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PAYLOAD OPTIMIZATION OF MULTISTAGE LAUNCH VEHICLES

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SUMMARY

The methods of the calculus of variations are used to maximize payload capability for multistage launch vehicles. The method of solution uses the Lagrange multipliers to determine the optimum thrust direction profile, as well as to construct partial derivatives of payload with respect to the stage propellant loadings and a booster steering parameter. These derivatives are used to terminate the stages and/or as terminal equations to be satisfied. Maximum payload can thus be achieved with a single solution, rather than with a family of parametric results. Constant thrust and specific-impulse operation is assumed for each upper stage (booster thrust and specific impulse vary with atmospheric pressure), and structure weight can be either fixed or a linear function of the stage propellant loading. Two-dimensional flight in a central, inverse-square gravitational field is assumed.

Numerical results are presented for two- and three-stage launch vehicles flown to circular orbit and Earth escape, respectively. Parametric results are presented and compared with the overall optimum solution obtained by use of the variational technique.

INTRODUCTION

A problem that frequently arises in trajectory optimization studies is that of determining the maximum payload capability of a multistage launch vehicle flown to a prescribed set of burnout conditions. If all vehicle parameters are specified, the problem reduces to that of finding the optimum steering profile. In many cases, however, not all these parameters are specified, and those left unspecified can be varied to maximize payload.

A typical situation that occurs in the design of future launch vehicles is one in which the propulsion system (thrust and propellant flow rate) is specified, but some or all the stage propellant loadings are left unspecified. The unspecified propellant loadings generally can be varied to achieve maximum payload capability for the vehicle.

An additional optimizing parameter frequently is available in the booster steering program. Since the booster stage operates in the atmosphere, the booster thrust direction profile is shaped to minimize aerodynamic heating and loads and is not available for complete optimization. A single degree of freedom remains, however, corresponding to the magnitude of a short pitchover phase following the initial vertical rise. This degree of freedom, sometimes called the booster kick angle, determines the amount of trajectory lofting during boost phase. Since the upper stages operate essentially under vacuum conditions, the steering program for these stages is available for complete optimization.

Many authors (e.g., refs. 1 to 7) have treated the problem of optimizing the stage propellant loadings of multistage vehicles. None of these authors, however, has attempted to optimize the steering program for these vehicles. Others (e.g., refs. 8 to 10) have used the calculus of variations to optimize the steering program for various rocket vehicles. In particular, reference 11 treats the problem of optimizing the steering program of a multistage launch vehicle. Reference 11, however, does not consider the problem of optimizing the stage propellant loadings or booster kick angle.

Recently, Mason, Dickerson, and Smith (ref. 12) have considered the problem of simultaneously optimizing the steering program and the stage propellant loadings of a multistage launch vehicle. These authors followed the approach of Denbow (ref. 13) and Hunt and Andrus (ref. 14) in formulating the variational problem.

The present report was written concurrently with reference 12 and presents a method which allows the propellant loadings, booster kick angle, and upper-stage steering program to be simultaneously optimized. The variational approach is somewhat different from that used in reference 12. By following the method of reference 15, the maximizing functional is written as the sum of the final payload and a constraint integral for each of the upper stages. The resulting boundary equations supply partial derivatives of payload with respect to the unspecified parameters. These derivatives are then used, along with the required burnout conditions, as terminal equations to be satisfied. The analysis does not require that all the stage propellant loadings (or booster kick angle) be optimized. Equations are developed for optimizing payload with respect to any combination of unspecified parameters.

The variational equations for optimizing vehicle parameters have been incorporated into a digital computer program used previously at Lewis for parametric launch-vehicle studies. This computer program is not discussed in detail; however, numerical results are presented for two- and three-stage launch vehicles flown to circular orbit and Earth escape, respectively, to demonstrate the feasibility of the variational approach. Parametric results are presented showing the variation of payload with propellant loadings and booster kick angle. The resulting payload envelopes are then compared with the overall optimum points generated directly by means of the variational technique in order to verify the equations. Along with the results, the procedures used to obtain numerical

results are briefly discussed.

ANALYSIS

The problem to be solved is to determine the maximum payload capability of an N-stage launch vehicle flown to a specified set of burnout conditions. The analysis admits atmospheric effects during booster phase but assumes vacuum operation for all other stages. Because of these atmospheric effects, the booster steering program is assumed completely specified (e.g., zero angle of attack), except for the booster kick angle. The upper-stage steering program, however, is unconstrained and is determined to maximize payload. The calculus of variations is used for this purpose.

Each of the upper stages is assumed to operate at a fixed (constant) value of thrust and propellant flow rate. These values may be zero, so that coast phases are admitted. The structure mass for each stage is assumed to be a function of the stage propellant loading, defined by

$$m_s = m_H + km_p$$

where m_s is the total structure mass, m_H is the fixed mass, m_p is the stage propellant mass, and k is the propellant sensitive mass fraction. (All symbols are defined in appendix A.)

In addition to the variational trajectory, provision is also made for adding an additional velocity increment Δv_I after the desired orbit conditions are achieved. This velocity increment is achieved by use of the final stage for propulsion. The amount of propellant required for this maneuver is calculated by use of the standard impulsive velocity equations.

Variational Problem

Since the booster steering program is not subject to complete optimization, the booster stage is not treated in the following Euler-Lagrange equations. The booster degrees of freedom (propellant loading and kick angle) are included by allowing variations in the position and velocity at second-stage ignition. The associated equations, along with the equations for optimizing upper-stage propellant loadings, are treated in the boundary equations resulting from the variational analysis.

The variational problem to be solved is that of finding the upper-stage thrust program which maximizes the payload capability of an N-stage launch vehicle for given boundary conditions. This problem can be formulated as a generalized Bolza problem.
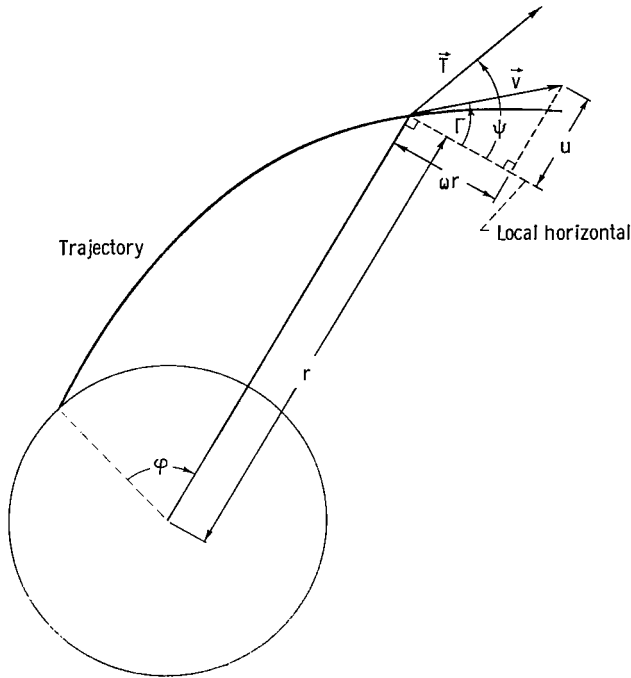
Figure 1. - Definition of problem variables.

By following the treatment in reference 15, the functional to be minimized can be written as

$$J = g + \sum_{i=2}^{N} \int_{t_{i-1}}^{t_i} F_i \, dt \qquad (1)$$

where g is a function of initial and final conditions to be minimized and F_i consists of a set of constraints to be applied to each of the upper stages, added together with the aid of Lagrange multipliers. For this problem, the payload is to be maximized, so that

$$g = -m_{PL} \qquad (2)$$

The constraint equations are

$$f_{1i} = \dot{u} + \frac{\mu}{r^2} - \omega^2 r - \frac{T_i}{m} \sin \psi = 0 \qquad (3a)$$

$$f_{2i} = r\dot{\omega} + 2u\omega - \frac{T_i}{m} \cos \psi = 0 \qquad (3b)$$

$$f_{3i} = \dot{r} - u = 0 \qquad (3c)$$

$$f_{4i} = \dot{\varphi} - \omega = 0 \qquad (3d)$$

$$f_{5i} = \dot{m} + \beta_i = 0 \qquad (3e)$$

applicable on the interval

$$t_{i-1} \leq t \leq t_i$$

for $i = 2, \ldots, N$

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Equations (3a) to (3d) are the two-body equations of motion written in two-dimensional polar coordinates with an inverse square force field acting. The thrust direction ψ and the state variables are defined in figure 1. Equation (3e) defines the propellant flow rate.

Equations (3) are combined to give

$$F_i = \sum_{j=1}^{5} \lambda_{ji} f_{ji} \qquad i = 2, \ldots, N \qquad (4)$$

where λ_{ji} are undetermined Lagrange multipliers, which are functions of time since the constraint equations must be satisfied at all points of the trajectory.

Euler-Lagrange Equations

As shown in reference 16, a necessary condition for g to be minimized is that the Euler-Lagrange equations be satisfied. The Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial F_i}{\partial \dot{x}_j}\right) = \frac{\partial F_i}{\partial x_j} \qquad i = 2, \ldots, N; \ j = 1, \ldots, 6 \qquad (5)$$

where x_j are the problem variables

$$\left. \begin{aligned} x_1(t) &= u \\[6pt] x_2(t) &= \omega \\[6pt] x_3(t) &= r \\[6pt] x_4(t) &= \varphi \\[6pt] x_5(t) &= m \\[6pt] x_6(t) &= \psi \end{aligned} \right\} \qquad (6)$$

The Euler-Lagrange equations for the present problem can be written explicitly by use of equations (3) and (4):

$$\dot{\lambda}_1 = 2\omega\lambda_2 - \lambda_3 \tag{7a}$$

$$\dot{\lambda}_2 = -2\omega\lambda_1 + \frac{u}{r}\lambda_2 - \frac{\lambda_4}{r} \tag{7b}$$

$$\dot{\lambda}_3 = -\left(\frac{2\mu}{r^3} + \omega^2\right)\lambda_1 + \dot{\omega}\lambda_2 \tag{7c}$$

$$\dot{\lambda}_4 = 0 \tag{7d}$$

$$\dot{\lambda}_5 = \frac{T}{m^2}(\lambda_1 \sin\psi + \lambda_2 \cos\psi) \tag{7e}$$

$$(\lambda_1 \cos\psi - \lambda_2 \sin\psi)\frac{T}{m} = 0 \tag{7f}$$

Equations (7) (and subsequent equations) apply separately to each of the upper stages. The subscript i has been omitted for simplicity.

Equation (7f) determines the thrust direction (for $T \neq 0$):

$$\tan\psi = \frac{\lambda_1}{\lambda_2} \tag{8a}$$

$$\sin\psi = \pm\frac{\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}} \tag{8b}$$

$$\cos\psi = \pm\frac{\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}} \tag{8c}$$

The uncertainty in sign in equations (8b) and (8c) corresponds to an equivalent uncertainty of 180° in the thrust direction. The choice of sign will be determined later in the analysis.

