ON THE THEORY OF EVOLUTION OF COMPLETELY MIXED STARS WITH MASS LOSS

Richard Stothers

Summary

The consequences of substituting the $\mathfrak{M}$-L-μ (mass-luminosity-mean molecular weight) relation of stellar structure theory in place of the "empirical" $\mathfrak{M}$-L and $\mathfrak{M}$-μ relations are investigated for the Russian theory of evolution of completely mixed stars with mass loss. The mass loss rate becomes a free parameter, but stars still evolve along the main sequence. It is found that by theoretically evolving the initial luminosity function, the observed luminosity functions of early-type galactic clusters may be reasonably well reproduced with a variety of mass loss rates and ages. Hence the luminosity function *per se* does not give definite evidence for the mode of stellar evolution (homogeneous or inhomogeneous), rate of mass loss, or age of a cluster.

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1. Introduction.— According to a theory developed primarily by the Russian astronomers, the upper main sequence on the H-R diagram is a locus of stars undergoing evolution with complete mixing and substantial mass loss. Inhomogeneous evolution would take place in a few, slowly rotating massive stars (to explain the observed Trumpler turnoff in open clusters) and in stars evolved to a sufficiently low mass so that rotation and mass loss are ineffective (to explain the subgiant and giant branches in globular clusters). However, the majority of stars observed today will have evolved <u>down</u> the main sequence from some initial mass lying within the range of an assumed universal initial mass function (see, e.g., Masevich & Parenago 1950, Idlis 1957, Masevich 1959, Fesenkov & Idlis 1959).

The rate of mass loss has been taken to be Fesenkov's (1949) law

$$\frac{d\mathfrak{M}}{dt} = - k L, \qquad (1)$$

and the mass-luminosity relation to be that derived from observations,

[redacted]

$$\frac{L}{L_\odot} = \left(\frac{\mathfrak{M}}{\mathfrak{M}_\odot}\right)^4 . \qquad (2)$$

From the rate of hydrogen depletion in a completely mixed star and a relation between the mass and mean molecular weight, Idlis (1957) was able to derive the value of the constant k. Using this value of k and integrating equation (1), Usher (1963) evolved Idlis's initial luminosity function and found disagreement with the observed luminosity functions of two typical galactic clusters. In particular, an unobserved accumulation of stars was predicted by the theory. Roberts (Henyey 1960) had earlier come to a similar conclusion.

However, the semi-empirical basis on which the relation between mass and mean molecular weight depends is highly uncertain (cf. Section 2). If we drop this explicit relation and, instead, allow k to be a free parameter, stars will still evolve along the main sequence (cf. Section 3). In this paper we shall investigate the consequences of evolution in the unconstrained scheme.

2. $\mathfrak{M}$-μ relation.—Idlis (1957) used a semi-empirical relation between $\mathfrak{M}$ and μ to obtain the mass-loss coefficient k. Although this relation, $\mu \sim \mathfrak{M}^{-0.225}$ (Severny 1954), was

obtained with the help of published abundance analyses of three stars (including the sun), the evidence is generally held to be inconclusive. For instance, Strömgren (1963) finds adequate fits to the observed main sequence using the same "normal" chemical composition in models of B, A, and F stars; in particular, the position of Sirius A is predicted with X = 0.70. From available observational data, Ezer and Cameron (1964) find a value X = 0.74 for the sun; their use of this value gives reasonable results for a model of the present sun constructed at the end of a series of pre-main sequence contraction models. However, observations of B stars yield X = 0.60 (Aller 1961), i.e. an increase of mean molecular weight with increasing mass. This is presumably due to the age difference between the low and high mass stars, and reflects the chemical evolution of the Galaxy.

Ideally, the $\mathfrak{M}$-μ relation should be determined for a single cluster, wherein the stars are roughly coeval. If μ is constant and the cluster luminosity function is not the Idlis function, then evolution does not proceed with complete mixing.

3. Evolution along the main sequence.—A spread in luminosity for stars of the same spectral type is observed on the upper main sequence. Some of the spread is due to observational errors, such as uncertain extinction corrections in regions of patchy absorption. From spectroscopic

evidence, a high percentage of binaries occurs among early-type stars, causing spuriously high luminosities. Different rotations may also cause luminosity differences. Finally, evolutionary effects will result in departures from the initial main sequence.

