-0.35355E 04  0.35355E 04  0.          -1.00000E 04  0.          0.35355E 04 -0.35355E 04  0.          -0.          -0.
-0.          -0.40964E 04  0.18432E 05 -0.81388E 03  0.13608E 04 -0.13608E 04 -0.27217E 04
    ROW   24
 0.          0.           0.           0.           0.           0.           -0.          -0.          -1.00000E 04 -0.
 0.35355E 04 -0.35355E 04 -0.          0.           -0.          -0.          0.           -0.          -0.          -0.
-0.          -0.27217E 04 -0.81388E 03  0.18979E 05  0.27217E 04 -0.27217E 04 -0.54433E 04
    ROW   25
 0.          0.           0.           0.           0.           0.           0.           0.           0.           0.
 0.          0.           -0.13608E 04 -0.13608E 04  0.27217E 04 -0.13608E 04  0.13608E 04 -0.27217E 04 -0.13608E 04 -0.13608E 04
-0.27217E 04 -0.13608E 04  0.13608E 04  0.27217E 04  0.54433E 04  0.          0.
    ROW   26
 0.          0.           0.           0.           0.           0.           0.           0.           0.           0.
 0.          0.           -0.13608E 04 -0.13608E 04  0.27217E 04  0.13608E 04 -0.13608E 04  0.27217E 04 -0.13608E 04 -0.13608E 04
-0.27217E 04  0.13608E 04 -0.13608E 04 -0.27217E 04  0.           0.54433E 04 -0.
    ROW   27
 0.          0.           0.           0.           0.           0.           0.           0.           0.           0.
 0.          0.           0.27217E 04  0.27217E 04 -0.54433E 04 -0.27217E 04  0.27217E 04 -0.54433E 04 -0.27217E 04 -0.27217E 04
-0.54433E 04  0.27217E 04 -0.27217E 04 -0.54433E 04  0.           -0.          0.21773E 05
WEIGHT MATRIX
 0.15000E 02  0.15000E 02  0.15000E 02  0.15000E 02  0.15000E 02  0.15000E 02  0.15000E 02  0.15000E 02  0.15000E 02  0.15000E 02
 0.15000E 02  0.15000E 02  0.15000E 02  0.15000E 02  0.15000E 02  0.15000E 02  0.15000E 02  0.15000E 02  0.15000E 02  0.15000E 02
 0.15000E 02  0.15000E 02  0.15000E 02  0.15000E 02  0.15000E 02  0.15000E 02  0.15000E 02
LOAD MATRIX
    LOAD   1
 0.          0.           0.15000E 03  0.           0.           0.15000E 03  0.           0.           0.15000E 03  0.
 0.          0.15000E 03  0.           0.           0.15000E 03  0.           0.           0.15000E 03  0.           0.
 0.15000E 03  0.          0.           0.15000E 03  0.           0.           0.15000E 03
    LOAD   2
 0.          0.15000E 02 -0.15000E 02  0.           0.15000E 02  0.           0.           0.15000E 02  0.           0.
 0.15000E 02 -0.15000E 02  0.          0.30000E 02 -0.15000E 02  0.           0.30000E 02 -0.15000E 02  0.           0.30000E 02
 0.          0.           0.30000E 02  0.           0.           0.45000E 02 -0.75000E 01
    LOAD   3
 0.          -0.23077E 01 -0.75000E 01  0.          -0.23077E 01  0.75000E 01  0.          -0.23077E 01  0.75000E 01  0.
-0.23077E 01 -0.75000E 01  0.          0.12692E 02 -0.75000E 01  0.           0.12692E 02 -0.75000E 01  0.          0.12692E 02
 0.75000E 01  0.          0.12692E 02  0.75000E 01  0.           0.27692E 02  0.