Equations (7a) to (7d) must be integrated, along with the equations of motion (eqs. (3a) to (3d)) to determine the thrust direction and the optimum trajectory. It is shown later that equation (7e) need not be integrated. Equation (7d) is easily integrated

to give

$$\lambda_4 = \text{Constant} \tag{9}$$

First Integral

Since the function F does not contain the independent variable (time) explicitly, a first integral to the Euler-Lagrange equations exists (for each stage), which can be stated as (ref. 16)

$$F - \sum_{j=1}^{6} \frac{\partial F}{\partial \dot{x}_j} \dot{x}_j = C \tag{10}$$

Equation (10) can be written explicitly by use of equations (3) and (4):

$$C - \left(\frac{\mu}{r^2} - \omega^2 r\right)\lambda_1 - 2u\omega\lambda_2 + u\lambda_3 + \omega\lambda_4 - \beta\lambda_5 + \frac{T}{m}(\lambda_1 \sin \psi + \lambda_2 \cos \psi) = 0 \tag{11}$$

Equation (11) is used later in the analysis to eliminate both C and λ_5 from the boundary equations.

Weierstrass Condition

The uncertainty in sign in equations (8b) and (8c) can be resolved by applying the necessary condition of Weierstrass (ref. 16). Following the development in reference 17, this condition can be stated as $E \geq 0$ for a minimum, where

$$E = F\left(x_j^*, \dot{x}_j^*\right) - F(x_j, \dot{x}_j) - \sum_{j=1}^{6} \left(\dot{x}_j^* - \dot{x}_j\right) \frac{\partial F}{\partial \dot{x}_j} \tag{12}$$

The x_j correspond to the minimizing values, which differ from the x_j^* by a finite but admissible amount. For the present problem, only ψ is subject to such a variation. Equation (12) can be evaluated to give

$$-\frac{T}{m}(\lambda_1 \sin \psi^* + \lambda_2 \cos \psi^*) + \frac{T}{m}(\lambda_1 \sin \psi + \lambda_2 \cos \psi) \geq 0 \tag{13}$$

Equations (8b) and (8c) are now used along with

$$\left.\begin{aligned}\sin \psi^* &= \mp \frac{\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}} \\[2em] \cos \psi^* &= \mp \frac{\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}}\end{aligned}\right\} \tag{14}$$

to give equation (13) in the form

$$\pm \frac{2T}{m} \sqrt{\lambda_1^2 + \lambda_2^2} \geq 0 \tag{15}$$

Equation (15) implies that the plus sign must be chosen in equations (8b) and (8c).

$$\sin \psi = \frac{\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}} \tag{8b1}$$

$$\cos \psi = \frac{\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}} \tag{8c1}$$

Weierstrass-Erdmann Corner Condition

The boundary conditions on the Lagrange multipliers at staging points can be derived with the aid of the Weierstrass-Erdmann corner condition (ref. 16). This condition states that $\partial F / \partial \dot{x}_j$ (j = 1, . . ., 6) must be continuous at such corners. For the present problem,

$$\left.\begin{array}{c} \dfrac{\partial F}{\partial \dot{u}} = \lambda_1 \\[2em] \dfrac{\partial F}{\partial \dot{\omega}} = r\lambda_2 \\[2em] \dfrac{\partial F}{\partial \dot{r}} = \lambda_3 \\[2em] \dfrac{\partial F}{\partial \dot{\varphi}} = \lambda_4 \\[2em] \dfrac{\partial F}{\partial \dot{m}} = \lambda_5 \\[2em] \dfrac{\partial F}{\partial \dot{\psi}} = 0 \end{array}\right\} \tag{16}$$

All the multipliers are thus continuous across staging, hence continuous throughout the trajectory. Equations (16) also imply the continuity of the thrust direction and rate ψ and $\dot{\psi}$ since these variables are functions of the state conditions and of the Lagrange multipliers (appendix B); however, the constant of integration C is not, in general, continuous. From equations (11) and (16) the discontinuity in C is given by

$$\Delta C = -\sqrt{\lambda_1^2 + \lambda_2^2}\ \Delta\!\left(\frac{T}{m}\right) + \lambda_5\,\Delta\beta \tag{17}$$

Transversality Equation

The relation between changes in boundary conditions and changes in J is expressed by the general transversality equation (ref. 16). For this problem, the transversality equation can be written

$$dJ = \sum_{i=2}^{N}\left[\left(F_i - \sum_{j=1}^{6} \dot{x}_j\,\frac{\partial F_i}{\partial \dot{x}_j}\right)dt + \sum_{j=1}^{6} \frac{\partial F}{\partial \dot{x}_j}\,dx_j\right]_{t_{i-1}}^{t_i} + dg \tag{18}$$

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This equation can be written explicitly by using the definition of F (eqs. (3) and (4)) and the first integral (eq. (10)).

$$dJ = \sum_{i=2}^{N} \left(C_i \, dt + \lambda_1 \, du + r\lambda_2 \, d\omega + \lambda_3 \, dr + \lambda_4 \, d\varphi + \lambda_5 \, dm_i \right)_{t_{i-1}}^{t_i} + dg \qquad (19)$$

The subscript i has been used with C and dm since these variables may be discontinuous at staging points.

Boundary Equations

If some of the problem variables (state conditions or control variables) are not specified, values should be chosen which minimize J (or, equivalently, maximize payload).

According to reference 16, minimizing J is accomplished by setting dJ (eq. (19)) equal to zero. Equation (19) has the form

$$dJ = \sum_{j=1}^{m} G_j \, dx_j = 0 \qquad (20)$$

If the m problem variables x_j are all independent, dJ will vanish if, and only if, each term on the right side of equation (20) is independently set equal to zero. For specified variables x_j, the allowable variation dx_j is zero; for unspecified x_j, the coefficient G_j must be set equal to zero. Equation (20) can thus be interpreted as a total differential of J, and

$$G_j = \frac{\partial J}{\partial x_j} \qquad (21)$$

Equation (19) is not suitable for this interpretation, since the variables are not all independent. In the following section, the dependent variables in equation (19) are eliminated by expressing the dependence explicitly.

Consider first the terms in equation (19) involving the variations of the state variables.

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$$\sum_{i=2}^{N} \left(\lambda_1 \, du + r\lambda_2 \, d\omega + \lambda_3 \, dr + \lambda_4 \, d\varphi \right)\Big|_{t_{i-1}}^{t_i} = - \sum_{i=2}^{N-1} \Big[(\lambda_1 \, du)^+ - (\lambda_1 \, du)^- + (r\lambda_2 \, d\omega)^+ - (r\lambda_2 \, d\omega)^-$$

$$+ (\lambda_3 \, dr)^+ - (\lambda_3 \, dr)^- + (\lambda_4 \, d\varphi)^+ - (\lambda_4 \, d\varphi)^- \Big]_{t=t_i}$$

$$+ (\lambda_1 \, du + r\lambda_2 \, d\omega + \lambda_3 \, dr + \lambda_4 \, d\varphi)_{t=t_N^-} - (\lambda_1 \, du + r\lambda_2 \, d\omega + \lambda_3 \, dr + \lambda_4 \, d\varphi)_{t=t_1^+} \qquad (22)$$

where the superscripts $-$ and $+$ refer to conditions before and after staging, respectively. The state variables are continuous throughout the trajectory, so that

$$\left(dx_j \right)_{t=t_i^+} = \left(dx_j \right)_{t=t_i^-} \qquad i = 2, \ldots, N-1; \; j = 1, \ldots, 4$$

Since the Lagrange multipliers are also continuous (eqs. (16)), the terms before and after staging in equation (22) are equal and cancel.

At $t = t_1$, the variations in state conditions are due to the allowable variations in booster burning time and kick angle.

$$\left(dx_j \right)_{t=t_1^+} = \frac{\partial x_j}{\partial \tau_1} d\tau_1 + \frac{\partial x_j}{\partial \alpha} d\alpha \qquad j = 1, \ldots, 4 \qquad (23)$$

where α is the booster kick angle and τ_1 is the booster burning time. (In general, τ_i is the burning time of stage i, that is, $\tau_i = t_i - t_{i-1}$.) The partial derivatives in equation (23) are evaluated by using a numerical approximation method, which is discussed later.

The state variations at $t = t_N^-$, as expressed in equation (22), may or may not be independent, depending on the specification of the desired burnout conditions. Some typical burnout requirements are presented in appendix C. For generality, the variables x_j are expressed in terms of a set of generalized (independent) state variables, η_k, $k = 1, \ldots, 4$, so that

$$dx_j = \sum_{k=1}^{4} \frac{\partial x_j}{\partial \eta_k} d\eta_k \qquad j = 1, \ldots, 4 \qquad (24)$$

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Equations (23) and (24) are combined with equation (22) to give

$$\sum_{i=2}^{N} (\lambda_1\, du + r\lambda_2\, d\omega + \lambda_3\, dr + \lambda_4\, d\varphi)\Big|_{t_{i-1}}^{t_i} = \sum_{j=1}^{4} \left(\lambda_1\frac{\partial u}{\partial \eta_j} + r\lambda_2\frac{\partial \omega}{\partial \eta_j} + \lambda_3\frac{\partial r}{\partial \eta_j} + \lambda_4\frac{\partial \varphi}{\partial \eta_j}\right)_{t=t_N} d\eta_j$$

$$-\left(\lambda_1\frac{\partial u}{\partial \alpha} + r\lambda_2\frac{\partial \omega}{\partial \alpha} + \lambda_3\frac{\partial r}{\partial \alpha} + \lambda_4\frac{\partial \varphi}{\partial \alpha}\right)_{t=t_1} d\alpha - \left(\lambda_1\frac{\partial u}{\partial \tau_1} + r\lambda_2\frac{\partial \omega}{\partial \tau_1} + \lambda_3\frac{\partial r}{\partial \tau_1} + \lambda_4\frac{\partial \varphi}{\partial \tau_1}\right)_{t=t_1} d\tau_1$$

$$(25)$$

The superscripts + and − are omitted in equation (25) and subsequent equations. The expressions to be evaluated at t_1 and t_N should be evaluated at t_1^+ and t_N^-, respectively.

