SOME GENERAL CONSIDERATIONS CONCERNING THE DESTABILIZING EFFECT IN NONCONSERVATIVE SYSTEMS*

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ABSTRACT

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A linear system with N degrees of freedom subjected to a set of nonconservative forces which depend linearly on generalized coordinates is considered. Several theorems concerning the destabilizing effect induced by the presence of small forces which depend on generalized velocities are established. These forces may be due to viscous damping and gyroscopic effects.

Author

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1. Introduction

The destabilizing effect of viscous damping in a linear, dynamic system with two degrees of freedom, subjected to nonconservative forces which depend on generalized coordinates, was first noted by Ziegler [1]*. This problem was further explored by Bolotin [2], and Herrmann and Jong [3,4]. However, this effect was not placed into the framework of theorems of sufficient generality. Nor was it shown whether a more general system with many degrees of freedom can also exhibit such behavior. Moreover, from examples worked out in [1,2,3,4], it is not clear whether other velocity dependent forces, (such as Coriolis forces on vibrating pipes conveying fluid, or purely gyroscopic forces [5,6]), can have similar effects.

The purpose of the present study is to prove that the critical load of an undamped system with N degrees of freedom subjected to nonconservative forces, which are linear functions of the generalized coordinates, is an upper bound for the critical load of the same system when, in addition, some sufficiently small forces which are linear functions of the generalized velocities are also present. Therefore, it is concluded that in a general nonconservative system with N degrees of freedom not only slight viscous damping but all sufficiently small velocity dependent forces may have a destabilizing effect.

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*Numbers in brackets refer to references listed at the end of this paper.

2. Statement of the Problem

We consider an N-degree-of-freedom, linear, dynamic system represented by generalized coordinates q_j ; $j = 1,2,\ldots,N$. The system is assumed to be holonomic and autonomous, and is subjected to a set of generalized forces, $Q_j = Q_j(F)$; $j = 1,2,\ldots,N$, which are defined as linear functions of a real, finite parameter F. This parameter, $(0 \leq F < \infty)$, is associated with the magnitude of the externally applied forces; $Q_j = 0$ for $F = 0$.

Let $q_j = \dot{q}_j = 0$, $(j = 1,2,\ldots,N \; ; \; \dot{q}_j \equiv \frac{dq_j}{dt})$, be the equilibrium state of the system. With $M = [M_{jk}]$ the generalized mass matrix, and $V = \frac{1}{2} \sum_{j,k=1}^{N} \bar{K}_{jk} \, q_j \, q_k$ the strain energy function, the equations of motion of the undamped system may be written as

$$M_{jk} \ddot{q}_k + \bar{K}_{jk} q_k = Q_j \; ; \; j,k = 1,2\ldots,N \, , \tag{1}$$

where the summation convention on all repeated indices is implied and will be employed in the sequel.

Let us assume that the generalized forces, Q_j , are given as linear functions of the generalized coordinates;

$$Q_j = FK_{jk} \, q_k \; ; \; j,k = 1,2,\ldots,N \, ,$$

where $K = [K_{jk}]$ is a nonsymmetric matrix, and F a real, finite parameter. For F = 0, (1) represent the equations of free oscillation of the undamped system which we assume to possess N distinct, non-zero frequencies.

In conjunction with (1) we shall consider the following linear system

$$M_{jk}\ \ddot{q}_k + \epsilon\ G_{jk}\ \dot{q}_k + \bar{K}_{jk}\ q_k = Q_j\ ;\ j = 1,2,\ldots,N\ , \tag{2}$$

where ϵ is an infinitesimal quantity, $G = [G_{jk}]$ a general non-symmetric matrix with prescribed constant elements. For $\epsilon = 0$, equations (2) reduce to equations (1).

In the following sections we shall prove that the critical load of system (1) is an upper bound for the critical load of system (2) when $O(\epsilon^2)$ can be neglected in comparison with $O(\epsilon)$.

3. Destabilizing Effect of Slight Damping

Let us first consider the effect of slight viscous damping. Thus we assume that $G = [G_{jk}]$ is a symmetric, positive definite matrix.

We take solutions of (1) and (2) in the form $q_k = A_k e^{i\omega t}$; $i = \sqrt{-1}$, and obtain from (1) and (2), respectively

$$-\omega^2 M_{jk} A_k + (\bar{K}_{jk} - FK_{jk})A_k = 0 , \tag{3}$$

$$-\omega^2 M_{jk} A_k + \epsilon(i\omega)G_{jk} A_k + (\bar{K}_{jk} - FK_{jk})A_k = 0 , \tag{4}$$

$j,k = 1,2,\ldots,N$.