        TIME AFTER GENERATING LOAD AND WEIGHT MATRICES = 005/05
```

TIME AFTER STIFFNESS MATRIX INVERSION = 005405

STATIC DISPLACEMENTS

| JT | LOAD1 | LOAD2 | LOAD3 | LOAD4 | LOAD5 | LOAD6 |
|---|---|---|---|---|---|---|
| 1 | 0. | 0. | 0. | | | |
| | 0. | 0. | 0. | | | |
| | 0. | 0. | 0. | | | |
| 2 | 0. | 0. | 0. | | | |
| | 0. | 0. | 0. | | | |
| | 0. | 0. | 0. | | | |
| 3 | 0. | 0. | 0. | | | |
| | 0. | 0. | 0. | | | |
| | 0. | 0. | 0. | | | |
| 4 | 0. | 0. | 0. | | | |
| | 0. | 0. | 0. | | | |
| | 0. | 0. | 0. | | | |
| 5 | 0.01538317 | -0.02531287 | -0.00284465 | | | |
| | 0.03006235 | 0.05494349 | 0.01979345 | | | |
| | 0.02778610 | -0.01256081 | -0.00606607 | | | |
| 6 | 0.01939500 | 0.00647399 | 0.00323569 | | | |
| | 0.02917826 | 0.04615866 | 0.01654229 | | | |
| | 0.02667625 | 0.01706974 | 0.00799934 | | | |
| 7 | 0.01682069 | 0.00266050 | 0.00155147 | | | |
| | 0.03574745 | 0.05684254 | 0.02189341 | | | |
| | 0.03824945 | 0.02229279 | 0.01085492 | | | |
| 8 | 0.01136804 | -0.00408927 | -0.00222205 | | | |
| | 0.00386674 | 0.10113487 | 0.03497049 | | | |
| | 0.03532609 | -0.01775166 | -0.00640767 | | | |
| 9 | -0.00526717 | 0.00511878 | 0.00143541 | | | |
| | 0.02410492 | 0.12664784 | 0.05335533 | | | |
| | 0.05163705 | -0.02327973 | -0.01121195 | | | |
| 10 | 0.00789956 | -0.00431980 | -0.00199059 | | | |
| | 0.03842554 | 0.12686644 | 0.05407221 | | | |
| | 0.04321418 | -0.01594307 | -0.00793725 | | | |
| 11 | 0.00708516 | 0.00341342 | 0.00214157 | | | |
| | 0.04057192 | 0.12179944 | 0.05191859 | | | |
| | 0.04004222 | 0.02761615 | 0.01391250 | | | |
| 12 | -0.00448448 | -0.00130942 | -0.00120694 | | | |
| | 0.02625130 | 0.13133085 | 0.05489407 | | | |
| | 0.05769064 | 0.02218695 | 0.01266314 | | | |
| 13 | -0.01172735 | 0.00417117 | 0.00134132 | | | |
| | 0.03337994 | 0.13248150 | 0.08331100 | | | |
| | 0.05866387 | 0.00191315 | 0.00181630 | | | |

TIME AFTER STATIC DISPLACEMENT CALCULATIONS = 005405

STATIC MEMBER LOADS

| JTA | JTB | LOAD1 | LOAD2 | LOAD3 | LOAD4 | LOAD5 | LOAD6 |
|---|---|---|---|---|---|---|---|
| 1 | 2 | -0. | -0. | -0. | | | |
| 1 | 4 | 0. | 0. | 0. | | | |
| 1 | 6 | 266.76 | 170.69 | 79.99 | | | |
| 2 | 3 | 0. | 0. | 0. | | | |
| 2 | 6 | 36.41 | 53.22 | 23.82 | | | |
| 2 | 7 | 382.49 | 222.93 | 108.55 | | | |
| 3 | 4 | 0. | 0. | 0. | | | |
| 3 | 5 | 62.02 | -36.24 | -16.12 | | | |
| 3 | 7 | 12.51 | -172.75 | -55.19 | | | |
| 3 | 8 | 353.26 | -122.52 | -64.08 | | | |
| 4 | 5 | 277.88 | -125.61 | -60.68 | | | |
| 4 | 6 | -12.51 | -145.45 | -42.71 | | | |
| 5 | 6 | 8.85 | 87.85 | 32.51 | | | |
| 5 | 7 | -21.24 | 30.37 | 11.48 | | | |
| 5 | 8 | -40.15 | 12.24 | 6.23 | | | |
| 5 | 10 | 54.26 | -33.82 | -18.69 | | | |
| 5 | 11 | 8.73 | -133.39 | -60.72 | | | |
| 6 | 7 | -25.74 | -37.63 | -16.84 | | | |
| 6 | 11 | 133.66 | 105.47 | 59.13 | | | |
| 7 | 9 | 8.73 | 121.16 | 46.98 | | | |