Since, in addition, our knowledge of stellar masses is very incomplete (Schwarzschild 1958), the specification of unique $\mathfrak{M}$-L and $\mathfrak{M}$-μ relations seems to be unwarranted. Now exact calculations show that, for a wide range of $\mathfrak{M}$ and X, homogeneous main sequence models fall close to the observed main sequence (Blackler 1958, Iben 1963). Furthermore, Gamow (1938) has shown that, without mass loss, completely mixed stars will evolve up the main sequence until the hydrogen has been practically exhausted. Therefore we shall adopt a $\mathfrak{M}$-L-X relation as being more realistic for evolutionary calculations.

The form of this relation must be gotten from theory. Eddington's (1926) mass-luminosity law is strictly applicable only to stars in which $\kappa L(r)/\mathfrak{M}(r)$ is a constant; in theory, therefore, only to gravitationally contracting stars of extremely high mass (and hence high radiation pressure). In this case it will be given by

$$L \sim \frac{\mathfrak{M}}{1 + X}, \tag{3}$$

where electron scattering dominates the opacity. This may be considered as a limiting case. For ordinary massive stars with the same opacity source, the dimensionless envelope parameter C (Schwarzschild & Härm 1958) yields

$$L \sim \frac{\mu^4}{1 + X} \mathfrak{M}^3. \tag{4}$$

For stars in which bound-free absorption processes also occur, combination of Kushwaha's (1957) parameters A and C yields

$$L \sim \frac{\mu^8}{1 + X} \mathfrak{M}^5. \tag{5}$$

The empirical result that $L \sim \mathfrak{M}^4$ along most of the main sequence is confirming evidence of the validity of these essentially dimensional arguments.

Comparison of the results for detailed models of homogeneous stars (derived from a variety of sources) may also be used to determine the mass and chemical composition dependences, whenever the range in X for a given mass is large enough. The adopted dependences are given in Table 1, where λ and ν are defined by

$$L \sim (1 + X)^{-\nu} \mathfrak{M}^{\lambda}. \tag{6}$$

It may be noted that $\mu \sim (1 + X)^{-1.4}$ for small values of Z. The bolometric magnitude listed for each mass in Table 1 refers to an (initial) hydrogen abundance of about 0.7 (Henyey, LeLevier, & Levée 1959).

4. Basic Equations.—The rate of hydrogen depletion in a completely mixed star is given by

$$\frac{d(X\,\mathfrak{M})}{dt} = -\frac{L}{E} - \xi X, \tag{7}$$

where $E = 6.0 \times 10^{18}$ erg/gm and ξ is the rate of mass loss in the form of corpuscular radiation. The full rate of mass loss is

$$\frac{d\,\mathfrak{M}}{dt} = -\xi - \frac{L}{c^2}. \tag{8}$$

If ξ becomes as small as L/c^2 (the loss due to radiant energy), then $d\,\mathfrak{M}/dt$ may, as usual, be neglected on an evolutionary time scale. Therefore we omit L/c^2 entirely and obtain from equations (7) and (8)

$$\frac{dX}{dt} = -\frac{L}{E\,\mathfrak{M}}. \tag{9}$$

The form of ξ will be assumed to have a power-dependence on L:

$$\frac{d\mathfrak{m}}{dt} = - k L^{\iota}. \tag{10}$$

Finally, the mass-luminosity law may be written generally as

$$\frac{L}{L_o} = \left(\frac{1 + X}{1 + X_o}\right)^{-\nu} \left(\frac{\mathfrak{m}}{\mathfrak{m}_o}\right)^{\lambda}, \tag{11}$$

where a zero subscript refers to the initial epoch.

Equations (9), (10), and (11), together with the initial conditions

$$\text{at } t = 0: \quad \mathfrak{m} = \mathfrak{m}_o, \ L = L_o, \ X = X_o, \tag{12}$$

determine the evolution of the star completely.