Inasmuch as a suitable form for the state variation terms in equation (19) has been obtained, the mass (and payload) variation terms are now considered. Since the propellant flow rate for each stage is constant, the variables m_i are not independent and can be expressed as functions of the burning times of the stages. Specifically,

$$m_i^0 = m_0 - \sum_{\ell=1}^{i-1}\left[(1 + k_\ell)\beta_\ell \tau_\ell + m_{H,\ell}\right] \qquad i = 1,\,\ldots,\,N \qquad (26a)$$

$$m_i^f = m_0 - \sum_{\ell=1}^{i-1}\left[(1 + k_\ell)\beta_\ell \tau_\ell + m_{H,\ell}\right] - \beta_i \tau_i \qquad i = 1,\,\ldots,\,N \qquad (26b)$$

where m_0 is the lift-off mass and $m_{H,\ell}$ and k_ℓ are the fixed hardware mass and propellant tank fraction, respectively, for each stage. The summations in equations (26) (and all other similar summations) are defined to be zero whenever the lower summation limit exceeds the upper limit. The superscripts 0 and f refer to conditions at the beginning and the end of each stage, respectively.

In order to calculate the final payload, the velocity impulse discussed earlier must first be considered. This velocity impulse is added after the desired orbit is achieved, using the Nth stage for propulsion. After this maneuver is completed, the Nth stage hardware (fixed and variable) is jettisoned. The propellant required to achieve the velocity impulse is

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$$\Delta m_I = m_N^f \left(1 - e^{-\Delta V_I \beta_N / T_N} \right)$$

and the final payload is

$$m_{PL} = m_N^f - (1 + k_N) \Delta m_I - k_N \beta_N \tau_N - m_{H,N}$$

The payload can be expressed as a function of burning times, by use of equation (26b).

$$m_{PL} = \gamma \left\{ m_0 - \sum_{\ell=1}^{N-1} \left[(1 + k_\ell) \beta_\ell \tau_\ell + m_{H,\ell} \right] - \beta_N \tau_N \right\} - k_N \beta_N \tau_N - m_{H,N} \qquad (27)$$

where γ has been defined as

$$\gamma = -k_N + (1 + k_N) e^{-\Delta V_I \beta_N / T_N} \qquad (28)$$

The variations dm_i^0, dm_i^f, and dm_{PL} can now be expressed in terms of the variations in burning times by differentiating equations (26a), (26b), and (27).

$$dm_i^0 = - \sum_{\ell=1}^{i-1} (1 + k_\ell) \beta_\ell \, d\tau_\ell \qquad (29a)$$

$$\left. \begin{array}{c} \\ \\ \end{array} \right\} \quad i = 1, \ldots, N$$

$$dm_i^f = - \sum_{\ell=1}^{i-1} (1 + k_\ell) \beta_\ell \, d\tau_\ell - \beta_i \, d\tau_i \qquad (29b)$$

$$dm_{PL} = -\gamma \sum_{\ell=1}^{N-1} (1 + k_\ell) \beta_\ell \, d\tau_\ell - (\gamma + k_N) \beta_N \, d\tau_N \qquad (29c)$$

By use of equations (29a) and (29b), the terms in equation (19) involving dm_i can be simplified and expressed in terms of the variations $d\tau_i$:

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$$\sum_{i=2}^{N} \left(\lambda_5 \, dm_i\right)_{t_{i-1}}^{t_i} = \sum_{i=2}^{N} \lambda_5^i \, dm_i^f - \sum_{i=2}^{N} \lambda_5^{i-1} \, dm_i^0$$

$$= \sum_{i=2}^{N-1} \lambda_5^i \left(dm_i^f - dm_{i+1}^0\right) + \lambda_5^N \, dm_N^f - \lambda_5^1 \, dm_2^0$$

$$= \sum_{i=2}^{N-1} \lambda_5^i k_i \beta_i \, d\tau_i - \lambda_5^N \sum_{i=1}^{N-1} (1 + k_i)\beta_i \, d\tau_i - \lambda_5^N \beta_N \, d\tau_N + \lambda_5^1 (1 + k_1)\beta_1 \, d\tau_1$$

$$(30)$$

where

$$\lambda_5^i \equiv \left(\lambda_5\right)_{t=t_i}$$

The continuity of λ_5 (from eqs. (16)) is used in equation (30). Also, the time variation terms in equation (19) can be expressed in terms of variations in the stage burning times:

$$\sum_{i=2}^{N} \left(C_i \, dt\right)_{t_{i-1}}^{t_i} = \sum_{i=2}^{N} C_i (dt_i - dt_{i-1}) = \sum_{i=2}^{N} C_i \, d\tau_i \tag{31}$$

Using the definition of g (eq. (2)) and equation (29c) yields the variation of g:

$$dg = \gamma \sum_{i=1}^{N-1} (1 + k_i)\beta_i \, d\tau_i + (\gamma + k_N)\beta_N \, d\tau_N \tag{32}$$

Since a suitable form for all the terms in equation (19) has been obtained, equations (25), (30), (31), and (32) are now substituted into equation (19), and the following result is obtained:

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$$dJ = \sum_{i=2}^{N} C_i \, d\tau_i + \sum_{j=1}^{4} \left(\lambda_1 \frac{\partial u}{\partial \eta_j} + r\lambda_2 \frac{\partial \omega}{\partial \eta_j} + \lambda_3 \frac{\partial r}{\partial \eta_j} + \lambda_4 \frac{\partial \varphi}{\partial \eta_j} \right)_{t=t_N} d\eta_j$$

$$- \left(\lambda_1 \frac{\partial u}{\partial \alpha} + r\lambda_2 \frac{\partial \omega}{\partial \alpha} + \lambda_3 \frac{\partial r}{\partial \alpha} + \lambda_4 \frac{\partial \varphi}{\partial \alpha} \right)_{t=t_1} d\alpha$$

$$- \left(\lambda_1 \frac{\partial u}{\partial \tau_1} + r\lambda_2 \frac{\partial \omega}{\partial \tau_1} + \lambda_3 \frac{\partial r}{\partial \tau_1} + \lambda_4 \frac{\partial \varphi}{\partial \tau_1} \right)_{t=t_1} d\tau_1$$

$$+ \sum_{i=2}^{N-1} \lambda_5^i k_i \beta_i \, d\tau_i - \lambda_5^N \sum_{i=1}^{N-1} (1 + k_i)\beta_i \, d\tau_i - \lambda_5^N \beta_N \, d\tau_N$$

$$+ \lambda_5^1 (1 + k_1)\beta_1 \, d\tau_1 + \gamma \sum_{i=1}^{N-1} (1 + k_i)\beta_i \, d\tau_i + (\gamma + k_N)\beta_N \, d\tau_N = 0 \qquad (33)$$

Combining terms yields

$$dJ = \sum_{i=2}^{N-1} \left[(1 + k_i)\beta_i \left(\gamma - \lambda_5^N \right) + C_i + \lambda_5^i k_i \beta_i \right] d\tau_i$$

$$+ \left[(1 + k_1)\beta_1 \left(\gamma - \lambda_5^N + \lambda_5^1 \right) - \lambda_1 \frac{\partial u}{\partial \tau_1} - r\lambda_2 \frac{\partial \omega}{\partial \tau_1} - \lambda_3 \frac{\partial r}{\partial \tau_1} - \lambda_4 \frac{\partial \varphi}{\partial \tau_1} \right]_{t=t_1} d\tau_1$$

$$+ \left[(\gamma + k_N)\beta_N + C_N - \beta_N \lambda_5^N \right] d\tau_N - \left(\lambda_1 \frac{\partial u}{\partial \alpha} + r\lambda_2 \frac{\partial \omega}{\partial \alpha} + \lambda_3 \frac{\partial r}{\partial \alpha} + \lambda_4 \frac{\partial \varphi}{\partial \alpha} \right)_{t=t_1} d\alpha$$

$$+ \sum_{j=1}^{4} \left(\lambda_1 \frac{\partial u}{\partial \eta_j} + r\lambda_2 \frac{\partial \omega}{\partial \eta_j} + \lambda_3 \frac{\partial r}{\partial \eta_j} + \lambda_4 \frac{\partial \varphi}{\partial \eta_j} \right)_{t=t_N} d\eta_j = 0 \qquad (34)$$

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The coefficient of $d\tau_i$ in equation (34) contains C_i and λ_5^N. Both of these can be eliminated by using the following identity for λ_5^N:

$$\lambda_5^N \equiv \sum_{\substack{\ell=i+1 \\ \beta_\ell \neq 0}}^{N} \left(\frac{C_\ell}{\beta_\ell} - \frac{C_\ell}{\beta_\ell} \right) + \sum_{\ell=i+1}^{N} \left(\lambda_5^\ell - \lambda_5^{\ell-1} \right) + \lambda_5^i$$

$$= - \sum_{\substack{\ell=i+1 \\ \beta_\ell \neq 0}}^{N} \left(\frac{C_\ell}{\beta_\ell} - \lambda_5^\ell \right) + \sum_{\substack{\ell=i+1 \\ \beta_\ell \neq 0}}^{N} \left(\frac{C_\ell}{\beta_\ell} - \lambda_5^{\ell-1} \right) + \sum_{\substack{\ell=i+1 \\ \beta_\ell = 0}}^{N} \left(\lambda_5^\ell - \lambda_5^{\ell-1} \right) + \lambda_5^i \tag{35}$$

Equation (7e) shows that $\dot{\lambda}_5 = 0$ for coast phases, so that for $T_i = 0$,

$$\lambda_5^i = \lambda_5^{i-1} \tag{36}$$

It is convenient to define

$$S_i^f = \frac{C_i}{\beta_i} - \lambda_5^i \qquad \beta_i \neq 0 \tag{37a}$$

$$S_i^0 = \frac{C_i}{\beta_i} - \lambda_5^{i-1} \qquad \beta_i \neq 0 \tag{37b}$$

$$S_i^f = 0 \qquad \beta_i = 0 \tag{37c}$$

$$S_i^0 = 0 \qquad \beta_i = 0 \tag{37d}$$

$$i = 2, \ldots, N$$

$$S_1^f = -\frac{1}{\beta_1} \left(\lambda_1 \frac{\partial u}{\partial \tau_1} + r\lambda_2 \frac{\partial \omega}{\partial \tau_1} + \lambda_3 \frac{\partial r}{\partial \tau_1} + \lambda_4 \frac{\partial \varphi}{\partial \tau_1} \right) \tag{37e}$$

With these definitions, equation (35) becomes

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$$\lambda_5^N = -\sum_{\ell=i+1}^{N}\left(S_\ell^f - S_\ell^0\right) + \lambda_5^i \tag{38}$$

By use of equations (37) and (38), equation (34) becomes (if it is assumed that the first and last stages are powered):

$$dJ \sum_{\substack{i=1\\ \beta_i \neq 0}}^{N-1}\left\{(1+k_i)\beta_i\left[\gamma + \sum_{\ell=i+1}^{N}\left(S_\ell^f - S_\ell^0\right)\right] + \beta_i S_i^f\right\}d\tau_i$$

$$+ \sum_{\substack{i=2\\ \beta_i=0}}^{N-1} C_i\, d\tau_i + \left[(\gamma + k_N)\beta_N + \beta_N S_N^f\right]d\tau_N - \left(\lambda_i\frac{\partial u}{\partial \alpha} + r\lambda_2\frac{\partial \omega}{\partial \alpha} + \lambda_3\frac{\partial r}{\partial \alpha} + \lambda_4\frac{\partial \varphi}{\partial \alpha}\right)_{t=t_1}d\alpha$$

$$+ \sum_{j=1}^{4}\left(\lambda_1\frac{\partial u}{\partial \eta_j} + r\lambda_2\frac{\partial \omega}{\partial \eta_j} + \lambda_3\frac{\partial r}{\partial \eta_j} + \lambda_4\frac{\partial \varphi}{\partial \eta_j}\right)_{t=t_N}d\eta_j = 0 \tag{39}$$

