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INTERACTION OF STRUCTURE AND LIQUID IN THE SOUND SUPPRESSOR SYSTEM

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DEFINITION OF SYMBOLS

Symbol	Definition
$x, y, z,$	Cartesian coordinate system
y_o	Excitation amplitude
Ω	Forcing circular frequency
ω_n	Natural circular frequency
t	Time
a	Container length
h	Fluid height
Φ	Velocity potential of liquid
g	Gravity constant
$\bar{x}(y, t)$	Free fluid surface elevation
p	Pressure distribution of liquid
$p_{y=o}$	Pressure of liquid at container wall $y = 0$
$p_{y=a}$	Pressure of liquid at container wall $y = a$
p_{bottom}	Pressure of liquid at container bottom $x = -h$
ρ	Liquid mass density
$\eta_n = \frac{\Omega}{\omega_n}$	Ratio of forcing frequency to natural frequency of n^{th} liquid mode
F_y	Fluid force in y-direction
α	Stiffness parameter

DEFINITION OF SYMBOLS (Cont'd)

Symbol	Definition
M_z	Moment of liquid about axis through center of gravity of undisturbed liquid
u, v	Velocity of liquid in x and y-direction respectively
m_n	Mass of n^{th} sloshing mode
m_o	Non-sloshing mass
h_n	Location of n^{th} sloshing mass (mass of n^{th} mode)
h_o	Location of non-sloshing mass
y(t)	Tank displacement
y_n	Displacement of n^{th} sloshing mass, m_n, with respect to the container wall
I_o	Moment of inertia of non-sloshing mass
T	Kinetic energy
V	Potential energy
k_n	Spring stiffness of spring attached to container wall and n^{th} sloshing mass, m_n
m	Total liquid mass, also first sloshing mass in the structural analysis
EI_y	Stiffness of one beam pair in y-direction
EI_z	Stiffness of one beam pair in z-direction
k_{yn}	Stiffness of n^{th} slosh spring in y-direction

DEFINITION OF SYMBOLS (Concluded)

Symbol	Definition
k_{zn}	Stiffness of n^{th} slosh spring in z-direction
$M = m_c + m_o + \frac{17}{35} m_{ss}$	Mass of y_1-degrees-of-freedom
ℓ_o	Distance of M from grade
$\delta_{iy_k\, z_\lambda}$	Influence coefficients
μ	Number of pairs of beams on one short side of the sound suppressor system
ν	Number of pairs of beams on one long side of the sound suppressor system
ω_1, ω_2	Natural frequencies of the coupled system structure-liquid
y_1, y_2	Response amplitude to harmonic excitation in y-direction
z_1, z_2	Response amplitude to harmonic excitation in z-direction
F_o	Wind force amplitude
D	Dissipation function
c_n	Damping coefficient of n^{th} sloshing mode
ζn	Damping factor of n^{th} sloshing mode
g_{ss}	Structural damping of sound suppressor structural system

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INTERACTION OF STRUCTURE AND LIQUID IN THE SOUND SUPPRESSOR SYSTEM

SUMMARY

An analysis of the interaction of structure and liquid of the Saturn V sound suppressor system has been performed. For this reason the liquid response in a rectangular container to translatory excitation has been determined and was described by a mechanical analogon. It was found that more than eighty percent of the liquid mass in the water reservoir sloshes for the fundamental sloshing mode. The natural frequencies of the liquid are extremely low; the fundamental frequency is in the vicinity of 0.2-0.3 or 0.4-0.5 rad/sec depending on the orientation of the excitation. An analysis of the interaction of the liquid and the sound suppressor structure revealed rather low coupled frequencies of 0.22 and 0.74 rad/sec in longitudinal direction (y-direction) and about 0.5 and 6.7 rad/sec in the cross direction (z-direction). It was found that an increase of stiffness does not pay off. Because the cross direction offers the largest structural area, the response of the system to some winds has been treated. Liquid and structural amplitudes are presented for a sinusoidal gust wind of 13 m/sec amplitude, for rectangular pulses of various durations as well as for exponentially decaying pulses of different decay magnitude.

SECTION I. INTRODUCTION

Through model tests employing live rocket engines it has been determined that the most reliable and effective means to suppress the high-intensity low-frequency noise generated by large space vehicle booster engines during test firings is the injection of large quantities of water near the engine nozzle exit. This reduces the kinetic energy of the high velocity gases. A reservoir of about five million gallon capacity is necessary to supply the water for the suppression of sound of the Saturn V rocket engines. The water must be supplied just below the elevation of the engine exit plane which is almost 100 feet above grade (Fig. 1).

The operation of the sound suppressor or some wind forces on the total system may cause destructive forces or uncontrollable conditions which could lead to the malfunction of the sound suppressor or even to the destruction of the total system. It is, therefore, necessary to investigate the motion of the liquid in the container and its coupling with the elastic support structure to reveal the integrity of the intended design and make recommendations for a final enhanced system.

In the following, the response of the liquid in the container due to translatory excitation is determined and is described as a mechanical analogon. From this investigation it will be learned how many of the liquid vibration modes will have to be included in the overall structural analysis. After the equations of motion of the system have been derived, the coupled frequencies will be determined for various stiffnesses of the support structure. Forced undamped and damped harmonic response of the system is studied and finally the response of the system due to various wind inputs is derived.

SECTION II. RESPONSE OF LIQUID TO TRANSLATORY EXCITATION

For the structural analysis of the sound suppressor system the response of the liquid in the reservoir due to some outside excitation has to be known. It is also important to know its natural frequencies. Because of the system design only translatory excitation of the liquid container is probable. For this reason, investigation is restricted to this type of excitation for a rectangular container of infinite width (Fig. 2). If the liquid is considered incompressible, irrotational and nonviscous, the velocity of the liquid can be described by the gradient of a velocity potential, which because of the continuity equation must be a solution of the Laplace equation

$$\nabla^2\Phi = 0 \qquad (2.1)$$

For translatory harmonic excitation in y-direction of the form $y(t) = y_o e^{i\Omega t}$, the boundary conditions are

$$\frac{\partial \Phi}{\partial y} = y_o i\Omega e^{i\Omega t} \qquad (2.2)$$

at the side walls y = 0 and a

$$\frac{\partial \Phi}{\partial x} = 0 \tag{2.3}$$

at the container bottom x = -h. The free fluid surface condition is given by the kinematic and dynamic conditions, expressing that the normal velocity of a fluid particle at the free fluid surface is equal to the normal velocity of the free liquid surface, and that the pressure at the free fluid surface is equal to the ambient gas pressure, i.e., $p = p_o$. The dynamic condition is obtained from the linearized unsteady Bernoulli equation and together with the linearized kinematic condition, yields the free fluid surface condition

$$\frac{\partial^2 \Phi}{\partial t^2} + g \frac{\partial \Phi}{\partial x} = 0 \tag{2.4}$$

at the free fluid surface x = 0. A transformation

$$\Phi (y, x, t) = \left[\phi(y, x) + i\Omega y_o y \right] e^{i\Omega t} \tag{2.5}$$

yields homogeneous wall boundary conditions. The problem that has to be solved now, therefore, is the Laplace equation $\nabla^2 \phi = 0$ for the disturbance potential ϕ with homogeneous wall boundary conditions.

$$\frac{\partial \phi}{\partial y} = 0 \tag{2.6}$$

at y = 0 and a

$$\frac{\partial \phi}{\partial x} = 0 \tag{2.7}$$

at x = -h and the transformed free surface condition

$$g \frac{\partial \phi}{\partial x} - \Omega^2 \phi = i\Omega^3 y_o y \tag{2.8}$$

at $x = 0$.

A solution satisfying the Laplace equation and the wall boundary condition (2.6 and 2.7) is given by

$$\phi(y, x) = \sum_{n=o}^{\infty} A_n \frac{\cosh\left[\frac{n\pi}{a}(x+h)\right]}{\cosh\left[\frac{n\pi h}{a}\right]} \cos\left(\frac{n\pi}{a} y\right)$$

where the values A_n are unknown coefficients that shall be determined from the free fluid surface condition (2.8). Expanding y into a Fourier cosine series

$$y = \frac{a}{2} - \frac{4a}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos\left[\frac{2n-1}{a}\pi y\right]}{(2n-1)^2}$$

the expression (2.8) yields

$$\sum_{n=o}^{\infty} A_n \left\{\frac{n\pi}{a} \tanh\left(\frac{n\pi h}{a}\right) - \Omega^2\right\} \cos\left(\frac{n\pi}{a} y\right) = i\Omega^3 y_o \left\{\frac{a}{2} - \frac{4a}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos\left[\frac{2n-1}{a}\pi y\right]}{(2n-1)^2}\right\}$$

from which with the square of the circular natural frequency (see Table 1)

$$\omega_n^2 = g \frac{n\pi}{a} \tanh\left(\frac{n\pi h}{a}\right) ; \qquad n = 1, 2, \ldots \tag{2.9}$$

(This is obtained for free oscillaticns of the liquid where y_o is identically zero. The equation [2.8] is, therefore, satisfied for $y_o = 0$ and $\Omega^2 = \omega^2 n(\omega_n^2)$, as can be seen from the last equation before equation [2.9]).

one obtains for the unknown values A_n the expressions:

$$A_o = iy_o \Omega \frac{a}{2} ; \quad A_{2n} = 0 \quad \text{and} \quad A_{2n-1} = \frac{4i\Omega^3 y_o a}{\pi^2 (2n-1)^2 (\Omega^2 - \omega_{2n-1}^2)} .$$

The disturbance potential is, therefore,

$$\phi(y, x) = -iy_o \Omega a \left\{ \frac{1}{2} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\Omega^2 \cosh\left[\frac{(2n-1)\pi}{a}(x+h)\right] \cos\left[\frac{(2n-1)\pi}{a} y\right]}{(\Omega^2 - \omega_{2n-1}^2)\,(2n-1)^2 \cosh\left[\frac{(2n-1)\pi}{a} h\right]} \right\}$$

and the velocity potential yields

$$\Phi(y, x, t) = i\Omega y_o e^{i\Omega t}$$

$$\left\{ (y - \frac{1}{2}) + \frac{4a}{\pi^2} \sum_{n=1}^{\infty} \frac{\eta_{2n-1}^2}{\left(\eta_{2n-1}^2 - 1\right)} \cdot \frac{\cosh\left[\frac{(2n-1)\pi}{a}(x+h)\right]}{\cosh\left[\frac{(2n-1)\pi}{a} h\right]} \cos\left[\frac{(2n-1)\pi}{a} y\right] \right\}. \tag{2.10}$$

The free fluid surface displacement, the pressure distribution and the velocity distribution, as well as the liquid force and moment can be determined from the velocity potential by differentiations and integrations. The free liquid surface displacement measured from the quiescent fluid surface is given by

$$\bar{x}(y, t) = \frac{\Omega^2}{g} y_o e^{i\Omega t} \left[\left(y - \frac{a}{2}\right) + \frac{4a}{\pi^2} \sum_{n=1}^{\infty} \frac{\eta_{2n-1}^2}{(\eta_{2n-1}^2 - 1)} \frac{\cos\left[\frac{(2n-1)\pi}{a} y\right]}{(2n-1)^2} \right]. \tag{2.11}$$

The pressure distribution $p = -\rho \partial\Phi/\partial t - \rho g x$ is given by

$$p = \rho\Omega^2 y_o e^{i\Omega t} \left\{ (y - \frac{a}{2} - \frac{4a}{\pi^2} \sum_{n=1}^{\infty} \frac{\cosh\left[\frac{(2n-1)\pi}{a}(x+h)\right] \eta_{2n-1}^2}{(2n-1)^2 \cosh\left[\frac{(2n-1)\pi}{a} h\right] (1 - \eta_{2n-1}^2)} \right.$$

$$\left. \cos\left[\frac{(2n-1)\pi}{a} y\right] \right\} - \rho g x .$$

At the tank walls y = o and y = a the pressure yields

$$p_{y=o} = -\rho\Omega^2 y_o e^{i\Omega t}\left\{\frac{a}{2} + \frac{4a}{\pi^2}\sum_{n=1}^{\infty} \frac{\cosh\left[\frac{(2n-1)\pi}{a}(x+h)\right]\eta^2_{2n-1}}{(2n-1)^2\cosh\left[\frac{(2n-1)\pi}{a}h\right](1-\eta^2_{2n-1})}\right\}$$

$$- \rho g x \tag{2.12}$$

and

$$p_{y=a} = \rho\Omega^2 y_o e^{i\Omega t}\left\{\frac{a}{2} + \frac{4a}{\pi^2}\sum_{n=1}^{\infty} \frac{\cosh\left[\frac{(2n-1)\pi}{a}(x+h)\right]\eta^2_{2n-1}}{(2n-1)^2\cosh\left[\frac{(2n-1)\pi}{a}h\right](1-\eta^2_{2n-1})}\right\}$$

$$- \rho g x \; . \tag{2.13}$$

At the container bottom x = -h the pressure distribution is given by

$$p_{bottom} = \rho\Omega^2 y_o e^{i\Omega t}\left\{\left(y - \frac{a}{2}\right) - \frac{4a}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos\left[\frac{(2n-1)\pi}{a}y\right]\eta^2_{2n-1}}{(2n-1)^2\cosh\left[\frac{(2n-1)\pi}{a}h\right](1-\eta^2_{2n-1})}\right\} + \rho g h \; . \tag{2.14}$$

Integration of the pressure components yields the fluid force and moment. The force in y-direction (i.e., the force per unit width) is

$$F_y = \int_{-h}^{o} (p_{y=a} - p_{y=o})\, dx$$

which yields with $m = \rho a h$ as the liquid mass per unit width the expression

$$F_y = m\Omega^2 y_o e^{i\Omega t}\left\{1 + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\tanh\left[\frac{(2n-1)\pi}{a}h\right]\eta_{2n-1}^2}{(2n-1)^3\,\pi\left(\frac{h}{a}\right)\,(1-\eta_{2n-1}^2)}\right\}. \tag{2.15}$$

The moment of the liquid about the center of gravity of the undisturbed fluid is given by

$$M_z = -\int_{-h}^{o}(p_{y=a} - p_{y=o})\left(x + \frac{h}{2}\right)dx - \int_{o}^{a} p_{bottom}\left(y - \frac{a}{2}\right)dy$$

and yields the expression

$$M_z = -ma\Omega^2 y_o e^{i\Omega t}\left[\frac{1}{12\left(\frac{h}{a}\right)}\right.$$

$$+\frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{\tanh\left(\frac{[2n-1]\pi}{a}h\right)+\frac{2}{(\pi h/a)(2n-1)}\left[\frac{2}{\cosh[(2n-1)\pi h/a]}-1\right]}{(2n-1)^3}$$

$$\left.\cdot\frac{\eta_{2n-1}^2}{(1-\eta_{2n-1}^2)}\right]. \tag{2.16}$$

The velocity distribution is given by

$$v = \frac{\partial\Phi}{\partial y} = i\Omega y_o e^{i\Omega t}$$

$$\left\{1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\eta_{2n-1}^2\cosh\frac{(2n-1)\pi}{a}(x+h)}{(1-\eta_{2n-1}^2)(2n-1)\cosh\left[\frac{(2n-1)\pi}{a}h\right]}\sin\left[\frac{(2n-1)\pi}{a}y\right]\right\}$$

which represents the velocity in y-direction, and

$$u = \frac{\partial \Phi}{\partial x} = -\frac{4i\Omega}{\pi} y_o e^{i\Omega t} \sum_{n=1}^{\infty} \frac{\eta^2_{2n-1} \sinh\left[\frac{(2n-1)\pi}{a}(x+h)\right]}{(1-\eta^2_{2n-1})(2n-1)\cosh\left[\frac{(2n-1)\pi}{a}h\right]} \cos\left[\frac{(2n-1)\pi}{a}y\right]$$

which is the velocity in x-direction.

From the results of this section the following is concluded: The natural frequency (2. 8) of the liquid is inversely proportional to the length of the reservoir, indicating that these values shall be very small. Since the liquid height ratio h/a is very small, the effect of the hyperbolic tangent function is pronounced for many of the lower vibration modes, and approaches with increasing mode number n the value of unity. The first term in the free fluid surface elevation (2. 10) is the surface plane elevation caused by the change of the acceleration vector during motion. The infinite series accounts for the liquid waves on the surface. In the fluid force expression (2. 15) the first term represents the inertial force of the liquid. In or close to resonance the liquid force can be a multiple of the inertial force, therefore endangering the total sound suppressor system. In the expression (2. 16) the first term of the liquid moment is identified as the moment caused by the shift of the center of gravity for a planar free fluid surface. In the velocity distribution v the first term represents the container motion. Omission of this term yields the velocity distribution in the container.

As can be seen from the previous results, the liquid motion in the container exhibits singularities at the resonances. However, since there is a little damping in the system, (due to internal and wall friction), a damping value should be introduced to limit the amplitudes at the resonances. This will be performed with the help of a mechanical analogy, which also will provide a simple method for the description of the liquid motion, and reduce the infinite number of degrees of freedom of the liquid system to a finite number that is pertinent for the dynamic behavior of the liquid.

SECTION III. MECHANICAL ANALOGY OF LIQUID BEHAVIOR

For the structural analysis a simple mechanical analogon for the description of the liquid motion is needed. This mechanical model, by comparison with the liquid theory, will then present the amount of sloshing modal masses and provide sufficient information of the number of liquid vibration modes that must be considered in the structural analysis. In a vibrating container the liquid oscillates in the close proximity of the free surface. Part of the liquid in the lower part of the container follows the motion of the container like a rigid body. To describe this motion with a simple mechanical analogy a spring-mass-system is employed. The sloshing mass m_n of the n^{th} vibration mode is connected with two springs of stiffness $k_{n/2}$ to the container walls y = o and y = a. Its location from the center of gravity of the quiescent liquid is denoted by h_n. The non-vibrating liquid in the lower part of the container is represented by a mass m_o of moment of inertia I_o. This mass is attached rigidly to the container at a height h_o below the center of gravity of the undisturbed liquid.

Once the equations of the mechanical analogy are derived, they will be compared with the fluid theory to obtain the mechanical values, such as m_n, m_o, k_n, h_n and h_o (Fig. 3).

A. Analytical Model for the Description of the Liquid Motion

The equations of motion of the mechanical analogy are derived with the help of the Lagrange equations. If y_n is the displacement of the n^{th} sloshing mass m_n with respect to the container wall y, the tank displacement in y-direction, and ϑ the rotation about the z-axis, the kinetic energy is then

$$T = \frac{m_o}{2} (\dot{y} + h_o\dot{\vartheta}) + \frac{1}{2} I_o \dot{\vartheta}^2 + \frac{1}{2} \sum_{n=1}^{\infty} m_n(\dot{y}_n + \dot{y} + h_n\dot{\vartheta})^2 \quad . \qquad (3.1)$$

The first two terms represent the kinetic energy of the mass m_o which is rigidly connected with the container. The series describes the kinetic energy of the modal masses m_n. The potential energy is given by

$$V = \frac{1}{2} g\vartheta^2 m_o h_o - \frac{1}{2} g\vartheta^2 \sum_{n=1}^{\infty} m_n h_n - g\vartheta \sum_{n=1}^{\infty} m_n y_n + \frac{1}{2} \sum_{n=1}^{\infty} k_n y_n^2 \quad . \tag{3.2}$$

Here, the first term represents the potential energy caused by the lifting of the mass m_o during rotation about the z-axis, while the second and third terms describe the same fact for the modal masses. The last term is the energy that is stored in the springs of stiffness k_n.

To make the mechanical analogy equivalent to the fluid system, the sum of the nonsloshing mass and the modal masses must be equal to the total liquid mass

$$m = m_o + \sum_{n=1}^{\infty} m_n \quad . \tag{3.3}$$

Furthermore, for small oscillations, the center of gravity of the liquid shifts in the first approximation horizontally only. Therefore,

$$m_o h_o = \sum_{n=1}^{\infty} m_n h_n \tag{3.4}$$

must be satisfied. Because of this condition, the first two terms of the potential energy cancel each other.

The equations of motion are now derived from the Lagrange equation

$$\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{q}_i}\right) + \frac{\delta V}{\delta q_i} = Q_i \tag{3.5}$$

where Q_i are the generalized forces and are

$$Q_y = -F_y \quad ; \quad Q_\vartheta = -M_z \quad ; \quad Q_{y_n} = 0 \tag{3.6}$$

and y, ϑ and y_n are the generalized coordinates. The equations of motion are, therefore,

$$m_o(\ddot{y} + h_o\ddot{\vartheta}) + \sum_{n=1}^{\infty} m_n(\ddot{y}_n + \ddot{y} + h_n\ddot{\vartheta}) = -F_y \tag{3.7}$$

$$I_o\ddot{\vartheta} + m_o h_o^2\ddot{\vartheta} - g\sum_{n=1}^{\infty} m_n h_n + \sum_{n=1}^{\infty} m_n h_n(\ddot{y}_n + h_n\ddot{\vartheta}) = -M_z \tag{3.8}$$

$$\ddot{y}_n + \omega_n^2 y_n = -\ddot{y} - h_n\ddot{\vartheta} + g\vartheta \quad ; \quad n = 1, 2, \ldots \tag{3.9}$$

where $\omega_n^2 = \frac{k_n}{m_n}$ (see 2.8). The first equation is the force equation, the second is the moment equation and the third represents the sloshing equations ($n = 1, 2, \ldots$).