| 7 | 12 | 194.41 | -1.06 | 20.08 | | | |
| 7 | 11 | 57.64 | 22.85 | 12.34 | | | |
| 8 | 9 | 163.11 | -110.26 | -48.04 | | | |
| 8 | 10 | 56.78 | -17.30 | -8.81 | | | |
| 8 | 12 | 0.00 | 21.21 | -3.26 | | | |
| 9 | 10 | -131.67 | 74.39 | 34.56 | | | |
| 9 | 12 | -21.46 | -46.83 | -15.39 | | | |
| 9 | 13 | 37.46 | -21.67 | -12.58 | | | |
| 10 | 11 | -21.46 | 50.67 | 21.54 | | | |
| 10 | 13 | 54.39 | -38.04 | -21.33 | | | |
| 11 | 12 | -115.70 | -47.23 | -33.49 | | | |
| 11 | 13 | 37.46 | 33.45 | 21.33 | | | |
| 12 | 13 | 54.39 | 17.07 | 12.58 | | | |
| 10 | 12 | -1.05 | -7.27 | -0.19 | | | |
| 5 | 9 | 15.99 | -11.44 | -4.17 | | | |

EQUILIBRIUM CHECK FOR LOADING 1

| JOINT | X1-DIRECTION | X2-DIRECTION | X3-DIRECTION |
|---|---|---|---|
| 1 | 0. | 0. | -416.7625 |
| 2 | 25.7441 | 0. | -558.2376 |
| 3 | 43.6581 | 8.8459 | -555.9648 |
| 4 | -0. | -8.8459 | -417.0351 |
| 5 | -0.0000 | 0.0000 | -0.0001 |
| 6 | -0.0000 | -0.0000 | -0.0000 |
| 7 | 0.0000 | -0.0000 | -0.0000 |
| 8 | 0.0000 | 0.0000 | -0.0000 |
| 9 | -105.0656 | 0.0000 | 0.0000 |
| 10 | 70.0534 | -0.0000 | 0.0000 |
| 11 | 59.6436 | 0.0000 | 0.0000 |
| 12 | -94.2376 | 0.0000 | 0.0000 |
| 13 | -0.0000 | 0.0000 | 0.0000 |

EQUILIBRIUM CHECK FOR LOADING 2

| JOINT | X1-DIRECTION | X2-DIRECTION | X3-DIRECTION |
|---|---|---|---|
| 1 | 0. | 0. | -170.6875 |
| 2 | 37.6349 | 0. | -260.5629 |
| 3 | -25.6254 | -122.1518 | 285.2938 |
| 4 | -0. | -102.6484 | 243.4565 |
| 5 | 0.0000 | -0.0000 | 0.0000 |
| 6 | 0.0000 | -0.0000 | -0.0000 |
| 7 | 0.0000 | 0.0000 | -0.0000 |
| 8 | -0.0000 | -0.0000 | 0.0000 |
| 9 | 57.4534 | -0.0000 | -0.0000 |
| 10 | -41.4797 | -0.0000 | 0.0000 |
| 11 | 17.4154 | 0.0001 | 0.0000 |
| 12 | -45.3986 | 0.0001 | -0.0000 |
| 13 | -0.0000 | 0.0001 | -0.0000 |

EQUILIBRIUM CHECK FOR LOADING 3

| JOINT | X1-DIRECTION | X2-DIRECTION | X3-DIRECTION |
|---|---|---|---|
| 1 | 0. | 17.3077 | -87.4934 |
| 2 | 16.8421 | 17.3077 | -132.8914 |
| 3 | -11.3965 | -21.7193 | 122.0001 |
| 4 | -0. | -12.8962 | 98.3846 |
| 5 | 0.0000 | 0.0000 | 0.0000 |
| 6 | 0. | -0.0000 | -0.0000 |
| 7 | 0.0000 | 0.0000 | -0.0000 |
| 8 | -0.0000 | -0.0000 | 0.0000 |
| 9 | 26.4749 | -0.0000 | -0.0000 |
| 10 | -19.4899 | -0.0000 | 0.0000 |
| 11 | 16.0524 | 0.0000 | 0.0000 |
| 12 | -28.4830 | 0.0000 | -0.0000 |
| 13 | 0.0000 | 0.0000 | -0.0000 |

TIME AFTER STATIC LOAD CALCULATIONS = ,005411

CONVERGENCE TO LOWEST 6 MODES

| MODE | EIGENVALUE | MAX. VECTOR CHANGE |
|---|---|---|
| 1 | 0.63994561E-01 | 0.000000007 |
| 1 | 0.63994562E-01 | 0.000000007 |
| 1 | 0.63994562E-01 | 0.000000007 |
| 1 | 0.63994562E-01 | 0.000000007 |
| 1 | 0.63994562E-01 | 0.000000007 |
| 1 | 0.63994561E-01 | 0.000000395 |
| 1 | 0.63994562E-01 | 0.000000097 |
| 2 | 0.11038709E-01 | 0.000665978 |
| 2 | 0.11038719E-01 | 0.000058271 |
| 2 | 0.11038719E-01 | 0.000013456 |
| 2 | 0.11038719E-01 | 0.000005573 |