The variations in equation (39) are all independent, so that the form of equation (20) has been achieved, with

$$G(\tau_i) = (1+k_i)\beta_i\left[\gamma + \sum_{\ell=i+1}^{N}\left(S_\ell^f - S_\ell^0\right)\right] + \beta_i S_i^f \qquad \beta_i \neq 0;\ i = 1,\ \ldots,\ N-1 \tag{40a}$$

$$G(\tau_i) = C_i \qquad \beta_i = 0;\ i = 2,\ \ldots,\ N-1 \tag{40b}$$

$$G(\tau_N) = (\gamma + k_N)\beta_N + \beta_N S_N^f \tag{40c}$$

$$G(\alpha) = -\left(\lambda_1\frac{\partial u}{\partial \alpha} + r\lambda_2\frac{\partial \omega}{\partial \alpha} + \lambda_3\frac{\partial r}{\partial \alpha} + \lambda_4\frac{\partial \varphi}{\partial \alpha}\right)_{t=t_1} \tag{40d}$$

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$$G(\eta_j) = \left(\lambda_1 \frac{\partial u}{\partial \eta_j} + r\lambda_2 \frac{\partial \omega}{\partial \eta_j} + \lambda_3 \frac{\partial r}{\partial \eta_j} + \lambda_4 \frac{\partial \varphi}{\partial \eta_j}\right)_{t=t_N} \qquad j = 1, \ldots, 4 \qquad (40e)$$

From equations (11) and (37),

$$S_i^f = \frac{\left[\left(\frac{\mu}{r^2} - \omega^2 r\right)\lambda_1 + 2u\omega\lambda_2 - u\lambda_3 - \omega\lambda_4 - \frac{T_i}{m_i^f}\sqrt{\lambda_1^2 + \lambda_2^2}\right]_{t=t_i}}{\beta_i} \qquad \beta_i \neq 0;\ i = 2, \ldots, N \qquad (41a)$$

$$S_i^0 = \frac{\left[\left(\frac{\mu}{r^2} - \omega^2 r\right)\lambda_1 + 2u\omega\lambda_2 - u\lambda_3 - \omega\lambda_4 - \frac{T_i}{m_i^0}\sqrt{\lambda_1^2 + \lambda_2^2}\right]_{t=t_{i-1}}}{\beta_i} \qquad \beta_i \neq 0;\ i = 2, \ldots, N \qquad (41b)$$

$$C_i = \left[\left(\frac{\mu}{r^2} - \omega^2 r\right)\lambda_1 + 2u\omega\lambda_2 - u\lambda_3 - \omega\lambda_4\right] \qquad \beta_i = 0;\ t_{i-1} \leq t \leq t_i \qquad (41c)$$

Since equations (41) do not contain C or λ_5, equation (7e) need not be integrated to evaluate equations (40), as indicated earlier.

Boundary Value Problem

The determination of an optimum trajectory requires the simultaneous integration of the equations of motion (eqs. (3a) to (3d)) and the Euler-Lagrange equations (eqs. (7a) to (7d)). A set of initial conditions (state variables and Lagrange multipliers) and staging times is required in order to start the integration and to specify the trajectory uniquely.

The trajectory thus generated must satisfy $N + 5$ final conditions, corresponding to the $N + 5$ independent problem variables in equation (39). For specified variables x_j, the final conditions have the form

$$x_j = x_{j,d} \qquad (42)$$

where the subscript d indicates the desired final value. For unspecified variables,

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equations (40) supply auxiliary final conditions with the form

$$G(x_j) = 0 \qquad (43)$$

Some of equations (42) are easily satisfied; for example, a specified burning time for any stage can be achieved simply by terminating that stage at the proper time during the integration.

An iteration is required in order to satisfy the nontrivial final conditions, and variable initial conditions (equal to the number of final conditions) must be available. For the present problem, the initial state conditions cannot be varied independently, since these variables are determined by the choice of booster burning time and kick angle. The Lagrange multipliers $(\lambda_i,\ i = 1,\ \ldots,\ 4)$, however, can be varied independently. In addition, the burning times of all stages being optimized and the booster kick angle (if it is being optimized) are available as variable initial conditions.

The initial and final conditions for the two-point boundary value problem can be listed as follows:

$$
\begin{array}{cc}
\text{Unknown initial condition} & \text{Desired final condition} \\
\lambda_1 & \eta_j(t_N) = \eta_{j,d} \\
\lambda_2 & \text{or} \qquad \Big\} \ j = 1,\ \ldots,\ 4 \\
\lambda_3 & \\
\lambda_4 & G(\eta_j) = 0 \\
\alpha & G(\alpha) = 0 \\
\tau_{n_1} & G\left(\tau_{n_1}\right) = 0 \\
\vdots & \vdots \\
\tau_{n_K} & G\left(\tau_{n_K}\right) = 0
\end{array}
\qquad (44)
$$

Equation (44) contains $K + 5$ unknown initial conditions (for K optimized stages, $K \leq N$) and an equal number of desired final conditions. The size of the iteration loop can be reduced by using the fact that equations (7) are homogeneous in the λ's. This implies that the choice of any one λ in equation (44) is arbitrary and serves only as a scale factor for the others. The value of this multiplier can be chosen to satisfy one of equations (40a) or (40c) appearing on the right side of equation (44). The choice of the arbitrary multiplier is made in appendix B.

The iteration size can be further reduced when $\lambda_4 = 0$ is a required final condition, which occurs when the travel angle is unspecified, as shown in appendix C. Since λ_4 is a constant (eq. (9)) and is continuous across staging (eq. (16)), this final condition can be satisfied at $t = t_1$; λ_4 is thus removed from both sides of equation (44).

Equation (40d) can also be evaluated at $t = t_1$, and the booster kick angle optimization can be removed from equation (44). This is accomplished by choosing one λ at $t = t_1$, such that equation (40d) is satisfied.

The burning times of optimized stages can sometimes be removed from the iteration, along with an equal number of equations (40a), (40b), and (40c). The principle involved is similar to that used in removing equation (40d): If all variables in the equation $G(\tau_\ell) = 0$ can be calculated at (or previous to) $t = t_\ell$, stage ℓ can be terminated (during stage ℓ) whenever the optimizing equation is satisfied. The number of optimized stages which can be thus removed depends on the characteristics of the particular problem.

Consider first two optimized powered stages τ_ℓ and τ_m, $\ell < m$, with $k_m \neq 0$. Equation (40a) must be set equal to zero for each stage:

$$(1 + k_\ell) \left[\gamma + \sum_{i=\ell+1}^{N} \left(S_i^f - S_i^0 \right) \right] + S_\ell^f = 0 \tag{45a}$$

$$(1 + k_m) \left[\gamma + \sum_{i=m+1}^{N} \left(S_i^f - S_i^0 \right) \right] + S_m^f = 0 \tag{45b}$$

These two equations can be combined to give

$$\frac{S_\ell^f}{1 + k_\ell} + \sum_{i=\ell+1}^{m-1} \left(S_i^f - S_i^0 \right) + \frac{k_m}{1 + k_m} S_m^f - S_m^0 = 0 \tag{46}$$

As indicated earlier, equation (45a) can be satisfied by the proper choice of the initial scale factor for the λ's. With equation (45a) satisfied, equations (45b) and (46) are equivalent; however, equation (46) contains terms which can all be calculated at, or prior to, stage m cutoff, and this equation can be used to terminate stage m. Specifically, stage m is terminated when

$$S_m = S_m^f = \frac{1 + k_m}{k_m} \left[S_m^0 - \frac{S_\ell^f}{1 + k_\ell} - \sum_{i=\ell+1}^{m-1} \left(S_i^f - S_i^0 \right) \right] \tag{47}$$

If $k_m = 0$, the term in S_m^f disappears from equation (46), and this equation can then be used to terminate stage $m - 1$:

$$S_{m-1}^f - S_m^0 = \sum_{i=\ell+1}^{m-1} S_i^0 - \sum_{i=\ell+1}^{m-2} S_i^f - \frac{S_\ell^f}{1 + k_\ell} \qquad m > \ell + 1 \qquad (48a)$$

$$S_m^0 - \frac{S_\ell^f}{1 + k_\ell} = 0 \qquad m = \ell + 1 \qquad (48b)$$

The terms on the left side of equations (48) are evaluated at stage $m - 1$ cutoff, and the terms on the right side of (48a) are evaluated during previous stages.

For the special case $m = N$, $k_N \neq 0$, equation (40c)

$$(\gamma + k_N)\beta_N + \beta_N S_N^f = 0 \qquad (49)$$

is used instead of equation (45b). For this case, the scale factor is calculated from equation (45a), after S_N^f is first eliminated by use of equation (49). This scale factor is then applied to equation (49) to give

$$S_N^f = -\frac{\gamma + k_N}{k_N}\left[\frac{S_\ell^f}{1 + k_\ell} - S_N^0 + \sum_{i=\ell+1}^{N-1}\left(S_i^f - S_i^0\right)\right] \qquad (50)$$

Equation (50) is used to terminate stage N.

If $m = 2$ and $\ell = 1$, equation (48b) becomes

$$S_2^0 - \frac{S_1^f}{1 + k_1} = 0 \qquad (51)$$

Equation (51) could be used to terminate the booster stage optimally. An alternative procedure, however, is to terminate the booster at a prespecified time and to choose the initial value of one of the Lagrange multipliers such that equation (51) is satisfied. This procedure is similar to that used in eliminating equation (40d) and is computationally more convenient, as will be shown in appendix B.

For coast phases to be optimized, equations (40b) supply criteria for termination of

the previous stage, similar to equations (48) and (51). For such cases, stage $\ell - 1$ is terminated when $C_\ell = 0$, where stage ℓ is the coasting stage to be optimized.

In general, use of the preceding equations can supply either zero, one, or two equations for termination of a stage. If no equation is supplied, the burning time of that stage must be used as an initial condition in the external iteration. If one or two equations for termination are available, the stage is terminated by use of one of the equations, and the additional equation (if available) is used as a desired final condition in the external iteration.