B. Solution of the Equations of the Model and Determination of the Mechanical Analogy Values

Since a pitching motion (i.e., oscillation about the z-axis) of the sound suppressor reservoir is unlikely, only the translatory excitation case is treated. If $\vartheta = 0$, the equations of motion yield

$$m\ddot{y} + \sum_{n=1}^{\infty} m_n\ddot{y}_n = -F_y \tag{3.10}$$

$$\sum_{n=1}^{\infty} m_n(h_n\ddot{y}_n - gy_n) = -M_z \tag{3.11}$$

$$\ddot{y}_n + \omega_n^2 y_n = -\ddot{y} \quad ; \quad (n = 1, 2, \ldots) \quad . \tag{3.12}$$

From the sloshing equation with $y = y_o e^{i\Omega t}$ and $y_n = Y_n e^{i\Omega t}$

$$Y_n = \frac{\Omega^2 y_o}{\omega_n^2 - \Omega^2} \qquad \text{or} \qquad y_n = \frac{\Omega^2 y}{\omega_n^2 - \Omega^2} \tag{3.13}$$

is obtained. Therefore,

$$\ddot{y}_n = \frac{\Omega^2 \ddot{y}}{\omega_n^2 - \Omega^2} \quad . \tag{3.14}$$

With this, the force equation yields

$$F_y = - m\ddot{y} \left[1 + \sum_{n=1}^{\infty} \frac{m_n}{m} \frac{\Omega^2}{(\omega_n^2 - \Omega^2)} \right] \quad . \tag{3.15}$$

The moment is then

$$M_z = - m\ddot{y} \sum_{n=1}^{\infty} \frac{m_n}{m} \frac{(g + h_n \Omega^2)}{(\omega_n^2 - \Omega^2)} \quad . \tag{3.16}$$

After this derivation and solution of the mechanical analogy, the mechanical values have to be determined. This will be performed by comparing the mechanical analogy with the results of the liquid theory (Section II). However, the moment equation (2.16) as obtained from the fluid theory is not compatible with the moment equation of the mechanical analogy (3.16) and has to be transformed.

The other expressions are then compared. First, the spring-stiffness k_n has to be chosen such that its ratio with the n^{th} sloshing mass m_n represents the square of the circular natural frequency:

$$\frac{k_n}{m_n} = \omega_n^2 = \frac{gn\pi}{a} \tanh\left(\frac{n\pi h}{a}\right) \quad ; \qquad \text{for } n = 1, 2, \ldots$$

Comparison of the forces (2. 15) and (3. 15) yields (Table 2)

$$\frac{m_{2n-1}}{m} = \frac{8}{\pi^3} \frac{\tanh\left[\frac{(2n-1)\pi}{a} h\right]}{(2n-1)^3 (h/a)} \qquad \text{for } n = 1, 2, \ldots \tag{3. 17}$$

and

$$\frac{m_{2n}}{m} = 0 \quad .$$

With

$$\frac{1}{12(h/a)} = \frac{8}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4 (h/a)}$$

the transformed moment reads

$$M_z = - m\ddot{y} \sum_{n=1}^{\infty} \frac{8a}{\pi^4} \frac{\tanh\left[\frac{(2n-1)\pi h}{a}\right]}{(h/a)(2n-1)^4 (\omega_{2n-1}^2 - \Omega^2)} \left[\frac{g(2n-1)\pi}{a} \right.$$

$$\left. + \Omega^2 \left\{\frac{\pi h}{2a}(2n-1) - 2 \tanh\left[\frac{(2n-1)\pi}{2a} h\right]\right\}\right]. \tag{3. 18}$$

Comparison with the moment equation (2. 16) and that of the mechanical analogy yields the relation for h_n

$$h_n = \frac{h}{2}\left\{1 - \frac{4a}{\pi h (2n-1)} \tanh\left[\frac{(2n-1)\pi}{a} h\right]\right\}. \tag{3. 19}$$

The magnitude of the modal masses and their location depends only on the geometry of the container. Therefore, it will be expected that the influence of the liquid in the reservoir upon the structural behavior of the sound suppressor system can be determined mainly by the geometry of the reservoir.

SECTION IV. STRUCTURAL ANALYSIS

For the proper design of the system, the knowledge of the interaction of the structure of the sound suppressor system and the liquid in the reservoir is of great importance. This is particularly true since more than eighty percent (four million gallons) of the liquid sloshes in the container. In the analysis the liquid in the container is described by the mechanical analogy (see Section III). Since the occurrence of longitudinal (in x-direction), pitching, yawing and roll motions of the container is very remote, analysis is restricted to the tranlational degrees of freedom in y and z direction. These, however, can be assumed as uncoupled. Thus, a two degree of freedom system is obtained if all higher sloshing masses are neglected and only the fundamental sloshing mode is retained. This is justified since the second sloshing mass is only 8. 6 percent of the total liquid mass. Since the mass of the support structure exhibits a rather large portion of the total mass of the system, it cannot be neglected. The amount of 17/35 of the support structure is added to the mass of the container structure m_c and the nonsloshing mass m_o because the support structure represents cantilever beams. The container is carried by ν pairs of cantilever beams on either of the long sides in the shown orientation (Fig. 4) and by μ pairs of cantilever beams at the small sides of the container (z-direction). The stiffness of these beams are EI_y in y-direction and EI_z in z-direction. The beams are supposed to be identical except for the orientation, which is exhibited in Figure 4. The value E is Young's modulus of elasticity, I is the geometric moment of inertia of the cross section and ℓ is the length of the beams. The sloshing mass of the n^{th} liquid mode is attached with springs of stiffness $k_{yn}/2$ and $k_{zn}/2$ to the container walls at a height h_n. The distance from the grade is denoted by L_n. The mass M composed of the container mass m_c, nonsloshing mass m_o and 17/35 of the support structure, i.e.,

$$M = m_c + m_o + 17/35\, m_{ss}$$

is located at a distance ℓ_o from the grade. Its displacement in y-direction is denoted by y_1 and in z-direction by z_1. The n^{th} slosh mass performs a displacement relative to the reservoir wall and is denoted by y_{n2} for motion in the y-direction and by z_{n2} for motion in the z-direction. If, as in the numerical evaluation, only the fundamental slosh mass is considered, its displacement relative to the container wall is denoted by y_2 for the y-direction and by z_2 for the z-direction.

A. Equations of Motion

Since the fluid mass that participates in the motion of the free fluid surface (i.e., the sloshing mass) decreases as $1/(2n-1)^3$, only one sloshing mass $m_1 = m$, attached at a location $L = \ell + h_o + h_1$ from the lower end of the beam, is considered (see Fig. 4). The sum of the mass of the container structure, the mass of the non-sloshing liquid and the effective mass of the support structure is denoted by M. The generalized coordinates are y_1 and y_2, where y_1 is the displacement of the mass M at the location ℓ and y_2 is the displacement of the sloshing mass m at the location L. For the derivation of the equations of motion the displacement influence coefficient method is employed. For this reason the displacement of the beam at the location x caused by a force F at the location ξ has to be determined. The moment at the location x caused by a force at the location ξ is given by

$$M = EIy'' = F(\xi - x) \qquad \text{for } x < \xi \ .$$

In assuming a uniform beam (i.e., constant stiffness EI), the displacement can easily be obtained by integration of

$$y'' = \frac{F}{EI}(\xi - x) \qquad \text{for } x < \xi$$

with the boundary conditions of zero slope and zero displacement at x = o in the case $x < \xi$. This yields a slope $(x < \xi)$

$$y'(x) = \frac{F}{2EI}\left[2\xi x - x^2\right]$$

and a displacement for $x < \xi$

$$y(x) = \frac{F}{6EI}\left[3\,\xi x^2 - x^3\right] \quad .$$

For $x > \xi$ the displacement is

$$y(x) = y(\xi) + y'(\xi)(x - \xi)$$

which yields $(x > \xi)$

$$y(x) = \frac{F}{6EI}\left\{3\xi x^2 - x^3 + (x - \xi)^3\right\} \quad .$$

The influence coefficients δ_{ij} are determined next. Applying a unit force at the location (1) yields at this location with $x = \xi = \ell$ a deflection

$$\delta_{11} = \frac{\ell^3}{3EI} \quad . \tag{4.1}$$

The deflection of the mass m (i.e., at the location [2]) is then with $x = L$ and $\xi = \ell$

$$\delta_{12} = \frac{1}{6EI}\left\{3\ell L^2 - L^3 + (L - \ell)^3\right\}$$

which yields

$$\delta_{12} = \delta_{21} = \frac{\ell^3}{6EI}\left\{3\left(\frac{L}{\ell}\right) - 1\right\} \quad . \tag{4.2}$$

A unit force applied at the location (2), i.e., at the mass point m, yields a deflection of that point by

$$\delta_{22} = \frac{kL^3 + 3EI}{3kEI} \tag{4.3}$$

This is obtained from the fact that the springs with stiffness k and $3EI/\ell^3$ are in series, which means that their influence numbers add. The equations of motion for the total system are now derived.

A unit force applied in y-direction at the location (1), i.e., the location of the mass M, yields at that location (1) a deflection $\delta_{1_{y_1y_1}}$ in y-direction and a deflection $\delta_{1_{y_1z_1}}$ in z-direction. At the location (2), i.e., the location of the sloshing mass, the displacement is $\delta_{1_{y_1y_2}}$ in y-direction and $\delta_{1_{y_1z_2}}$ in z-direction.

A unit force applied in z-direction at the location (1) yields the deflections $\delta_{1_{z_1y_1}}$ in y-direction and $\delta_{1_{z_1z_1}}$ in z-direction at the location (1), and $\delta_{1_{z_1y_2}}$ in y-direction and $\delta_{1_{z_1z_2}}$ in z-direction at the location (2). Similar results for the deflections are obtained by applying a unit force at the location (2) first in y and then in z-direction. Therefore, $\delta_{2_{y_2y_1}}$ is the deflection in y-direction at the location (1) caused by a unit force in y-direction, applied at the location (2). $\delta_{2_{y_2z_1}}$ is the deflection in z-direction at the location (1) caused by that unit force, $\delta_{2_{y_2y_2}}$ is the displacement in y-direction at the location (2) and $\delta_{2_{y_2z_2}}$ is the deflection in z-direction at the location (2) caused by a unit force in y-direction at the location (2). A unit force in z-direction at the location (2) yields a deflection $\delta_{2_{z_2y_1}}$ in y-direction and $\delta_{2_{z_2z_1}}$ in z-direction at the location (1) and $\delta_{2_{z_2y_2}}$ in y-direction and $\delta_{2_{z_2z_2}}$ in z-direction at the location (2). The static problem is therefore solved if the influence coefficients $\delta_{iz_jy_k}$ and $\delta_{iy_jz_k}$ (i, j, k = 1, 2) are known. The deflections caused by the forces $\vec{F}_1 = Y_1\vec{j} + Z_1\vec{k}$ at the location (1) and $\vec{F}_2 = Y_2\vec{j} + Z_2\vec{k}$ at the location (2) are, therefore:

$$y_1 = \delta_{1_{y_1y_1}} Y_1 + \delta_{1_{z_1y_1}} Z_1 + \delta_{2_{y_2y_1}} Y_2 + \delta_{2_{z_2y_1}} Z_2$$

$$z_1 = \delta_{1_{y_1z_1}} Y_1 + \delta_{1_{z_1z_1}} Z_1 + \delta_{2_{y_2z_1}} Y_2 + \delta_{2_{z_2z_1}} Z_2$$

$$y_2 = \delta_{1_{y_1y_2}} Y_1 + \delta_{1_{z_1y_2}} Z_1 + \delta_{2_{y_2y_2}} Y_2 + \delta_{2_{z_2y_2}} Z_2$$

$$z_2 = \delta_{1_{y_1z_2}} Y_1 + \delta_{1_{z_1z_2}} Z_1 + \delta_{2_{y_2z_2}} Y_2 + \delta_{2_{z_2z_2}} Z_2 \quad .$$

Since the beams are all parallel they act like parallel springs, thus having a spring stiffness of the sum of their stiffnesses.

Considered is one pair of beams of the system. If its geometric moment of inertia for bending in y-direction is given by I_y and in z-direction by I_z, the stiffness in y-direction of 2 μ pairs of beams on the short sides of the system and 2 ν pairs of beams on the long sides of the container is given by: $\frac{3EI_y}{\ell^3} 2\nu + \frac{3EI_z}{\ell^3} 2\mu$. The stiffness in z-direction yields: $\frac{3EI_y}{\ell^3} 2\mu + \frac{3EI_z}{\ell^3} 2\nu$.

For free oscillations the forces acting on the system are

$$Y_1 = -M\ddot{y}_1 \quad , \quad Z_1 = -M\ddot{z}_1$$

$$Y_2 = -m\ddot{y}_2 \quad , \quad Z_2 = -m\ddot{z}_2$$

and the equations of motion are given by

$$y_1 + \delta_{1_{y_1y_1}} M\ddot{y}_1 + \delta_{1_{z_1y_1}} M\ddot{z}_1 + \delta_{2_{y_2y_1}} m\ddot{y}_2 + \delta_{2_{z_2y_1}} m\ddot{z}_2 = 0 \tag{4.4}$$

$$z_1 + \delta_{1_{y_1z_1}} M\ddot{y}_1 + \delta_{1_{z_1z_1}} M\ddot{z}_1 + \delta_{2_{y_2z_1}} m\ddot{y}_2 + \delta_{2_{z_2z_1}} m\ddot{z}_2 = 0 \tag{4.5}$$

$$y_2 + \delta_{1_{y_1y_2}} M\ddot{y}_1 + \delta_{1_{z_1y_2}} M\ddot{z}_1 + \delta_{2_{y_2y_2}} m\ddot{y}_2 + \delta_{2_{z_2y_2}} m\ddot{z}_2 = 0 \tag{4.6}$$

$$z_2 + \delta_{y_1z_2} M\ddot{y}_1 + \delta_{1_{z_1z_2}} M\ddot{z}_1 + \delta_{2_{y_2z_2}} m\ddot{y}_2 + \delta_{2_{z_2z_2}} m\ddot{z}_2 = 0 \quad . \tag{4.7}$$

Since the support beams are chosen to be I-beams, thus exhibiting symmetry, the influence coefficients $\delta_{1_{z_1 y_1}}$, $\delta_{1_{y_1 z_1}}$, $\delta_{2_{z_2 y_1}}$, $\delta_{1_{y_2 z_1}}$, $\delta_{1_{z_1 y_2}}$, $\delta_{2_{z_2 y_2}}$, $\delta_{1_{y_1 z_2}}$, and $\delta_{2_{y_2 z_2}}$ vanish. This means that a force in y-direction does not create a deflection in z-direction. Mechanically this means that the coupling of the y and z-direction is zero. The equations of motion, therefore, split up into two pairs of equations and are

$$y_1 + \delta_{1_{y_1 y_1}} M\ddot{y}_1 + \delta_{2_{y_2 y_1}} m\ddot{y}_2 = 0$$

$$y_2 + \delta_{1_{y_1 y_2}} M\ddot{y}_1 + \delta_{2_{y_2 y_2}} m\ddot{y}_2 = 0 \tag{4.8}$$

and

$$z_1 + \delta_{1_{z_1 z_1}} M\ddot{z}_1 + \delta_{2_{z_2 z_1}} m\ddot{z}_2 = 0$$

$$z_1 + \delta_{1_{z_1 z_2}} M\ddot{z}_1 + \delta_{2_{z_2 z_2}} m\ddot{z}_2 = 0 \tag{4.9}$$

The influence coefficients with (4.1) through (4.3) are given by

$$\delta_{1_{y_1 y_1}} = \frac{\ell^3}{6E(\nu I_y + \mu I_z)} \quad , \quad \delta_{2_{y_2 y_1}} = \frac{\ell^3\left[3\left(\frac{L}{\ell}\right) - 1\right]}{12E(\nu I_y + \mu I_z)}$$

$$\delta_{1_{y_1 y_2}} = \frac{\ell^3\left[3\left(\frac{L}{\ell}\right) - 1\right]}{12E(\nu I_y + \mu I_z)} \quad , \quad \delta_{2_{y_2 y_2}} = \frac{k_y L^3 + 6E(\nu I_y + \mu I_z)}{6kyE(\nu I_y + \mu I_z)} \tag{4.10}$$

and

$$\delta_{1_{z_1 z_1}} = \frac{\ell^3}{6E(\nu I_z + \mu I_y)} \quad , \quad \delta_{2_{z_1 z_2}} = \frac{\ell^3\left[3\left(\frac{L}{\ell}\right) - 1\right]}{12E(\nu I_z + \mu I_y)}$$

$$\delta_{1_{z_2 z_1}} = \frac{\ell^3\left[3\left(\frac{L}{\ell}\right) - 1\right]}{12E(\nu I_z + \mu I_y)} \quad , \qquad \delta_{2_{z_2 z_2}} = \frac{k_z L^3 + 6E(\nu I_z + \mu I_y)}{6k_z E(\nu I_z + \mu I_y)} \quad . \tag{4.11}$$

As can be seen from the previous equations, the two pairs of equations of motion can be solved independently. Transformation of the displacement influence coefficient matrix

$$[\delta] \equiv \begin{bmatrix} \delta_{1_{y_1 y_1}} & \delta_{2_{y_2 y_1}} \\ \\ \delta_{1_{y_1 y_2}} & \delta_{2_{y_2 y_2}} \end{bmatrix} = \begin{bmatrix} \dfrac{\ell^3}{6\nu EI_y} & \dfrac{\ell^3\left[3\left(\frac{L}{\ell}\right) - 1\right]}{12\nu EI_y} \\ \\ \dfrac{\ell^3\left[3\left(\frac{L}{\ell}\right) - 1\right]}{12\nu EI_y} & \dfrac{k_y L^3 + 6\nu EI_y}{6k_y EI_y \nu} \end{bmatrix}$$

into the force influence coefficient matrix yields with $k_{\nu\lambda} = (-1)^{\lambda + \nu} \dfrac{\Delta_{\nu\lambda}}{\Delta}$ where Δ is the determinant of the displacement influence coefficients and $\Delta_{\nu\lambda}$ the subdeterminant to the term $(\lambda\nu)$.

$$[K] = \frac{1}{\Delta} \begin{bmatrix} 24EI_y \nu(k_y L^3 + 6EI_y \nu) & -12\nu EI_y k_y \ell^3 \left[3\left(\frac{L}{\ell}\right) - 1\right] \\ \\ -12\nu EI_y k_y \ell^3 \left[3\left(\frac{L}{\ell}\right) - 1\right] & 24\ell^3 \nu EI_y k_y \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ \\ k_{21} & k_{22} \end{bmatrix}$$

It is

$$\Delta = 4\ell^3(k_y L^3 + 6\nu EI_y) - \ell^6 k_y \left[3\left(\frac{L}{\ell}\right) - 1\right] .$$

The equations of motion in y-direction, therefore, can also be written as

$$M\ddot{y}_1 + k_{11}y_1 + k_{12}y_2 = 0$$

$$m\ddot{y}_2 + k_{21}y_1 + k_{22}y_2 = 0 \quad .$$

These equations exhibit static coupling.

B. Analysis of the Interaction of Structure and Liquid in the Reservoir

Since the equations of motion in y and z-direction are the same kind, the solution of one set is sufficient. The final results can be modified by writing just the value $z_i (i = 1, 2)$ instead of y_i, and the expressions $\delta_{iz_jz_k}$ instead of the influence coefficients $\delta_{iy_jy_k}$.

1. Free Oscillations.

For the free oscillations the solution is

$$y_1 = \left\{ \begin{matrix} A_1 \cos \omega t \\ B_1 \sin \omega t \end{matrix} \right\}$$

$$y_2 = \left\{ \begin{matrix} A_2 \cos \omega t \\ B_2 \sin \omega t \end{matrix} \right\}$$

which yield the frequency equation

$$\begin{vmatrix} 1 - \delta_{1_{y_1y_1}} M\omega^2 & - \delta_{2_{y_2y_1}} m\omega^2 \\ 1 - \delta_{1_{y_1y_2}} M\omega^2 & 1 - \delta_{2_{y_2y_2}} m\omega^2 \end{vmatrix} = 0 \quad .$$

This yields with $1/\omega^2 = \chi^2$ the expression

$$\chi^4 - \chi^2\left\{\delta_{1_{y_1y_1}} M + \delta_{2_{y_2y_2}} m\right\} + mM\left\{\delta_{1_{y_1y_1}}\delta_{2_{y_2y_2}} - \delta_{1_{y_1y_2}}\delta_{2_{y_2y_1}}\right\} = 0 \quad .$$

The solution of this equation is

$$\chi^2_{1/2} = \frac{1}{2}\left\{\delta_{1_{y_1y_1}} M + \delta_{2_{y_2y_2}} m\right\} \pm \frac{1}{2}\sqrt{(\delta_{1_{y_1y_1}} M - m\delta_{2_{y_2y_2}})^2 + 4mM\delta_{1_{y_1y_2}}\delta_{2_{y_2y_1}})} \tag{4. 12}$$

which gives the square of the natural circular frequencies

$$\omega_1^2 = 1/\chi_1^2 \quad \text{and} \quad \omega_2^2 = 1/\chi_2^2 \quad .$$

The Amplitude ratios are

$$\frac{A_{11}}{A_{21}} = \frac{\delta_{2_{y_2y_2}} m\omega_1^2 - 1}{\delta_{1_{y_1y_2}} M\omega_1^2} \equiv \frac{1}{\lambda_1} \quad ; \quad \frac{A_{12}}{A_{22}} = \frac{\delta_{2_{y_2y_2}} m\omega_2^2 - 1}{\delta_{1_{y_1y_2}} M\omega_2^2} \equiv \frac{1}{\lambda_2} \quad . \tag{4. 13}$$

These ratios are equal to the ratio of the displacements y_1/y_2 at these frequencies. With these functions

$$y_{11}\,(t) = A_{11} \cos \omega_1 t \qquad , \qquad y_{12}\,(t) = A_{12} \cos \omega_2 t$$

$$y_{21}\,(t) = \lambda_1 A_{11} \cos \omega_1 t \qquad , \qquad y_{22}\,(t) = \lambda_2 A_{12} \cos \omega_2 t$$

are obtained, which are particular solutions with the integration constants A_{11} and A_{12}. A solution of the form

$$y_1 = B_1 \sin \omega t$$

$$y_2 = B_2 \sin \omega t$$

yields the same frequency equation and mode shape numbers λ_1 and λ_2. The complete solution, therefore, is given by

$$y_1(t) = A_{11} \cos \omega_1 t + B_{11} \sin \omega_1 t + A_{12} \cos \omega_2 t + B_{12} \sin \omega_2 t$$

$$y_2(t) = \lambda_1 \left[A_{11} \cos \omega_1 t + B_{11} \sin \omega_1 t\right] + \lambda_2 \left[A_{12} \cos \omega_2 t + B_{12} \sin \omega_2 t\right]$$

where A_{11}, A_{12}, B_{11} and B_{12} are four integration constants which can be determined by the initial conditions at $t = o$. With

$$y_1(o) = y_{10} \quad , \quad \dot{y}_1(o) = \upsilon_{10}$$

$$y_2(o) = y_{20} \quad , \quad \dot{y}_2(o) = \upsilon_{20} \tag{4.14}$$

the solution is

$$y_1(t) = \left(\frac{y_{10}\lambda_2 - y_{20}}{\lambda_2 - \lambda_1}\right) \cos \omega_1 t + \left(\frac{\upsilon_{10}\lambda_2 - \upsilon_{20}}{\omega_1(\lambda_2 - \lambda_1)}\right) \sin \omega_1 t$$

$$+ \left(\frac{y_{20} - \lambda_1 y_{10}}{\lambda_2 - \lambda_1}\right) \cos \omega_2 t + \left(\frac{\upsilon_{20} - \lambda_1 \upsilon_{10}}{\omega_2(\lambda_2 - \lambda_1)}\right) \sin \omega_2 t \quad . \tag{4.15}$$

$$y_2(t) = \frac{\lambda_1}{(\lambda_2 - \lambda_1)} \left[(y_{10}\lambda_2 - y_{20}) \cos \omega_1 t + \frac{1}{\omega_1} (\upsilon_{10}\lambda_2 - \upsilon_{20}) \sin \omega_1 t\right]$$

$$+ \frac{\lambda_2}{(\lambda_2 - \lambda_1)} \left[(y_{20} - \lambda_1 y_{10}) \cos \omega_2 t \quad \frac{1}{\omega_2} (\upsilon_{20} - \lambda_1 \dot{\upsilon}_{10}) \sin \omega_2 t\right] \quad . \tag{4.16}$$

Similar results are obtained for the free oscillation in z-direction.