| 2 | 0.11038719E-01 | 0.000006467 |
| 2 | 0.11038719E-01 | 0.000001729 |
| 3 | 0.82409734E-02 | 0.006063089 |
| 3 | 0.82423080E-02 | 0.003602661 |
| 3 | 0.82424561E-02 | 0.001157068 |
| 3 | 0.82425051E-02 | 0.000185363 |
| 3 | 0.82425056E-02 | 0.000102922 |
| 4 | 0.67466675E-02 | 0.215138115 |
| 4 | 0.72383913E-02 | 0. 08249253 |
| 4 | 0.72394553E-02 | 0.002187192 |
| 4 | 0.72395213E-02 | 0.000354290 |
| 5 | 0.48421382E-02 | 0.107673283 |
| 5 | 0.40875161E-02 | 0.218936933 |
| 5 | 0.47534530E-02 | 0.118257985 |
| 6 | 0.40191810E-02 | 0.113201775 |
| 6 | 0.42004121E-02 | 0.040190138 |
| 1 | 0.63994562E-01 | 0.000000022 |
| 1 | 0.63994562E-01 | 0.000000007 |
| 1 | 0.63994562E-01 | 0.000000007 |
| 1 | 0.63994562E-01 | 0.000000015 |
| 1 | 0.63994562E-01 | 0.000000007 |
| 1 | 0.63994562E-01 | 0.000000440 |
| 1 | 0.63994562E-01 | 0.000000112 |
| 2 | 0.11038719E-01 | 0.000000082 |
| 2 | 0.11038719E-01 | 0.000000022 |
| 2 | 0.11038719E-01 | 0.000000007 |
| 2 | 0.11038719E-01 | 0.000000000 |
| 2 | 0.11038719E-01 | 0.000000097 |
| 2 | 0.11038719E-01 | 0.000000037 |
| 3 | 0.82425056E-02 | 0.000072084 |
| 3 | 0.82425058E-02 | 0.000007465 |
| 3 | 0.82425058E-02 | 0.000000700 |
| 3 | 0.8  25057E-02 | 0.000001565 |
| 3 | 0.82425057E-02 | 0.000001259 |
| 4 | 0.72395232E-02 | 0.000062875 |
| 4 | 0.72395232E-02 | 0.000010066 |
| 4 | 0.72395232E-02 | 0.000020303 |
| 4 | 0.72395232E-02 | 0.000017390 |
| 5 | 0.52037250E-02 | 0.001769274 |
| 5 | 0.52034867E-02 | 0.003032565 |
| 5 | 0.52036846E-02 | 0.001885715 |
| 6 | 0.42589100E-02 | 0.003401726 |
| 6 | 0.42591604E-02 | 0.001364447 |
| 1 | 0.63994562E-01 | 0.000000015 |
| 1 | 0.63994562E-01 | 0.000000007 |
| 1 | 0.63994562E-01 | 0.000000007 |
| 1 | 0.63994562E-01 | 0.000000007 |
| 1 | 0.6399 562E 01 | 0.000000007 |
| 1 | 0.63994562E-01 | 0.000000395 |
| 1 | 0.63994562E-01 | 0.000000127 |
| 2 | 0.11038720E-01 | 0.000000060 |
| 2 | 0.11038719E-01 | 0.000000030 |
| 2 | 0.11038719E-01 | 0.000000007 |
| 2 | 0.11038719E-01 | 0.000000007 |
| 2 | 0.11038719E-01 | 0.000000127 |
| 2 | 0.11038719E-01 | 0.000000052 |
| 3 | 0.82425057E-02 | 0.000000812 |
| 3 | 0.82425060E-02 | 0.000000201 |
| 3 | 0.82425058E-02 | 0.000000060 |
| 3 | 0.82425057E-02 | 0.000000320 |
| 3 | 0.82425058E-02 | 0.000000104 |
| 4 | 0.72395232E-02 | 0.000003271 |
| 4 | 0.72395213E-02 | 0.000000514 |
| 4 | 0.72395212E-02 | 0.000001036 |
| 4 | 0.72395233E-02 | 0.000001013 |
| 5 | 0.52038351E-02 | 0.000160731 |
| 5 | 0.52038332E-02 | 0.000277162 |
| 5 | 0.52038349E-02 | 0.000177965 |
| 6 | 0.42591950E-02 | 0.000437558 |
| 6 | 0.42591983E-02 | 0.000163138 |
| 1 | 0.63994561E-01 | 0.000000007 |
| 1 | 0.63994562E-01 | 0.000000007 |
| 1 | 0.63994562E-01 | 0.000000007 |
| 1 | 0.63994562E-01 | 0.000000007 |
| 1 | 0.63994562E-01 | 0.000000015 |
| 1 | 0.63994562E-01 | 0.000000402 |
| 1 | 0.63994562E-01 | 0.000000104 |
| 2 | 0.11038719E-01 | 0.000000045 |
