Ionospheric Research

NASA Grant No. NsG 134-61

Scientific Report

on

"Electron Density and the Electric Field in a Steady-State F-Layer"

by

Craig Comstock

January 16, 1966

Scientific Report No. 263

Ionosphere Research Laboratory

Submitted by: J. S. Nisbet (aw)
John S. Nisbet, Associate Professor of Electrical Engineering, Project Supervisor

Approved by: A. H. Waynick
A. H. Waynick, Director, Ionosphere Research Laboratory

The Pennsylvania State University

College of Engineering

Department of Electrical Engineering

TABLE OF CONTENTS

	Page
ABSTRACT	i
I. INTRODUCTION	1
II. THE BASIC EQUATIONS	1
III. THE PARTICLE DENSITIES	4
IV. THE ELECTRIC FIELD AND THE FLOW VELOCITIES	15
V. CONCLUSIONS	21
ACKNOWLEDGEMENTS	23
REFERENCES	24

ABSTRACT

14960

A steady state solution is obtained to the equation for the electron density in an isothermal one dimensional model of the F-layer. The solution is an extension of a previous result of Bowhill to include the effects of different ion and electron temperatures and a more realistic electron production term. Simultaneously a solution is obtained for the steady state electric field necessary to maintain charge neutrality. This result is new and somewhat surprising. Considerable attention is paid to the possible boundary conditions for such a model and their effect on the solution. The effect of a plane parallel approximation on the boundary conditions is also included.

Author

I. Introduction

In a recent paper Comstock [1965a] has derived a set of equations governing the diffusion of a three component (ions, electrons, and neutrals) gas in a spherically symmetric, steady state, isothermal planetary atmosphere, in the absence of a magnetic field. The distinctive features of these equations are: 1) that they include a differential equation for the electric field throughout the atmosphere; 2) that they are applicable to any degree of ionization from a weakly ionized atmosphere to a totally ionized atmosphere; and 3) that no a priori assumption has been made concerning the validity of ambipolar diffusion. Some of the properties of these equations have already been discussed [Comstock 1965 a, 1965b]. In this paper we specialize these equations to the case where the neutral to ion density ratio is between 10^4 and 1, which ratio is consistent with the major portion of the F-layer. We then solve for the electric field and the ion and electron velocities, subject to appropriate boundary conditions. The solutions to this idealized model ionosphere should give some insight into the physics of the earth's F-layer.

II. The Basic Equations

Our basic assumptions include the following. Each species, k, is isothermal at its own temperature T_k. The average drift velocities q_k are sufficiently small that the drift kinetic energy is small compared to the potential energy due to gravity. Everywhere except at the boundary of such an atmosphere, the gas is nearly electrically neutral (see [Comstock, 1965a] for a further discussion of this point). Lastly, the electric field E is derivable from a potential ϕ. Then throughout the following the subscripts i, e, n refer to ions, electrons and neutrals (atomic oxygen) respectively. We introduce the following definitions:

μ = mass ratio = m_e/m_i

$H = \dfrac{kT_n}{m_n g_o}$

g_o = the gravitational field at the reference height

$H_e = H/\mu$

$a = T_e/T_n$

$b = H_e g_o/H^2$

n_{eo} = the reference electron density

n_{no} = the reference neutral density.

We then define our dimensionless variables by

$\eta_e = n_e/n_{eo}$

$\eta_n = n_n/n_{no}$

$z = r/H$

$\varphi = e\phi/kT_n$

$\psi = \dfrac{1}{g_o} \dfrac{d}{dr} G(r)$

In the above G(r) is the gravitational potential due to the earth's gravitational field. We also write

$\dfrac{n_n}{\sigma_{in}}$ = the ion-neutral collision frequency

g(z) = the (known) ionization rate

h(z) = the (known) recombination rate.