2. Forced Harmonic Excitation.

For a forced oscillation with an excitation function $F_{\substack{oy\\z}} \sin \Omega t$ acting at the container, i.e., at the location (1), the equations of motion are:

$$y_1 + \delta_{1_{y_1 y_1}} M\ddot{y}_1 + \delta_{2_{y_2 y_1}} m\ddot{y}_2 = \delta_{1_{y_1 y_1}} F_{oy} \sin \Omega t$$

$$y_2 + \delta_{1_{y_1 y_2}} M\ddot{y}_1 + \delta_{2_{y_2 y_2}} m\ddot{y}_2 = \delta_{1_{y_1 y_2}} F_{oy} \sin \Omega t \qquad (4.17)$$

for forcing in y-direction, and for a forcing function in z-direction at the location (1) it is

$$z_1 + \delta_{1_{z_1 z_1}} M\ddot{z}_1 + \delta_{2_{z_2 z_1}} m\ddot{z}_2 = \delta_{1_{z_1 z_1}} F_{oz} \sin \Omega t$$

$$z_2 + \delta_{1_{z_1 z_2}} M\ddot{z}_1 + \delta_{2_{z_2 z_2}} m\ddot{z}_2 = \delta_{1_{z_1 z_2}} F_{oz} \sin \Omega t \qquad (4.18)$$

The steady state solution of the system (4.17) yields the response amplitudes

$$Y_1 = F_{oy} \frac{\delta_{1_{y_1 y_1}} [1 - m\Omega^2 \delta_{2_{y_2 y_2}}] + \delta_{1_{y_1 y_2}} \delta_{2_{y_2 y_1}} m\Omega^2}{mM\delta_{1_{y_1 y_1}} \delta_{2_{y_2 y_2}} (\Omega^2 - \omega_1^2)(\Omega^2 - \omega_2^2)} \qquad (4.19)$$

$$Y_2 = F_{oy} \frac{\delta_{2_{y_1 y_2}}}{mM\delta_{1_{y_1 y_1}} \delta_{2_{y_2 y_2}} (\Omega^2 - \omega_1^2)(\Omega^2 - \omega_2^2)} \qquad (4.20)$$

Similar results are obtained for the system (4.18) by just exchanging y_i and Y_i with z_i and Z_i.

3. Wind Response. The response of the system due to some wind loading is important for the proper design of the structure. For this reason, the effect of an arbitrary forcing function $F_y(t)$ and $F_z(t)$ in y and z-direction respectively is investigated.

The equations of motion for excitation in y-direction are given by

$$y_1 + \delta_{1_{y_1y_1}} M\ddot{y}_1 + \delta_{2_{y_2y_1}} m\ddot{y}_2 = \delta_{1_{y_1y_1}} F_y(t) \tag{4.21}$$

$$y_2 + \delta_{1_{y_1y_2}} M\ddot{y}_1 + \delta_{2_{y_2y_2}} m\ddot{y}_2 = \delta_{1_{y_1y_2}} F_y(t) \tag{4.22}$$

and for excitation in z-direction, the equations of motion are

$$z_1 + \delta_{1_{z_1z_1}} M\ddot{z}_1 + \delta_{2_{z_2z_1}} m\ddot{z}_2 = \delta_{1_{z_1z_1}} F_z(t) \tag{4.23}$$

$$z_2 + \delta_{1_{z_1z_2}} M\ddot{z}_1 + \delta_{2_{z_2z_2}} m\ddot{z}_2 = \delta_{1_{z_1z_2}} F_z(t) \quad . \tag{4.24}$$

These represent two linear inhomogeneous differential equations, of which the transient solution shall be determined for an arbitrary excitation. Since the system already might be in some kind of motion caused by the surge wave or some disturbance created by the firing of the rocket engines before the wind hits the system at a time $t = t_o$, the free oscillations during the time interval $0 \leq t \leq t_o$ are also treated in order to present properly the initial conditions at the time t_o.

a. Step function. For the derivation of an arbitrary time function input, the case of a suddenly applied constant force has to be treated first. Since the above systems of two differential equations are of the same form except for the constant coefficients, the derivation of the response is restricted to one system. Introducing for

$$\delta_{1_{y_1y_1}} \equiv \delta_1 \ , \qquad \delta_{2_{y_2y_1}} = \delta_{1_{y_1y_2}} \equiv \delta_{12} \quad \text{and} \quad \delta_{2_{y_2y_2}} \equiv \delta_2$$

the equations of motion that have to be treated for a forcing function (Fig. 5)

$$F(t) = \begin{cases} 0 & \text{for } 0 \le t \le t_o \\ F_o & \text{for } t \ge t_o \end{cases} \tag{4.25}$$

are given by

$$y_1 + \delta_1 M\ddot{y}_1 + \delta_{12} m\ddot{y}_2 = \delta_1 F(t) \tag{4.26}$$

$$y_2 + \delta_{12} M\ddot{y}_1 + \delta_2 m\ddot{y}_2 = \delta_{12} F(t) \quad . \tag{4.27}$$

For the time interval $0 \le t \le t_o$, the system is not disturbed and exhibits zeros on the righthand side. With the initial conditions at the time $t = 0$

$$y_1(o) = y_{10} \qquad \text{and} \qquad \dot{y}_1(o) = v_{10}$$

$$y_2(o) = y_{20} \qquad\qquad \dot{y}_2(o) = v_{20}$$

the solution yields for the time interval $0 \le t \le t_o$:

$$y_1 = \frac{1}{(\lambda_2 - \lambda_1)} \left\{ (y_{10}\lambda_2 - y_{20}) \cos \omega_1 t + \frac{1}{\omega_1} (v_{10}\lambda_2 - v_{20}) \sin \omega_1 t \right.$$

$$\left. + (y_{20} - \lambda_1 y_{10}) \cos \omega_2 t + \frac{1}{\omega_2} (v_{20} - \lambda_1 v_{10}) \sin \omega_2 t \right\} \tag{4.28}$$

$$y_2 = \frac{1}{(\lambda_2 - \lambda_1)} \left\{ \lambda_1 (y_{10}\lambda_2 - y_{20}) \cos \omega_1 t + \frac{\lambda_1}{\omega_1} (v_{10}\lambda_2 - v_{20}) \sin \omega_1 t \right.$$

$$\left. + \lambda_2 (y_{20} - \lambda_1 y_{10}) \cos \omega_2 t + \frac{\lambda_2}{\omega_2} (v_{20} - \lambda_1 v_{10}) \sin \omega_2 t \right\} . \tag{4.29}$$

At the time $t = t_o$, a constant force F_o (wind) is suddenly applied. The solution of the above differential equations in this time interval $(t = t_o)$ is

$$\left.\begin{aligned} y_1 &= A_{11} \cos \omega_1 t + B_{11} \sin \omega_1 t + A_{12} \cos \omega_2 t + B_{12} \sin \omega_2 t + \delta_1 F_o \\ y_2 &= \lambda_1 (A_{11} \cos \omega_1 t + B_{11} \sin \omega_1 t) + \lambda_2 (A_{12} \cos \omega_2 t + B_{12} \sin \omega_2 t) \\ &\quad + \delta_{12} F_o \end{aligned}\right\} t \geq t_o$$

where A_{11}, A_{12}, B_{11} and B_{12} are integration constants and $\delta_1 F_o$ and $\delta_{12} F_o$ are the particular solutions connected with the inhomogeneous part of the differential equations. The constants of integration can be determined by the new initial conditions at $t = t_o$ as obtained from the final values of displacement and velocity in the first time interval. This means that at the time the wind disturbance excites the system, it had already some displacement and velocity from a previous disturbance. The constants of integration finally are

$$A_{11} = \frac{(y_{10}\lambda_2 - y_{20})}{(\lambda_2 - \lambda_1)} - \frac{F_o(\delta_1\lambda_2 - \delta_{12})}{(\lambda_2 - \lambda_1)} (\cos \omega_1 t_o)$$

$$B_{11} = \frac{v_{10}\lambda_2 - v_{20}}{\omega_1(\lambda_2 - \lambda_1)} - \frac{F_o(\delta_1\lambda_2 - \delta_{12})}{(\lambda_2 - \lambda_1)} (\sin \omega_1 t_o)$$

$$A_{12} = \frac{y_{20} - \lambda_1 y_{10}}{(\lambda_2 - \lambda_1)} - \frac{F_o(\delta_{12} - \lambda_1\delta_1)}{(\lambda_2 - \lambda_1)} (\cos \omega_2 t_o)$$

$$B_{12} = \frac{v_{20} - \lambda_1 v_{10}}{\omega_2(\lambda_2 - \lambda_1)} - \frac{F_o(\delta_{12} - \lambda_1\delta_1)}{(\lambda_2 - \lambda_1)} (\sin \omega_2 t_o)$$

and the solution of the system yields for $t \geq t_o$

$$y_1 = \frac{(y_{10}\lambda_2 - y_{20})}{\lambda_2 - \lambda_1} \cos \omega_1 t + \frac{v_{10}\lambda_2 - v_{20}}{\omega_1(\lambda_2 - \lambda_1)} \sin \omega_1 t + \frac{y_{20} - \lambda_1 y_{10}}{\lambda_2 - \lambda_1} \cos \omega_2 t$$

$$+ \frac{v_{20} - \lambda_1 v_{10}}{\omega_2(\lambda_2 - \lambda_1)} \sin \omega_2 t + \delta_1 F_o - \frac{F_o(\delta_1\lambda_2 - \delta_{12})}{\lambda_2 - \lambda_1} \cos \omega_1(t - t_o)$$

$$- \frac{F_o(\delta_{12} - \lambda_1\delta_1)}{\lambda_2 - \lambda_1} \cos \omega_2(t - t_o) \qquad (4.30)$$

$$y_2 = \frac{\lambda_1(y_{10}\lambda_2 - y_{20})}{\lambda_2 - \lambda_1} \cos \omega_1 t + \frac{\lambda_1(v_{10}\lambda_2 - v_{20})}{\omega_1(\lambda_2 - \lambda_1)} \sin \omega_1 t$$

$$+ \frac{\lambda_2(y_{20} - \lambda_1 y_{10})}{\lambda_2 - \lambda_1} \cos \omega_2 t + \frac{\lambda_2(v_{20} - \lambda_1 v_{10})}{\omega_2(\lambda_2 - \lambda_1)} \sin \omega_2 t + \delta_{12} F_o$$

$$- \left\{ \frac{\lambda_1 F_o(\delta_1\lambda_2 - \delta_{12})}{\lambda_2 - \lambda_1} \cos \omega_1(t - t_o) \right\} - \left\{ \frac{\lambda_2 F_o(\delta_{12} - \lambda_1\delta_1)}{\lambda_2 - \lambda_1} \cos \omega_2(t - t_o) \right\}. \qquad (4.31)$$

It can be seen immediately that if the system was at rest at $t = 0$, i.e., $y_{10} = y_{20} = 0$; $v_{10} = v_{20} = 0$, the motion for $t \geq t_o$ is given by expressions in the second line of each formula. It represents oscillations with the circular frequency ω_1 and ω_2 about the new equilibrium positions $\delta_1 F_o$ for y_1 and $\delta_{12} F_o$ for y_2, indicating that the structure as well as the free fluid surface oscillate about an inclined position.

If the system is at rest until at a time $t = 0$ a wind <u>step function</u> of magnitude F_o is applied to the container, the system responds with

$$y_1 = F_o \left\{ \delta_1 - \frac{(\delta_1\lambda_2 - \delta_{12})}{\lambda_2 - \lambda_1} \cos \omega_1 t - \frac{(\delta_{12} - \lambda_1\delta_1)}{\lambda_2 - \lambda_1} \cos \omega_2 t \right\}. \qquad (4.32)$$

$$y_2 = F_o \left\{ \delta_{12} - \frac{\lambda_1(\delta_1\lambda_1 - \delta_{12})}{\lambda_2 - \lambda_1} \cos \omega_1 t - \frac{\lambda_2(\delta_{12} - \lambda_1\delta_1)}{\lambda_2 - \lambda_1} \cos \omega_2 t \right\}. \qquad (4.33)$$

b. Rectangular pulse. With these results a rectangular pulse of magnitude F_o can be obtained by subtracting the response of a step function applied at $t = t_1$ and of magnitude F_o. It is, therefore, the response of the system due to a rectangular pulse of magnitude F_o and duration t_1 seconds given by the equation (4.32) and (4.33) for $0 \leq t \leq t_1$ and for $t \geq t_1$ by

$$y_1 = F_o \left\{ \frac{\delta_1\lambda_2 - \delta_{12}}{(\lambda_2 - \lambda_1)} [\cos \omega_1(t - t_1) - \cos \omega_1 t] + \frac{(\delta_{12} - \lambda_1\delta_1)}{(\lambda_2 - \lambda_1)} [\cos \omega_2(t - t_1) - \cos \omega_2 t] \right\} \qquad (4.34)$$

$$y_2 = F_o \left\{ \frac{\lambda_1(\delta_1\lambda_2 - \delta_{12})}{(\lambda_2 - \lambda_1)} [\cos \omega_1 (t - t_1) - \cos \omega_1 t] + \frac{\lambda_2(\delta_{12} - \lambda_1\delta_1)}{\lambda_2 - \lambda_1} [\cos \omega_2(t - t_1) - \cos \omega_2 t] \right\} . \qquad (4.35)$$

Several elementary forcing functions shall be treated. First, the response of the system to a simple exponential asymptotic step $F_o(1 - e^{-\alpha t})$ shall be investigated. Then an exponentially decaying pulse of the form $F_o e^{-\beta t}$ will be treated, where the latter one represents for appropriately large F_o very closely the blast loading case.

c. The exponential asymptotic step. The equations of motion for the system disturbed by an exponential asymptotic step of the form (Fig. 6)

$$F(t) = \left\{ \begin{array}{ll} o & ; \text{ for } o \le t \le t_o \\ F_o(1 - e^{-\alpha(t - t_o)}) & ; \text{ for } t \ge t_o \end{array} \right\} \tag{4.36}$$

are given by

$$y_1 + \delta_1 M\ddot{y}_1 + \delta_{12} m\ddot{y}_2 = \left\{ \begin{array}{ll} o & ; \text{ for } o \le t \le t_o \\ \delta_1 F_o(1 - e^{-\alpha(t - t_o)}) & ; \text{ for } t \ge t_o \end{array} \right\} \tag{4.37}$$

$$y_2 + \delta_{12} M\ddot{y}_1 + \delta_2 m\ddot{y}_2 = \left\{ \begin{array}{ll} o & ; \text{ for } o \le t \le t_o \\ \delta_{12} F_o(1 - e^{-\alpha(t - t_o)}) & ; \text{ for } t \ge t_o \end{array} \right\} \tag{4.38}$$

The solution in the interval $0 \le t \le t_o$ is given by (4.28) and (4.29) if the initial conditions at $t = 0$ are given by $y_1(0) = y_{10}$, $y_2(0) = y_{20}$, $\dot{y}_1(0) = v_{10}$ and $\dot{y}_2(0) = v_{20}$. At the time $t = t_o$ the exponential asymptotic step commences. For $t \ge t_o$ the solution, therefore, is

$$y_1 = A_{11} \cos \omega_1 t + B_{11} \sin \omega_1 t + A_{12} \cos \omega_2 t + B_{12} \sin \omega_2 t + \delta_1 F_o$$
$$+ \frac{F_o\left\{\delta_{12}^2 \quad m\alpha^2 - \delta_1(1 + \delta_2 m\alpha^2)\right\} e^{-\alpha(t - t_o)}}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + mM\alpha^4(\delta_1\delta_2 - \delta_{12}^2)\right\}}$$

$$y_2 = \lambda_1(A_{11} \cos \omega_1 t + B_{11} \sin \omega_1 t) + \lambda_2(A_{12} \cos \omega_2 t + B_{12} \cos \omega_2 t)$$

$$+ \delta_{12} F_o \left[1 - \frac{e^{-\alpha(t - t_o)}}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + mM\alpha^4(\delta_1\delta_2 - \delta_{12}^2)\right\}} \right]$$

where A_{11}, B_{11}, A_{12} and B_{12} are integration constants and the last two terms in each equation represent the particular solution of the system. The constants of integration can be determined by the new initial conditions at the time $t = t_o$.