| 2 | 0.11038719E-01 | 0.000000037 |
| 2 | 0.11038719E-01 | 0.000000007 |
| 2 | 0.11038719E-01 | 0.000000015 |
| 2 | 0.11038719E-01 | 0.000000186 |
| 2 | 0.110 8719E-01 | 0.000000067 |
| 3 | 0.82425056E-02 | 0.000000233 |
| 3 | 0.82425057E-02 | 0.000000037 |
| 3 | 0.82425057E-02 | 0.000000052 |
| 3 | 0.82425058E-02 | 0.000000283 |
| 3 | 0.82425056E-02 | 0.000000113 |
| 4 | 0.72395231E-02 | 0.000000194 |
| 4 | 0.72395232E-02 | 0.000000052 |
| 4 | 0.72395233E-02 | 0.000000492 |
| 4 | 0.72395232E-02 | 0.000000149 |

```
5      0.52038362E-02         0.000014663
5      0.52038363E-02         0.000025883
5      0.52038363E-02         0.000017382
6      0.42591969E-02         0.000035897
6      0.42591991E-02         0.000013232
1      0.63994562E-01         0.000000007
1      0.63994562E-01         0.000000007
1      0.63994562E-01         0.000000015
1      0.63994562E-01         0.000000007
1      0.63994567E-01         0.000000007
1      0.63994562E-01         0.000000440
1      0.63994562E-01         0.000000112
2      0.11038720E-01         0.000000052
2      0.11038719E-01         0.000000022
2      0.11038719E-01         0.000000015
2      0.11038719E-01         0.000000007
2      0.11038720E-01         0.000000156
2      0.11038719E-01         0.000000030
3      0.82425057E-02         0.000000231
3      0.82425054E-02         0.000000030
3      0.82425056E-02         0.000000045
3      0.82425057E-02         0.000000238
3      0.82425057E-02         0.000000082
4      0.72395233E-02         0.000000127
4      0.72395234E-02         0.000000104
4      0.72395233E-02         0.000000477
4      0.72395233E-02         0.000000201
5      0.52038362E-02         0.000002407
5      0.52038363E-02         0.000004441
5      0.52038363E-02         0.000003144
6      0.42591990E-02         0.000004329
6      0.42591990E-02         0.000001580
7      0.303970 8E 04         0.000000037
1      0.15626737E 02         0.000000030
1      0.15626757E 02         0.000000827
2      0.90590271E 02         0.000000425
2      0.90590271E 02         0.000000589
3      0.12132239E 03         0.000000931
3      0.12132237E 03         0.000001013
4      0.13813073E 03         0.000001386
4      0.13813064E 03         0.000001565
5      0.19216610E 03         0.000001609
5      0.19216595E 03         0.000001602
6      0.23478604E 03         0.000000551
6      0.23478594E 03         0.000001423
CONDITION NUMBER IS 0.1945E 03
COMPARISON OF EIGENVALUES OF A AND C
     0.689983E-05      0.253320E-06      0.938773E-06      0.238419E-06      0.417233E-06      0.447035E-06

          TIME AFTER EIGENVALUE COMPUTATIONS = 005421
FREQUENCIES
        12.37        29.78        34.46        36.77        43.37        47.94
MODE SHAPES
 JT   MODE1      MODE2      MODE3      MODE4      MODE5      MODE6
  1  0.         0.         0.         0.         0.         0.