Then the basic equations governing our atmosphere become (Comstock [1965a])

$$\nabla^2 \eta_e + \frac{d\eta_e}{dz}\left(\frac{\psi}{1+a} - \frac{d}{dz}\ln\eta_n\right) - \frac{\psi}{1+a}\eta_e \frac{d}{dz}\ln\eta_n$$

$$= + (h(z)n_e - g(z)n_{no}\eta_n)\frac{n_n}{2\sigma_{in}}\frac{1}{\mu b n_{eo}(1+a)} \tag{1}$$

$$\nabla^2\varphi + \left(\frac{d}{dz}\ln\eta_n - \frac{d}{dz}\ln\eta_e\right)\left(\frac{d}{dz}\ln\eta_e + \frac{\psi a}{a+1}\right)$$

$$= -\frac{1}{\eta_e}\frac{n_n}{2\sigma_{in}}\frac{(n_{eo}\eta_e h(z) - g(z)\eta_n n_{no})}{\mu b n_{eo}(1+a)}, \tag{2}$$

$$\nabla^2\eta_n - \frac{1}{\eta_n}\left(\frac{d\eta_n}{dz}\right)^2 = \frac{1}{2}\frac{n_n}{\sigma_{in}}\frac{1}{abn_{no}}(g(z)\,n_n - n_e h(z)) \tag{3}$$

$$q_e = \frac{-2\mu bH}{n_n/\sigma_{in}}\left[\left\{E + a\frac{d}{dz}\ln\eta_e\right\}\frac{d\lambda}{2\mu} + \left\{\psi - E + \frac{d}{dz}\ln\eta_e\right\}\right] \tag{4}$$

$$q_i = -\frac{2\mu bH}{n_n/\sigma_{in}}\left[\psi + (1+a)\frac{d}{dz}\ln\eta_e\right] \tag{5}$$

The production term g(z) is taken from Nisbet [1963]. It is a model devised so as to include the depletion of radiation with depth in a crude, but useable form. If the radiation intensity per unit area on a large sphere of radius z_o is $g(z_o)$ then

$$g(z) = \begin{cases} \dfrac{g(z_o)z_o^2}{z^2}\left\{1 - k\sec\Theta\displaystyle\int_z^{z_o} n_n \exp\left[\frac{B}{x}(\alpha-1)\right]\frac{dx}{x^2}\right\}, & g(z) > 0 \\ 0, & g(z) \le 0 \end{cases}$$

$$= \begin{cases} \dfrac{g(z_o)z_o^2}{z^2} \left\{ q_m - p_m \exp\left[\dfrac{B(\alpha-1)}{z} \right] \right\}, & , \; g(z) > 0 \\ 0 & , \; g(z) \leq 0 \end{cases}$$

The recombination term $h(z)\, n_e$ is taken as a linear function of the electron density. The function $h(z)$ is proportional to the (known) density of molecular oxygen, assumed in static equilibrium, according to the well known charge exchange recombination mechanism in the F-layer.

It is the solution to this set of equations, (1)-(5), which we wish to discuss here.

III. The Particle Densities

We see that we can solve the equations (1) and (3) for the electron and neutral densities independently of the electric field. The equation for the neutral density is the simplest, and thus shall be solved first.

Considering first the right hand side of equation (3) and using the data typical of the ionosphere, we see that the ionization and recombination terms are of the same order of magnitude, but that each is of the order of 10^{-3}. So we shall neglect the right hand side of (3). Then the solution of the neutral density distribution is the familiar barotropic solution in spherical coordinates

$$\eta_n = A \quad \exp \frac{B}{z} = \exp\left[\frac{B}{z} - \frac{B}{z_o} \right] , \tag{6}$$

where A and B are arbitrary constants of integration.

Then equations (1) and (2) become

$$\nabla^2 \eta_e + \frac{d\eta_e}{dz} \left(\frac{1}{1+a} + 1\right) \frac{B}{z^2} + \frac{1}{(1+a)} \left(\frac{B}{z^2}\right)^2 \eta_e$$

$$= - \alpha_1 \eta_n^2 g(z) + \beta_1 h(z) \eta_e \eta_n \qquad (7)$$

$$\nabla^2 \varphi = \frac{d}{dz} \ln \left(\frac{\eta_e}{\eta_n}\right) \left(\frac{Ba}{(1+a)z^2} + a \frac{d}{dz} \ln \eta_e\right) + \alpha_1 a \frac{\eta_n^2}{\eta_e} g(z)$$

$$- \beta_1 ah(z)\eta_n \qquad (8)$$

If the recombination coefficient β_1 and the ionization coefficient α_1 were small enough to be ignored, then the electron equation (7) would be a very simple equation to solve. It is easily seen that the solution would be

$$\eta_e = A_1 \exp\left(\frac{B}{z}\right) + A_2 \exp\left(\frac{B}{(1+a)z}\right) . \qquad (9)$$

That is, the purely diffusive motion of the ion-electron gas consists of a linear combination of two diffusive terms, one with an effective molecular weight of the neutrals and the second with an effective molecular weight of the static neutral plasma. We note that, as long as $|A_2| > |A_1|$, then at sufficiently great heights, the solution (9) becomes dominated by the second term. That is, the statement that at sufficiently great heights the ion-electron plasma is essentially in diffusive equilibrium does not determine either of the two constants in (9). We note also that the two solutions which make up equation

(9) are monotonic functions of height. Thus to chose either coefficient zero is to insist that such a solution admit no peak.