These initial conditions are the final values for the displacement and the velocity in the first time interval. The constants of integration finally are

$$A_{12} = \frac{y_{20} - \lambda_1 y_{10}}{\lambda_2 - \lambda_1} + F_o \frac{\cos \omega_2 t_o}{(\lambda_2 - \lambda_1)}$$

$$\cdot \left\{ \delta_1\lambda_1 + \lambda_1 \frac{[\delta_{12}^2 m\alpha^2 - \delta_1(1 + m\alpha^2\delta_2)]}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + Mm\alpha^4(\delta_1\delta_2 - \delta_{12}^2)\right\}} \right.$$

$$\left. - \delta_{12}\left[1 - \frac{1}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^4(\delta_1\delta_2 - \delta_{12}^2)\right\}}\right]\right\} + \frac{F_o \sin \omega_2 t_o}{(\lambda_2 - \lambda_1)\omega_2}$$

$$\cdot \left\{ \lambda_1 \frac{[\delta_{12}^2 m\alpha^2 - \delta_1(1 + m\alpha^2\delta_2)]}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + Mm\alpha^4(\delta_1\delta_2 - \delta_{12}^2)\right\}} \right.$$

$$\left. + \delta_{12}\left[\frac{1}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + Mm\alpha^4(\delta_1\delta_2 - \delta_{12}^2)\right\}}\right]\right\}$$

$$B_{12} = \frac{\upsilon_{20} - \lambda_1\upsilon_{10}}{\omega_2(\lambda_2 - \lambda_1)} + \frac{F_o \sin \omega_2 t_o}{(\lambda_2 - \lambda_1)}$$

$$\cdot \left\{ \delta_1\lambda_1 + \lambda_1 \frac{[\delta_{12}^2 m\alpha^2 - \delta_1(1 + m\alpha^2\delta_2)]}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + Mm\alpha^4(\delta_1\delta_2 - \delta_{12}^2)\right\}} \right.$$

$$\left. - \delta_{12}\left[1 - \frac{1}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + Mm\alpha^4(\delta_1\delta_2 - \delta_{12}^2)\right\}}\right]\right\} - \frac{F_o \alpha \cos \omega_2 t_o}{(\lambda_2 - \lambda_1)}$$

$$\cdot \left\{ \lambda_1 \frac{[\delta_{12}^2 m\alpha^2 - \delta_1(1 + m\alpha^2\delta_2)]}{\left\{1 + \delta + M\alpha^2 + \delta_2 m\alpha^2 + Mm\alpha^4(\delta_1\delta_2 - \delta_{12}^2)\right\}} \right.$$

$$\left. + \delta_{12}\left[\frac{1}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + Mm\alpha^4(\delta_1\delta_2 - \delta_{12}^2)\right\}}\right]\right\}$$

$$A_{11} = \frac{y_{10}\lambda_2 - y_{20}}{(\lambda_2 - \lambda_1)} - \frac{F_o \cos \omega_1 t_o}{\lambda_2 - \lambda_1}$$

$$\cdot \left\{ \delta_2\lambda_2 + \lambda_2 \frac{\delta_{12}{}^2 m\alpha^2 - \delta_1(1 + \delta_2 m\alpha^2)}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + Mm\alpha^4(\delta_1\delta_2 - \delta_{12}{}^2)\right\}} \right.$$

$$\left. - \delta_{12}\left[1 - \frac{1}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + Mm\alpha^4(\delta_1\delta_2 - \delta_{12}{}^2)\right\}}\right]\right\} - \frac{F_o \alpha \sin \omega_1 t_o}{\omega_1(\lambda_2 - \lambda_1)}$$

$$\cdot \left\{ \lambda_2 \frac{\delta_{12}{}^2 m\alpha^2 - \delta_1(1 + m\alpha^2\delta_2)}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + Mm\alpha^4(\delta_1\delta_2 - \delta_{12}{}^2)\right\}} \right.$$

$$\left. + \delta_{12} \frac{1}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + Mm\alpha^4(\delta_1\delta_2 - \delta_{12}{}^2)\right\}} \right\}$$

$$B_{11} = \frac{(v_{10}\lambda_2 - v_{20})}{\omega_1(\lambda_2 - \lambda_1)} - \frac{F_o \sin \omega_1 t_o}{(\lambda_2 - \lambda_1)}$$

$$\cdot \left\{ \delta_2\lambda_2 + \lambda_2 \frac{\delta_{12}{}^2 m\alpha^2 - \delta_1(1 + \delta_2 m\alpha^2)}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + Mm\alpha^4(\delta_1\delta_2 - \delta_{12}{}^2)\right\}} \right.$$

$$\left. - \delta_{12}\left[1 - \frac{1}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + Mn\alpha^4(\delta_1\delta_2 - \delta_{12}{}^2)\right\}}\right]\right\} + \frac{F_o \alpha \cos \omega_1 t_o}{\omega_1(\lambda_2 - \lambda_1)}$$

$$\cdot \left\{ \lambda_2 \frac{\delta_{12}{}^2 m\alpha^2 - \delta_1(1 + m\alpha^2\delta_2)}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + Mm\alpha^4(\delta_1\delta_2 - \delta_{12}{}^2)\right\}} \right.$$

$$\left. + \frac{\delta_{12}}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + Mm\alpha^4(\delta_1\delta_2 - \delta_{12}{}^2)\right\}} \right\}$$

and the solution of the system yields for $t \geq t_o$ the expressions:

$$y_1 = \frac{(y_{10}\lambda_2 - y_{20})}{(\lambda_2 - \lambda_1)} \cos \omega_1 t + \frac{(v_{10}\lambda_2 - v_{20})}{\omega_1(\lambda_2 - \lambda_1)} \sin \omega_1 t + \frac{(y_{20} - \lambda_1 y_{10})}{(\lambda_2 - \lambda_1)} \cos \omega_2 t$$

$$+ \frac{(v_{20} - \lambda_1 v_{10})}{\omega_2(\lambda_2 - \lambda_1)} \sin \omega_2 t + \delta_1 F_o$$

$$+ \frac{F_o\left\{\delta_{12}{}^2 m\alpha^2 - \delta_1(1 + \delta_2 m\alpha_2{}^2)\right\} e^{-\alpha(t - t_o)}}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + mM\alpha^4(\delta_1\delta_2 - \delta_{12}{}^2)\right\}} - \frac{F_o \cos \omega_1(t - t_o)}{(\lambda_2 - \lambda_1)}$$

$$\cdot \left\{\delta_2\lambda_2 + \lambda_2 \frac{\delta_{12}{}^2 m\alpha^2 - \delta_1(1 + \delta_2 m\alpha^2)}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + mM\alpha^4(\delta_1\delta_2 - \delta_{12}{}^2)\right\}} - \delta_{12}\left[1\right.\right.$$

$$\left.\left. - \frac{1}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + mM\alpha^4(\delta_1\delta_2 - \delta_{12}{}^2)\right\}}\right]\right\} + \frac{F_o \alpha \sin \omega_1(t - t_o)}{\omega_1(\lambda_2 - \lambda_1)}$$

$$\cdot \left\{\lambda_2 \frac{\delta_{12}{}^2 m\alpha^2 - \delta_1(1 + \delta_2 m\alpha^2)}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + mM\alpha^4(\delta_1\delta_2 - \delta_{12}{}^2)\right\}}\right.$$

$$\left. + \delta_{12} \frac{1}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + mM\alpha^4(\delta_1\delta_2 - \delta_{12}{}^2)\right\}}\right\} + \frac{F_o \cos \omega_2(t - t_o)}{(\lambda_2 - \lambda_1)}$$

$$\cdot \left\{\delta_1\lambda_1 + \lambda_1 \frac{\delta_{12}{}^2 m\alpha^2 - \delta_1(1 + \delta_2 m\alpha^2)}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + mM\alpha^4(\delta_1\delta_2 - \delta_{12}{}^2)\right\}} - \delta_{12}\left[1\right.\right.$$

$$\left.\left. - \frac{1}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + mM\alpha^4(\delta_1\delta_2 - \delta_{12}{}^2)\right\}}\right]\right\} - \frac{F_o \alpha \sin \omega_2(t - t_o)}{\omega_2(\lambda_2 - \lambda_1)}$$

$$\cdot \left\{ \lambda_1 \frac{\delta_{12}^2 m\alpha^2 - \delta_1(1 + \delta_2 m\alpha^2)}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + mM\alpha^4(\delta_1\delta_2 - \delta_{12}^2)\right\}} \right.$$

$$\left. + \delta_{12} \frac{1}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + mM\alpha^4(\delta_1\delta_2 - \delta_{12}^2)\right\}} \right\} \tag{4.39}$$

$$y_2 = \frac{\lambda_1(y_{10}\lambda_2 - y_{20})}{(\lambda_2 - \lambda_1)} \cos \omega_1 t + \frac{\lambda_1(v_{10}\lambda_2 - v_{20})}{\omega_1(\lambda_2 - \lambda_1)} \sin \omega_1 t$$

$$+ \frac{\lambda_2(y_{20} - \lambda_1 y_{10})}{(\lambda_2 - \lambda_1)} \cos \omega_2 t + \frac{\lambda_2(v_{20} - \lambda_1 v_{10})}{\omega_2(\lambda_2 - \lambda_1)} \sin \omega_2 t + \delta_{12} F_o \left\{ 1 \right.$$

$$\left. - \frac{e^{-\alpha(t - t_o)}}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + mM\alpha^4(\delta_1\delta_2 - \delta_{12}^2)\right\}} \right\} - \frac{F_o \lambda_1 \cos \omega_1(t - t_o)}{(\lambda_2 - \lambda_1)}$$

$$\cdot \left\{ \delta_1\lambda_2 + \lambda_2 \frac{\delta_{12}^2 m\alpha^2 - \delta_1(1 + \delta_2 m\alpha^2)}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + mM\alpha^4(\delta_1\delta_2 - \delta_{12}^2)\right\}} - \delta_{12} \left[1 \right. \right.$$

$$\left. \left. - \frac{1}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + mM\alpha^4(\delta_1\delta_2 - \delta_{12}^2)\right\}} \right] \right\} + \frac{F_o \lambda_1 \alpha \sin \omega_1(t - t_o)}{\omega_1(\lambda_2 - \lambda_1)}$$

$$\cdot \left\{ \lambda_2 \frac{\delta_{12}^2 m\alpha^2 - \delta_1(1 + \delta_2 m\alpha^2)}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + mM\alpha^4(\delta_1\delta_2 - \delta_{12}^2)\right\}} \right.$$

$$\left. + \delta_{12} \frac{1}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + mM\alpha^4(\delta_1\delta_2 - \delta_{12}^2)\right\}} \right\} + \frac{F_o \lambda_2 \cos \omega_2(t - t_o)}{(\lambda_2 - \lambda_1)}$$

$$\cdot\left\{\delta_1\lambda_1 + \lambda_1 \frac{\delta_{12}^2 m\alpha^2 - \delta_1(1 + \delta_2 m\alpha^2)}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + mM\alpha^4(\delta_1\delta_2 - \delta_{12}^2)\right\}} - \delta_{12}\left[1\right.\right.$$

$$\left.\left. - \frac{1}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + mM\alpha^4(\delta_1\delta_2 - \delta_{12}^2)\right\}}\right]\right\} - \frac{F_o \lambda_2 \alpha \sin \omega_2(t - t_o)}{\omega_2(\lambda_2 - \lambda_1)}$$

$$\cdot\left\{\lambda_1 \frac{\delta_{12}^2 m\alpha^2 - \delta_1(1 + \delta_2 m\alpha^2)}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + mM\alpha^4(\delta_1\delta_2 - \delta_{12}^2)\right\}}\right.$$

$$\left. + \delta_{12} \frac{1}{\left\{1 + \delta_1 M\alpha^2 + \delta_2 m\alpha^2 + mM\alpha^4(\delta_1\delta_2 - \delta_{12}^2)\right\}}\right\} . \qquad (4.40)$$

For $\alpha = 0$ the differential equations exhibit no excitation and the solution is that of the free oscillation. For small values α the forcing function approaches the value F_o very slowly, while with an increase in magnitude of the excitation function it approaches more and more the step function. For $\alpha \rightarrow \infty$ the previous results should, therefore, exhibit in the limit the results of the excitation of the system with a step function.

It can be seen immediately that if the system was at rest at $t = 0$, the motion for $t \geq t_o$ is given by the expressions (4.39) and (4.40) if the first line is omitted.

d. Exponentially decaying pulse. The equations of motion for the system disturbed by a wind which is described by an exponentially decaying pulse starting at the time $t = t_o$ with an amplitude F_o (see Fig. 6) are given by

$$y_1 + \delta_1 M\ddot{y}_1 + \delta_{12} m\ddot{y}_2 = \begin{Bmatrix} o & \text{if} & 0 \leq t \leq t_o \\ \delta_1 F_o e^{-\beta(t - t_o)} & \text{if} & t > t_o \end{Bmatrix}$$

$$y_2 + \delta_{12} M \ddot{y}_1 + \delta_2 m \ddot{y}_2 = \begin{cases} o & \text{if} \quad 0 \le t \le t_o \\ \delta_{12} F_o e^{-\beta(t - t_o)} & \text{if} \quad t > t_o \end{cases} . \tag{4.42}$$

The solution in the time interval $0 \le t \le t_o$ is given by (4.28) and (4.29) if the initial conditions at $t = 0$ are given by $y_1(0) = y_{10}$, $y_2(0) = y_{20}$, $\dot{y}_1(0) = v_{10}$ and $\dot{y}_2(0) = v_{20}$. The solution of the system yields for $t \ge t_o$ the expressions

$$y_1 = \left(\frac{y_{10}\lambda_2 - y_{20}}{\lambda_2 - \lambda_1}\right)\cos \omega_1 t + \left(\frac{v_{10}\lambda_2 - v_{20}}{\omega_1(\lambda_2 - \lambda_1)}\right)\sin \omega_1 t + \left(\frac{y_{20} - \lambda_1 y_{10}}{\lambda_2 - \lambda_1}\right) \cos \omega_2 t$$

$$+ \left(\frac{v_{20} - \lambda_1 v_{10}}{\omega_2(\lambda_2 - \lambda_1)}\right) \sin \omega_2 t + \frac{F_o\left\{\delta_1(1 + \delta_2 m\beta^2) - \delta_{12}^2 m\beta^2\right\}e^{-\beta(t - t_o)}}{\left\{(1 + \delta_1 m\beta^2)(1 + \delta_2 m\beta^2) - mM\beta^4\delta_{12}^2\right\}}$$

$$+ \frac{F_o \cos \omega_1(t - t_o)}{(\lambda_2 - \lambda_1)} \left\{\lambda_2 \frac{\delta_{12}^2 m\beta^2 - \delta_1(1 + \delta_2\beta^2)}{\left\{(1 + \delta_1 m\beta^2)(1 + \delta_2 m\beta^2) - mM\beta^4\delta_{12}^2\right\}}\right.$$

$$\left. + \delta_{12} \frac{1}{\left\{(1 + \delta_1 m\beta^2)(1 + \delta_2 m\beta^2) - mM\beta^4\delta_{12}^2\right\}}\right\} - \frac{F_o \beta \sin \omega_1(t - t_o)}{\omega_1(\lambda_2 - \lambda_1)}$$

$$\cdot \left\{\lambda_2 \frac{\delta_{12}^2 m\beta^2 - \delta_1(1 + \delta_2 m\beta^2)}{\left\{(1 + \delta_1 m\beta^2)(1 + \delta_2 m\beta^2) - mM\beta^4\delta_{12}^2\right\}}\right.$$

$$\left. + \delta_{12} \frac{1}{\left\{(1 + \delta_1 m\beta^2)(1 + \delta_2 m\beta^2) - mM\beta^4\delta_{12}^2\right\}}\right\} - \frac{F_o \cos \omega_2(t - t_o)}{(\lambda_2 - \lambda_1)}$$

$$\cdot \left\{\lambda_1 \frac{\delta_{12}^2 m\beta^2 - \delta_1(1 + m\beta^2\delta_2)}{\left\{(1 + \delta_1 m\beta^2)(1 + \delta_2 m\beta^2) - mM\beta^4 \delta_{12}^2\right\}}\right.$$

$$+ \delta_{12} \frac{1}{\left\{(1 + \delta_1 m\beta^2)(1 + \delta_2 m\beta^2) - mM\beta^4\delta_{12}{}^2\right\}}\Bigg\} + \frac{F_o \beta \sin \omega_2(t - t_o)}{\omega_2(\lambda_2 - \lambda_1)}$$

$$\cdot \left\{\lambda_1 \frac{\delta_{12}{}^2 m\beta^2 - \delta_1(1 + m\beta^2\delta_2)}{\left\{(1 + \delta_1 m\beta^2)(1 + \delta_2 m\beta^2) - mM\beta^4\delta_{12}{}^2\right\}}\right.$$

$$\left. + \delta_{12} \frac{1}{\left\{(1 + \delta_1 m\beta^2)(1 + \delta_2 m\beta^2) - mM\beta^4\delta_{12}{}^2\right\}}\right\} \qquad (4.43)$$

$$y_2 = \frac{\lambda_1(y_{10}\lambda_2 - y_{20})}{(\lambda_2 - \lambda_1)} \cos \omega_1 t + \frac{\lambda_1(v_{10}\lambda_2 - v_{20})}{\omega_1(\lambda_2 - \lambda_1)} \sin \omega_1 t + \frac{\lambda_2(y_{20} - \lambda_1 y_{10})}{(\lambda_2 - \lambda_1)} \cos \omega_2 t$$

$$+ \frac{\lambda_2(v_{20} - \lambda_1 v_{10})}{\omega_2(\lambda_2 - \lambda_1)} \sin \omega_2 t + \frac{F_o \delta_{12} e^{-\beta(t - t_o)}}{\left\{(1 + M\beta^2\delta_1)(1 + m\beta^2\delta_2) - mM\beta^4\delta_{12}{}^2\right\}}$$

$$+ \left\{\lambda_2 \frac{\delta_{12}{}^2 m\beta^2 - \delta_1(1 + \delta_2 m\beta^2)}{\left\{(1 + M\beta^2\delta_1)(1 + m\beta^2\delta_2) - mM\beta^4\delta_{12}{}^2\right\}}\right.$$

$$\left. + \delta_{12} \frac{1}{\left\{(1 + M\beta^2\delta_1)(1 + m\beta^2\delta_2) - mM\beta^4\delta_{12}{}^2\right\}}\right\} \left\{\frac{F_o \lambda_1 \cos \omega_1(t - t_o)}{(\lambda_2 - \lambda_1)}\right.$$

$$\left. - \frac{F_o \lambda_1 \beta \sin \omega_1(t - t_o)}{\omega_1(\lambda_2 - \lambda_1)}\right\} + \left\{\lambda_1 \frac{\delta_{12}{}^2 m\beta^2 - \delta_1(1 + \delta_2 m\beta^2)}{\left\{(1 + M\beta^2\delta_1)(1 + m\beta^2\delta_2) - mM\beta^4\delta_{12}{}^2\right\}}\right.$$

$$\left. + \delta_{12} \frac{1}{\left\{(1 + M\beta^2\delta_1)(1 + m\beta^2\delta_2) - mM\beta^4\delta_{12}{}^2\right\}}\right\}$$

$$\left\{- \frac{F_o \lambda_2 \cos \omega_2(t - t_o)}{(\lambda_2 - \lambda_1)} + \frac{F_o \beta \lambda_2 \sin \omega_2(t - t_o)}{\omega_2(\lambda_2 - \lambda_1)}\right\} \qquad (4.44)$$

For $\beta = 0$ this yields the response of the system due to a step function at $t = t_o$. For increasing β the excitation function approaches the value zero more rapidly. If the system was at rest at $t = 0$ the motion is described by the above expressions by omitting the first line.

e. Arbitrary wind build-up. The results of these investigations can now be used to describe an arbitrary, suddenly applied case of excitation. If at a time $t = t_o$ a force $F(t)$ is suddenly applied, the response of the system can be found by the superposition of all effects due to an infinitesimal change in the force (cross-hatched area in Fig. 7). The response of the system due to this part is the same as that of a step function suddenly applied at $t = \tau$ and of the magnitude $\frac{dF}{d\tau} d\tau = F'(\tau) d\tau$. By integration, the effect of the total change in $F(t)$ from $t = t_o$ on is obtained. In addition, the constant part due to $F(t_o)$ must be superimposed. The problem that has to be solved now is given by the system of differential equations

$$y_1 + \delta_1 M\ddot{y}_1 + \delta_{12} m\ddot{y}_2 = \begin{Bmatrix} o & \text{if } 0 \le t \le t_o \\ \delta_1 F(t) & \text{if } t > t_o \end{Bmatrix} \tag{4.45}$$

$$y_1 + \delta_{12} M\ddot{y}_1 + \delta_2 m\ddot{y}_2 = \begin{Bmatrix} o & \text{if} \quad 0 \le t \le t_o \\ \delta_1 F(t) & \text{if} \quad t > t_o \end{Bmatrix} . \tag{4.46}$$

The solution of this system in the time interval $0 \le t \le t_o$ is the same as that obtained previously (4.28) and (4.29). For $t > t_o$ the solution is composed of the various effects of the strips $F'(\tau) d\tau$ in addition to that of the constant force $F(t_o)$. The response due to $F'(\tau) d\tau$ is given by

$$dy_1 = F'(\tau) \left\{ \delta_1 - \frac{(\delta_1 \lambda_2 - \delta_{12})}{(\lambda_2 - \lambda_1)} \cos \omega_1 (t - \tau) - \frac{(\delta_{12} - \lambda_1 \delta_1)}{(\lambda_2 - \lambda_1)} \cos \omega_2 (t - \tau) \right\} d\tau$$

$$dy_2 = F'(\tau)\left\{\delta_{12} - \frac{\lambda_1(\delta_1\lambda_2 - \delta_{12})}{(\lambda_2 - \lambda_1)} \cos\omega_1(t-\tau) - \frac{\lambda_2(\delta_{12} - \lambda_1\delta_1)}{(\lambda_2 - \lambda_1)} \cdot \cos\omega_2(t-\tau)\right\} d\tau \quad .$$

Superimposing the response of the step function of magnitude $F(t_o)$ with the integrated parts of the force $F(t)$ above $F(t_o)$ yields the total response

$$\begin{aligned} y_1 = {} & \frac{(y_{10}\lambda_2 - y_{20})}{(\lambda_2 - \lambda_1)} \cos\omega_1 t + \frac{(v_{10}\lambda_2 - v_{20})}{\omega_1(\lambda_2 - \lambda_1)} \sin\omega_1 t + \frac{(y_{20} - \lambda_1 y_{10})}{(\lambda_2 - \lambda_1)} \cos\omega_2 t \\ & + \frac{(v_{20} - \lambda_1 v_{10})}{\omega_2(\lambda_2 - \lambda_1)} \sin\omega_2 t + F(t_o)\left\{\delta_1 - \frac{(\delta_1\lambda_2 - \delta_{12})}{(\lambda_2 - \lambda_1)} \cos\omega_1(t - t_o) \right. \\ & \left. - \frac{(\delta_{12} - \lambda_1\delta_1)}{(\lambda_2 - \lambda_1)} \cos\omega_2(t - t_o)\right\} + \int_{t_o}^{t}\left\{\delta_1 - \frac{(\delta_1\lambda_2 - \delta_{12})}{(\lambda_2 - \lambda_1)} \cos\omega_1(t-\tau) \right. \\ & \left. - \frac{(\delta_{12} - \lambda_1\delta_1)}{(\lambda_2 - \lambda_1)} \cos\omega_2(t-\tau)\right\} F'(\tau)\, d\tau \end{aligned} \tag{4.47}$$

and

$$\begin{aligned} y_2 = {} & \frac{\lambda_1(y_{10}\lambda_2 - y_{20})}{(\lambda_2 - \lambda_1)} \cos\omega_1 t + \frac{\lambda_1(v_{10}\lambda_2 - v_{20})}{\omega_1(\lambda_2 - \lambda_1)} \sin\omega_1 t \\ & + \frac{\lambda_2(y_{20} - \lambda_1 y_{10})}{(\lambda_2 - \lambda_1)} \cos\omega_2 t + \frac{\lambda_2(v_{20} - \lambda_1 v_{10})}{\omega_2(\lambda_2 - \lambda_1)} \sin\omega_2 t \\ & + F(t_o)\left\{\delta_{12} - \frac{\lambda_1(\delta_1\lambda_2 - \delta_{12})}{(\lambda_2 - \lambda_1)} \cos\omega_1(t - t_o) - \frac{\lambda_2(\delta_{12} - \delta_1\lambda_1)}{(\lambda_2 - \lambda_1)} \cos\omega_2(t - t_o)\right\} \end{aligned}$$

$$+ \int_{t_o}^{t} \left\{ \delta_{12} - \frac{\lambda_1(\delta_1\lambda_2 - \delta_{12})}{(\lambda_2 - \lambda_1)} \cos \omega_1(t - \tau) \right.$$

$$\left. - \frac{\lambda_2(\delta_{12} - \lambda_1\delta_1)}{(\lambda_2 - \lambda_1)} \cos \omega_2(t - \tau) \right\} F'(\tau)\, d\tau \tag{4.48}$$