Systems (3) and (4) are each a set of linear, homogeneous equations in A_k. They have, therefore, nontrivial solutions if and only if the determinant of the coefficients of A_k, in each set, is equal to zero. These conditions yield

$$\det \left| a_{jk} \right| = 0 , \tag{5}$$

$$\det \left| a_{jk} + \epsilon(i\omega)G_{jk} \right| = 0 , \tag{6}$$

where $a_{jk} = -\omega^2 M_{jk} + (\bar{K}_{jk} - FK_{jk})$, and $\det \left| a_{jk} \right|$ denotes the determinant of the matrix $[a_{jk}]$.

For $F = 0$, equation (5) yields the natural frequencies of the free vibration of the undamped system. We assume that these frequencies, $(\omega_1^2 < \omega_2^2 < \ldots < \omega_N^2)$, are distinct and non-zero. We now increase F and assume that for a certain value of F, say F_e , equation (5) yields a double non-zero frequency. Let us suppose that, for $F = F_e$, ω_1^2 is equal to ω_2^2 (see Figure 1(a)), while all other (N-2) frequencies of the system are distinct and non-zero. If F is now increased beyond this critical value F_e, equation (5) will yield a pair of complex conjugate roots and, consequently, the system will oscillate with an exponentially increasing amplitude (flutter). We shall refer to F_e as the undamped critical load.

Let us now consider equation (6). For $F = 0$, the roots of this equation are all located on the left-hand side of the imaginary axis in the complex $i\omega$ plane. As we increase F, one of these roots approaches the imaginary line, and for a certain value of F, say F_d, equation (6) yields a real value for ω (see Figure 1(b)). If F is now increased beyond this critical value F_d, one of the roots of (6) becomes complex with negative imaginary part. The system, therefore, loses stability by flutter. We shall refer to F_d as the damped critical load.

In the sequel we will first study a system with two degrees of freedom and then extend our results to more general systems.

A. Systems with two degrees of freedom

We expand the frequency equation of the damped system as follows

$$\det \left| a_{jk} + \epsilon(i\omega)G_{jk} \right| = \det \left| a_{jk} \right| + \epsilon(i\omega) \sum_{j,k=1}^{2} G_{jk}\, a^{kj} - \epsilon^2 \omega^2 \det \left| G_{jk} \right| ; \quad j,k = 1,2 , \tag{7}$$

where a^{kj} is the cofactor [7] of the element a_{jk} in the $\det | a_{jk} |$. Moreover, we assume that $\det | G_{jk} | \neq 0$ (the case of $\det | G_{jk} | = 0$ will be discussed later). Then, for $\det | G_{jk} |$ finite and ϵ of infinitesimal order, we may neglect the last term on the right-hand side of equation (7) and obtain

$$\det|a_{jk}| + \epsilon(i\omega) \sum_{j,k=1}^{2} G_{jk}\, a^{kj} = 0 \;, \quad j,k = 1,2 \;. \tag{8}$$

Theorem 1:

The undamped critical load, F_e , is an upper bound for the damped critical load, F_d , when $O(\epsilon^2)$ can be neglected in comparison with $O(\epsilon)$.

Proof:

For $F = F_d$, equation (8) has one real root, $\omega = \bar{\omega}$, and one complex root with positive imaginary part. Therefore, for $F = F_d$ and $\omega = \bar{\omega}$, $\det|a_{jk}|$ and $\sum_{j,k=1}^{2} G_{jk}\, a^{kj}$ are both real and we must have

$$\det|a_{jk}| = 0 \;, \tag{9}$$

$$\sum_{j,k=1}^{2} G_{jk}\, a^{kj} = 0 \;. \tag{10}$$

However, $\det|a_{jk}| = 0$ cannot admit real roots if $F > F_e$. Therefore $F_d \leq F_e$.

Let us note that F_d can equal F_e if and only if the real root of (10) can be made equal to the double root of (9) for $F = F_e$. This, of course, depends on the other parameters of the system and may not always be achieved [4].

We now retract to equation (7) and consider the case when

$\det | G_{jk} | = 0$. The frequency equation of the damped system with two degrees of freedom is now given by equation (8), independently of the order of magnitude of ϵ . Following the line of reasoning similar to that used in the proof of theorem 1, we conclude that the critical load of the undamped system is an upper bound for that of the damped system, no matter what the order of magnitude of ϵ may be. Therefore, we state the following theorem.

Theorem 2:

> The critical load, F_e , of the undamped system with two degrees of freedom is an upper bound for the critical load, F_d , of the damped system for all finite values of ϵ when $\det | G_{jk} | = 0$.