We make these comments on this obviously unrealistic solution at this point because it turns out that many of the same features will appear in the solution to the full equation (7). We also note that from equations (6) and (9) we can suspect (and will later confirm) that the atmosphere we wish to describe is, of necessity, of finite thickness. If the atmosphere is to have a peak in the electron density, and to approach diffusive equilibrium at great heights, then at the lower end the electron density must drop below 10^{-4} times the neutral density. On the other hand, the electron density decreases, at great heights, more slowly than the neutral, and so will eventually exceed the neutral density. Thus at both the upper and lower extremes we violate the bounds on the density ratios under which assumption we derived our basic equations. Thus equations (1)-(5) can be used only to discuss atmospheres of finite thickness.

We now proceed to try to solve the equations (1), (2), (4), (5). Unfortunately for analytical purposes, the recombination and ionization terms in (7) cannot be ignored. Equation (7) is then a variable coefficient, linear differential equation, whose solution is not known in terms of standard functions. However, we can take advantage of the fact that our model atmosphere is necessarily a finite atmosphere, of relatively thin nature. The major difficulty in solving (7) and (8) is the variable coefficient nature of the equation. We thus expand the coefficients in a power series about some middle point in the atmosphere. It is convenient to chose this point as the point of maximum ionization rate. This is essentially equivalent to considering a plane-parallel, stratified atmosphere. In doing such an expansion, we must be careful that we do not misinterpret our boundary conditions. In particular, to apply boundary conditions at $\pm\infty$ will probably lead to false conclusions, and certainly is not physically meaningful.

If we let

$$\zeta = z - z_o \tag{10}$$

and expand the coefficients, neglecting terms of the order of $(\zeta/z_o)^2$, then our equations reduce to

$$\frac{d^2\eta_e}{d\zeta^2} + \left(\frac{1}{1+a} + 1\right)\frac{d\eta_e}{d\zeta} + \left(\frac{1}{1+a} - \beta_1 h(\zeta)\exp(-\zeta)\right)\eta_e = -\alpha_1 g(\zeta)\exp(-2\zeta), \tag{11}$$

$$\frac{dE}{d\zeta} = \frac{d}{d\zeta}\ln\left[\eta_e \exp(\zeta)\right]\frac{a}{a+1} + \frac{d}{d\zeta}\ln\eta_e + \alpha_1 g(\zeta)\frac{\exp(-2\zeta)}{\eta_e}$$

$$- \beta_1 h(\zeta)\exp(-\zeta) \tag{12}$$

where

$$g(\zeta) = g(z_o)(1 - p_m \exp\left[-(\alpha - 1)\zeta\right]\ ) \tag{13}$$

$$h(\zeta) = \exp(-a_1\zeta). \tag{14}$$

In the above a_1 is the reciprocal ratio of the scale height of the molecular constituent which is involved in the recombination mechanism to the scale height of the neutral constituent and $p_m = kn_{no}\sec\Theta/(a-1)$. Equation (11) is similar to one studied by

Bowhill [1962] and others. With the substitution

$$x = \exp -(a_1 + 1)\,\zeta \tag{15}$$

we can reduce equation (11) to one whose solution is readily expressible in terms of Hankel functions of imaginary argument and the associated Lommel functions (See Bowhill [1962]).

$$x^2 \frac{d^2\eta_e}{dx^2} + (1-2a_2)x\frac{d\eta_e}{dx} + (a_2^2 - \frac{1}{4}\nu^2)\eta_e - \beta_2 x \eta_e$$

$$= -\alpha_2\left[x^{2/(a_1+1)} - p_m x^{(\alpha+1)/(a_1+1)}\right], \tag{16}$$

where

$$\left.\begin{aligned} a_2 &= \frac{2+a}{2(a+1)(1+a_1)} \\ \nu &= \frac{a}{(a+1)(1+a_1)} \\ \beta_2 &= \frac{\beta_1}{(a_1+1)^2} \\ \alpha_2 &= \frac{\alpha_1}{(a_1+1)^2}\,. \end{aligned}\right\} \tag{17}$$