5. Solutions.—Let us introduce the following non-dimensional variables:

$$m = \frac{\mathfrak{m}}{\mathfrak{m}_o}, \ \ell = \frac{L}{L_o}, \ x = \frac{1 + X}{1 + X_o}, \ \tau = t \frac{L_o}{(1 + X_o) E \mathfrak{m}_o}. \tag{13}$$

Then the basic equations become

$$\frac{dx}{d\tau} = - \frac{\ell}{m}, \ \frac{dm}{d\tau} = - K \ell^{\iota}, \ \ell = x^{-\nu} m^{\lambda}, \tag{14}$$

with $K = k\,E(1 + X_o)L_o^{\,\iota-1}$ and

$$\text{at } \tau = 0: \quad m = 1, \; \ell = 1, \; x = 1. \tag{15}$$

In general, equations (14) may be integrated analytically for m, although a simple quadrature is in some cases necessary to obtain τ. We shall here write down the general solutions.

<u>Case</u> $\iota \neq 1$

$$\ell = x^{-\nu}\, m^{\lambda} \tag{16}$$

$$m = \left[1 + K \frac{\lambda(1-\iota)}{\nu(1-\iota)+1} \left\{x^{\nu(1-\iota)+1} - 1\right\}\right]^{1/\lambda(1-\iota)} \tag{17}$$

$$\tau' = \int_x^1 x^{\nu} \left[1 + K \frac{\lambda(1-\iota)}{\nu(1-\iota)+1} \left\{x^{\nu(1-\iota)+1} - 1\right\}\right]^{(\lambda-1)/\lambda(\iota-1)} dx \tag{18}$$

<u>Case</u> $\iota = 1$

$$\ell = x^{-\nu}\, e^{\lambda K(x - 1)} \tag{19}$$

$$m = e^{K(x - 1)} \tag{20}$$

$$\tau = \frac{1}{\nu + 1}\left(1 - x^{\nu + 1}\right) \qquad (\lambda = 1) \tag{21}$$

or

$$\tau = \sum_{i=0}^{\nu} \frac{\nu!}{(\nu-i)!} \frac{1}{\{(\lambda-1)K\}^{i+1}} \left[x^{\nu-i}\, e^{(\lambda-1)K(1-x)} - 1\right] \quad (\lambda>1) \tag{22}$$

for integral values of ν.

If no mass loss takes place $(k = 0)$,

$$\ell = x^{-\nu}, \; m = 1, \; \tau = \frac{1}{\nu + 1}\left(1 - x^{\nu+1}\right). \qquad (23)$$

This time scale is the same as for $L \sim \mathfrak{M}$ $(\lambda = 1)$, because in both cases the mean rate of energy generation, $L/\mathfrak{M}$, depends only on X.

The case considered by Usher (1963) corresponds formally to the neglect of changing chemical composition $(\nu = 0)$,

$$x = 1, \; \ell = m^{\lambda}, \; \tau = \frac{1}{K(\lambda-1)}\left(m^{1-\lambda} - 1\right), \qquad (24)$$

since, under the assumption of equation (2), the mass dependence of X does not affect τ and need not be specified except to determine k.

If the mass loss is extensive enough, the luminosity will initially decrease. The minimum value of K for which the luminosity never increases is

$$K = \frac{\nu}{\lambda} x^{-1}. \qquad (25)$$

Since $x \geq 0.5$, we have that $K \gtrsim 2\nu/\lambda$ for L never to rise during hydrogen-burning. However, if $K < \nu/\lambda x$, L will return to its initial value when x is given by

$$\ln x = \frac{\lambda}{\nu} K(x - 1). \qquad (26)$$

6. <u>Initial luminosity function</u>.— We have adopted Idlis's (1957) initial luminosity function Ψ_{bol}, which exhibits a large maximum near $M_{bol} = 0$ and a smaller maximum near $M_{bol} = -4.5$. This function agrees reasonably well with functions derived by Masevich (1956) and by Salpeter (1955), in the treatment which Idlis gives them. However, Usher (1963) found that Sandage's (1957a) recomputation of the Salpeter function predicted too few of the brightest stars. The reason seems to be not that Sandage's function differs somewhat from Salpeter's original function, but that the evolutionary theory used to obtain Ψ_{bol} differs for Sandage's (and Salpeter's) function and for Idlis's rederived Salpeter function.