B. System with N degrees of freedom

The proof of theorem 1 was an immediate consequence of a property of the frequency equation of the damped system with two degrees of freedom. The problem becomes more complicated if the system has more than two degrees of freedom. However, one may still use a similar line of reasoning.

We expand equation (6), collect the terms of equal power in ϵ , and obtain

$$\det | a_{jk} + \epsilon(i\omega)G_{jk} | = \det | a_{jk} | + \epsilon(i\omega) \sum_{j,k=1}^{N} G_{jk} a^{kj} + O(\epsilon^2) + \dots , \quad j,k = 1,2,\dots,N . \qquad (11)$$

The first term on the right-hand side of this equation is a polynomial of degree N in ω^2 and may be written as

$$\det | a_{jk} | \equiv P(\omega^2) \equiv P_N \omega^{2N} + P_{N-1} \omega^{2(N-1)} + \dots + P_o .$$

Similarly, the term $\sum_{j,k=1}^{N} G_{jk} a^{kj}$, which is a polynomial of degree (N-1) in ω^2, can be written as

$$\sum_{j,k=1}^{N} G_{jk} a^{kj} \equiv R(\omega^2) \equiv R_{N-1} \omega^{2(N-1)} + R_{N-2} \omega^{2(N-2)} + \dots + R_o .$$

Therefore, equation (11) becomes

$$\det | a_{jk} + \epsilon(i\omega)G_{jk} | = P(\omega^2) + i\epsilon\omega R(\omega^2) + O(\epsilon^2) + \dots . \tag{12}$$

We neglect $O(\epsilon^2)$ and higher in equation (12) and obtain

$$P(\omega^2) + i\epsilon\omega R(\omega^2) = 0 , \tag{13}$$

for the frequency equation of system (2). We now set $\omega = \lambda + i\epsilon\gamma$ and substitute into $P(\omega^2)$ and $R(\omega^2)$ to obtain

$$P(\lambda + i\epsilon\gamma) = P(\lambda^2) + i\epsilon\gamma \left[2\lambda \frac{dP(\lambda^2)}{d(\lambda^2)}\right] + O(\epsilon^2) + \dots ,$$

$$R(\lambda + i\epsilon\gamma) = R(\lambda^2) + O(\epsilon) + \dots .$$

Therefore, equation (13) becomes

$$P(\lambda^2) + i\epsilon\lambda \left[2\gamma \frac{dP(\lambda^2)}{d(\lambda^2)} + R(\lambda^2)\right] + O(\epsilon^2) + \dots = 0.$$

Neglecting terms of order higher than ϵ , we must have

$$P(\lambda^2) = 0 ,$$

$$\gamma = - \frac{R(\lambda^2)}{2P'(\lambda^2)} ,$$

$$P'(\lambda^2) = \frac{dP(\lambda^2)}{d(\lambda^2)} \neq 0 . \tag{14}$$

The first equation in (14) is the frequency equation of the undamped system and the second equation defines, to the first order of approximation in ϵ , the effect of slight damping on the frequencies of the undamped system. The constraint given by $P'(\lambda^2) \neq 0$ indicates that the perturbation method breaks down when $P(\lambda^2) = 0$ admits double roots. For $F = 0$, the roots of equation $P(\lambda^2) = 0$ are all real and distinct. Thus in this case, to the first order of approximation in ϵ , the roots of equation (13) are

$$\pm \omega_k = \pm \lambda_k + i\epsilon\gamma_k ,$$

$$\gamma_k = - \frac{R(\lambda^2{}_k)}{2P'(\lambda^2{}_k)} ; \quad k = 1,2,\ldots,N , \tag{15}$$

where λ_k^2 are real, positive roots of $P(\lambda^2) = 0$. The system can perform oscillations only with an exponentially decaying amplitude and, therefore, all γ_k ; $k = 1,2,\ldots,N$ are positive, real numbers.

We shall now assume that the damped system is stable for all $F < F_d$ and consider the following cases:

(a) $\quad F < F_d < F_e$,

(b) $\quad F_e < F < F_d$; $F_e < F_d$.

For case (a), $P(\lambda^2) = 0$ yields N distinct roots. From equations

(15) we then obtain γ_k ; $k = 1,2,\ldots,N$, which are, by our assumption, all positive, real numbers.