The solution to the homogeneous portion of (16) is

$$\eta_{ec} = A_1\, x^{a_2} I_\nu (2\sqrt{\beta_2 x}) + A_2\, x^{a_2} I_{-\nu} (2\sqrt{\beta_2 x}). \tag{18}$$

Since very small x corresponds to large z, the Taylor series expansion of (18) enables us to determine that the second solution $I_{-\nu}$ corresponds to the exponential solution

$$A_2 x^{a_2} I_{-\nu}(2\sqrt{\beta_2 x}) \sim A_2 \exp(B/(1+a)z), \tag{19}$$

while the first solution I_ν corresponds to the neutral solution (6). The essential details of this calculation are in Bowhill [1962]. We note that the homogeneous portion of equation (16) includes the recombination term. Now the condition of strict diffusive equilibrium at infinity would correspond to choosing $A_1 = 0$. Since equation (18) is the complete solution to (16) when the ionization rate is zero, say at night, then we see that no peak can exist, since the two solutions in (18) are monotonic functions of ζ. Thus the choice $A_1 = 0$ for one boundary condition is equivalent to placing the entire burden of maintaining a peak in the electron density on the ionizing radiation. Thus the choice $A_1 = 0$ is inconsistent with the observed phenomenon that a peak in ion density exists at night in the absence of ionizing radiation. We note also that, as in the previous models, the condition that the solution tend asymptotically toward diffusive equilibrium is automatically satisfied for any non-zero choice of the arbitrary coefficients since it is equivalent to

$$\lim_{x \to 0} \frac{I_\nu}{I_{-\nu}} = 0 = \lim_{x \to 0} x^{2\nu} . \tag{20}$$

Thus these considerations do not give a physically reasonable definitive choice for the arbitrary coefficients in (18). Our only conclusion is that $A_1 = 0$ is not permissible.

At the other end, by considering the asymptotic expansion for the Hankel functions for large ζ we can determine the nature of the solution (18) near the bottom of our atmosphere. But the functions $I_{\pm\nu}(y)$ diverge exponentially for large values of their arguments, and since y itself is an exponential, then the solution (18) has the asymptotic value

$$\eta_{ec} = \exp\left(2\sqrt{\beta_2}\exp - \left(\frac{\left\{1+a_1\right\}\zeta}{2}\right)\right)\left\{F(\zeta)\right\} \tag{21}$$

where $F(\zeta)$ is a non-vanishing function of ζ. But η_{ec} is the solution to the electron density distribution in the absence of ionizing radiation and including the effects of recombination. To have an exponentially increasing solution in the presence of a recombination mechanism which itself is becoming more and more effective as the altitude decreases is not physically reasonable. Thus the only solution to (18) which gives an acceptable solution below the region of maximum ionization rate is $A_1 = A_2$, so that the solution (18) can be written

$$\eta_{ec} = A_1 x^{a_2} K_\nu \left(2\sqrt{\beta_2 x}\right). \tag{22}$$

This solution does not fit the condition of strict diffusive equilibrium above the peak. Thus there is a second argument that strict diffusive equilibrium cannot be a realistic boundary condition, contrary to the assumptions of Bowhill and others. We note also that equation (22) has a peak in the interior, independent of the fact that there is no ionizing radiation. The recombination alone is sufficient to create a peak in the ion density distribution. (Of course, to maintain a steady state, non-zero electron density distribution with only recombination present, there must be a flux of electrons and ions across one of the boundaries. And we shall see later that there is a flux across the outer boundary.)

Reverting to the inhomogeneous equation (16) the full solution is expressible in terms of Hankel functions and associated Lommel functions (see Watson [1952]). A particular solution N_1 of (16) can be written as

$$\frac{N_1}{x^{a_2}} = a_4 \mathcal{S}_{\mu_4,\,\nu}(y) - a_5 p_m \mathcal{S}_{\mu_5,\,\nu}(y)$$

$$= a_4 \left(s_{\mu_4,\nu}(y) + b_4(-\nu) I_{-\nu}(2\sqrt{\beta_2 x}) + b_4(\nu) I_\nu(2\sqrt{\beta_2 x}) \right)$$

$$- a_5 p_m \left(s_{\mu_5,\,\nu}(y) + b_5(-\nu) I_{-\nu}(2\sqrt{\beta_2 x}) + b_5(\nu) I_\nu(2\sqrt{\beta_2 x}) \right), \tag{23}$$