Integrating by parts yields then

$$y_1 = \frac{(y_{10}\lambda_2 - y_{20})}{(\lambda_2 - \lambda_1)} \cos \omega_1 t + \frac{(v_{10}\lambda_2 - v_{20})}{\omega_1(\lambda_2 - \lambda_1)} \sin \omega_1 t + \frac{(y_{20} - \lambda_1 y_{10})}{(\lambda_2 - \lambda_1)} \cos \omega_2 t$$

$$+ \frac{(v_{20} - \lambda_1 v_{10})}{\omega_2(\lambda_2 - \lambda_1)} \sin \omega_2 t + \frac{\omega_1(\delta_1\lambda_2 - \delta_{12})}{(\lambda_2 - \lambda_1)} \sin \omega_1 t \int_{t_o}^{t} F(\tau) \cos \omega_1 \tau\, d\tau$$

$$- \frac{\omega_1(\delta_1\lambda_2 - \delta_{12})}{(\lambda_2 - \lambda_1)} \cos \omega_1 t \int_{t_o}^{t} F(\tau) \sin \omega_1 \tau\, d\tau$$

$$+ \frac{\omega_2(\delta_{12} - \lambda_1\delta_1)}{(\lambda_2 - \lambda_1)} \sin \omega_2 t \int_{t_o}^{t} F(\tau) \cos \omega_2 \tau\, d\tau$$

$$- \frac{\omega_2(\delta_{12} - \lambda_1\delta_1)}{(\lambda_2 - \lambda_1)} \cos \omega_2 t \int_{t_o}^{t} F(\tau) \sin \omega_2 \tau\, d\tau \tag{4.49}$$

and

$$y_2 = \frac{\lambda_1(y_{10}\lambda_2 - y_{20})}{(\lambda_2 - \lambda_1)} \cos \omega_1 t + \frac{\lambda_1(v_{10}\lambda_2 - v_{20})}{\omega_1(\lambda_2 - \lambda_1)} \sin \omega_1 t$$

$$+ \frac{\lambda_2(y_{20} - \lambda_1 y_{10})}{(\lambda_2 - \lambda_1)} \cos \omega_2 t + \frac{\lambda_2(v_{20} - \lambda_1 v_{10})}{\omega_2(\lambda_2 - \lambda_1)} \sin \omega_1 t$$

$$+\frac{\omega_1\lambda_1(\delta_1\lambda_2 - \delta_{12})}{(\lambda_2 - \lambda_1)} \sin \omega_1 t \int_{t_o}^{t} F(\tau) \cos \omega_1 \tau d\tau$$

$$-\frac{\omega_1\lambda_1(\delta_1\lambda_2 - \delta_{12})}{(\lambda_2 - \lambda_1)} \cos \omega_1 t \int_{t_o}^{t} F(\tau) \sin \omega_1 \tau d\tau$$

$$+\frac{\omega_2\lambda_2(\delta_{12} - \lambda_1\delta_1)}{(\lambda_2 - \lambda_1)} \sin \omega_2 t \int_{t_o}^{t} F(\tau) \cos \omega_2 \tau d\tau$$

$$-\frac{\omega_2\lambda_2(\delta_{12} - \lambda_1\delta_1)}{(\lambda_2 - \lambda_1)} \cos \omega_2 t \int_{t_o}^{t} F(\tau) \sin \omega_2 \tau d\tau \tag{4.50}$$

If the system was at rest at the time $t = 0$ the response for $t \geq t_o$ is given by the above formulae by omitting the first four terms in each equation.

SECTION V. DAMPED VIBRATIONS OF THE SYSTEM

Especially in the harmonically forced vibration case the displacements of the sound suppressor can be rather large around the natural frequencies of the system. Since the undamped case exhibits singularities at the natural frequencies, damping must be included to find the maximum amplitudes.

A. Introduction of Damping

The results of the previously derived potential theory of fluid oscillations are only valid and applicable if the exciting frequency is not too close to the resonances and if the excitation amplitude is not too large. The latter condition can be assumed to be satisfied; thus, linearized theory is justified. In the vicinity of the resonances, however, fluid forces occur which are a multiple of the inertial force of the liquid. These areas, of course, represent the important frequency range in which the motion of the liquid will have its most pronounced effect upon the structure of the sound suppressor system. The introduction of a damping factor is, therefore, of importance for the determination of the liquid forces in these frequency ranges. It is introduced as a linear viscous damping of which the damping factor is determined by experiments. This actually represents an equivalent linear damping, which, in the case of the sound suppressor system, is of very small magnitude ($\zeta_n = 0.001$ to 0.01).

To account for damping in the liquid a dissipation function

$$D = \frac{1}{2} \sum_{n=1}^{\infty} c_n \dot{y}_n^2$$

has to be introduced in the description of the liquid motion by the analytical model. This yields an additional term in the sloshing equations (3.9) which with

$$c_n = 2\zeta_n \omega_n m_n$$

now reads

$$\ddot{y}_n + 2\zeta_n \omega_n \dot{y}_n + \omega_n^2 y_n = -\ddot{y} \tag{5.1}$$

and yields a liquid force

$$F_y = m\ddot{y} \left[1 + \sum_{n=1}^{\infty} \frac{m_n}{m} \frac{\Omega^2}{(\omega_n^2 - \Omega^2 + 2i\zeta_n \omega_n \Omega)} \right] \tag{5.2}$$

and a moment of the fluid

$$M_z = -m\ddot{y} \sum_{n=1}^{\infty} \frac{m_n}{m} \frac{\left(g + h_n \Omega^2\right)}{(\omega_n^2 - \Omega^2 + 2i\zeta_n \omega_n \Omega)} \quad . \tag{5.3}$$

After introduction of the linear damping terms into the mechanical model, the values for the damped liquid force and moment can be obtained by merely introducing in the resonance terms instead of $(\omega_n^2 - \Omega^2)$ the expression $(\omega_n^2 - \Omega^2 + 2i\zeta_n \omega_n \Omega)$. The structural damping can be introduced as a linear viscous damping and is represented as a dashpot with a damping coefficient c_{ss} at the location of the mass M.

Therefore, the equations of motion (4. 8) are with damping, in y-direction

$$y_1 + \delta_{1_{y_1y_1}}(M\ddot{y}_1 + c_{ss}^{(y)}\dot{y}_1) + \delta_{2_{y_2y_1}}(m\ddot{y}_2 + c_{ss}^{(y)}\dot{y}_2) = 0 \quad (= \delta_{1_{y_1y_1}}F_{oy}\sin\Omega t) \tag{5. 4}$$

$$y_2 + \delta_{1_{y_1y_2}}(M\ddot{y}_1 + c_{ss}^{(y)}\dot{y}_1) + \delta_{2_{y_2y_2}}(m\ddot{y}_2 + c_{ss}^{(y)}\dot{y}_2) = 0 \quad (= \delta_{1_{y_1y_2}}F_{oy}\sin\Omega t) \tag{5. 5}$$

and similarly, in z-direction

$$z_1 + \delta_{1_{z_1z_1}}(M\ddot{z}_1 + c_{ss}^{(z)}\dot{z}_1) + \delta_{2_{z_2z_1}}(M\ddot{z}_2 + c_{ss}^{(z)}\dot{z}_2) = 0 \quad (= \delta_{1_{z_1z_1}}F_{oz}\sin\Omega t) \tag{5. 6}$$

$$z_2 + \delta_{1_{z_1z_2}}(M\ddot{z}_1 + c_{ss}^{(z)}\dot{z}_1) + \delta_{2_{z_2z_2}}(M\ddot{z}_2 + c_{ss}^{(z)}\dot{z}_2) = 0 \quad (= \delta_{1_{z_1z_2}}F_{oz}\sin\Omega t) . \tag{5. 7}$$

For the free oscillation problem the right hand sides of these equations are zero, while for the forced vibration problem (excitation again acting at the container) the right hand sides of these differential equations are represented by the expressions in the parentheses.

The damping coefficient c_{ss} of the structural damping is given by

$$c_{ss} = g_{ss}M\omega_{ss} = Mg_{ss}\sqrt{\frac{3EI}{M\ell^3}}$$

which in y-direction is

$$c_{ss}^{(y)} = g_{ss}M\sqrt{\frac{6\nu EI_y}{M\ell^3}} \tag{5. 8}$$

and in z-direction is

$$c_{ss}^{(z)} = g_{ss} M \sqrt{\frac{6\mu EI_z}{M\ell^3}} \qquad (5.9)$$

The value g_{ss} is the structural damping and probably has a value in the range $0.005 \le g_{ss} \le 0.02$.

The damping coefficient c_s of the sloshing liquid is given by

$$c_s = 2m\omega_s \zeta_s$$

which in y-direction is

$$c_s = 2m\omega_{sy} \zeta_s \qquad (5.10)$$

and in z-direction is

$$c_s = 2m\omega_{sz} \zeta_s \qquad (5.11)$$

B. Damped Sinusoidal Response

To obtain the steady-state solution for the damped system caused by some harmonic excitation, complex solution methods are employed. With $\delta_{1_{y_1y_1}} \equiv \delta_1$, $\delta_{2_{y_1y_2}} \equiv \delta_{12}$ and $\delta_{2_{y_2y_2}} \equiv \delta_2$, the two simultaneous differential equations (5.4) and (5.5) can be written in the form

$$\ddot{y}_1 M\delta_1 + \ddot{y}_2 m\delta_{12} + \delta_1 c_{ss}^{(y)} \dot{y}_1 + \delta_{12} c_s^{(y)} \dot{y}_2 + y_1 = \delta_1 F_{oy} e^{i\Omega t} \qquad (5.12)$$

$$\ddot{y}_1 M\delta_{12} + \ddot{y}_2 m\delta_2 + \delta_{12} c_{ss}^{(y)} \dot{y}_1 + \delta_2 c_s^{(y)} \dot{y}_2 + y_2 = \delta_{12} F_{oy} e^{i\Omega t} \tag{5.13}$$

The steady-state solution of this system is

$$y_1 = \overline{Y}_1 e^{i\Omega t}$$

$$y_2 = \overline{Y}_2 e^{i\Omega t}$$

which inserted in the above equations yields two complex algebraic equations for the complex amplitudes $\overline{Y}_1$ and $\overline{Y}_2$:

$$\overline{Y}_1(1 - M\delta_1\Omega^2) + i\Omega\overline{Y}_1\delta_1 c_{ss}^{(y)} + i\Omega\overline{Y}_2\delta_{12} c_s^{(y)} - m\delta_{12}\Omega^2\overline{Y}_2 = \delta_1 F_{oy}$$

$$\overline{Y}_1 i\Omega\delta_{12} c_{ss}^{(y)} - M\delta_{12}\Omega^2\overline{Y}_1 + \overline{Y}_2(1 - m\delta_2\Omega^2) + i\Omega\overline{Y}_2\delta_2 c_s^{(y)} = \delta_{12} F_{oy}$$

which yields

$$\overline{Y}_1 = \frac{\delta_1 F_{oy}(1 - m\delta_2\Omega^2 + c_s^{(y)}\delta_2 i\Omega) - \delta_{12} F_{oy}(\delta_{12} c_s^{(y)} i\Omega - m\delta_{12}\Omega^2)}{(1 - M\delta_1\Omega^2 + i\Omega\delta_1 c_{ss}^{(y)})(1 - m\delta_2\Omega^2 + i\Omega\delta_2 c_s^{(y)}) - (\delta_{12} c_s^{(y)} i\Omega - m\delta_{12}\Omega^2)(\delta_{12} c_{ss}^{(y)} i\Omega - M\delta_{12}\Omega^2)}$$

$$\overline{Y}_2 = \frac{\delta_{12} F_{oy}(1 - M\delta_1\Omega^2 + i\Omega\delta_1 c_{ss}^{(y)}) - \delta_1 F_{oy}(i\Omega\delta_{12} c_{ss}^{(y)} - M\delta_{12}\Omega^2)}{(1 - M\delta_1\Omega^2 + i\Omega\delta_1 c_{ss}^{(y)})(1 - m\delta_2\Omega^2 + i\Omega\delta_2 c_s^{(y)}) - (\delta_{12} c_s^{(y)} i\Omega - m\delta_{12}\Omega^2)(\delta_{12} c_{ss}^{(y)} i\Omega - M\delta_{12}\Omega^2)}$$

Since the amplitude of the structure and liquid rather than the phase relation is of interest, only the absolute value of the amplitudes is investigated:

$$\left| \frac{\overline{Y}_1}{F_{oy}} \right| = \sqrt{\frac{c^2 + d^2}{a^2 + b^2}} \qquad (5.14)$$

$$\left| \frac{\overline{Y}_2}{F_{oy}} \right| = \frac{\delta_{12}}{\sqrt{a^2 + b^2}} \qquad (5.15)$$

where

$$a = 1 - \Omega^2 \left[M\delta_1 + m\delta_2 + 2mM\omega_s^{(y)} \sqrt{\frac{6EI_y \alpha\nu}{M\ell^3}}\, g_{ss}^{(y)} \zeta_s^{(y)} \left[\delta_1\delta_1 - \delta_{12}^2\right] \right]$$

$$+ mM\Omega^2 \left(\delta_1\delta_2 - \delta_{12}^2 \right)$$

$$b = \Omega \Bigg\{ \delta_1 M g_{ss}^{(y)} \sqrt{\frac{6EI_y \alpha\nu}{M\ell^3}} + 2\delta_2 m \omega_s^{(y)} \zeta_s^{(y)} - \Omega^2 \Bigg[mM\delta_1\delta_2 g_{ss}^{(y)} \sqrt{\frac{6EI_y \alpha\nu}{M\ell^3}}$$

$$+ 2\delta_1\delta_2 mM\omega_s^{(y)} \zeta_s^{(y)} - \delta_{12}^2 \left(mMg_{ss}^{(y)} \sqrt{\frac{6EI_y \alpha\nu}{M\ell^3}} + 2mM\omega_s^{(y)} \zeta_s^{(y)} \right) \Bigg] \Bigg\}$$

$$c = \delta_1(1 - m\delta_2\Omega^2) + m\delta_{12}^2\Omega^2$$

$$d = 2m\Omega\omega_s^{(y)} \zeta_s^{(y)} (\delta_1\delta_1 - \delta_{12}^2)$$

A parameter α has been included to show the effect of stiffness variation.

C. Damped Liquid Oscillations

The introduction of linear damping into the results of the liquid analysis (Section II) requires in the resonance terms the value $(1 - \eta_n^2 + 2i\zeta_n\eta_n)$ instead of $(1 - \eta_n^2)$. Some of the expressions in Section II, however, cannot be used in their present form. A transformation with

$$y - \frac{a}{2} = -\frac{4a}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos\left[\frac{(2n-1)}{a}\pi y\right]}{(2n-1)^2}$$

yields a free fluid surface elevation

$$\bar{x}(y,t) = \frac{4a\Omega^2}{\pi^2 g}y_o e^{i\Omega t}\sum_{n=1}^{\infty}\frac{\cos\left[\frac{(2n-1)}{a}\pi x\right]}{(\eta^2_{2n-1} - 1 + 2i\zeta_{2n-1}\eta_{2n-1})(2n-1)^2}. \qquad (5.16)$$

The pressure distributions can be kept in their original form by replacing into them the expression $(1 - \eta_n^2 + 2i\zeta_n\eta_n)$ instead of $(1 - \eta_n^2)$. The damped liquid force is

$$F_y = m\Omega^2 y_o e^{i\Omega t}\left[1 + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\tanh\left[\frac{(2n-1)\pi h}{a}\right]\eta^2_{2n-1}}{(2n-1)^3\pi\frac{h}{a}\left(1-\eta^2_{2n-1} + 2i\zeta_{2n-1}\eta_{2n-1}\right)}\right].$$

(5.17)

The moment with

$$\frac{1}{12\frac{h}{a}} = \frac{8}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{\left(\frac{h}{a}\right)(2n-1)^4}$$

is

$$M_z = \frac{4ma}{\pi^3} \Omega^2 y_o e^{i\Omega t} \sum_{n=1}^{\infty} \frac{\left[(2n-1) \tanh\left(\frac{(2n-1)}{a} \pi h\right) + \frac{4}{\pi \frac{h}{a} \cosh\left[\frac{(2n-1)\pi h}{a}\right]}\right] \eta_{2n-1}^2 - \frac{2}{\pi \frac{h}{a}}}{(2n-1)^4 \left(\eta_{2n-1}^2 - 1 + 2i\zeta_{2n-1}\eta_{2n-1}\right)} . \tag{5.18}$$

SECTION VI. NUMERICAL EVALUATION AND CONCLUSIONS

A. Liquid Motion and Mechanical Analogon

The natural frequencies of the liquid in the container are well described by the formula (2.9). They are indirectly proportional to the square root of the length of the container. Table 1 exhibits the frequencies that come into play during excitation in y-direction and z-direction if one chooses for the length of the container $a_1 = 250$ feet and $a_2 = 150$ feet respectively. Since no roll excitation is involved in the structural analysis only these frequencies are used. Since the liquid height is rather small compared to the length and width of the container the hyperbolic tangent function plays an important role.

The circular frequencies, of which thirty are presented in Table 1, exhibit rather small values. In y-direction they range from $\omega_1 = 0.22$ to $\omega_{30} = 3.48$ rad/sec and in z-direction from $\omega_1 = 0.43$ to $\omega_{30} = 4.83$ rad/sec. Of course, the fundamental frequency $\omega_1 = 0.22$ rad/sec for h_1/a_1 or $\omega_1 = 0.28$ rad/sec for h_3/a_1 and $\omega_1 = 0.43$ rad/sec for h_1/a_2 or $\omega_1 = 0.53$ rad/sec for h_3/a_2 play the most important role since the largest sloshing mass is connected with those.

The values of the liquid height were chosen $h_1 = 10$ feet, $h_2 = 12$ feet and $h_3 = 16$ feet. The sloshing liquid masses can be obtained from the mechanical analogon and are represented in Table 2 for various liquid heights and container excitation directions. The results show that the first sloshing mode is the most dominant and that eighty percent of the liquid mass sloshes in the first vibration mode. This is an extremely large value and requires special precautions. If four million gallons of water will oscillate, the system will experience very large forces and moments produced by the oscillating liquid. The sloshing mass of the second mode is only about eight percent of the total liquid mass and only one-tenth of that of the first mode, while the third sloshing mode exhibits only one to three percent of the total liquid mass participating in an oscillation. These magnitudes suggest, of course, that only the very large first sloshing mode has to be considered for a stability and response analysis. The water in the container can, therefore, be described by a simple mechanical model consisting of a mass point attached with springs at the tank wall, such that the ratio of spring stiffness to sloshing mass k/m represents the square of the first circular natural frequency ω_1^2 of the liquid in the container.

Before proceeding to the interaction of structure and liquid a few results of the fluid response to translatory excitation of the container are presented. The pressure distribution along the container bottom is given for various fluid heights and forcing frequencies in y- and z-direction. Figures 8 through 11 exhibit the pressure $(p_{bottom} - \rho gh)/\rho a y_o e^{i\Omega t}$. The maximum pressure is exhibited at left and right container walls and assumes for an excitation amplitude of y_o feet a value of about $(55y_o + 62.4)$ lbs/ft^2, which yields for an excitation amplitude of $y_o = 1$ ft a value of 679 lbs/ft^2. This happens if the forcing frequency is $\Omega = 1.1\omega_1$. For a larger fluid height of $h_2 = 13$ feet the maximum pressure at the container bottom for one foot excitation amplitude is about 880 lbs/ft^2. For excitation in z-direction where the container exhibits a length of $a_2 = 150$ feet, the maximum pressure at the container bottom is 744 lbs/ft^2 for a liquid height of $h_1 = 10$ feet and 960 lbs/ft^2 for a liquid height of $h_2 = 13$ feet (Fig. 11). Again, these pressures are based on one-foot excitation amplitude and a forcing frequency close to resonance, $\Omega = 1.1\omega_1$. The response of the fluid force (lbs/ft) and liquid moment [ft - lbs/ft] are exhibited in Figures 12 and 13. At resonance, of course, they exhibit singularities since no damping has been included in the results of the theory. For a very small damping a peak value of the force in y-direction of about 3.63×10^6 lbs is obtained in the first resonance for a one-foot excitation amplitude, while the liquid moment exhibits at this location a value of about 6×10^8 ft-lbs about the center of gravity of the undisturbed liquid.

B. Interaction of Structure and Liquid

Here the free and forced oscillations have been treated. Furthermore, the response of the system to various wind pulses has been performed.

1. Free Oscillations. The knowledge of the coupled frequencies and mode shapes is of fundamental interest. In addition, the effect of the change of stiffness has been investigated and it was found that an increased stiffness would not pay off. Table 3 exhibits the values for the coupled natural frequencies $\omega_{1;2}$ of the sound suppressor system for the orientation in the z-direction. The

value α is a stiffness parameter exhibiting for unity in its magnitude the nominal stiffness and for $\alpha = 2$ twice the stiffness of the support structure.

It was found that the natural frequencies in y-direction for a liquid height of sixteen feet are extremely low (0.22 rad/sec) and that an increase in stiffness does not particularly pay off. In z-direction, where the sound suppressor system exhibits its largest area against a wind, the lowest natural frequency is presented in Table 3 and is very close to the uncoupled fundamental natural frequency of the liquid. It is reached monotonically ($\omega_1 = 0.5319$) with increasing stiffness of the structure. The second coupled natural frequency exhibits larger values (a little more than one cps for nominal stiffness of the support structure) and shows that a doubling of the stiffness of the support structure increases this frequency only by about forty percent. This frequency approaches with increasing stiffness the value infinity.