For case (b), $P(\lambda^2) = 0$ has, at least, one pair of complex conjugate roots. We denote these roots by $\bar{\lambda}_{1,2} = (\alpha \pm i\beta)$ and from (15) obtain

$$\pm \bar{\omega}_{1,2} = \pm (\alpha \pm i\beta) - i\epsilon \frac{R[(\alpha \pm i\beta)^2]}{2P'[(\alpha \pm i\beta)^2]} ,$$

which indicates that, for $F > F_e$, the damped system admits, at least, one complex frequency with negative imaginary part. This, therefore, contradicts the assumption that the system is stable for $F_d > F_e$. We are thus forced to take $F_d \le F_e$ in order to remove the contradiction.

Let us note that, for $F = F_d = F_e$, equations (15) can be used only for the distinct roots of $P(\lambda^2) = 0$. The perturbation method, which was introduced in this section, breaks down if $P'(\lambda^2) = O(\epsilon)$ while $R(\lambda^2)$ is non-zero. We shall not, however, concern ourselves with a detailed study of this case here and simply admit the possibility of $F_d = F_e$. In fact, as $F_d > F_e$ renders the damped system unstable, we can only conclude that $F_d \le F_e$. Therefore, we may state the following theorem.

Theorem 3:

> The critical load, F_e , of the undamped system (1) is an upper bound for the critical load, F_d , of the associated slightly damped system when ϵ is sufficiently small.

4. The General Case

For an arbitrarily specified damping matrix $G = [G_{jk}]$, system (2) may become self-exciting. That is, for an infinitely small value of F, the frequency equation of this system may possess complex roots with negative imaginary parts. In these cases we shall agree to define $F_d = 0$ as the damped critical load of the system.

On the other hand, the frequency equation of system (2) may yield roots with only positive imaginary parts for $F = O(\epsilon)$. This indicates that the damped system is stable for small values of the load parameter F. However, as we increase F, one of these roots may move toward the imaginary line in the $i\omega$ plane. Therefore, for a certain value of F, say F_d, the frequency equation of system (2) may yield a non-zero, real root. In this case, if we then increase F beyond this critical value F_d, the frequency equation will have a root with negative imaginary part and the damped system will flutter. We shall refer to F_d as the damped critical load. On the basis of the above preliminaries it is now possible to follow the same chain of arguments outlined in the previous section and establish the following more general theorems.

Theorem 4:

> The critical load, F_e, of the undamped system (1) is an upper bound for the critical load, F_d, of the damped system (2) when ϵ is sufficiently small. $G = [G_{jk}]$ need not be a symmetric, positive definite matrix.

Theorem 5:

The critical load, F_e , of the undamped system (1) with $N = 2$ is an upper bound for the critical load, F_d , of the associated damped system for all finite values of ϵ when $\det |G_{jk}| = 0$. $G = [G_{jk}]$ need not be a symmetric, positive definite matrix.

5. Concluding Remarks

From the above results we immediately conclude that, in a linear system with N degrees of freedom, subjected to nonconservative forces, not only slight viscous damping but all sufficiently small velocity dependent forces have, in general, a destabilizing effect. Moreover, the damped critical load, F_d , is highly dependent upon the structure of the matrix $G = [G_{jk}]$ but is always bounded from above by the undamped critical load F_e . This indicates that, even at the limit as $\epsilon \to 0$, F_d is in general less than F_e . Let us explore this point in more detail for a system with two degrees of freedom.

For ϵ finite, the steady state motion of the system is possible if the frequency of the oscillation satisfies the following equations (see equation (7)):

$$\det | a_{jk} | - \epsilon^2 \omega^2 \det | G_{jk} | = 0 ,$$

$$\sum_{j,k=1}^{2} G_{jk} a^{kj} = 0 . \tag{16}$$

In this case, one may solve the second equation in (16) for ω as a function of F and then substitute the result into the first equation to obtain a relationship between F and ϵ . In this manner a stability curve, in the F-ϵ plane, may be constructed (see Figure 2). However, from theorem 1 we immediately conclude that, in general, this curve suffers a finite discontinuity at $\epsilon = 0$. This means that, although for $\epsilon = 0$ the critical load is F_e , for $\epsilon = 0^+$ the critical load is given by F_d which is, in general, less than F_e. Therefore, the point F_e is, in general, an isolated point in the F-ϵ plane (Figure 2). In reference [3] an appropriate physical interpretation of this phenomenon is given by introducing the concept of the degree of instability. An interpretation based on the energy of the system is also given in [8] for systems with more than two degrees of freedom.