where

$$a_j = -4\alpha_2 (4\beta_2)^{-\gamma_j} \exp(\gamma_j \pi i)$$

$$\mu_j = 2(\gamma_j) - 1$$

$$b_j(\nu) = -4^{(\gamma_j - 1)} \frac{\left(\frac{\mu_j + \nu - 1}{2}\right)! \cdot \left(\frac{\mu_j - \nu - 1}{2}\right)!}{\sin \pi \nu} \cos\left(\frac{\mu_j + \nu}{2}\right) e^{\pi i \nu/2}$$

$$\gamma_4 = \frac{3a+2}{2(1+a)(1+a_1)}$$

$$\gamma_5 = \frac{2\alpha(a+1)+a}{2(1+a)(1+a_1)}$$

$$y = -2i\sqrt{\beta_2 x}$$

Using asymptotic expansions of the terms in (23) we will be able to identify the behavior of this particular solution above and below the point $\zeta = 0$. The asymptotic expansion for the functions $\mathcal{S}_{\mu,\nu}(y)$ (see Watson [1952]) gives the first term of the series as

$$N_1 \sim \frac{\alpha_2 g_\infty}{\beta_2} \left\{ \exp\left[(a_1 - 1)\zeta\right] - p_m \exp\left[(a_1 - \alpha)\zeta\right] \right\}. \tag{24}$$

This expansion is asymptotic for $\zeta \to -\infty$, that is, below the peak in the ionization rate. The particular solution (24) tends to zero as long as the effective molecular weight of the molecular species involved in the recombination is greater than the effective molecular weight of the ionizable constituent, which condition holds for the ionosphere. We note that the particular solution (24) has only an exponential approach to zero, while the complementary solution (22) goes as exp [exp ζ]. Thus the particular solution will dominate the solution to (16) sufficiently far below the ionization rate peak. We also note that, because of the nature of our expression for the ionizing radiation, the expression (24) will become negative at the point where

$$p_m = \exp\left[(\alpha - 1)\zeta\right]. \tag{25}$$

Thus equation (25) can be used ot obtain a crude approximation to the location of the bottom of the ionosphere. Typical values for the terms involved give a value of $\zeta = -1.7$.

Thus the ion-electron density drops to zero in less than 2 scale heights ($\sim$ 100 km) below the point of peak ionization rate. This is the order of magnitude of answer which is observed. This answer is expected to be only a crude approximation for two reasons. First, the expression for the ionization term is itself a crude model. Secondly, the value -1.7 is obtained by using the only first term of an asymptotic expansion, valid for $\zeta \rightarrow -\infty$. It is of interest to check whether only one term is sufficient.

A sufficient condition that the asymptotic representation is given to a reasonable degree of accuracy by the first term alone is that the second term in the asymptotic series by negligible in comparison with the first. This requirement can be reduced to

$$- (a \;\; + 1)\zeta >> \left\{ \begin{array}{l} \ln\left[\frac{a_1}{\beta_2} \quad (a_1 - \frac{a}{a+1})\right] \\ \\ \ln\left[\frac{a_1 - (\alpha - 1)}{\beta_2} \quad (a_1 - [\alpha - 1] - \frac{a}{1+a})\right] \end{array} \right\} . \tag{26}$$

Using typical values for the F-layer, we see that this inequality holds for values of ζ slightly less than 1. Thus the asymptotic expression is a valid representation for (23) at distances of only one scale height below the peak in the ionizing radiation.

To obtain the behavior of the particular solution (23) above the peak we obtain the Taylor series solution for the functions $S_{\mu,\nu}(y)$, using Watson [1952] again. We see that

$$N_1 \sim \alpha_2 g_\infty \left\{ \frac{\exp[-2\zeta]}{1+\frac{a}{a+1}} - \frac{p_m \exp[-(\alpha+1)\zeta]}{\alpha(\alpha + \frac{a}{1+a})} \right\} + \text{terms in } g_\infty I_{\pm\nu}(y) , \tag{27}$$

The condition for neglecting the next term in the Taylor series is given by

$$y^2 << (\mu + 3)^2 - \nu^2 , \tag{28}$$

which can be written

$$(a_1 + 1)\zeta >> \left\{ \begin{matrix} \ln\left[\dfrac{\beta_2}{(2+a_1)(2+a_1+\frac{a}{a+1})} \right] \\ \ln\left[\dfrac{\beta}{(\alpha+1+a_1)(\alpha+1+a+\frac{a}{a+1})} \right] \end{matrix} \right\} . \tag{29}$$