2. Sinusoidally Forced Oscillations. Since a wind in the z-direction is more likely to build up in a regular fashion and since the area obstructing the wind perpendicular to the z-direction is the largest, this direction is chosen for the numerical evaluation of the response of the system to a sinusoidal excitation force. Although a pure sinusoidal wind of amplitude 13 m/sec is quite unlikely for a longer time period, the results will reveal some interesting facts about the influence of the damping of the structure and liquid. All wind data are based on the American Standards Association (ASA) paper A58.1 - 1955, "American Standard Building Code Requirements for Minimum Design Loads in Buildings and other Structures," and the American Society of Civil Engineers (ASCE) paper No. 3269, "Wind Forces on Structures."

The response curves for which graphs have been omitted here show that the response of structure as well as liquid is very small as long as the forcing frequency Ω is not in the immediate neighborhood of one of the natural frequencies ω_1 or ω_2. An increase in structural damping decreases the response around the resonance peak of the structure considerably, but has hardly any noticeable influence upon the magnitude of the liquid response. The same is true for a change of liquid damping. An increase of the damping of the liquid will change (decrease) the peak length of the liquid response in the immediate vicinity of the natural frequency of the liquid, but does not effect the response of the structure.

The following values exhibit the effect of the change of liquid and structural damping upon the peak values for the deflection of the structure (z_1 max) and the amplitude of the liquid (z_2 max) at the container wall.

	$\zeta = 0.001$		$\zeta = 0.01$		$\zeta = 0.02$		$\zeta = 0.05$	
	g=0.001	g=0.02	g=0.001	g=0.02	g=0.001	g=0.02	g=0.001	g=0.02
$z_{1\,max}$	50"	2.5"	49.5"	2.5"	49.5"	2.5"	49.5"	2.5"
$z_{2\,max}$	9.6"	9.6"	5.3"	5"	3"	3"	1.25"	1.25"

Most likely the system will exhibit a liquid damping factor ζ between $\zeta = 0.01$ and $\zeta = 0.03$ and a structural damping of at least $g = 0.02$. A maximum displacement of the container in the most unfavorable sinusoidal wind excitation case would be of a magnitude of 2.5 inches, while the liquid height would exhibit a magnitude of about five inches.

3. Wind Response. The response of the sound suppressor system has been numerically evaluated for various wind inputs in z-direction. An average value for the wind pressure of 50 psf has been employed. Figures 14 through 37 exhibit the response of an exponentially decaying pulse for various stiffness parameters α and various degrees of decay. The equivalent wind force is $F_{z\,eq.} = 1.1 \times 10^6$ lbs. It is assumed here that the system at the time $t = 0$ was at rest and that at this time a wind of equivalent force $F_{z\,eq.}\, e^{-\beta t}$ hits the sound suppressor system. For $\beta = 0$ the step function is obtained (i.e., a wind force which at $t = 0$ suddenly assumes a value $F_{z\,eq.}$ and remains at this value). The figures exhibit the displacement of the structure $(z_1/F_{z\,eq.})$ in ins/lbs and the liquid amplitude $(z_2/2.5F_{z\,eq.})$ in ins/lbs, where the factor 2.5 stems from the conversion of the displacement z_2 to the liquid amplitude at the container wall. The main results are presented in Table 4.

It can be seen that for the case of excitation by a step function $(\beta = 0)$, the structure performs oscillations and exhibits a maximum displacement of about 2.76 inches for half the nominal stiffness $(\alpha = \frac{1}{2})$. The liquid amplitude for these cases of stiffness are 7.2 inches for half the nominal stiffness, 3.6 inches for nominal stiffness, 1.77 inches for twice the stiffness and 0.7 inches for five times the structural stiffness of the support structure.

For $\beta = 0.1$ a slight decay of wind is experienced. Here, the response of structure and liquid can be seen for the various stiffness cases in Figures 22 through 29. The maximum structural amplitudes are 2.7 inches for half the nominal stiffness of the support structure, 1.3 inches for nominal stiffness, 0.66 inches for twice the stiffness and 0.28 inches for five times the nominal stiffness. The liquid exhibits a maximum amplitude of 5.5 inches for $\alpha = \frac{1}{2}$, 2.8 inches for $\alpha = 1$, 1.4 inches for $\alpha = 2$ and 0.55 inches for $\alpha = 5$.

Similar results are obtained for an exponentially decaying wind with $\beta = 0.5$. This wind already reaches half its initial value in about 1.6 seconds in contrast to the case $\beta = 0.1$ where it takes 7 seconds for the wind to exhibit half its initial magnitude. The results of the response of the container and the liquid are exhibited in Figures 30 through 37. The maximum amplitudes reached for the structure are 2.3 inches for $\alpha = \frac{1}{2}$, 1.2 inches for $\alpha = 1$, 0.63 inches for $\alpha = 2$ and 0.23 inches for $\alpha = 5$. The maximum liquid amplitudes are 2.8 inches for $\alpha = \frac{1}{2}$, 1.4 inches for $\alpha = 1$, 0.7 inches for $\alpha = 2$ and 0.28 inches for $\alpha = 5$. The values in the parenthesis behind the maximum values in the table indicate the values which are assumed after the pulse has ceased to be effective.

The response of the support structure and the liquid to a rectangular pulse of $t = 2$, 4 and 10 seconds duration is exhibited in Figures 38 through 61 for various stiffness parameters ($\alpha = \frac{1}{2}$, 1, 2 and 5). A wind is suddenly applied and ceases to act at a certain time t_1. It may be noted that the pulse duration and the natural frequency have an important influence upon the magnitude of the vibration after the pulse has been completed. This can be seen from the following investigation. The solution for the motion of the system after the pulse has been completed is given by

$$z_1 = A_1[\cos \omega_1(t - t_1) - \cos \omega_1 t] + A_2[\cos \omega_2(t - t_1) - \cos \omega_2 t]$$

$$z_2 = \lambda_1 A_1[\cos \omega_1(t - t_1) - \cos \omega_1 t] + \lambda_2 A_2[\cos \omega_2(t - t_1) - \cos \omega_2 t]$$

where A_1 and A_2 are some given amplitudes. Combining the terms in the parenthesis to one trigonometric function yields

$$\cos \omega_1(t - t_1) - \cos \omega_1 t = C \sin (\omega_1 t - \alpha)$$

where

$$C \sin \alpha = \sin \omega_1 t_1 = 2 \sin \frac{\omega_1 t_1}{2} \cos \frac{\omega_1 t_1}{2}$$

$$C \cos \alpha = [1 - \cos \omega_1 t_1] = 2 \sin^2 \frac{\omega_1 t_1}{2}$$

which gives the amplitude and phase relation

$$C = 2 \left| \sin \frac{\omega_1 t_1}{2} \right| \qquad \text{and} \qquad \alpha = \frac{\omega_1 t_1}{2} \ .$$

Therefore

$$z_1 = 2A_1 \left| \sin \frac{\omega_1 t_1}{2} \right| \sin \left(\omega_1 t - \frac{\omega_1 t_1}{2} \right) + 2A_2 \left| \sin \frac{\omega_2 t_1}{2} \right| \sin \left(\omega_2 t - \frac{\omega_2 t_1}{2} \right)$$

$$z_2 = 2\lambda_1 A_1 \left| \sin \frac{\omega_1 t_1}{2} \right| \sin \left(\omega_1 t - \frac{\omega_1 t_1}{2} \right) + 2\lambda_2 A_2 \left| \sin \frac{\omega_2 t_1}{2} \right| \sin \left(\omega_2 t - \frac{\omega_2 t_1}{2} \right) \ .$$

From this it can be seen that the magnitude of the vibration after the pulse has been completed depends on the value $\left| \sin \frac{\omega_1 t_1}{2} \right|$ and $\left| \sin \frac{\omega_2 t_1}{2} \right|$. If those values happen to be small values the response will be small. The main results are given in Table 4, while the more detailed response can be seen in Figures 38 through 61.

The response for a rectangular wind pulse of various durations ($t_1 = 2$, 4, 10 sec.) for $\alpha = \frac{1}{2}$ is exhibited in Figures 38 through 43. The maximum amplitude of the container is 2.76 inches in all three pulse cases. After the pulse has been completed the structure oscillates with about the same amplitude in the case of pulse duration $t_1 = 2$ seconds, while it drastically has changed its amplitude to a value of 0.3 inches in the case of a pulse duration $t_1 = 4$ seconds. For a pulse duration of $t_1 = 10$ seconds the amplitude remains 2.76 inches. The liquid amplitudes for these cases are in the same sequence given by 3.6 inches, 6.2 inches and 7.2 inches.

After the pulse is completed the liquid amplitude remains about the same for the cases of pulse duration $t_1 = 2$ and 4 seconds, while for $t_1 = 10$ seconds it changes to the small value of 3.5 inches. For nominal stiffness ($\alpha = 1$) the results can be seen in Figures 44 through 49. For a pulse duration of $t_1 = 2$ seconds the structure exhibits a maximum deflection of 1.4 inches and assumes the value of 0.6 inches after the pulse has been completed. For four seconds pulse duration the amplitude is 1.4 inches; the structure then shows also a value of 1.4 inches for the pulse duration of ten seconds, after which it performs oscillations of maximum amplitude of 1.2 inches. The liquid exhibits in the first case of a pulse duration of 2 seconds the amplitude of 1.8 inches, while for $t_1 = 4$ seconds, it is 3.1 inches and finally for a pulse duration of $t_1 = 10$ seconds the amplitude of the liquid is 3.6 inches during pulse duration and 1.7 inches after the pulse has been completed.

For twice the nominal stiffness ($\alpha = 2$) the structure exhibits in all three pulse cases 0.7 inches, while after the pulse has been completed the amplitude is 0.01 inches for $t_1 = 2$ and 4 seconds and 0.03 inches for $t_1 = 10$ seconds. The liquid exhibits values of 0.9 inches throughout in the case $t_1 = 2$ seconds, 1.5 inches throughout in the case $t_1 = 4$ seconds, and 1.8 inches during pulse duration of length $t_1 = 10$ seconds and 0.8 inches after the pulse has been completed. (See Figs. 50 through 55).

For five times the nominal stiffness ($\alpha = 5$) the structure exhibits in all three pulse cases the displacement 0.3 inches during the duration of the pulse, while after the pulse has been completed an amplitude of 0.2 inches for $t_1 = 2$ seconds, 0.3 inches for $t_1 = 4$ seconds and 0.23 inches for $t_1 = 10$ seconds exists. The liquid exhibits an amplitude of 0.36 inches throughout for $t_1 = 2$ seconds, 0.62 inches throughoutfor $t_1 = 4$ seconds, 0.72 inches during pulse duration $t_1 = 10$ seconds, 0.32 inches after the pulse has been completed (Figs. 56 through 61).

Numerical Values

liquid mass $\quad m = 0.865 \times 10^5 \dfrac{\text{lbs sec}^2}{\text{inch}}$

liquid weight $\quad W = 3.34 \times 10^7$ lbs weight of support structure $W_{ss} = 5.76 \times 10^6$ lbs

mass of support structure $m_{ss} = 1.5 \times 10^4 \dfrac{\text{lbs sec}^2}{\text{inch}}$

weight of container $W_c = 3.58 \times 10^6$ lbs

mass of container $m_c = 0.926 \times 10^4 \dfrac{\text{lbs sec}^2}{\text{inch}}$

fluid heights: $h_1 = 10$ feet

$h_2 = 13$ feet

$h_3 = 16$ feet

container dimensions: $a_1 = 250$ feet

$a_2 = 130$ feet

pairs of support beams: $\nu = 16$; $\mu = 8$

Young's modulus of elasticity: $E = 2.1 \times 10^7$ lbs/inch^2

geometric moment of inertia of each pair of I-beams: $I_y = 7348$ (inch)^4
$I_z = 1 \times 10^6$ (inch)^4

distance of the location of the sloshing mass from ground level: $\ell = 90$ feet $= 1.08 \times 10^3$ inches

distance of the nonsloshing mass: $L = 91$ feet $= 1.09 \times 10^3$ inches

SECTION VII. CONCLUSIONS AND RECOMMENDATIONS

According to the previously presented results it can be seen that because eighty percent of the liquid (4×10^6 gallons) sloshes in the first vibration mode, the problem of interaction of structure and liquid can be a quite serious one. In y-direction the structure is a little weak and exhibits large deflections if excited in resonance. A regular pattern for a wind input is unlikely because the tower obstructs a clean build-up of winds; therefore, a slight stiffening is recommended. In z-direction a stiffening does not pay off, since the design of the support structure seems adequate to withstand various wind pulses.

George C. Marshall Space Flight Center,
National Aeronautics and Space Administration,
Huntsville, Alabama, September 30, 1965.

TABLE I. NATURAL CIRCULAR FREQUENCIES

natural circular frequency	y-direction			z-direction			
ω_n	h_1/a_1	h_2/a_1	h_3/a_1	h_1/a_2	h_2/a_2	h_3/a_2	fluid height ratio
ω_1	0.2248	0.2258	0.2832	0.4293	0.4683	0.5319	
ω_2	0.4461	0.5051	0.5555	0.8354	0.9305	1.0044	
ω_3	0.6608	0.7422	0.8087	1.2024	1.3104	1.3837	
ω_4	0.8663	0.9634	1.0380	1.5243	1.6358	1.6851	
ω_5	1.0609	1.1666	1.2425	1.8028	1.8880	1.9306	
ω_6	1.2435	1.3515	1.4236	2.0439	2.1104	2.1389	
ω_7	1.4136	1.5191	1.5842	2.2548	2.3042	2.3223	
ω_8	1.5716	1.6710	1.7276	2.4421	2.4774	2.4886	
ω_9	1.7180	1.8090	1.8569	2.6111	2.6358	2.6425	
ω_{10}	1.8538	1.9353	1.9748	2.7660	2.7829	2.7869	
ω_{11}	1.9797	2.0515	2.0835	2.9098	2.9213	2.9236	
ω_{12}	2.0970	2.1592	2.1848	3.0450	3.0526	3.0539	
ω_{13}	2.2066	2.2598	2.2801	3.1730	3.1780	3.1788	
ω_{14}	2.3094	2.3545	2.3704	3.2951	3.2984	3.2988	
ω_{15}	2.4062	2.4441	2.4565	3.4123	3.4144	3.4147	
ω_{16}	2.4979	2.5295	2.5390	3.5251	3.5265	3.5267	
ω_{17}	2.5851	2.6113	2.6186	3.6342	3.6351	3.6352	
ω_{18}	2.6683	2.6899	2.6955	3.7400	3.7406	3.7406	
ω_{19}	2.7480	2.7657	2.7700	3.8427	3.8431	3.8431	
ω_{20}	2.8247	2.8392	2.8424	3.9427	3.9429	3.9429	
ω_{21}	2.8987	2.9105	2.9129	4.0402	4.0403	4.0403	
ω_{22}	2.9703	2.9798	2.9817	4.1353	4.1354	4.1354	
ω_{23}	3.0379	3.0474	3.0488	4.2283	4.2283	4.2283	

TABLE I. (Concluded)

natural circular frequency	y-direction			z-direction			
ω_n	h_1/a_1	h_2/a_1	h_3/a_1	h_1/a_2	h_2/a_2	h_3/a_2	fluid height ratio
ω_{24}	3.1072	3.1135	3.1145	4.3192	4.3193	4.3193	
ω_{25}	3.1730	3.1780	3.1788	4.4083	4.4084	4.4084	
ω_{26}	3.2372	3.2412	3.2418	4.4956	4.4957	4.4957	
ω_{27}	3.2999	3.3031	3.3036	4.5813	4.5813	4.5813	
ω_{28}	3.3613	3.3639	3.3642	4.6654	4.6654	4.6654	
ω_{29}	3.4214	3.4235	3.4238	4.7479	4.7479	4.7479	
ω_{30}	3.4805	3.4821	3.4823	4.8291	4.8291	4.8291	

TABLE II. SLOSH MASS RATIO

	y-direction			2n - 1/m z-direction		
n	h_1/a_1	h_2/a_1	h_3/a_1	h_1/a_2	h_2/a_2	h_3/a_2
1	0.806	0.803	0.7998	0.795	0.785	0.772
2	0.086	0.083	0.081	0.077	0.070	0.064
3	0.029	0.027	0.025	0.022	0.019	0.016

TABLE III. COUPLED NATURAL FREQUENCIES
($\alpha \equiv$ STIFFNESS PARAMETER)

α	ω_{1_z}	ω_{2_z}
$\frac{1}{2}$	0.528	4.76
1	0.53	6.69
2	0.53	9.43
5	0.53	14.89

TABLE IV. WIND RESPONSE

Decaying Pulse:

β	z_1	z_2	z_1	z_2	z_1	z_2	z_1	z_2
0	2.76"	7.2"	1.4"	3.6"	0.7"	1.77"	0.3"	0.7"
0.1	2.7"	5.5"	1.3"(0.7")	2.8"(1.2")	0.66"(0.38")	1.4"(0.9")	0.28"(0.14")	0.55"(0.36")
0.5	2.3"(1.3")	2.8"(2.7")	1.2"(0.7")	1.4"(1.3")	0.63"(0.34")	0.7"(0.7")	6.23"(0.14")	0.28"(0.27")

Rectangular Pulse:

t_1 \ α	$\frac{1}{2}$	$\frac{1}{2}$	1	1	2	2	5	5
	z_1	z_2	z_1	z_2	z_1	z_2	z_1	z_2
2	2.76"	3.6"	1.4"(0.6")	1.8"	0.7"(0.01")	0.9"	0.3"(0.2")	0.36"
4	2.78"(0.3")	6.2"	1.4"(1")	3.1"	0.7"(0.01")	1.5"	0.3"	0.62"
10	2.76"	7.2"(3.5")	1.4"(1.2")	3.6"(1.7")	0.7"(0.03")	1.8"(0.8")	0.3"(0.23")	0.72"(0.32")