One other simplification of the two-point boundary value problem is possible when stage N is being optimized but cannot be terminated by use of equation (50) (this occurs when $k_N = 0$). For this case, stage N can be terminated when one of the desired orbit conditions (appendix C) is satisfied; τ_N and one of the orbit conditions are thus eliminated from equation (44). It is important, however, to choose an orbit condition which increases or decreases monotonically during flight, so that the desired value will always be achieved. Energy and velocity are two examples of such terminating variables.

Choice of Initial Conditions

In order to facilitate convergence of the two-point boundary value problem, it is desirable to start the iteration process with initial guesses which are as close as possible to the converged values. It is difficult to guess values for the Lagrange multipliers. It is possible, however, to use the thrust direction ψ and its derivative $\dot\psi$ as variable initial conditions, and to calculate the initial values of two of the multipliers from these variables. The conversion equations are derived in appendix B:

$$\lambda_1 = \lambda_0 \sin \psi \tag{B4}$$

$$\lambda_2 = \lambda_0 \cos \psi \tag{B5}$$

$$\lambda_3 = \frac{\lambda_0}{\cos \psi} \left(2\omega - \dot\psi - \frac{u}{r} \sin \psi \cos \psi\right) + \frac{\lambda_4}{r} \tan \psi \qquad \psi \neq \frac{\pi}{2} \tag{B6}$$

where λ_0 is the arbitrary (positive) scale factor. If either the booster burning time or kick angle is being optimized internally, the appropriate optimizing equation is used to calculate ψ rather than one of the multipliers. When both the booster burning time and kick angle are optimized internally, ψ and τ_1 are calculated from the optimizing equations. An iteration is required to calculate these variables, and the equations and procedures are presented in appendix B.

Examples

The equations and procedures derived in the preceding sections can best be illustrated with the aid of examples:

(1) A three-stage vehicle, flown to circular orbit at a specified radius with the travel angle unspecified, where all three stages and the booster kick angle are available for optimization and $k = 0$ for stages 2 and 3.

Equations (40a) and (40c) must be satisfied for the optimization of the propellant loadings:

$$(1 + k_1)\beta_1 \left[\gamma + \sum_{i=2}^{3} \left(S_i^f - S_i^0 \right) \right] + \beta_1 S_1^f = 0 \tag{52a}$$

$$(1 + k_2)\beta_2 \left(\gamma + S_3^f - S_3^0 \right) + \beta_2 S_2^f = 0 \tag{52b}$$

$$(\gamma + k_3)\beta_3 + \beta_3 S_3^f = 0 \tag{52c}$$

Combining these equations as in equation (48b) gives (if $k_2 = k_3 = 0$):

$$S_2^0 - \frac{S_1^f}{1 + k_1} = 0 \tag{53a}$$

$$S_3^0 - S_2^f = 0 \tag{53b}$$

Equations (53) are used in place of (52b) and (52c). Since the booster kick angle is also being optimized, equations (40d) and (53a) are used to calculate ψ and τ_1, as shown in appendix B. Equation (53b) is used to terminate stage 2 internally, and equation (52a) is satisfied by the proper choice of the scale factor λ_0. Since no optimizing equation is available for terminating the final stage, this stage is terminated when the desired final velocity is achieved.

The final orbit requirements for this problem are given in appendix C by equation (C3). Since λ_4 can be removed from the iteration and the desired velocity has been used to terminate stage 3, only two initial and final conditions remain for the external iteration:

$$
\left.\begin{array}{cc}
\text{Initial} & \text{Final} \\
\text{conditions} & \text{conditions} \\[2em]
\dot{\psi} & r = r_d \\[2em]
\alpha & u = 0
\end{array}\right\} \tag{54}
$$

(2) Same as example (1) but with a fixed second-stage burning time (propellant loading): For this case, equations (52a) and (52c) must be satisfied, and these equations are combined as in equation (48a) to give

$$
S_2^f - S_3^0 = S_2^0 - \frac{S_1^f}{1 + k_1} \tag{55}
$$

Equation (55) cannot be used to terminate stage 2, since this stage must be terminated when $t - t_1 = \tau_2$. Also, equation (53a) is not available to terminate stage 1, so that τ_1 must be placed in the iteration. Equation (40d) is used to calculate ψ. As in the previous example, equation (52a) is satisfied by the choice of scale factor, and stage 3 is terminated on final velocity. The iteration variables for this example are

$$
\left.\begin{array}{cc}
\text{Initial conditions} & \text{Final conditions} \\[2em]
\tau_1 & u = 0 \\[2em]
\alpha & r = r_d \\[2em]
\dot{\psi} & S_2^f - S_3^0 = S_2^0 - \dfrac{S_1^f}{1 + k_1}
\end{array}\right\} \tag{56}
$$

If $k_3 \neq 0$ in this example, equation (50) would be used instead of equation (55):

$$
S_3^f = -\frac{\gamma + k_3}{k_3}\left(\frac{S_1^f}{1 + k_1} - S_3^0 + S_2^f - S_2^0\right) \tag{57}
$$

Equation (57) would be used (instead of the final velocity) to terminate stage 3, and the final conditions become

$$u = 0$$

$$r = r_d \tag{58}$$

$$\omega = \sqrt{\frac{\mu}{r_d^3}}$$

while the initial conditions remain the same as in equation (56).

PROCEDURE AND IMPLEMENTATION

In order to obtain numerical results, the equations derived in the preceding sections were programed for solution on an IBM 7094 computer. Equations (3) and (7) are integrated by using a variable step-size Runge-Kutta integration scheme with an error control to minimize truncation effects (ref. 18).

The two-point boundary value problem of equation (44) is overcome by using a multivariable Newton-Raphson iteration scheme. With this method, changes in final conditions are assumed to be related to changes in initial conditions by first-order, finite-difference equations:

$$\delta Y = M \ \delta X \tag{59}$$

where δX and δY are n-vectors (with an $n \times n$ iteration assumed) denoting differences in initial and final conditions, and M is an $n \times n$ matrix of partial derivatives of final conditions with respect to initial conditions:

$$M_{jk} = \frac{\partial Y_j}{\partial X_k} \approx \frac{\Delta Y_j}{\Delta X_k} \tag{60}$$

The matrix M is obtained by integrating a reference trajectory and n independent perturbed trajectories, so that the equation

$$\Delta Y = M \ \Delta X \tag{61}$$

is obtained where ΔX is an $n \times n$ matrix of differences in initial conditions such that ΔX_{jk} is the difference of the jth initial condition (from its value on the reference trajectory) on the kth perturbed trajectory, and ΔY is an equivalent matrix of differences

in final conditions. Since the first guesses of initial conditions will not, in general, lead to a converged solution, the guesses are improved by using the equation

$$\delta Y_r = M \, \delta X_r \tag{62}$$

where the subscript r indicates differences between reference and desired values. The predicted changes in initial conditions δX_r are obtained by combining equations (61) and (62):

$$\delta X_r = \Delta X \, \Delta Y^{-1} \, \delta Y_r \tag{63}$$

Equation (63) is used to predict the initial conditions for each iteration cycle, and the iteration proceeds until the desired final conditions are satisfied within some prespecified tolerance.

Criteria for Comparison and Termination

Since the final payload is to be optimized, it is desirable that the payload error, as well as the errors in final state conditions, be within tolerance before the iteration is terminated. A measure of the error in payload is obtained from the following equation:

$$\delta m = \left[\sum_{j=1}^{n} \left(\frac{\partial m}{\partial Y_j} \, \delta Y_j \right)^2 \right]^{1/2} \tag{64}$$

Equation (64) is also used to determine whether or not the iteration is converging. If the value of δm on the new reference trajectory is larger than the δm on the previous reference trajectory, it is likely that the magnitude of δX_r is so large that the higher order terms ignored in equation (59) have become important. In such cases, a new trajectory is flown with a reduced value of δX_r, and this procedure is repeated until the value of δm is less than that of the previous reference trajectory. When δm has decreased, the associated trajectory is used as the new reference trajectory, and the iteration proceeds.

Booster Table

In many trajectory studies, the booster stage remains fixed while other parameters

(such as upper-stage thrust levels and orbit parameters) are allowed to vary. Because of the large number of cases that can be run with the same booster stage, it is convenient to integrate a family of boost trajectories, and to store the burnout points in table form. The required booster burnout conditions can then be obtained from the table (by use of some interpolation scheme), rather than from an integration of new booster trajectories for each case. With lift-off thrust-to-weight ratio fixed, for example, two booster degrees of freedom are available: kick angle and burning time. The booster table is thus a two parameter family of burnout points (radius, velocity, flight path angle, and weight).

For a problem involving a new booster stage, three booster trajectories are initially flown at different kick angles in the estimated region of interest. For each of these trajectories, the state variables are stored at fixed time intervals, ranging from minimum to maximum propellant loading. Other values of kick angle are flown and added to the table as required. During each computer run, the new booster data developed is stored and added to the table, so that after a series of computer runs using the same booster, a comprehensive table of booster burnout points is available.

A two-dimensional second-order interpolation scheme is used to determine the burn-out conditions for intermediate values of kick angle and burning time. The partial derivatives required in equations (40a) and (40d) are also determined, by differentiating the second-order interpolation polynomial. The interpolation accuracy depends on the table spacing as well as on the amount and form of variation between table points. In order to cover the region of interest with a minimum of trajectories, the initial kick angle spacing is large. On the other hand, time spacing is small, since only one trajectory is required to obtain a complete set of time points for each kick angle.

In order to determine whether or not the interpolation accuracy is acceptable, an additional booster trajectory is flown after convergence is achieved by using the converged values of burning time and kick angle. The upper stages are then reflown with the exact booster burnout conditions. The payload and burnout conditions from the exact flight are then compared with the results of the interpolated converged flight. If agreement is acceptable, the interpolated results are retained and the problem is finished. If agreement is not acceptable, new booster trajectories are flown and added to the table (in the region of convergence) by using valves of kick angle halfway between the previous table points. The flight is then reconverged with the modified table. The interpolation accuracy is tested as before, and the entire process is repeated (if necessary) until acceptable agreement is reached.

Convergence Properties

When the problem to be solved is not well known to the analyst, it is difficult to obtain initial conditions which are close to their converged values. On the other hand, the

equations for optimal staging are sometimes very strongly coupled to the initial conditions. For this reason, it is desirable to optimize the steering program and booster kick angle first, before attempting to optimize burning times.

For a three-stage vehicle to be completely optimized (booster kick angle and all three burning times optimized), the procedure is as follows: First, the booster and second-stage burning times are held fixed at initial estimated values while the booster kick angle and third-stage burning time are optimized. The converged values of kick angle, pitch rate, and third-stage burning time are then used as first guesses, and the second- and third-stage burning times are optimized in the second case. Finally, all three stages (and the booster kick angle) are optimized in the third case.

This procedure has been used with good results in solving problems which were previously unfamiliar (as in the results presented). Of course, when the problem to be solved represents a small perturbation from a previously solved problem, complete optimization can be accomplished in one pass.