For the values typical of the ionosphere these inequalities hold again for $\zeta = 1$, and so the Taylor series expression for the particular solution is valid everywhere more than one scale height above the peak of the ionizing radiation. We see from (27) that the particular solution N_1 consists of a term which is essentially proportional to the ionization term plus two diffusive terms. The first represents the contribution to the electron density of the ionizing radiation, without the action of diffusion, and the second two terms represent the redistribution of the electrons by the diffusive and recombination mechanisms. We note that the recombination coefficient β_2 influences the particular solution above the peak only indirectly, in that none of the three terms in the expansion (27) is directly proportional to the recombination term, in contrast to below the peak. However, it would be incorrect to conclude from this, as Bowhill [1962] has done, that the recombination coefficient does not effect the formation of the electron peak (which is above the ionization peak), since, as we have seen, a peak exists in the complementary solution for the electron density in the absence of ionizing radiation, due to the recombination. The recombination coefficient enters indirectly in the coefficients of the second two, diffusive, terms in (27).

Thus the particular solution (23) obeys the one boundary condition we have applied, namely that the electron density vanish sufficiently far below the ionization peak. Thus the solution to (11) is given by the sum of (23) and (22), that is

$$\eta_e = A_1 x^{a_2} K_\nu (2\sqrt{\beta_2 x}) + N_1 (x). \tag{30}$$

The fact that the atmosphere is finite and thus the lower boundary is not at $\zeta = -\infty$ does not alter this lower boundary condition since all other solutions to (11) involve $I_\nu (2\sqrt{\beta_2 x})$ which is exponentially growing. This is a physically impossible situation in the presence of both a decreasing ionization and an increasing recombination.

Thus the solution to the electron density distribution is reduced to determining the constant A_1 in (30). To do so we must decide on another boundary condition. Unfortunately there is no unequivocal choice for this condition. Many authors [eg: Bowhill [1962]] chose diffusive equilibrium. Others [Nisbet [1963]] chose a given flux at a specified height. In this paper we decide to defer the question until we have investigated the solution for the electric field and those for the flow velocities.

IV. The Electric Field and the Flow Velocities

By eliminating the ionization and recombination terms between equations (7) and (8) we can obtain

$$\nabla^2 \varphi = - a \nabla^2 \ln \eta_e . \tag{31}$$

This has the solution

$$E = -a \frac{d}{dz} \ln \eta_e + \frac{A}{z^2} , \tag{32}$$

and the diffusion velocities can be written

$$q_e = - \left[\left\{ E + a \frac{d}{dz} \ln \eta_e \right\} \frac{d\lambda}{2\mu} + \left\{ \psi - E + \frac{d}{dz} \ln \eta_e \right\} \right] \frac{2\mu bH}{n_n / \sigma_{in}} \tag{33}$$

$$q_i = - \left\{ \psi + (1+a) \frac{d}{dz} \ln \eta_e \right\} \frac{2\mu bH}{n_n / \sigma_{in}} \tag{34}$$

where $d\lambda$ is the ratio of ion-neutral to ion-electron collision frequencies. Thus

$$q_e = q_i - \frac{bH}{n_e / \sigma_{ie}} \frac{A}{z^2} . \tag{35}$$

Then from (32) the condition of strict diffusive equilibrium at infinity means that $A = 0$, and thus from (35) the ion and electron velocities would be equal throughout. Equations (33) and (34) show that there is a downward flux of particles from the top of the atmosphere. This can be seen by noting that for static equilibrium, or for strict diffusive equilibrium,

$$\frac{d}{dz} \ln \eta_e = \frac{-1}{(1+a)} \psi$$

and

$$E = \frac{-a}{1+a} \psi ,$$

so that for an ionosphere with a peak (where $\frac{d}{dz} \ln \eta_e$ vanishes) the parentheses in (34) and in (33) are always positive.

We now consider equations (32)-(35) in more detail. If we take as a condition that there be no net current at any one point, then there is no current anywhere, and thus, in (35) and (32),

$$A = 0. \tag{36}$$

Then from (32) we see that

$$E = - a \frac{d}{dz} \ln \eta_e . \tag{37}$$

Thus, at the peak in electron density, the electric field vanishes; below the peak the electric field changes sign, and actually increases the effective gravitational field on the ions (instead of reducing it to half the field as a simple, isothermal, static analysis gives). Crudely, equation (37) says that the electric field, in magnitude, is equal to "a" times the slope of the electron density profile (measured in neutral scale heights). Thus the electric field, below the electron density peak, can be several times the gravitational field.