     0.         0.         0.         0.         0.         0.
     0.         0.         0.         0.         0.         0.
  2  0.         0.         0.         0.         0.         0.

     0.         0.         0.         0.         0.         0.
     0.         0.         0.         0.         0.         0.
  3  0.         0.         0.         0.         0.         0.
     0.         0.         0.         0.         0.         0.
     0.         0.         0.         0.         0.         0.
  4  0.         0.         0.         0.         0.         0.
     0.         0.         0.         0.         0.         0.
     0.         0.         0.         0.         0.         0.
  5 -0.02562743 -0.02810322  0.57066567  0.11458231  0.1782[illegible]  0.77999969
     0.30728967  0.02515277  0.55150928  1.00000000 -0.18457[illegible] -0.05948[illegible]
    -0.05765129  0.04201912  0.15488976  0.26626224  0.16955020 -0.20198698
  6  0.04310413 -0.03164522  0.45143711 -0.66593612  0.73091545  0.12906925
     0.25752379 -0.03444349  0.48674731  0.90134669 -0.10255742 -0.07059892
     0.09969166 -0.04748153  0.05303090  0.09408917  0.21303938 -0.03170930
  7  0.02208707 -0.02174541  0.51014087 -0.79665640  0.70332668  0.14045765
     0.32539389  0.04869127  0.48133633 -0.12136800  0.25230478 -0.70506795
     0.13061886 -0.10986023  0.26768382 -0.27230422  0.10506120 -0.23334500
  8 -0.02075914 -0.03412094  0.62777955  0.06794161  0.06866439  1.00000000
     0.60008002  1.00000000  0.13866267 -0.15009185 -0.08724477  0.39294405
    -0.05058569  0.09659041  0.47443626  0.05079796  0.08499274  0.03373050
  9  0.01576062 -0.00368737 -0.10376333  0.02236422  0.09109525  0.36971948
     0.70830122  0.10336562 -0.07489931 -0.44318029  0.06947991 -0.49886666
    -0.10869638  0.04278910  0.78016703  0.06145129  0.09124414 -0.21498333
 10 -0.02305474 -0.05145338  0.11152916  0.44715174  0.12481312  0.09623892
     0.70546477 -0.17372541  0.13192586  0.61512093 -0.06153110  0.14156865
    -0.06882717  0.08306455  0.11977845  0.60174435  0.36569196 -0.49617331
 11  0.02142024 -0.04173765 -0.04222752 -0.11514223  0.37815405  0.17684583
     0.67618928 -0.19743756  0.10777669  0.78883633 -0.00428823  0.14388577
     0.16 25122 -0.09872479 -0.19778872  0.15198433  0.29315392 -0.09534554
 12 -0.00589481  0.11889928 -0.03239428 -0.12521704 -0.02777345 -0.28115080
     0.73836164  0.17371415 -0.01126443 -0.42508519 -0.08527055 -0.36538447
     0.12747931 -0.34515382  0.39586257 -0.31215496  0.01583693 -0.33318163
 13  0.02491737  0.16700893 -1.00000000  0.96618061  1.00000000  0.10182600
     1.00000000 -0.44855596 -0.55408062 -0.61586219 -0.00281190  0.50844740
     0.02664920 -0.10632525  0.33294874  0.22555431  0.26999233 -0.29213370

                              GENERALIZED SPRING MATRIX

   0.97341806E 03    -0.22268295E-03    -0.37956238E-03    -0.45776367E-04    0.70571899E-04    0.13542175E-03