Perturbation Size

As discussed earlier, a finite difference procedure is used to obtain the partial derivatives of final conditions with respect to initial conditions. The perturbation sizes used for the initial conditions must be carefully selected: if the perturbations are too small, round-off and truncation errors inherent in the integration scheme cause large errors in the final differences; if the perturbations are too large, nonlinear effects become important, and incorrect local derivatives are obtained. The difficulty in choosing the proper perturbation sizes is magnified by the strong coupling of the partial derivatives to the optimal staging equations.

In order to obtain accurate local partial derivatives, an automatic perturbation size checking scheme was implemented in the program. With this scheme, the perturbation sizes are adjusted (when required) so that the payload error difference (eq. (64)) between reference and perturbed flights is between fixed limits. Generally, the perturbation sizes must be adjusted whenever the optimization criteria are changed.

RESULTS

In order to demonstrate the validity of the equations and the feasibility of the variational technique, parametric results are presented and compared with the overall optimum solutions obtained by using the variational technique. Two- and three-stage launch vehicles were optimized by use of the equations developed. The results presented include a two-stage vehicle flown to circular orbit and a three-stage vehicle flown to Earth escape

through a circular parking orbit. In these results, both fixed and variable hardware weights were used, and the propellant loadings and booster kick angle optimized. Parametric results are also presented, which show the variation of payload with propellant loadings and kick angle.

Vehicle Definition

The vehicle chosen for this study is a hypothetical three-stage launch vehicle consisting of two chemical stages and one nuclear stage. The assumptions on propulsion and weights are listed in table I. The first stage is based on the Saturn SI-C stage, consisting of five F-1 engines using RP-LOX propellants. Each engine delivers 1.5 million pounds of thrust at sea level, with a specific impulse of 264 seconds. These data are used for illustrative purposes only and are not necessarily consistent with present SI-C values. The second stage consists of one M-1 engine using liquid hydrogen and liquid oxygen as propellants. This stage operates at a vacuum thrust of 1.5 million pounds and a vacuum specific impulse of 428 seconds. The third stage is a nuclear stage, with estimated thrust of 250 000 pounds and a specific impulse of 850 seconds.

As explained earlier, the booster burnout conditions are determined from a table as functions of burning time and kick angle. The launch thrust-to-weight ratio is fixed at 1.25, so that the launch weight is 6 million pounds. The booster stage is flown vertically for 15 seconds, at which time the relative velocity vector is instantaneously tipped to the

TABLE I. - LAUNCH VEHICLE PROPULSION

AND WEIGHT DATA

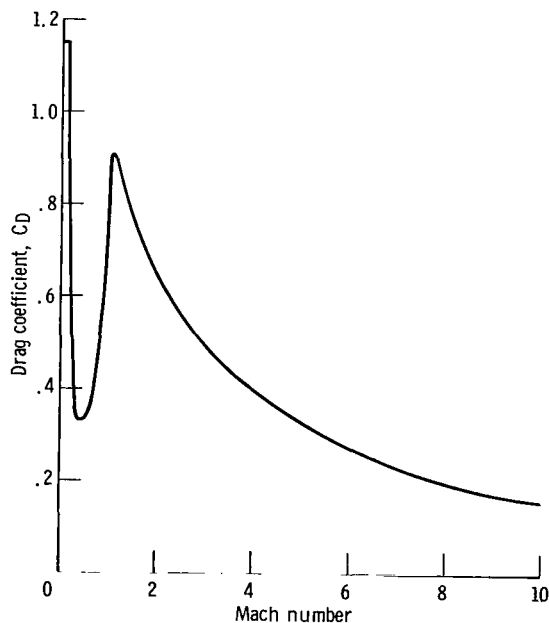|	Stage		
	First	Second	Third
Thrust, lb	7.5×10^6 (Sea level)	1.5×10^6	2.5×10^5
Specific impulse, sec	264 (Sea level) 305 (Vacuum)	428	850
Fixed hardware weight, lb	245,000	70,000	35,000
Propellant sensitive fraction	0.030	0.033	0.120
Drag reference area, sq ft	855	-------	-------

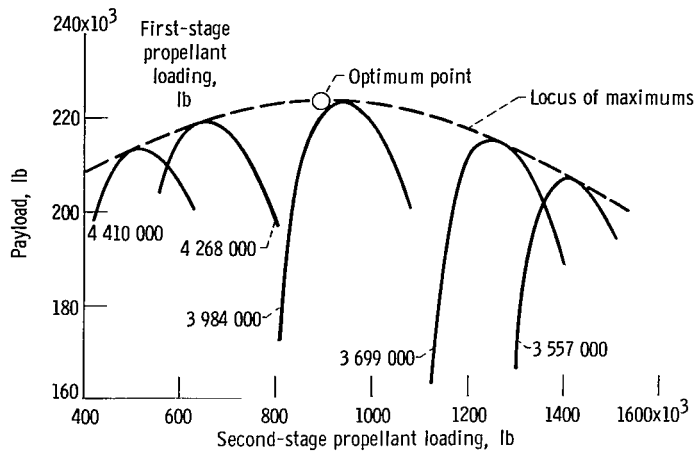Figure 2. - Drag coefficient as function of Mach number.

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Figure 3. - Payload capability as function of second-stage propellant loading. Three stages to Earth escape via 121-nautical-mile parking orbit; variable tanks; first-stage propellant loading at optimum point, 4 028 000 pounds; third-stage propellant loading at optimum point, 317 000 pounds.
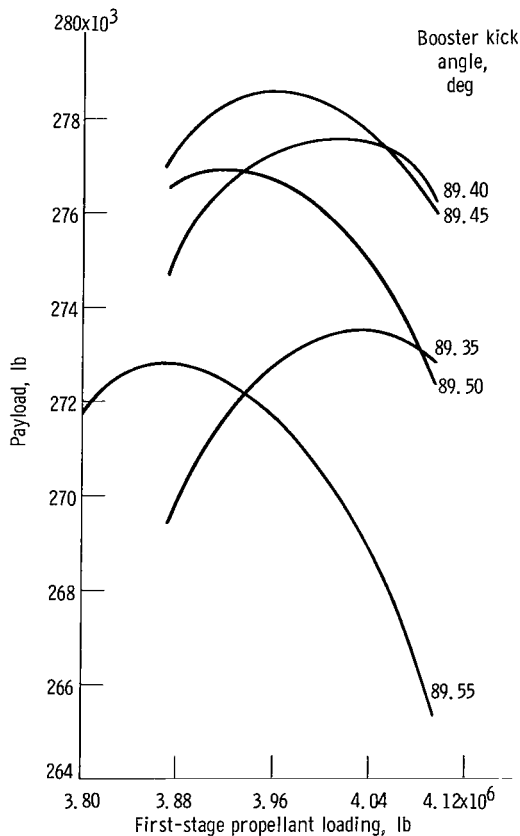

Figure 4. - Payload capability as function of first-stage propellant loading. Two stages to 121-nautical-mile orbit.

desired kick angle (measured from the horizontal) and azimuth heading. A launch azimuth of 90° is assumed. Following the initial vertical rise and pitchover, the booster is flown with a zero angle of attack steering program.

The booster trajectory simulation includes a detailed oblate Earth gravity model (ref. 19). The Air Research and Development Command atmosphere model (ref. 20) is assumed. Booster drag is calculated as a function of Mach number by using the drag coefficient curve presented in figure 2 (p. 29).

The second and third stages use the calculus of variations steering program, as explained earlier. A spherical Earth with no atmospheric forces is assumed for these stages.

Numerical Results

A typical propellant tank sizing study was conducted with the vehicle defined in table I. In this study, the stages are assumed to have variable tank size, and the optimum propellant capacities are determined by flying possible missions of interest.

The first mission is flown to Earth escape energy with three stages. This mission is typical of lunar and planetary probes and orbiters. A parking orbit ascent mode is used, wherein all three stages are used to enter a parking orbit of a
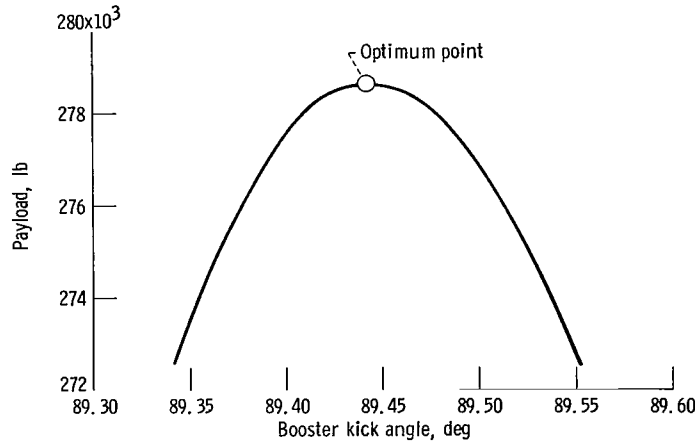
Figure 5. - Payload capability as function of booster kick angle. Propellant loadings optimized; two stages to 121-nautical-mile orbit; first-stage propellant loading at optimum point, 3 980 000 pounds; second-stage propellant loading at optimum point, 1 276 000 pounds.

121-nautical-mile altitude (similar to the Apollo mission), after which the third stage is burned to escape. The second burning of the third stage is assumed to be impulsive, with a velocity increment of 10 563 feet per second. The maximum payload capability for this mission is obtained by optimizing the booster kick angle as well as the propellant loadings of the three stages. By use of the equations derived earlier, maximum payload can be obtained with a single (converged) solution, represented by the optimum point in figure 3. Actually, since the hypothetical vehicle used for illustration is not typical of multistage chemical vehicles, good estimates of initial conditions (in this case, α, τ_1, and $\dot{\psi}$) were not known. The three-step convergence procedure discussed earlier was thus used.

With the parametric procedure, the maximum payload is obtained as the envelope of payload points with all possible combinations of propellant loadings and booster kick angle. The increased effort (number of solutions) required for this procedure is obvious. In addition to the parametric curves shown in figure 3, each point on the curves had to be obtained at an optimum kick angle, which required an additional family of curves (not shown in the figure), with kick angle as the independent parameter. As can be seen from figure 3, the payload capability and optimized propellant loading obtained from the variational procedure agree with the parametric results.