We note also that there is no inherent requirement in this formulation that A be chosen zero. That is, there is no internal inconsistency in requiring electrical neutrality and still not requiring ambipolar diffusion. This topic has been the subject of several recent papers (Chandra [1964], Kendall [1964]). In a recent paper Comstock [1965b] has pointed out that in the region of low ionization, just below the region under consideration, a vertical current may be necessary to maintain a steady state. If this were so, then by continuity a current would have to exist in this region also. Thus the constant A in (35) would be determined by this current. However, the choice of a non-zero A does not alter any of our above conclusions except in the actual location of the zero electric field. [There would, of course, have to be some mechanism outside the region under consideration to provide a closed path for the return of such a current.] Outside of this consideration there is then no compelling physical boundary condition which forces any one choice for the arbitrary constant A in the solution (32) for the electric field.

Turning to the flow velocities, equation (34) shows that the condition of diffusive equilibrium for the electron density is equivalent to zero ion velocity. Thus the ion flux at any one point is the physically logical condition for determining the other constant A_1 in the electron density solution (30). A common choice is no ion flux at infinity,

$$q_i(\infty) = 0 . \tag{38}$$

This would require that the electron density distribution be in strict diffusive equilibrium at infinity. However, since our solution for the electron density distribution is in terms of a plane parallel coordinate system, a great deal of care must be taken in applying this condition. We proceed as follows: using the Taylor series expansion for our answer, valid some distance above the region $\zeta \approx 0$, we obtain

$$\eta_e \approx [-A_1 + a_4 b_4(\nu) - p_m a_5 b_5(\nu)] \frac{(\beta_2)^{\nu}}{\nu!} \exp[-\zeta]$$

$$+ [A_1 + a_4 b_4(-\nu) - p_m a_5 b_5(-\nu)] \frac{(\beta_2)^{-\nu}}{(-\nu)} \exp\left(\frac{-\zeta}{1+a}\right)$$

$$- \alpha_2 g_\infty \left\{ \frac{\exp(-2\zeta)}{1 + \frac{a}{1+a}} - \frac{p_m \exp(-[\alpha + 1]\zeta)}{\alpha\left(\alpha + \frac{a}{1+a}\right)} \right\} \tag{39}$$

This can be written

$$\eta_e \approx \exp[-\zeta/(1+a)] \Big\{ c_1 + c_2 \exp[-a\zeta/(1+a)]$$

$$-c_3 \exp[-(1+2a)\zeta/(1+a)] + c_4 \exp\left[-\frac{[+a(1+a)\zeta]}{(1+a)}\right] \Big\} \tag{40}$$

For sufficiently large ζ, all of the terms in the bracket, except c_1, vanish, regardless of the values of c_2, c_3, or c_4. However, the fact that all the solutions of (11) are of the form $\exp(-k\zeta)$ is a consequence of the plane-parallel approximation. The solution to the simplified equations in spherical coordinates are of the form $\exp(k/r)$, which form does not vanish for large r. Thus we assume that the solution to the correct problem, in spherical coordinates, would not be in terms of vanishing exponentials.

The boundary condition of strict diffusive equilibrium is given normally by the equation

$$\frac{1}{\eta_e} \frac{d\eta_e}{d\zeta} = \frac{-\psi}{1+a} . \tag{41}$$

Because of this vanishing of the exponential solutions the equation (41) is satisfied for any choice of A_1 in (39). Thus, because of our approximation to plane-parallel coordinates, another way to express the boundary condition must be found. Bowhill [1962], Yonezawa [1958] and others have interpreted it to mean that the coefficient of exp $(-\zeta)$ should vanish. This is equivalent to requiring that at great heights any particles created by the ionizing radiation can diffuse only with the effective molecular weight of the ion-electron plasma. The author sees no physical justification for this distinction.

A more realistic requirement is that the constant A_1 be chosen so that all terms in equation (39) vanish, except the term exp $[-\zeta/1+a)]$, at the point where the ion velocity is to vanish. This really cannot be the point $\zeta = \infty$, since our plane parallel approximation must fail for large values of ζ. Let ζ_m be this point; then

$$A_1 = -a_4 b_4(\nu) + p_m a_5 b_5(\nu) + \alpha_2 f_\infty \, \nu ! \beta_2^{-\nu} \left\{ \frac{\exp(-\zeta_m)}{1 + \frac{a}{1+a}} - \frac{p_m \exp(-\alpha\zeta_m)}{\alpha(\alpha + \frac{a}{1+a})} \right\} \tag{42}$$

is the condition that the ion flux vanish at the point ζ_m.