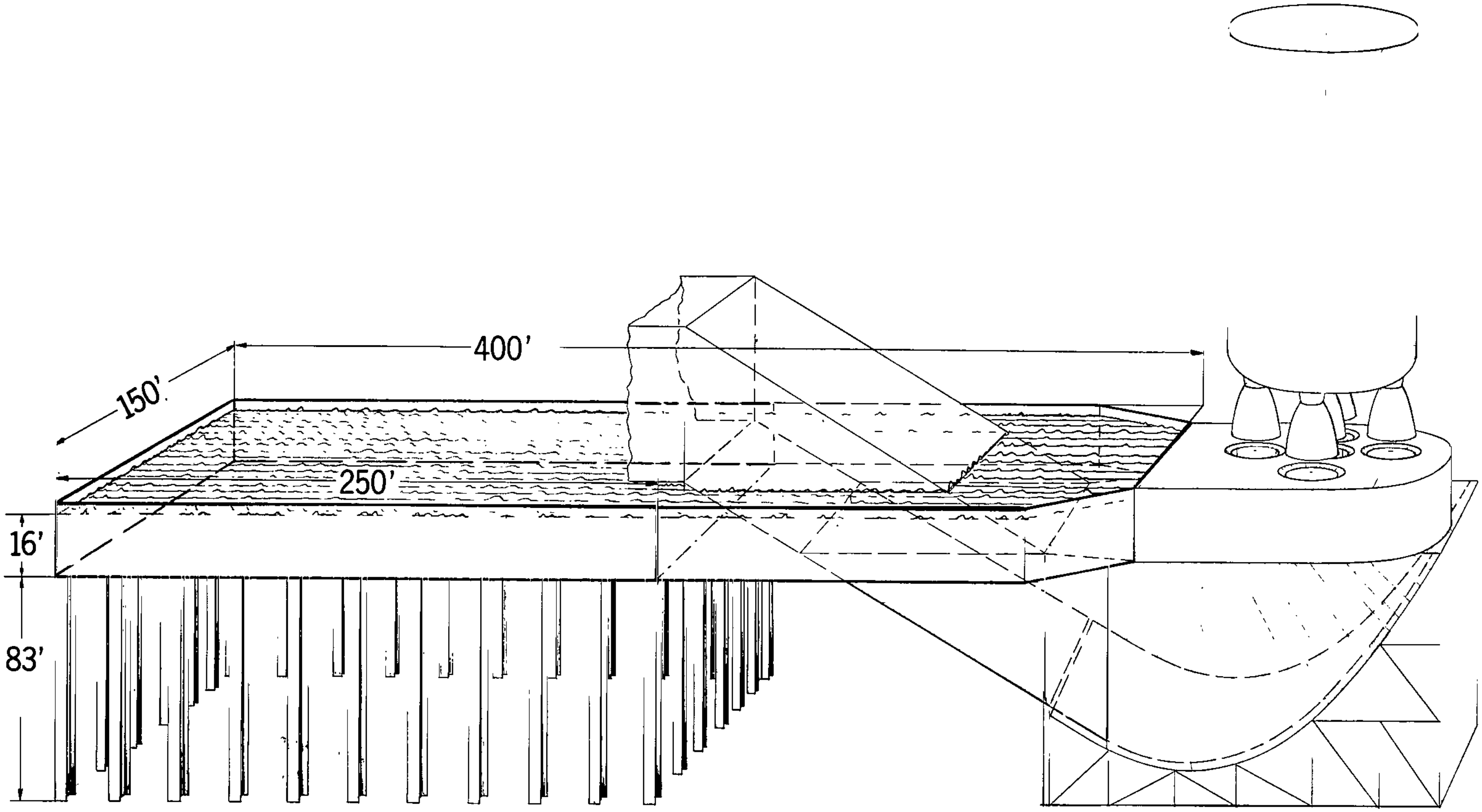

FIGURE 1. SCHEMATIC OF TANK SYSTEM, SATURN V SOUND SUPPRESSOR

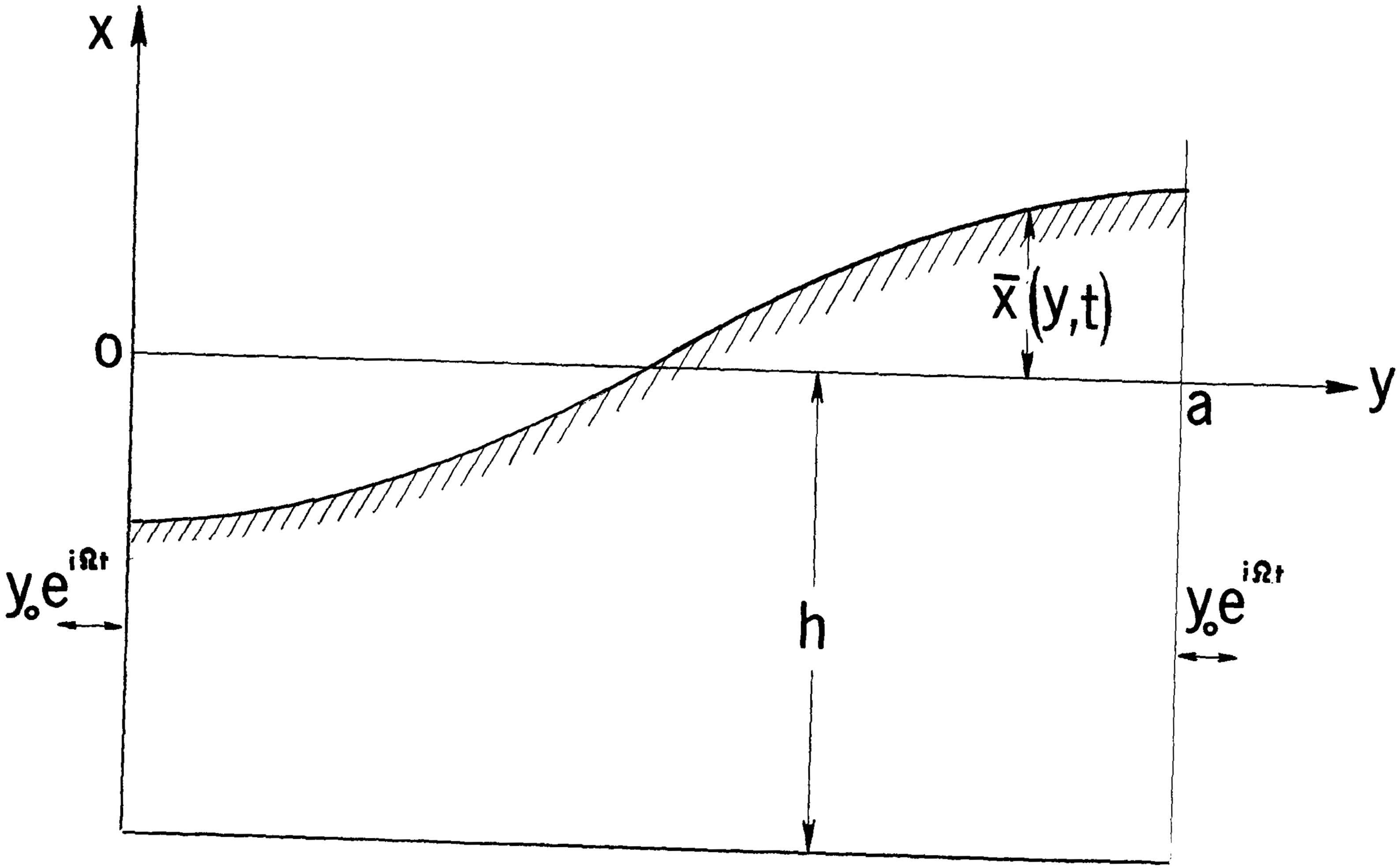

FIGURE 2. TANK GEOMETRY AND COORDINATE SYSTEM

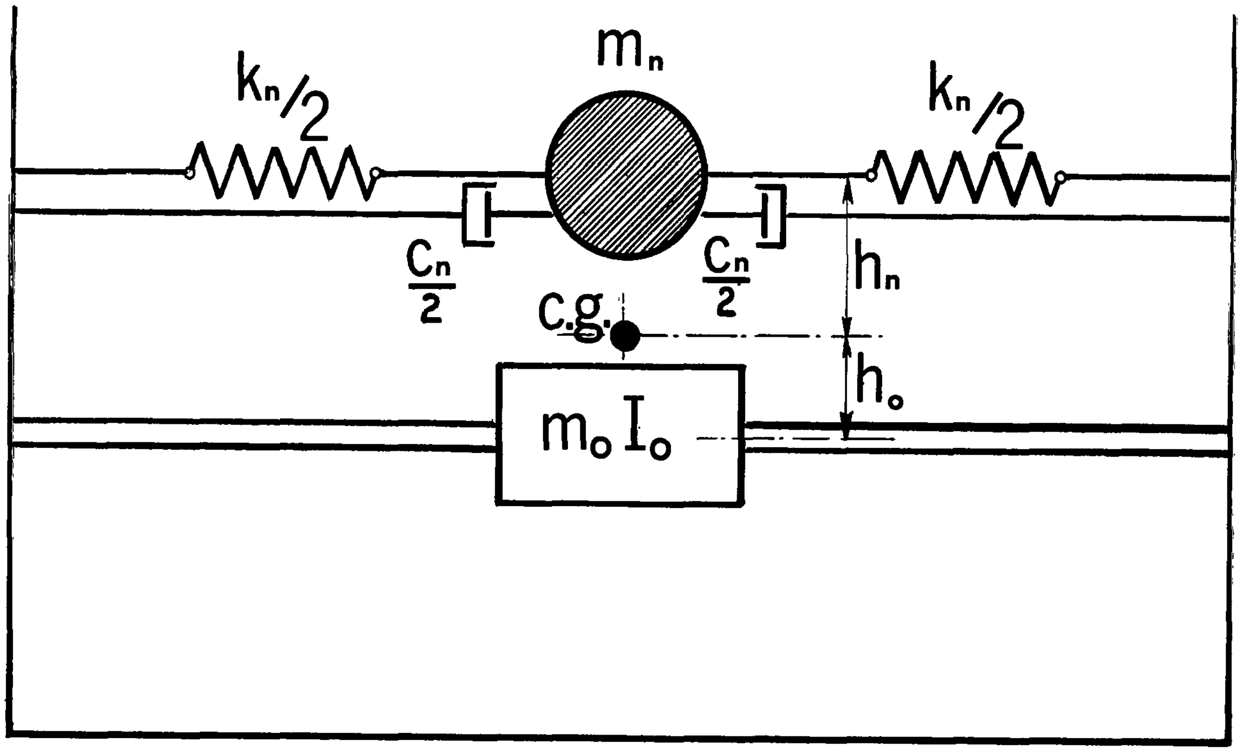

FIGURE 3. MECHANICAL MODEL

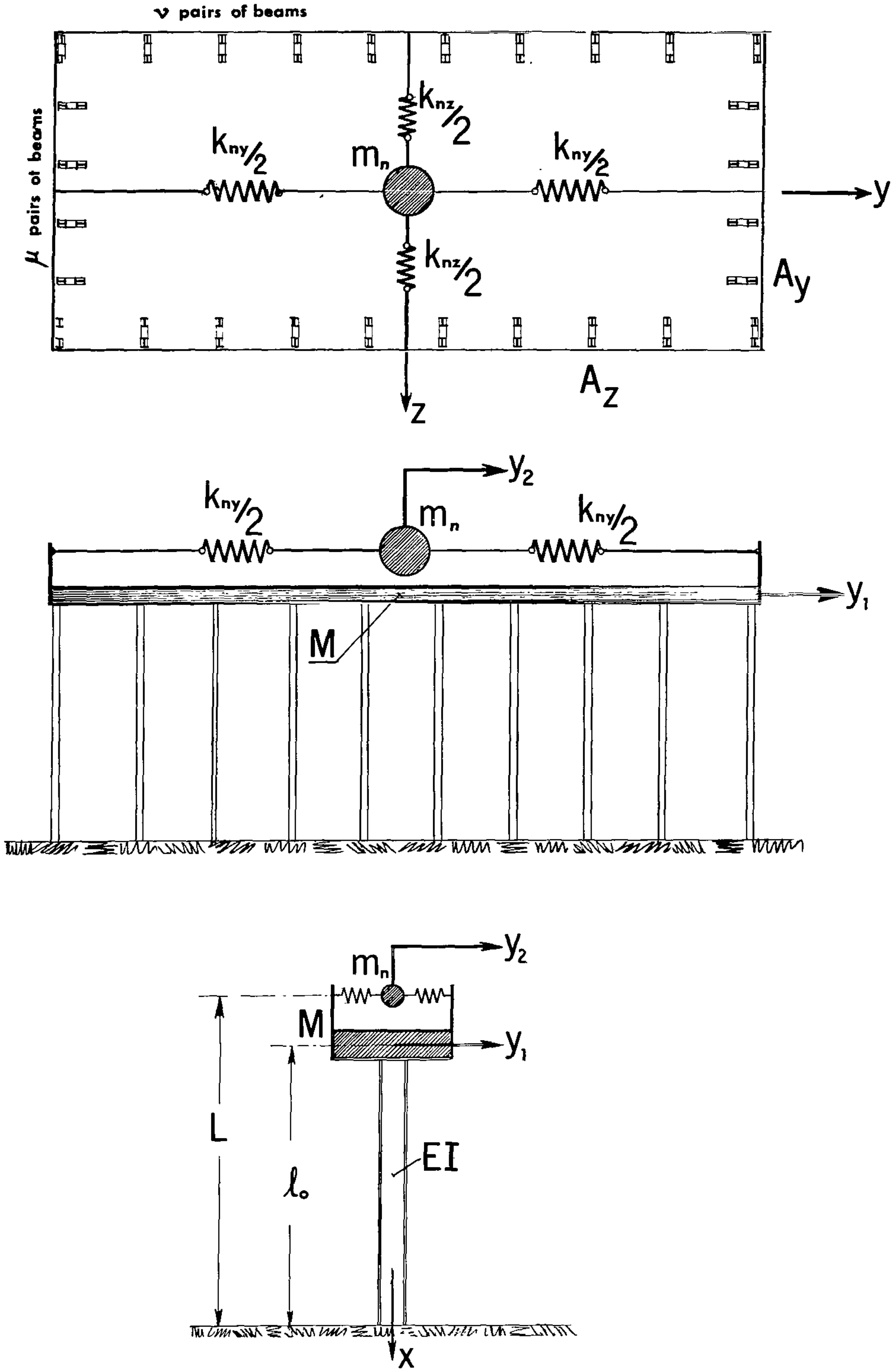

FIGURE 4. MECHANICAL SYSTEM

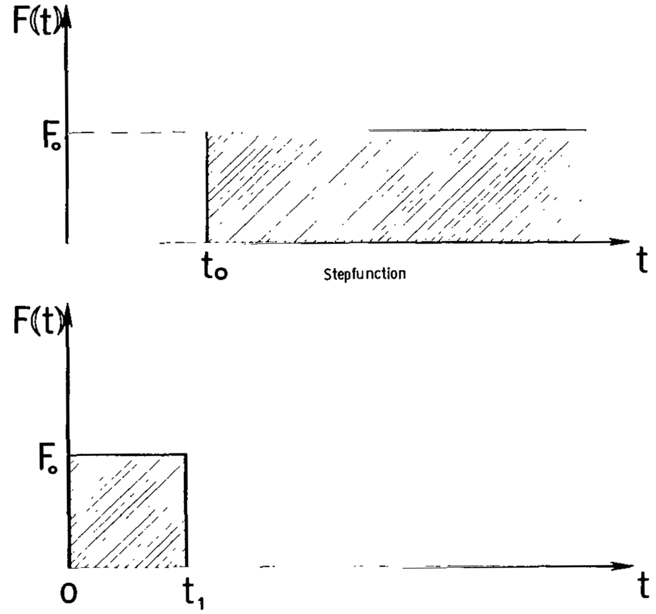

FIGURE 5. STEP FUNCTION AND RECTANGULAR PULSE

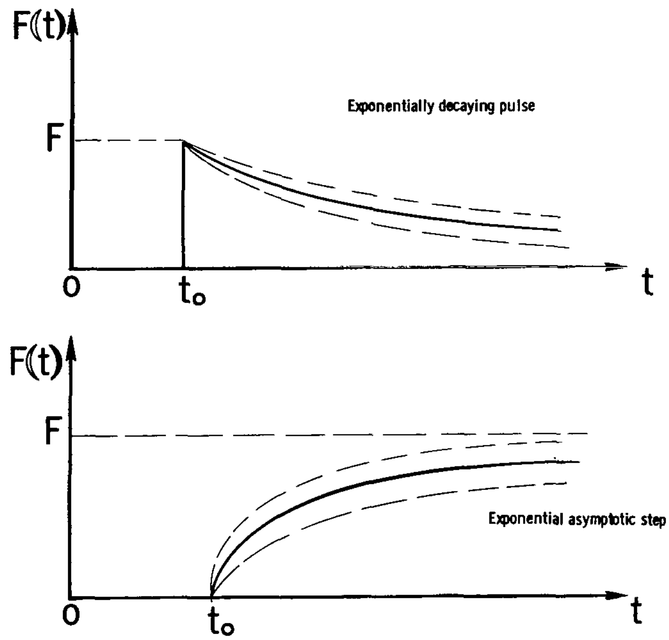

FIGURE 6. DISTURBING FUNCTIONS

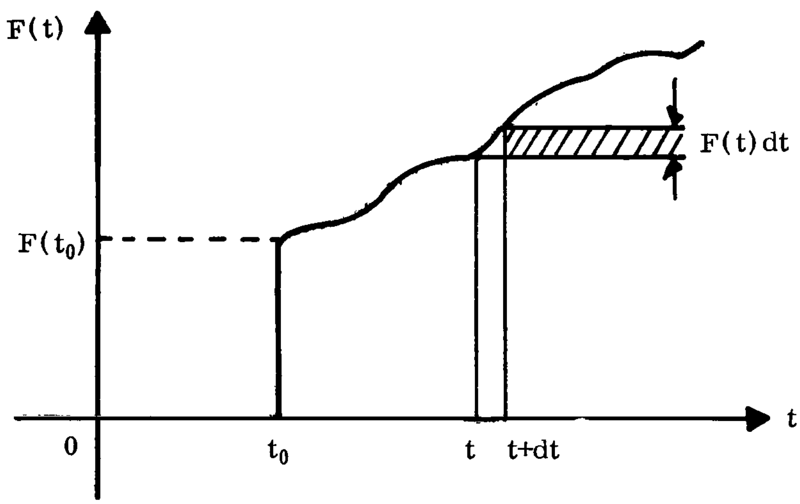

FIGURE 7. ARBITRARY WIND BUILD-UP

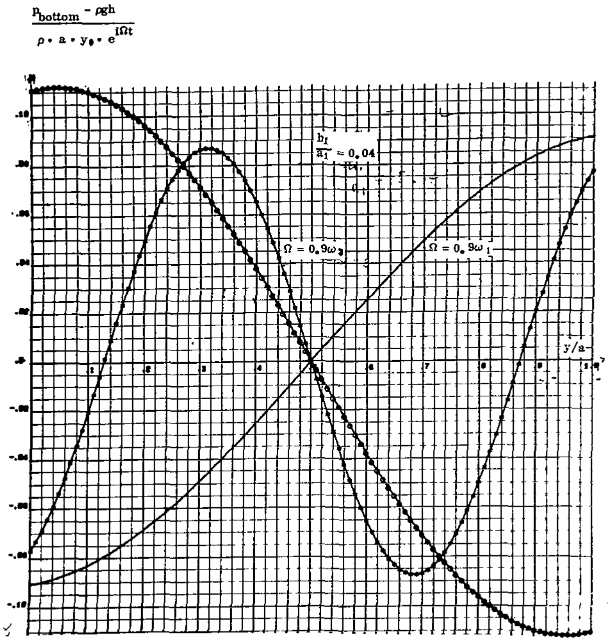

FIGURE 8. PRESSURE DISTRIBUTION AT CONTAINER BOTTOM FOR EXCITATION IN Y-DIRECTION

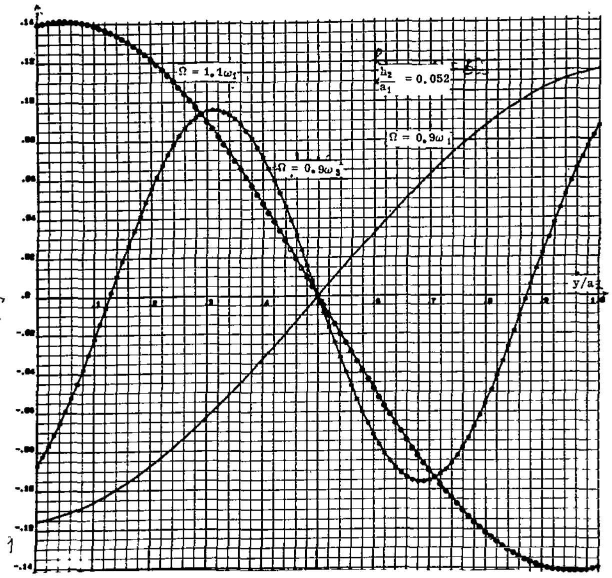

FIGURE 9. PRESSURE DISTRIBUTION AT CONTAINER BOTTOM FOR EXCITATION IN Y-DIRECTION

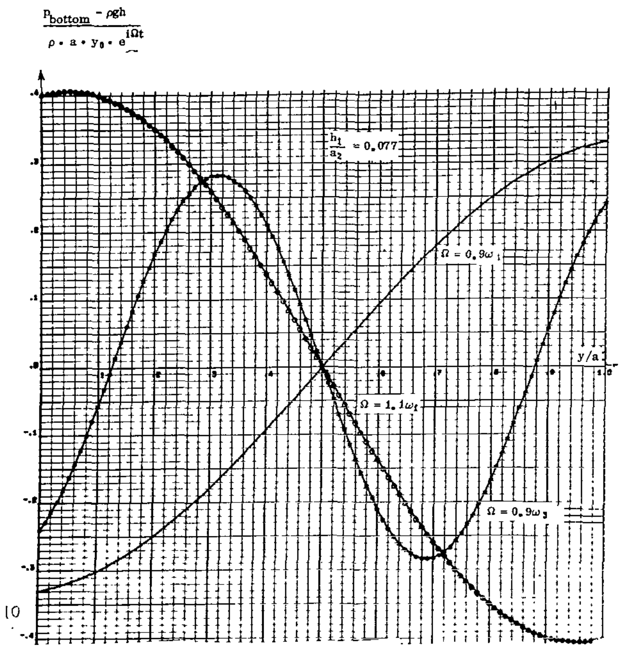