Idlis also compares his Ψ_{bol} with observations of the Cygnus and Orion associations. These comparisons and others made by Usher with data from Trumpler's unpublished catalogue of galactic clusters give reasonably good agreement with the Idlis function. Walker's (1956, 1957, 1961) extensive work on extremely young clusters offers, in particular, further evidence of the secondary maximum. Van den Bergh's (1957) luminosity function does not show the secondary maximum, but he smoothed his data considerably and investigated mainly clusters which contain stars later than B5 and which show some evidence of evolution. The raw observed data of van den Bergh and of Sandage, however, do show a secondary maximum at $M_V = -2.0$ to -2.5. Finally, the work of these two authors demonstrates the universality of the initial luminosity function among existing galactic clusters.

7. Evolution of initial luminosity function.—The consequences of evolving the initial Idlis function with the data given in Table 1 may be seen on Figure 1. (It was assumed that stars disappear from the main sequence when $X < 0.1$.) Results are shown for values of $k \times 10^{18}$ equal to 0, 0.22, and 1.0 gm/erg, and for $k \times 10^{18}/L_{13}$ equal to 1.0 gm-sec/erg^2, where L_{13} is the initial luminosity of a star of 13 $\mathfrak{M}_{\odot}$.

The first case corresponds to homogeneous evolution without mass loss. The initial rapid brightening of the more massive stars is evident as hydrogen becomes quickly consumed. Their subsequent disappearance from the main sequence occurs simultaneously with the slow brightening of the less massive stars. As a result, after 5×10^8 years a central accumulation of stars appears around $M_{bol} = -3$.

The choice of $k = 0.22 \times 10^{-18}$ for the second case was directed by a desire to cause an initial decrease in L and then an increase back to L_o at half the initial hydrogen content (cf. equation (26)). However, it should be noted that this will not occur on the same time scale for stars of different masses. We have adopted Fesenkov's law (equation (1)) for the form of the mass loss rate. From the first case without mass loss, the most massive stars begin to disappear at 2.5×10^7 years. Mass loss prolongs their lifetime up to 5×10^7 years, but during this time they will have dimmed and brightened successively. Consequently the luminosity function shows no apparent change until their hydrogen is completely exhausted. Even smaller amounts of dimming and brightening account for the stationary behavior of the low-mass stars. Because of the coarseness of the adopted Ψ_{bol}, it was impossible to find a value of k that would make Φ_{bol} actually show an initial decrease, subsequent increase, and final decrease at high luminosities.

Idlis's (1957) semi-empirical value of $k = 1 \times 10^{-18}$ corresponds to a mass loss which will dominate the evolution. Adopting Fesenkov's law for the form of the loss rate, we obtain the results for the third case in Figure 1. On account of such an extensive loss, hydrogen depletion in the most massive stars is only fifty per cent even after 5×10^{8} years. The result is that the luminosity function loses weight continually at the brighter magnitudes, while the secondary maximum gains and moves to fainter magnitudes. By 5×10^{7} years the accumulation toward $M_{bol} = 0$ has eliminated the secondary maximum entirely. A new minimum appearing after 1×10^{8} years is caused by the accelerating evolution of stars initially at $M_{bol} = 0$. These results are in many ways similar to those derived by Usher (1963) using only equations (1) and (2). The adoption of varying λ and ν, however, does cause the temporary disappearance of the central minimum in the luminosity function.

The fourth case in Figure 1 illustrates the choice of a mass loss rate proportional to L^2. The numerical coefficient corresponds to that used in the previous case but is normalized to the initial luminosity of the most massive stars (13 $\mathfrak{M}_\odot$). The effect is to deplete significantly only the mass of the more massive stars and hence to lower their luminosities. By 5×10^8 years their hydrogen has been exhausted, but the onset of hydrogen depletion in the stars of lower mass begins to increase these luminosities somewhat; a shift of the maximum in the luminosity function from $M_{bol} = 0$ to $M_{bol} = -2$ is the result.

Idlis's rate of mass loss corresponds to about 10^{-6} $\mathfrak{M}_\odot$/year for the most massive stars evolving at their maximum (initial) luminosity. Underhill's (1960) value of 10^{-7} $\mathfrak{M}_\odot$/year derived from observations of Be shell stars is probably a very generous estimate. Hence Idlis's rate may be taken as an upper limit.