Then the solution for the electron density distribution is

$$\eta_e = x^{a_2}\left\{-A_1\, K_\nu(2\sqrt{\beta_2 x}) + a_4 \mathcal{S}_{\mu_4,\nu}(y) - p_m a_5 \mathcal{S}_{\mu_5,\nu}(y)\right\} . \tag{43}$$

The ion velocity everywhere is then given by equation (34) and is downward throughout the atmosphere. The electron velocity cannot be determined until the constant A is determined. Until more information is known about the electric field in the atmosphere, we have no way to determine this constant.

V. Conclusions

The electron density distribution obtained here agrees in form with those obtained by other authors, (see Bowhill [1962], and references therein). By including a more realistic electron production term the fit of this model to the photoequilibrium curve at low altitudes is somewhat better. (see Bowhill [1962], figure 2). It is seen that the critical condition, from a theoretical point of view, is the boundary condition at the top of the ionosphere. Fortunately, from a numerical point of view, the production and loss terms at great heights are sufficiently small that the numerical value of the solution is not sensitive to whether diffusive equilibrium is taken to mean that the plasma diffuses strictly at the plasma "mean molecular" weight, or whether, as most people take it, the plasma has no diffusion at the neutral molecular weight.

The analytical form of the electron density in terms of the Hankel functions of imaginary argument shows why there has been difficulty trying to integrate the diffusion equations numerically downward from the top of the ionosphere. Rishbeth and Barron [1960] found a change of 1 part in 10^{-4} in their initial conditions at the top would cause an exponential divergence in the solution at the bottom. Since the Hankel functions in the analytical solution here are basically exponential functions of their arguments below the electron production peak, this is to be expected.

It is seen that the electron density solution is determined by the ion flux at great height, independent of the electron flux. Thus a vertical current is compatible with, and has no first order effect on, the usually obtained electron density. Any current which exists is seen to be directly related to the electric field in the ionosphere. The presence, or absence, of such a current cannot be settled within the framework of the present formulation, but must be taken as a boundary condition.

Even in the absence of any current the electric field in the ionosphere required to maintain charge neutrality is quite different from the usual static approximation. The electric field is seen to be as much as 3-4 times the gravitational field below the F-layer peak and of the opposite sign from the usual static approximation.

Since the electric field in the F-layer is influential in determining the loss of high energy particles trapped on the magnetosphere, further exploration of this point should be of interest.

Acknowledgements

This work was partially supported under NSF Grant GP-2226 with Harvard University and partially by NASA Grant NsG-134-61 with the Pennsylvania State University.

References

Bowhill, S. A. (1962), The Formation of the Daytime Peak of the Ionospheric F2-Layer, J. Atmos. Terr. Phys., 24, 503, (1962); also Pennsylvania State University, Ionosphere Research Laboratory, Report No. 154.

Chandra, S. (1964), Plasma Diffusion in the Ionosphere, J. Atmos. Terr. Phys., 26, 113, (1964).

Comstock, C., (1965a) A Three Fluid Model Ionosphere, Ph.D. Thesis, DEAP Report No. 2, Harvard University, 1965.

Comstock, C., (1965b), A Three Fluid Ionospheric Model, Submitted to J.G.R.

Kendall, P. C., (1964), On the Equations Governing Diffusion in the F2-Layer, J. Atmos. Terr. Phys. 26, 625, (1964).

Nisbet, J. S., (1963), Factors Controlling the Shape of the Upper F Region under Daytime Equilibrium Conditions, J.G.R. 68, 6099, (1963); also Pennsylvania State University, Ionosphere Research Laboratory Report No. 194.

Rishbeth, H. and D. W. Barron (1960), Equilibrium Electron Distribution in the Ionospheric F2-Layer, J. Atmos. Terr. Phys. 18, 234, (1960).

Watson, G. N., (1952), A Treatise on Bessel Functions, Cambridge University Press. (1952)

Yonezawa, T, (1958), On the Influence of Electron-Ion Diffusion Exerted on the Formation of the F2-Layer, J. Rad. Res. Japan, 5, 165, (1958).