Another mission of interest is two stages flown to a circular orbit, with the use of the first and second stages from table I (p. 29). Results for this mission are presented in figures 4 and 5. In figure 4, with the use of the parametric procedure, payload capability is presented as a function of first-stage propellant loading for various booster kick-angles. The maximum payload for each kick angle is determined from the figure and presented as a function of kick angle in figure 5. The case flown by use of the variational
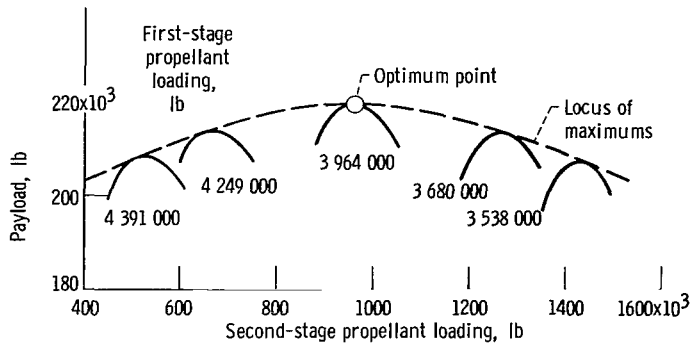
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Figure 6. - Payload capability as function of second-stage propellant loading. Three stages to Earth escape via 121-nautical-mile parking orbit; fixed tanks; first-stage propellant loading at optimum point, 3 964 000 pounds; third-stage propellant loading at optimum point, 310 000 pounds.

technique is represented by the optimum point in figure 5, and the payload obtained is in agreement with the envelope, as before.

Frequently, the propellant tanks of the various stages are sized for one mission and fixed at these values for all other missions. Propellant loadings can still be optimized in such cases but with the restriction that the propellant loadings cannot exceed the propellant capacities of the tanks.

In figure 6, the three-stage Earth escape mission is reoptimized with fixed tanks for the first and second stages. The tank weights and propellant capacities used are based on optimum values for the two-stage orbit mission (figs. 4 and 5). Since the maximum propellant capacities of the first and second stages were not exceeded in the optimum case, the propellant loadings of these stages were off-loaded to the optimum valves. The parametric results and the variational point are compared (as in fig. 3, p. 30), and the results are in agreement.

CONCLUDING REMARKS

A technique is presented which allows simultaneous optimization of the thrust direction profile and vehicle control parameters for multistage launch vehicles. The agreement of parametric and optimum results presented in figures 3 to 6 (pp. 30 to 32) demonstrates the correctness of the optimizing equations.

The amount of effort (and computer time) saved by the variational technique in determining the maximum payload capability can be seen by referring to the figures. In figure 3, for example, approximately 80 parametric data points are required to optimize the three-stage propellant loadings and booster kick angle (not shown in the figure). When good initial guesses are available, the computer time required to obtain the overall optimum solution is about the same as for each parametric solution. The time saving is therefore proportional to the number of parametric cases required for complete optimization. Even when good initial conditions are not available (this was the case in fig. 3), three cases are usually sufficient to obtain the optimum solution (as described earlier), and the time saving is still substantial.

The technique presented is, of course, not limited to the parameters (propellant

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loadings and booster kick angle) used herein. Other possible parameters (for launch vehicle optimization problems) are launch thrust-to-weight ratio and thrust levels for the various stages. In addition, the technique should be applicable to other types of optimization problems.

Lewis Research Center,
    National Aeronautics and Space Administration,
        Cleveland, Ohio, October 8, 1965.

APPENDIX A

SYMBOLS

a   functions defined in eqs. (B10), ft/sec^2

b   functions defined in eqs. (B13), $sq\ ft/(sec^2)(rad)$

C   constant of integration

E   Weierstrass excess function

e   eccentricity

F $\displaystyle\sum_{j=1}^{5} \lambda_j f_j$

f   constraint equation

G $\partial J/\partial x$

g   function of initial and final conditions to be minimized

H   energy per unit mass, $sq\ ft/sec^2$

J   functional to be minimized by variational methods

k   propellant sensitive mass fraction

M   matrix relating changes in final and initial conditions

m   mass, slugs

p   semilatus rectum, ft

r   radius, ft

S   functions defined in eqs. (37)

T   thrust, lb

t   time, sec

u   radial velocity, ft/sec

v   velocity, ft/sec

x   problem variable

α booster kick angle, rad

β mass flow rate, slug/sec

Γ flight path angle, rad

γ function defined in eq. (28)

ϵ argument of perigee, rad

η generalized state variable

θ true anomaly, rad

λ Lagrange multiplier

λ_0 arbitrary scale factor

μ Earth force constant, $cu\ ft/sec^2$

τ burning time, sec

φ polar angle, rad

ψ thrust direction, rad

ω angular velocity, rad/sec

Subscripts:

d   desired value

H   fixed hardware

I   impulsive

i   stage number

j   variable number

k   variable number

ℓ stage number

N   last stage

PL   payload

p   propellant

r   difference between reference and desired values

s   structure

0   initial

Superscripts:

f   end of stage

i   refers to $t = t_i$

0   beginning of stage

·   derivative with respect to time

*   finite, but allowable variation of the variable

+   after staging

−   before staging

→   vector

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APPENDIX B

CONVERSION EQUATIONS

Calculation of Lagrange Multipliers

In order to avoid the difficulty associated with guessing initial values of the Lagrange multipliers, equations are derived which express two of the multipliers in terms of the pitch attitude ψ and rate $\dot{\psi}$.

Equations (8a), (8b1) and (8c1) give ψ in terms of the Lagrange multipliers:

$$\tan \psi = \frac{\lambda_1}{\lambda_2} \tag{8a}$$

$$\sin \psi = \frac{\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}} \tag{8b1}$$

$$\cos \psi = \frac{\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}} \tag{8c1}$$

Differentiating equation (8a) yields the pitch rate

$$\dot{\psi} \sec^2 \psi = \frac{\lambda_2 \dot{\lambda}_1 - \lambda_1 \dot{\lambda}_2}{\lambda_2^2} \tag{B1}$$

which becomes, with the use of equations (8c1), (7a), and (7b),

$$\dot{\psi} = 2\omega - \frac{\left(\lambda_2 \lambda_3 + \frac{u}{r} \lambda_1 \lambda_2 - \frac{\lambda_1 \lambda_4}{r} \right)}{\lambda_1^2 + \lambda_2^2} \tag{B2}$$

The discussion following equation (44) shows that one of the multipliers can be picked arbitrarily. It is convenient to pick λ_0 as the scale factor, where

$$\lambda_0 = \sqrt{\lambda_1^2 + \lambda_2^2} > 0 \tag{B3}$$

With this choice,

$$\lambda_1 = \lambda_0 \sin \psi \tag{B4}$$

$$\lambda_2 = \lambda_0 \cos \psi \tag{B5}$$

Substituting equations (B3), (B4), and (B5) into (B2) and solving for λ_3 gives

$$\lambda_3 = \frac{\lambda_0}{\cos \psi} \left(2\omega - \dot{\psi} - \frac{u}{r} \sin \psi \cos \psi \right) + \frac{\lambda_4}{r} \tan \psi \qquad \psi \neq \frac{\pi}{2} \tag{B6}$$

The multipliers λ_1, λ_2, and λ_3 can now be calculated from ψ, $\dot{\psi}$, λ_4, and λ_0 by using equations (B4), (B5), and (B6).

Internal Optimization

The analysis of the boundary value problem showed that the booster kick angle and, in certain cases, the booster burning time can be optimized without iteration. In such cases, the optimizing equation can be used to calculate ψ or $\dot{\psi}$; one set of initial and final conditions is thus eliminated from the iteration.

The equation for optimizing the booster kick angle is

$$\lambda_1 \frac{\partial u}{\partial \alpha} + r\lambda_2 \frac{\partial \omega}{\partial \alpha} + \lambda_3 \frac{\partial r}{\partial \alpha} + \lambda_4 \frac{\partial \varphi}{\partial \alpha} = 0 \tag{40d}$$

or, in terms of ψ and $\dot{\psi}$,

$$\lambda_0 \sin \psi \frac{\partial u}{\partial \alpha} + r\lambda_0 \cos \psi \frac{\partial \omega}{\partial \alpha} + \left[\frac{\lambda_0}{\cos \psi} \left(2\omega - \dot{\psi} - \frac{u}{r} \sin \psi \cos \psi \right) + \frac{\lambda_4}{r} \tan \psi \right] \frac{\partial r}{\partial \alpha} + \lambda_4 \frac{\partial \varphi}{\partial \alpha} = 0$$

$$\tag{B7}$$

Equation (B7) can be solved for $\dot{\psi}$:

$$\dot{\psi} = 2\omega - \frac{u}{r} \sin \psi \cos \psi + \frac{\lambda_4}{\lambda_0} \frac{\sin \psi}{r} + \frac{\cos \psi}{\dfrac{\partial r}{\partial \alpha}} \left(\sin \psi \frac{\partial u}{\partial \alpha} + r \cos \psi \frac{\partial \omega}{\partial \alpha} + \frac{\lambda_4}{\lambda_0} \frac{\partial \varphi}{\partial \alpha} \right) \qquad \text{(B8)}$$

Use of equation (B8) to calculate $\dot{\psi}$, however, leads to serious convergence problems, since this equation is an extremely nonlinear function of ψ. Instead, $\dot{\psi}$ is used as a variable initial condition in the external iteration and equation (40d) is used to calculate ψ. Although a closed form solution for ψ is not available (unless $\lambda_4 = 0$), an iteration can be performed which converges rapidly. The procedure is as follows:

(1) Guess ψ.

(2) Calculate λ_1, λ_2, and λ_3 from equations (B4), (B5), and (B6).

(3) Calculate $\Delta = \lambda_1 \dfrac{\partial u}{\partial \alpha} + r \lambda_2 \dfrac{\partial \omega}{\partial \alpha} + \lambda_3 \dfrac{\partial r}{\partial \alpha} + \lambda_4 \dfrac{\partial \varphi}{\partial \alpha}$.

(4) Iterate steps (1) to (3) to obtain $\Delta = 0$.

The booster burning time can be optimized internally when

(1) Stage 2 is a powered stage (β_2, $T_2 \neq 0$) also being optimized and $k_2 = 0$

(2) Stage 2 is an optimized coast phase

The optimizing equations for these two cases are

$$S_2^0 - \frac{S_1^f}{1 + k_1} = 0 \qquad \text{(51)}$$

and

$$C_2 = 0 \qquad \text{(40b)}$$

If equations (37e), (41b), and (41c) are used and it is noted that $\partial r / \partial \tau_1 = u$ and $\partial \varphi / \partial \tau_1 = \omega$, equations (51) and (40b) become

$$a_1 \lambda_1 + a_2 \lambda_2 + a_3 \lambda_3 + a_4 \lambda_4 = a_5 \lambda_0 \qquad \text{(B9)}$$

where

$$
\left.
\begin{aligned}
a_1 &= \left(\frac{\mu}{r^2} - \omega^2 r\right) + \frac{\beta_2 \frac{\partial u}{\partial \tau_1}}{(1 + k_1)\beta_1} \\[2em]
a_2 &= 2u\omega + \frac{\beta_2 r \frac{\partial \omega}{\partial \tau_1}}{(1 + k_1)\beta_1} \\[2em]
a_3 &= \left[\frac{\beta_2}{(1 + k_1)\beta_1} - 1\right] u \\[2em]
a_4 &= \left[\frac{\beta_2}{(1 + k_1)\beta_1} - 1\right] \omega \\[2em]
a_5 &= \frac{T_2}{m_2^0}
\end{aligned}
\right\}
\qquad (B10)
$$

Equation (B9) can be expressed in terms of ψ and $\dot\psi$ as in equation (B8):

$$
\dot\psi = 2\omega - \frac{u}{r}\sin\psi\cos\psi + \frac{\lambda_4}{\lambda_0}\frac{\sin\psi}{r} + \frac{\cos\psi}{a_3}\left(a_1\sin\psi + a_2\cos\psi + \frac{\lambda_4}{\lambda_0}a_4 - a_5\right) \qquad (B11)
$$

As in equation (B8), however, $\dot\psi$ is a very nonlinear function of ψ, and equation (B9) must instead be solved for ψ in terms of $\dot\psi$. Again, a closed form solution is not available, and the iteration proceeds as previously described, except that

$$
\Delta = a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_4 - a_5\lambda_0
$$

for this case.