8. Comparison with observed luminosity functions.— Data for four galactic clusters (h Persei, the Pleiades, Praesepe, and the Hyades) were taken from Sandage (1957a). For consistency with the adopted Ψ_{bol}, bolometric corrections were based on Idlis's work, as follows: from his Figure 1 the relation between spectrum and M_{bol} may be read; then Parenago's relation between spectrum and B.C.

(Zonn & Rudnicki 1959) gives us the required M_V-B.C. relation. For $M_V < -4$, the bolometric correction was assumed to be -3.

Figure 2 shoes normalized luminosity functions for the four galactic clusters down to $M_V \approx +2$, with the omission of two extremely bright stars in h Persei. Comparison of the luminosity function of h Persei with Figure 1 shows reasonable agreement with almost any slightly evolved theoretical function. Indeed, even on the basis of inhomogeneous evolution Hayashi & Cameron (1962) derive the young age of $< 2 \times 10^7$ years. However, the tertiary maximum around $M_{bol} = -7$ suggest inhomogeneous evolution for at least some of the stars.

The luminosity function of the Pleiades may be reasonably well reproduced by the cases $0.5 - 1 \times 10^8$ years for $k = 0$ (no mass loss) or by the case 1×10^7 years for $k = 10^{-18}$ (maximum mass loss). The former age is, of course, close to that given directly by the assumption of inhomogeneous evolution (Sandage 1957b), since stars do not move far from the main sequence until the central hydrogen content is low, anyway.

Praesepe and the Hyades show considerable evidence of evolution, and on the mass loss theory ($k = 10^{-18}$) must have ages of 5×10^7 and 1×10^8 years, respectively. However, in order for all the massive stars to have evolved away and for stars of lower mass to have brightened to $M_{bol} = 0$, the ages of these two clusters must be in excess of 5×10^8 years in the case of no mass loss ($k = 0$). Sandage (1957b) gives $\sim 1 \times 10^9$ years for these clusters on the inhomogeneous theory.

We conclude that cluster luminosity functions per se may not be used to decide for or against the theory of homogeneous evolution with mass loss, since any observed function may be reasonably well reproduced on this theory. Moreover, it does not seem possible to determine k and the age uniquely, by using such functions. On the other hand, the theory of inhomogeneous evolution without mass loss makes definite predictions of the luminosity function. Good evidence exists that this theory predicts correctly (Hayashi & Cameron 1962). Further observational arguments may also be raised in its favor (Stothers 1963b).

Apart from an effectively negligible observed rate of mass loss (Stothers 1963a), two other direct observational tests may be made of the Russian theory. A $\mathfrak{M}$-μ

relation should exist among the members of a star cluster, and improved cluster expansion ages should yield definitive ages, which then narrow considerably the choice of comparative theoretical luminosity functions.

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Goddard Institute for Space Studies,
National Aeronautics and Space Administration,
New York, New York:

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TABLE 1

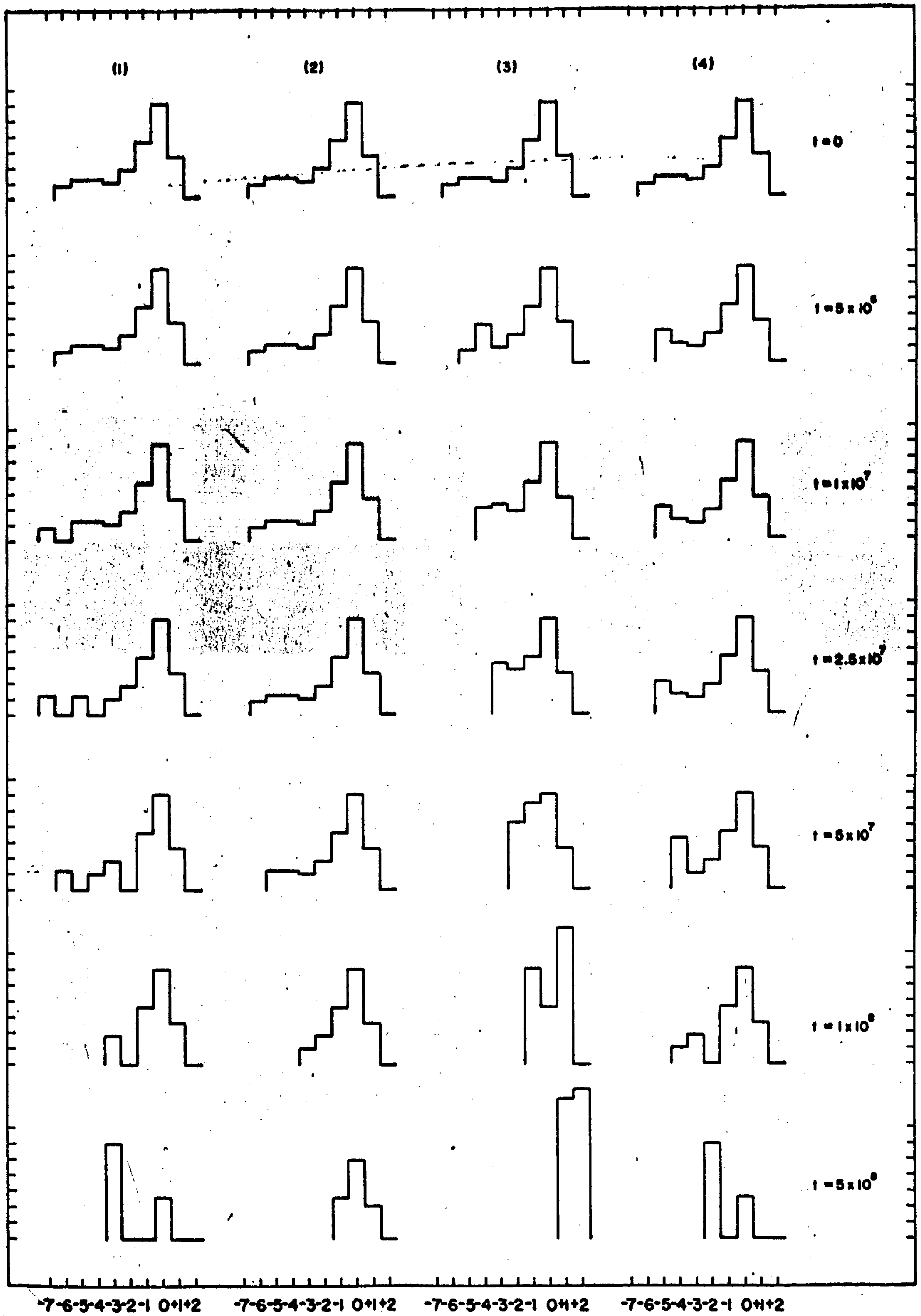
Adopted initial values

M_{bol}	$\mathfrak{M}/\mathfrak{M}_\odot$	λ	ν	Ψ_{bol}
-6	13	3	6	0.04
-5	10	3	6	0.06
-4	7.5	3	6	0.06
-3	5.5	3.5	7	0.05
-2	4.5	3.5	7	0.09
-1	3.5	3.5	7	0.18
0	2.5	4	9	0.30
+1	2.0	4	9	0.13
+2	1.7	4	9	0.00