  -0.49781799E-03     0.20923037E 04    -0.83923340E-04     0.28228760E-03    0.10681152E-03   -0.29754639E-03
  -0.96130371E-03    -0.11444092E-03     0.83654881E 04    -0.79345703E-03   -0.13275146E-02   -0.16479492E-02
  -0.81062317E-03     0.48828125E-03    -0.10528564E-02     0.13539269E 05    0.25634766E-02   -0.32043457E-03
   0.99182129E-04     0.21362305E-03    -0.23651123E-02     0.31280518E-02    0.79067302E 04   -0.30517578E-03
   0.84495544E-03    -0.64849854E-03    -0.25939941E-02    -0.21362305E-03   -0.35095215E-03    0.13553165E 05

                              NORMALIZED SPRING MATRIX

   0.09999999E 01    -0.16472652E-06    -0.14041918E-06    -0.13311672E-07    0.26854921E-07    0.3[illegible]360171E-07
  -0.36825372E-06     0.09999999E 01    -0.20059714E-07     0.53037350E-07    0.26260759E-07   -0.55875567E-07
  -0.35563450E-06    -0.27354156E-07     0.09999999E 01    -0.74555581E-07   -0.16322821E-06   -0.15476680E-06
  -0.23572753E-06     0.91740281E-07    -0.98929571E-07     0.09999999E 01    0.24776118E-06   -0.23654914E-07
```

| | | | | | |
|---|---|---|---|---|---|
| 0.37741911E-07 | 0.52521519E-07 | -0.29080888E-06 | 0.30232764E-06 | 0.09999999E 01 | -0.29480255E-07 |
| 0.24558579E-06 | -0.12178008E-06 | -0.24361441E-06 | -0.15769943E-07 | -0.33[illegible]02293E-07 | 0.09999999E 01 |

GENERALIZED WEIGHT MATRIX

| | | | | | |
|---|---|---|---|---|---|
| 0.55893996E 02 | 0.33993274E-07 | 0.24214387E-06 | -0.[illegible]803342E-06 | -0.40978193E-07 | -0.14062971E-06 |
| 0.33993274E-07 | 0.23096367E 02 | -0.74505806E-07 | 0.55879354E-07 | 0.74505806E-07 | 0.11175871E-06 |
| 0.24214387E-06 | -0.74505806E-07 | 0.68952616E 02 | -0.59604645E-07 | -0.14901161E-06 | -0.11920929E-06 |
| -0.10803342E-06 | 0.55879354E-07 | -0.59604645E-07 | 0.98017911E 02 | 0.52899122E-06 | 0.3725[illegible] |
| -0.40978193E-07 | 0.74505806E-07 | -0.14901161E-06 | 0.52899122E-06 | 0.41145349E 02 | -0.10430813E-06 |
| -0.14062971E-06 | 0.11175871E-06 | -0.11920929E-06 | 0.37252903E-07 | -0.10430813E-06 | 0.57725675E 02 |

NORMALIZED WEIGHT MATRIX

| | | | | | |
|---|---|---|---|---|---|
| 0.09999999E 01 | 0.94610334E-09 | 0.39004572E-08 | -0.14595629E-08 | -0.85449562E-09 | -0.24757682E-08 |
| 0.94610334E-09 | 0.09999999E 01 | -0.18669939E-08 | 0.11744291E-08 | 0.24168956E-08 | 0.30607300E-08 |
| 0.39004572E-08 | -0.18669939E-08 | 0.09999999E 01 | -0.72502338E-09 | -0.27975918E-08 | -0.18895144E-08 |
| -0.14595629E-08 | 0.11744291E-08 | -0.72502338E-09 | 0.09999999E 01 | 0.83298151E-08 | 0.49524817E-09 |
| -0.85449562E-09 | 0.24168956E-08 | -0.27975918E-08 | 0.83298151E-08 | 0.09999999E 01 | -0.21402931E-08 |
| -0.24757682E-08 | 0.30607300E-08 | -0.18895144E-08 | 0.49524817E-09 | -0.21402931E-08 | 0.09999999E 01 |

TIME AFTER DYNAMIC DISPLACEMENTS = 005422

## ACKNOWLEDGMENT

The problem was programmed by J. Heath and L. Schmele of the Computer Applications Section (JPL).