If the booster burning time and kick angle are both being optimized internally, equations (40d) and (B9) are available for the calculation of any two initial conditions. It is convenient to use these equations to calculate ψ and τ_1 and to leave $\dot\psi$ and α as variable initial conditions in the external iteration.

Equations (B9) and (40d) are combined to eliminate λ_3, after which the resulting

equation is used with equation (B3) to solve for λ_1 and λ_2:

$$\lambda_1 = \frac{b_1 b_3 - b_2 \sqrt{\left(b_1^2 + b_2^2\right)\lambda_0^2 - b_3^2}}{b_1^2 + b_2^2} \qquad \text{(B12a)}$$

$$\lambda_2 = \frac{b_2 b_3 + b_1 \sqrt{\left(b_1^2 + b_2^2\right)\lambda_0^2 - b_3^2}}{b_1^2 + b_2^2} \qquad \text{(B12b)}$$

where

$$b_1 = a_1 \frac{\partial r}{\partial \alpha} - a_3 \frac{\partial u}{\partial \alpha} \qquad \text{(B13a)}$$

$$b_2 = a_2 \frac{\partial r}{\partial \alpha} - a_3 r \frac{\partial \omega}{\partial \alpha} \qquad \text{(B13b)}$$

$$b_3 = \left(a_3 \frac{\partial \varphi}{\partial \alpha} - a_4 \frac{\partial r}{\partial \alpha}\right)\lambda_4 + a_5 \lambda_0 \frac{\partial r}{\partial \alpha} \qquad \text{(B13c)}$$

The choice of sign in front of the radical in equations (B12) was made to obtain a thrust direction between 0° and 90°. Substituting equations (B4) and (B5) into (B6) gives

$$\lambda_3 = \frac{\lambda_0^2}{\lambda_2} (2\omega - \dot{\psi}) - \frac{u}{r} \lambda_1 + \frac{\lambda_1 \lambda_4}{r \lambda_2} \qquad \text{(B14)}$$

An iteration is required to calculate τ_1, which proceeds as follows:
    (1) Guess τ_1.
    (2) Calculate λ_1, λ_2, and λ_3 from equations (B12) and (B14).

    (3) Calculate $\Delta = \lambda_1 \dfrac{\partial u}{\partial \alpha} + r\lambda_2 \dfrac{\partial \omega}{\partial \alpha} + \lambda_3 \dfrac{\partial r}{\partial \alpha} + \lambda_4 \dfrac{\partial \varphi}{\partial \alpha}$.

    (4) Iterate steps (1) to (3) to obtain $\Delta = 0$.
Equations (B12) and (40d) are equivalent to equations (B3), (B9), and (40d), and thus, the optimum values of ψ and τ_1 are obtained when the iteration converges. The value of ψ is calculated from equations (B12), by use of (B4) and (B5):

$$\sin \psi = \frac{b_1 b_3 - b_2 \sqrt{\left(b_1^2 + b_2^2\right)\lambda_0^2 - b_3^2}}{\lambda_0 \left(b_1^2 + b_2^2\right)} \tag{B15a}$$

$$\cos \psi = \frac{b_2 b_3 + b_1 \sqrt{\left(b_1^2 + b_2^2\right)\lambda_0^2 - b_3^2}}{\lambda_0 \left(b_1^2 + b_2^2\right)} \tag{B15b}$$

The value of ψ obtained from equations (B15) is independent of the value of λ_0 chosen, provided that λ_4 is scaled exactly as λ_0.

APPENDIX C

ORBIT REQUIREMENTS

A two-dimensional orbit is completely specified by any four independent orbit parameters. In many cases, however, one or more of these parameters is left unspecified and can be used to maximize the payload. For such cases, equations (40e) supply auxiliary boundary conditions to be satisfied, in order to guarantee optimum values of the unspecified variables.

A typical example is the requirement of a circular orbit at a specified radius, with the orbit injection point unspecified. For this case, the required final conditions can be stated as

$$\left.\begin{aligned} r &= r_d \\[6pt] u &= 0 \\[6pt] \omega &= \sqrt{\frac{\mu}{r_d^3}} \end{aligned}\right\} \tag{C1}$$

where r, u, and ω (and all other variables in this appendix) should be evaluated at $t = t_N^-$. Equations (40e) are written (with r, u, ω, and φ as independent variables)

$$\left.\begin{aligned} G(u) &= \lambda_1 \\[6pt] G(\omega) &= r\lambda_2 \\[6pt] G(r) &= \lambda_3 \\[6pt] G(\varphi) &= \lambda_4 \end{aligned}\right\} \tag{C2}$$

Since φ is not specified, $\lambda_4 = 0$ is the auxiliary boundary equation, and the required final conditions are

$$r = r_d$$

$$u = 0$$

$$\left. \omega = \sqrt{\frac{\mu}{r_d^3}} \right\} \quad \text{(C3)}$$

$$\lambda_4 = 0$$

If, in addition to the travel angle, the radius is also unspecified, equation (C2) gives $\lambda_3 = 0$ as an additional auxiliary boundary equation, and

$$u = 0$$

$$\left. \omega = \sqrt{\frac{\mu}{r_d^3}} \right\} \quad \text{(C4)}$$

$$\lambda_3 = 0$$

$$\lambda_4 = 0$$

are the required final conditions.

Frequently, the orbit is specified in terms of energy and flight path angle. For such cases, H, Γ, r, and φ can be used as independent variables, where

$$H = \frac{1}{2}(u^2 + r^2\omega^2) - \frac{\mu}{r}$$

and

$$\left. \right\} \quad \text{(C5)}$$

$$\tan \Gamma = \frac{u}{\omega r}$$

Solving for u and ω yields

$$\left.\begin{array}{l} u = \sqrt{2\left(H + \dfrac{\mu}{r}\right)} \sin \Gamma \\[4mm] \omega = \dfrac{1}{r}\sqrt{2\left(H + \dfrac{\mu}{r}\right)}\cos \Gamma \end{array}\right\} \tag{C6}$$

If the indicated partial derivatives are taken, equations (40e) become

$$\left.\begin{array}{l} G(H) = \left(\dfrac{u}{v^2}\lambda_1 + \dfrac{\omega r}{v^2}\lambda_2\right) \\[4mm] G(\Gamma) = (\omega r \lambda_1 - u\lambda_2) \\[4mm] G(r) = \left[-\dfrac{\mu u}{v^2 r^2}\lambda_1 - \left(\omega + \dfrac{\mu \omega}{rv^2}\right)\lambda_2 + \lambda_3\right] \\[4mm] G(\varphi) = \lambda_4 \end{array}\right\} \tag{C7}$$

If energy and flight path angle are specified (radius and travel angle optimized) the required final conditions are

$$\left.\begin{array}{l} \dfrac{1}{2}(u^2 + \omega^2 r^2) - \dfrac{\mu}{r} = H_d \\[4mm] \tan^{-1}\left(\dfrac{u}{\omega r}\right) = \Gamma_d \\[4mm] -\dfrac{\mu u}{v^2 r^2}\lambda_1 - \left(\omega + \dfrac{\mu \omega}{rv^2}\right)\lambda_2 + \lambda_3 = 0 \\[4mm] \lambda_4 = 0 \end{array}\right\} \tag{C8}$$

or, if energy is specified and all other variables are optimized, the required final conditions become

$$\frac{1}{2}\left(u^2 + r^2\omega^2\right) - \frac{\mu}{r} = H_d$$

$$\omega r \lambda_1 - u\lambda_2 = 0$$

$$-\frac{\mu u}{v^2 r^2}\lambda_1 - \left(\omega + \frac{\mu\omega}{rv^2}\right)\lambda_2 + \lambda_3 = 0 \qquad \qquad \text{(C9)}$$

$$\lambda_4 = 0$$

The last three of equations (C9) are equivalent to

$$\psi = \Gamma$$

$$\dot{\psi} = \dot{\Gamma} \qquad\qquad \text{(C10)}$$

$$\lambda_4 = 0$$

The case of an elliptic orbit is discussed in reference 8, in which the orbit elements e, p, θ and ϵ are chosen as the independent variables:

$$u = \sqrt{\frac{\mu}{p}}\, e \sin \theta$$

$$\omega = \sqrt{\frac{\mu}{p^3}}\, (1 + e \cos \theta)^2$$

$$r = \frac{p}{1 + e \cos \theta} \qquad\qquad \text{(C11)}$$

$$\varphi = \theta + \epsilon$$

For this case, equations (40e) become

$$G(e) = \frac{u}{e}\lambda_1 + \frac{2r^2\omega\cos\theta}{p}\lambda_2 - \frac{r^2\cos\theta}{p}\lambda_3$$

$$G(p) = \frac{r}{p}\lambda_3 - \frac{u}{2p}\lambda_1 - \frac{3r\omega}{2p}\lambda_2$$

$$G(\theta) = \frac{u\lambda_1}{\tan\theta} - \frac{2r^2 e\omega\sin\theta}{p}\lambda_2 + \frac{r^2 e\sin\theta}{p}\lambda_3 + \lambda_4$$

$$G(\epsilon) = \lambda_4$$

(C12)

Frequently, the eccentricity e and the semilatus rectum p of the ellipse are specified, while the true anomaly θ and argument of perigee ϵ are left open for optimization. For this case, the required final conditions are

$$p = \frac{r^4\omega^2}{\mu} = p_d$$

$$e = \sqrt{1 + \frac{u^2 + \omega^2 r^2 - 2\frac{\mu}{r}r^4\omega^2}{\mu^2}} = e_d$$

$$\frac{u\lambda_1}{\tan\theta} - \frac{2r^2 e\omega\sin\theta}{p}\lambda_2 + \frac{r^2 e\sin\theta}{p}\lambda_3 = 0$$

$$\lambda_4 = 0$$

(C13)

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