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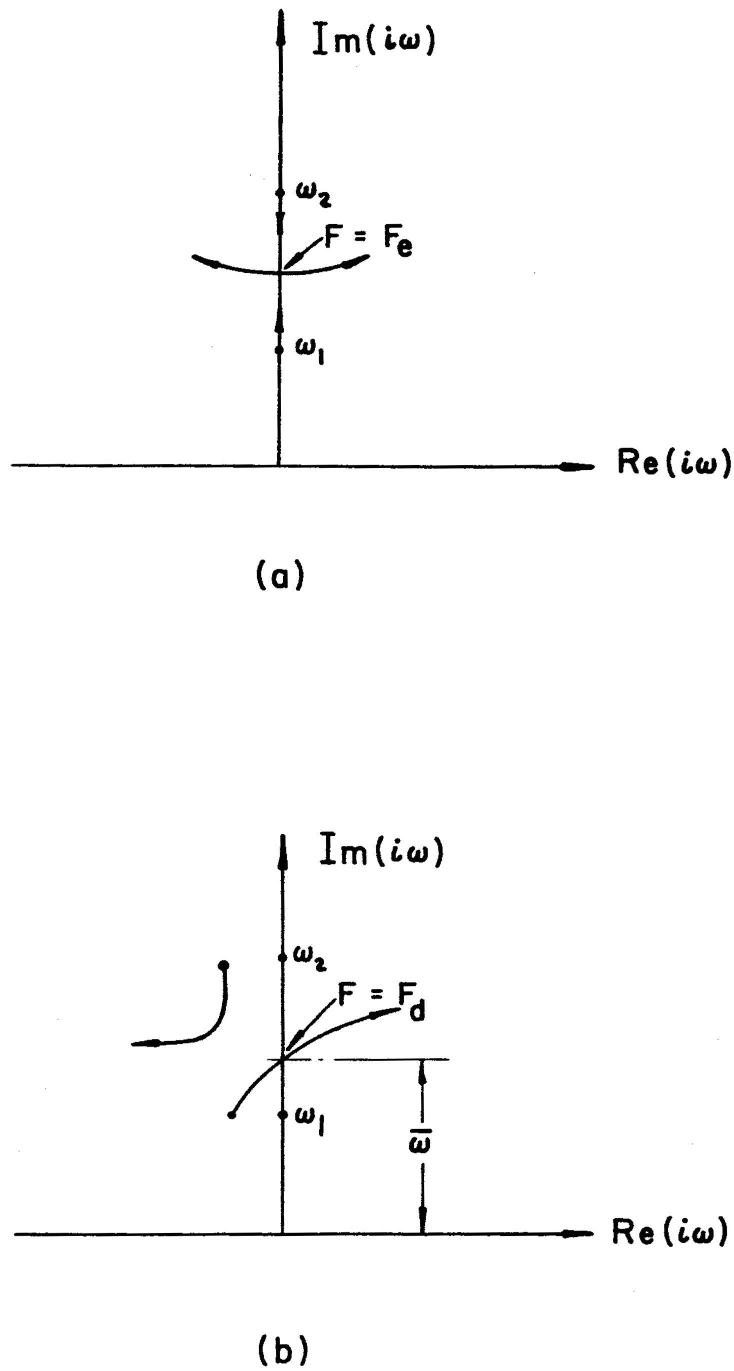
Im($i\omega$)
ω_2
$F = F_e$
ω_1
Re($i\omega$)
(a)
Im($i\omega$)
ω_2
$F = F_d$
ω_1
$\bar{\omega}$
Re($i\omega$)
(b)

Figure 1

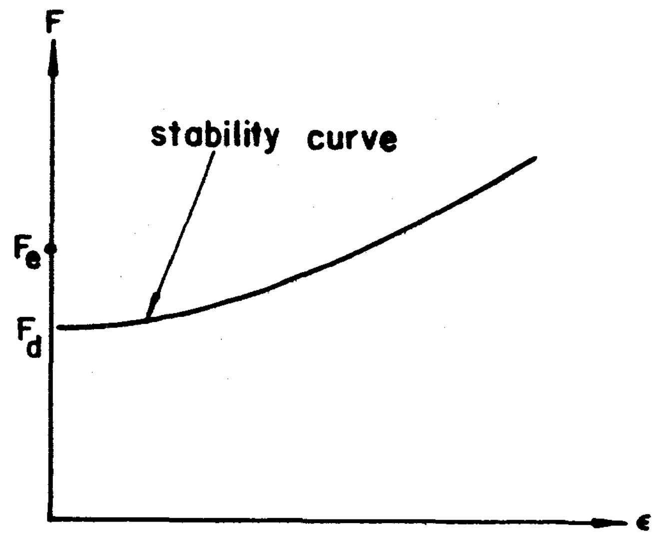

Figure 2