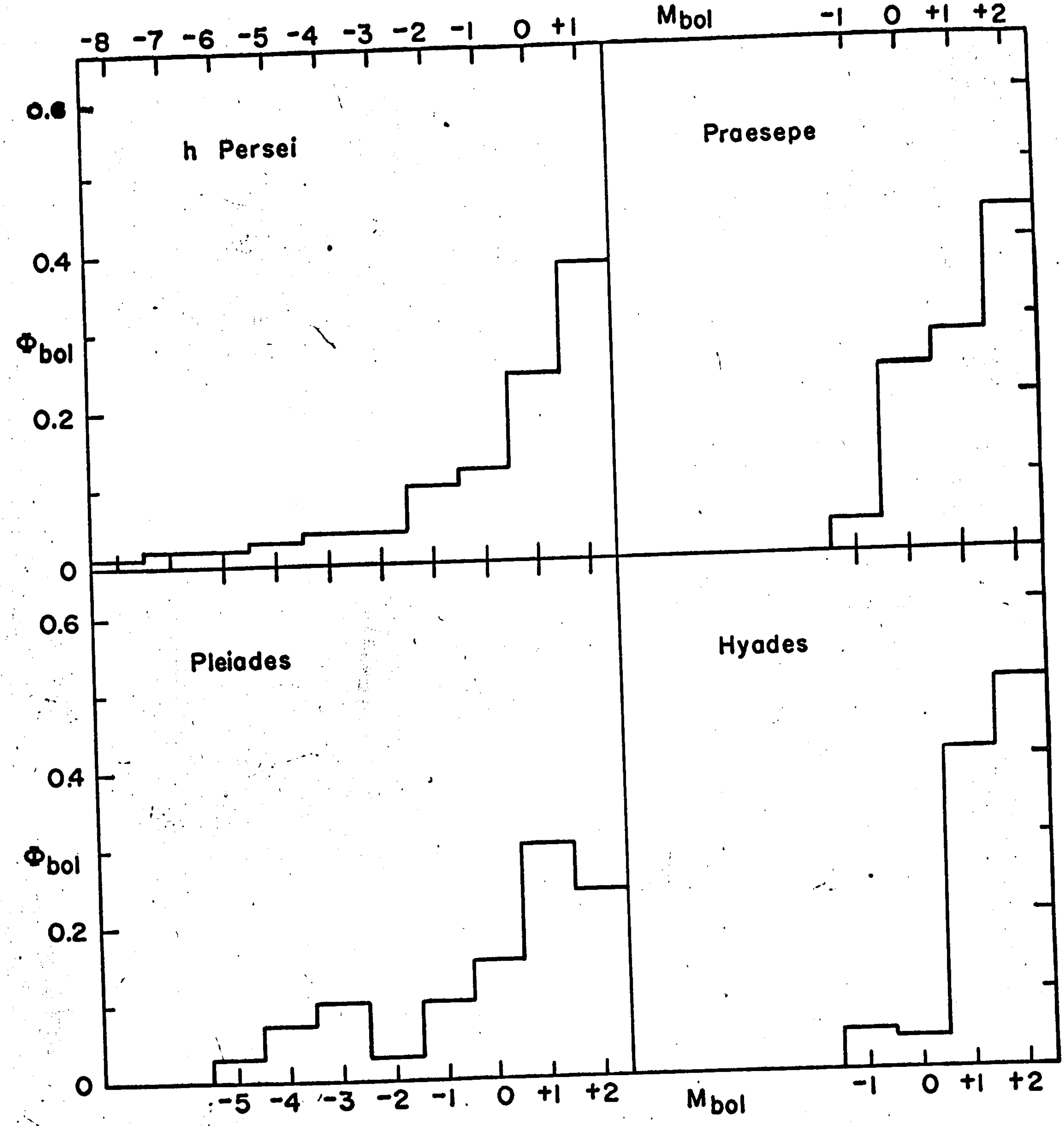
FIGURE CAPTIONS

Fig. 1. — Evolution of Idlis's initial luminosity function for the following values of the mass loss parameters: $\iota = 1$, (1) $k = 0$, (2) $k = 0.22 \times 10^{-18}$, (3) $k = 1 \times 10^{-18}$; and $\iota = 2$, (4) $k = 1 \times 10^{-18}\ L_{13}$. Ordinate intervals are equal to 0.05. Time, t, is given in years.

Fig. 2. — Normalized luminosity functions of four galactic clusters.



(1)
(2)
(3)
(4)
t = 0
t = 5 x 10^6
t = 1 x 10^7
t = 2.5 x 10^7
t = 5 x 10^7
t = 1 x 10^8
t = 5 x 10^8
-7-6-5-4-3-2-1 0+1+2
-7-6-5-4-3-2-1 0+1+2
-7-6-5-4-3-2-1 0+1+2
-7-6-5-4-3-2-1 0+1+2



Mbol
-8 -7 -6 -5 -4 -3 -2 -1 0 +1
-1 0 +1 +2
0.6
0.4
0.2
0
Φbol
h Persei
Praesepe
0.6
0.4
0.2
0
Φbol
Pleiades
Hyades
-5 -4 -3 -2 -1 0 +1 +2
Mbol
-1 0 +1 +2