FIGURE 10. PRESSURE DISTRIBUTION AT CONTAINER BOTTOM FOR EXCITATION IN Z-DIRECTION

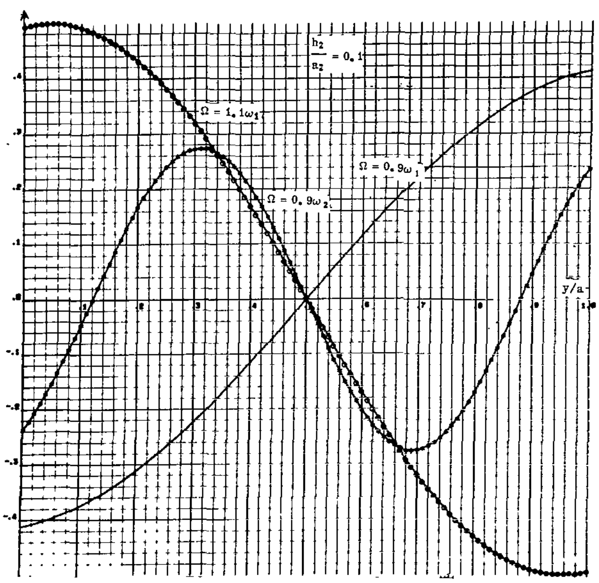

FIGURE 11. PRESSURE DISTRIBUTION AT CONTAINER BOTTOM FOR EXCITATION IN Z-DIRECTION

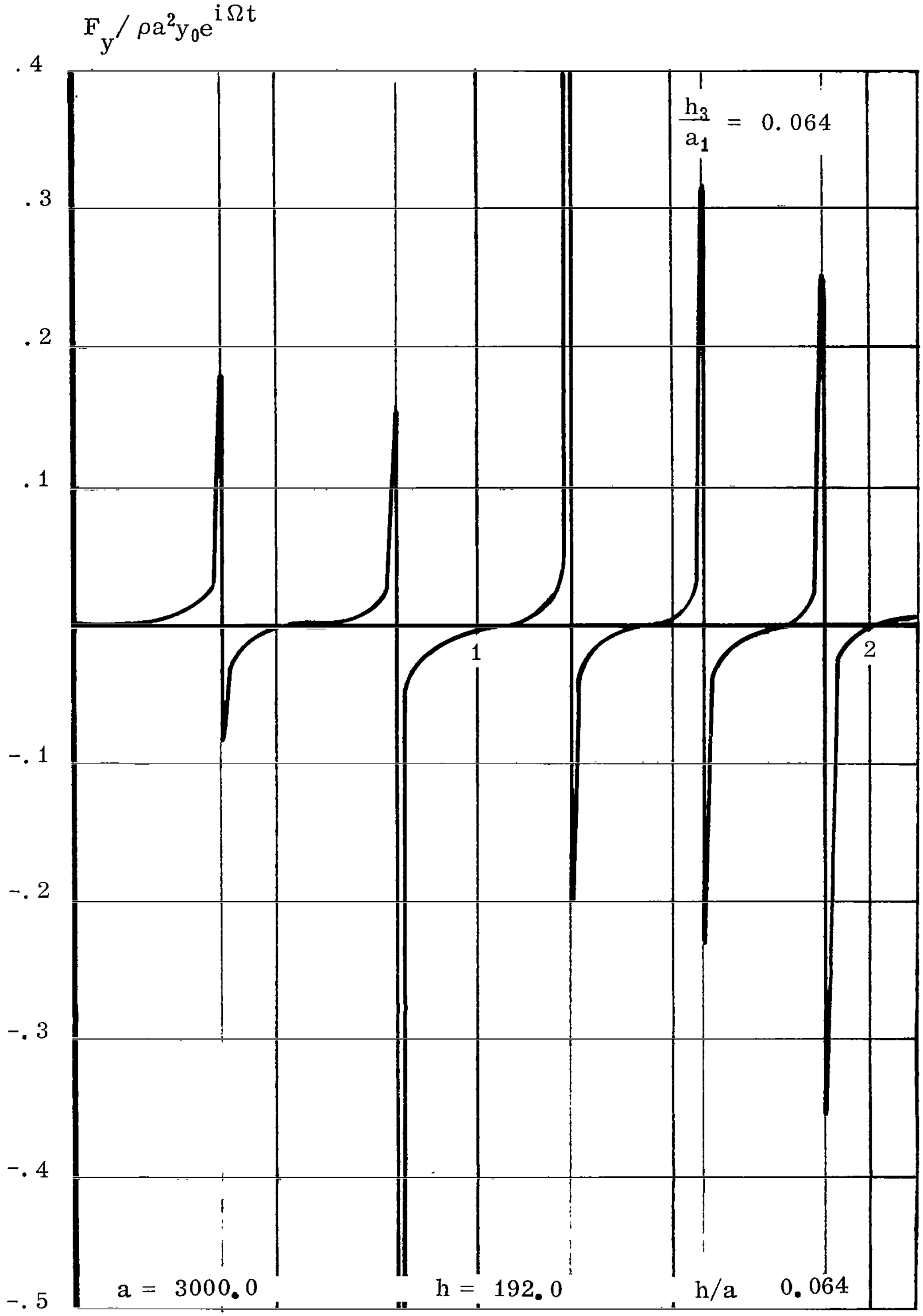

FIGURE 12. RESPONSE OF FLUID FORCE TO HARMONIC EXCITATION

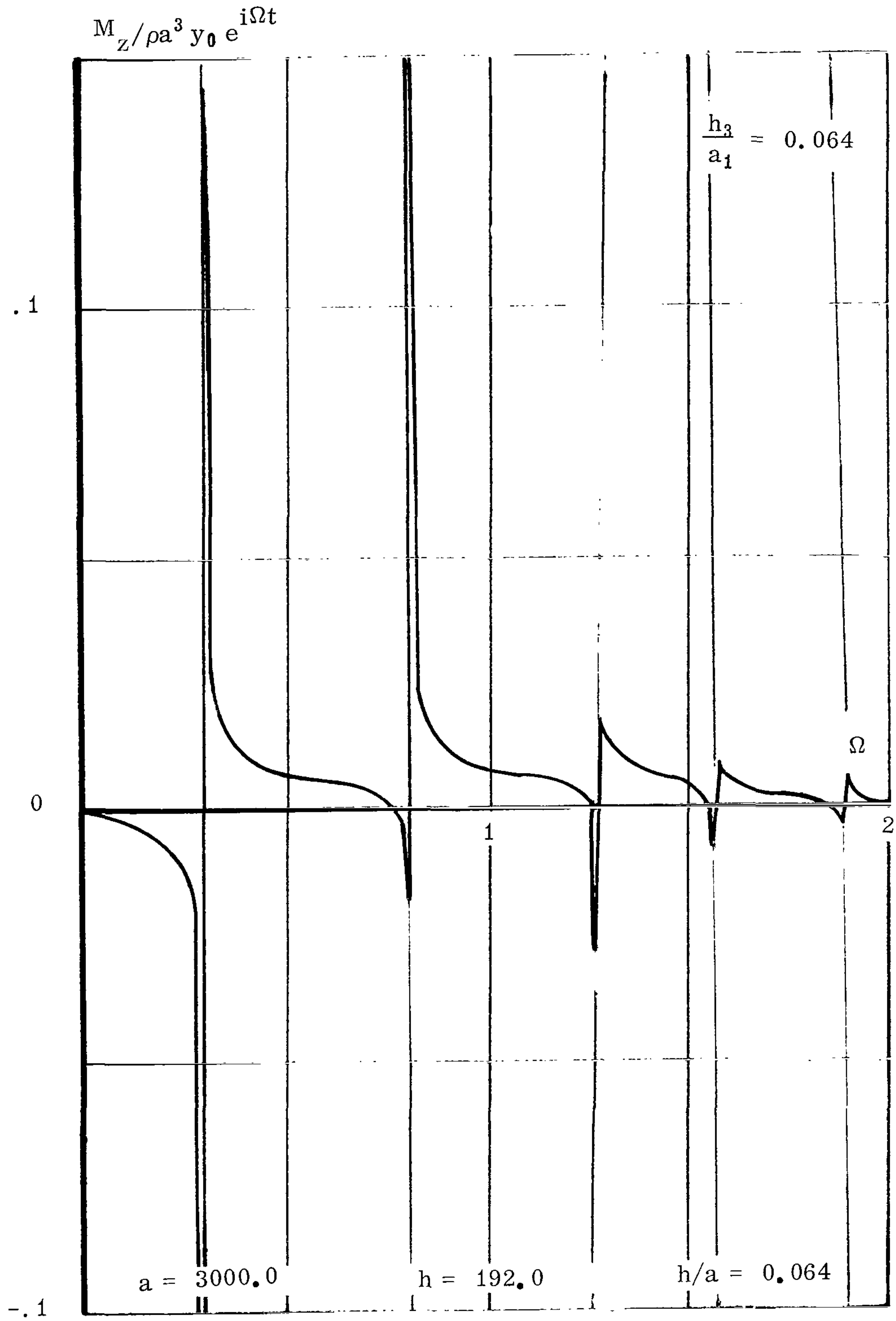

FIGURE 13. RESPONSE OF LIQUID MOMENT TO HARMONIC EXCITATION

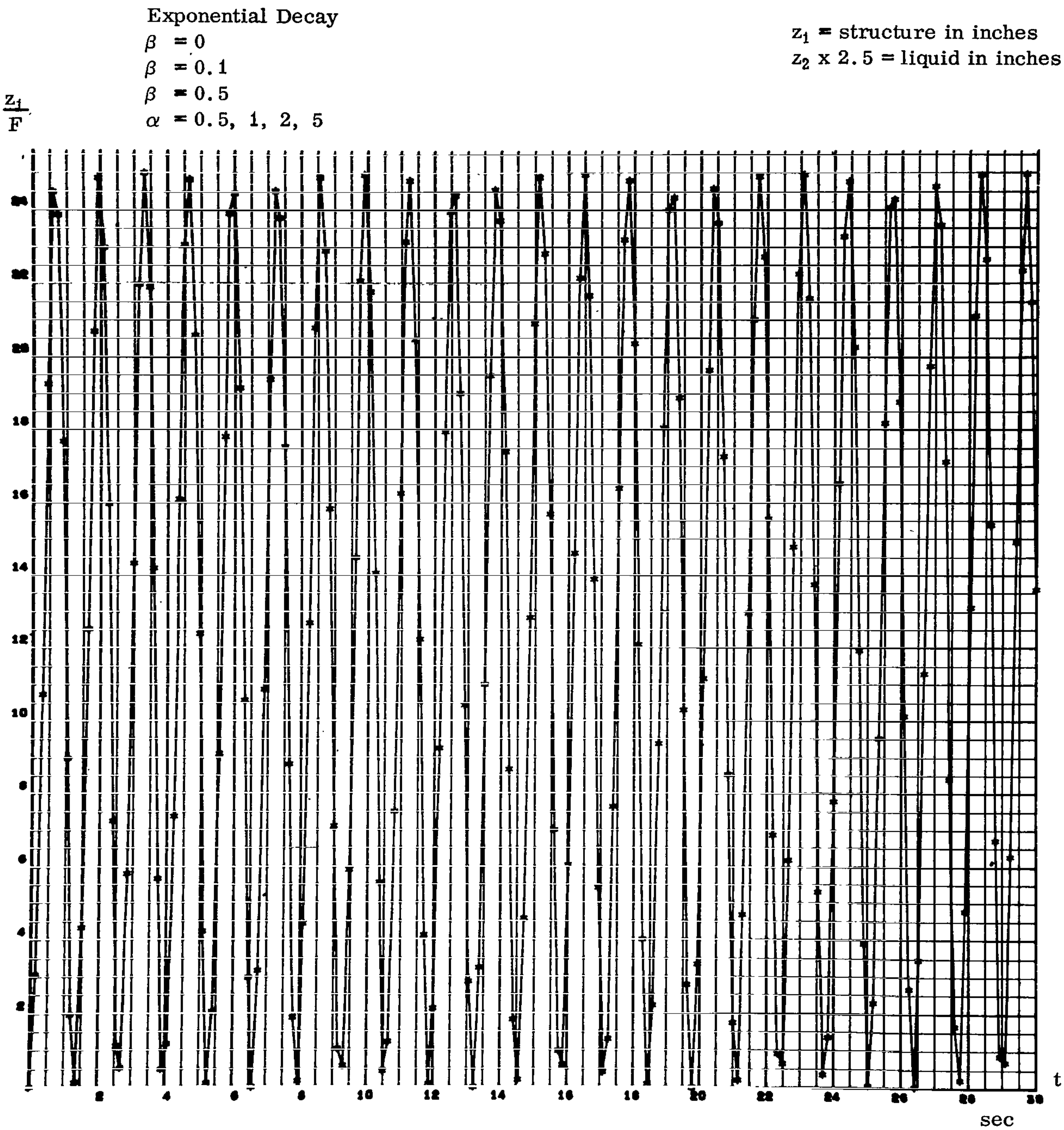

FIGURE 14. RESPONSE OF STRUCTURE TO STEP FUNCTION ($\alpha = \frac{1}{2}$)

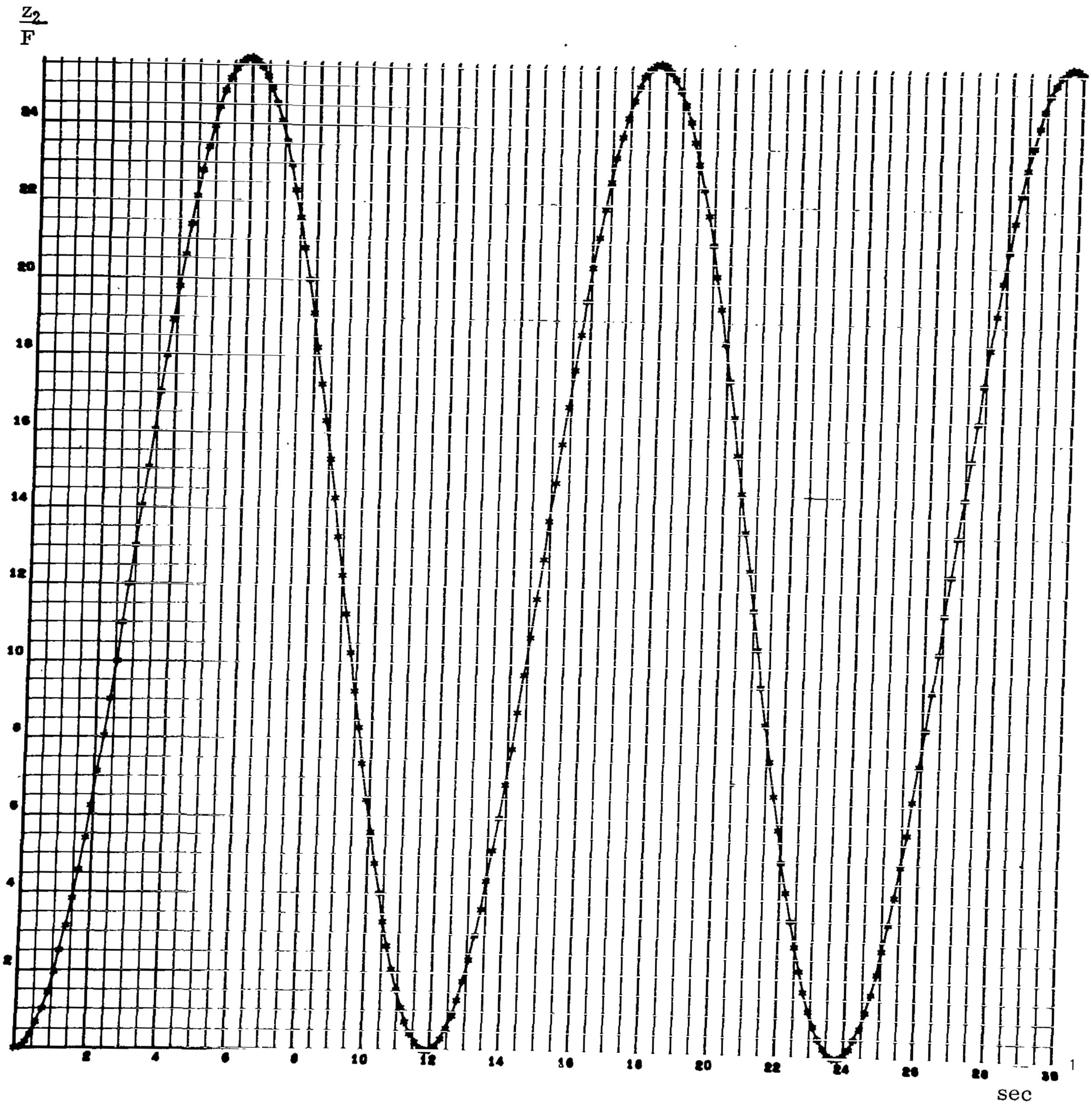

FIGURE 15. RESPONSE OF LIQUID TO STEP FUNCTION ($\alpha = \frac{1}{2}$)

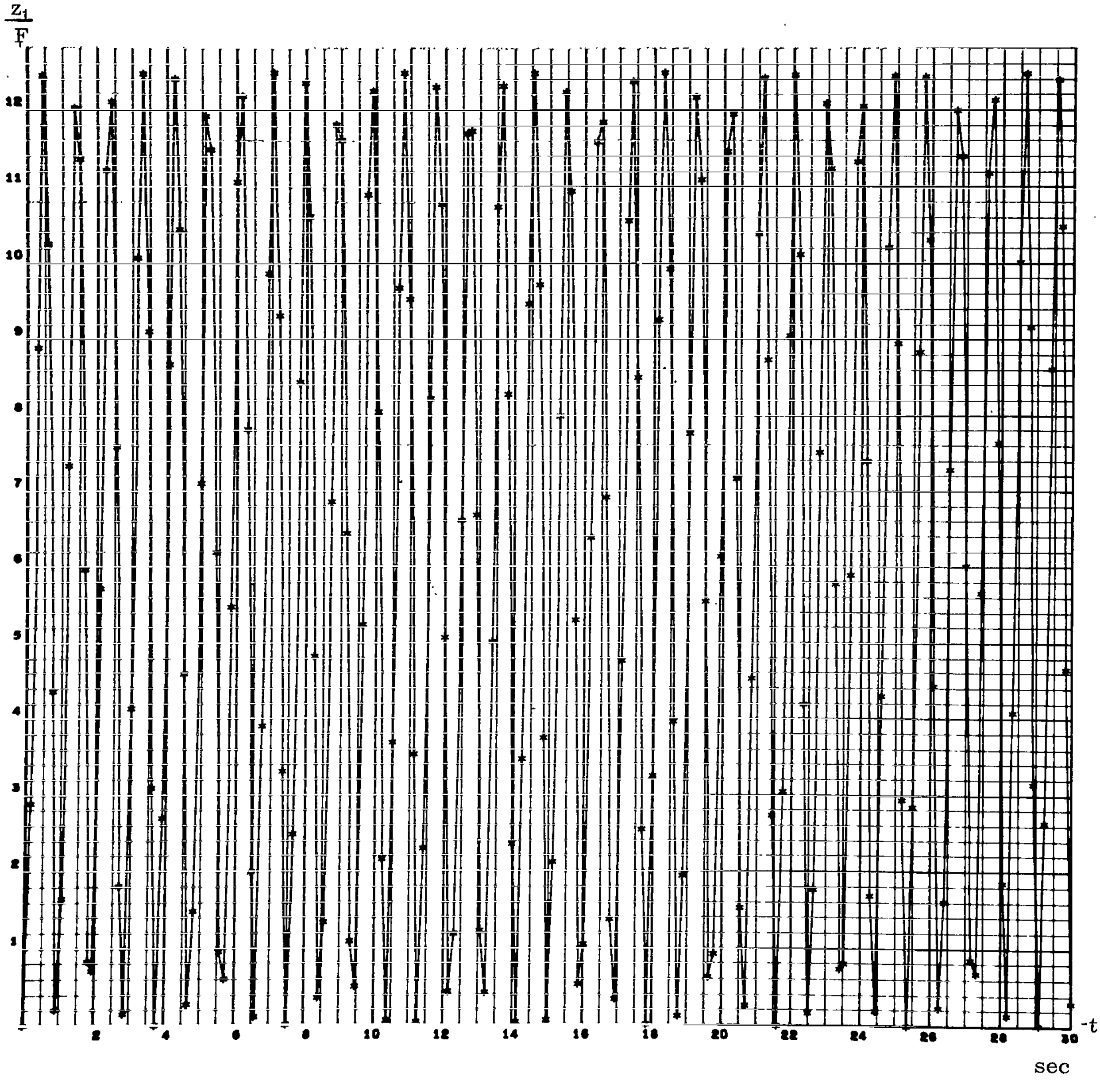

FIGURE 16. RESPONSE OF STRUCTURE TO STEP FUNCTION (α = 1)

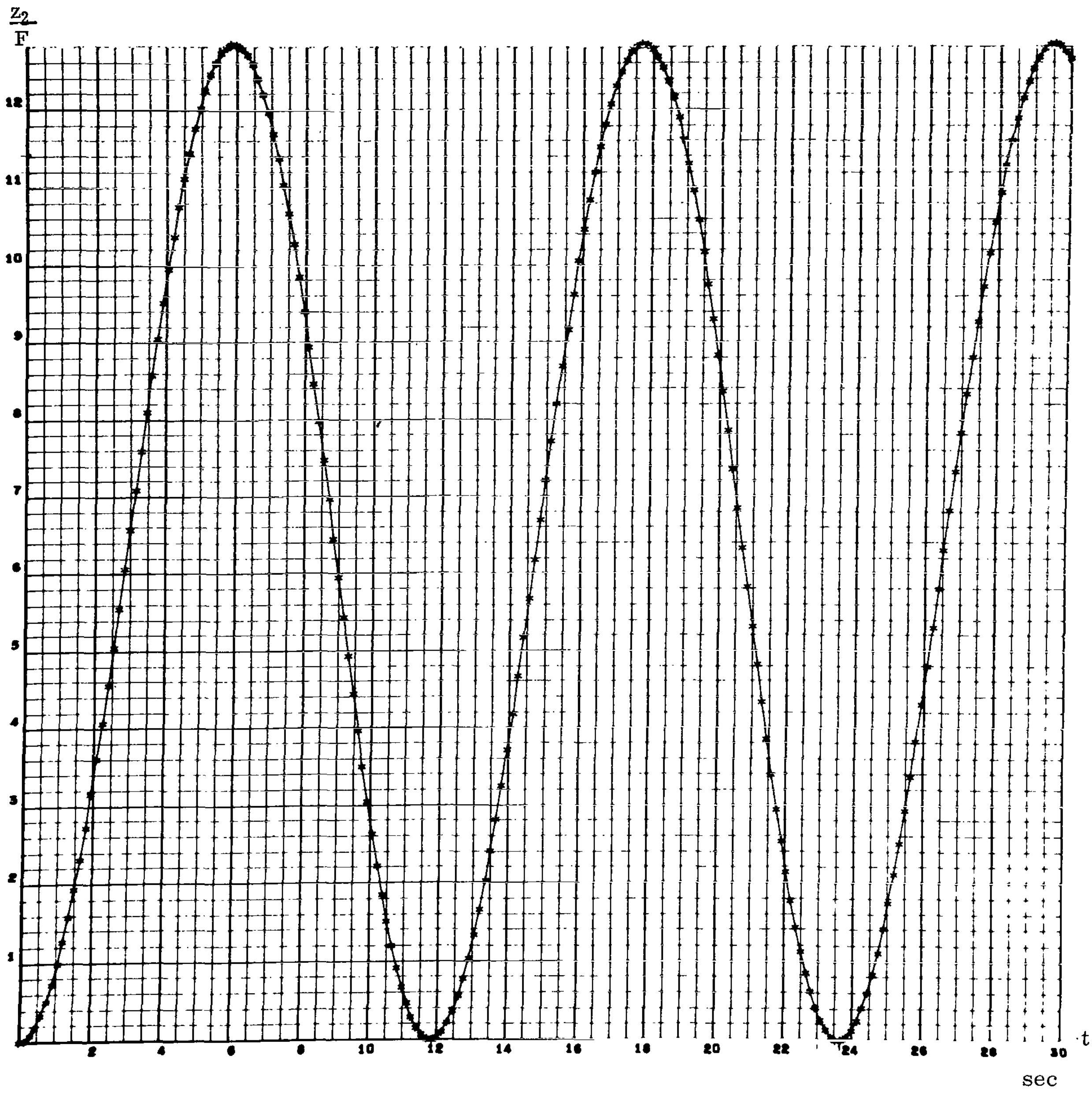

FIGURE 17. RESPONSE OF LIQUID TO STEP FUNCTION (α = 1)

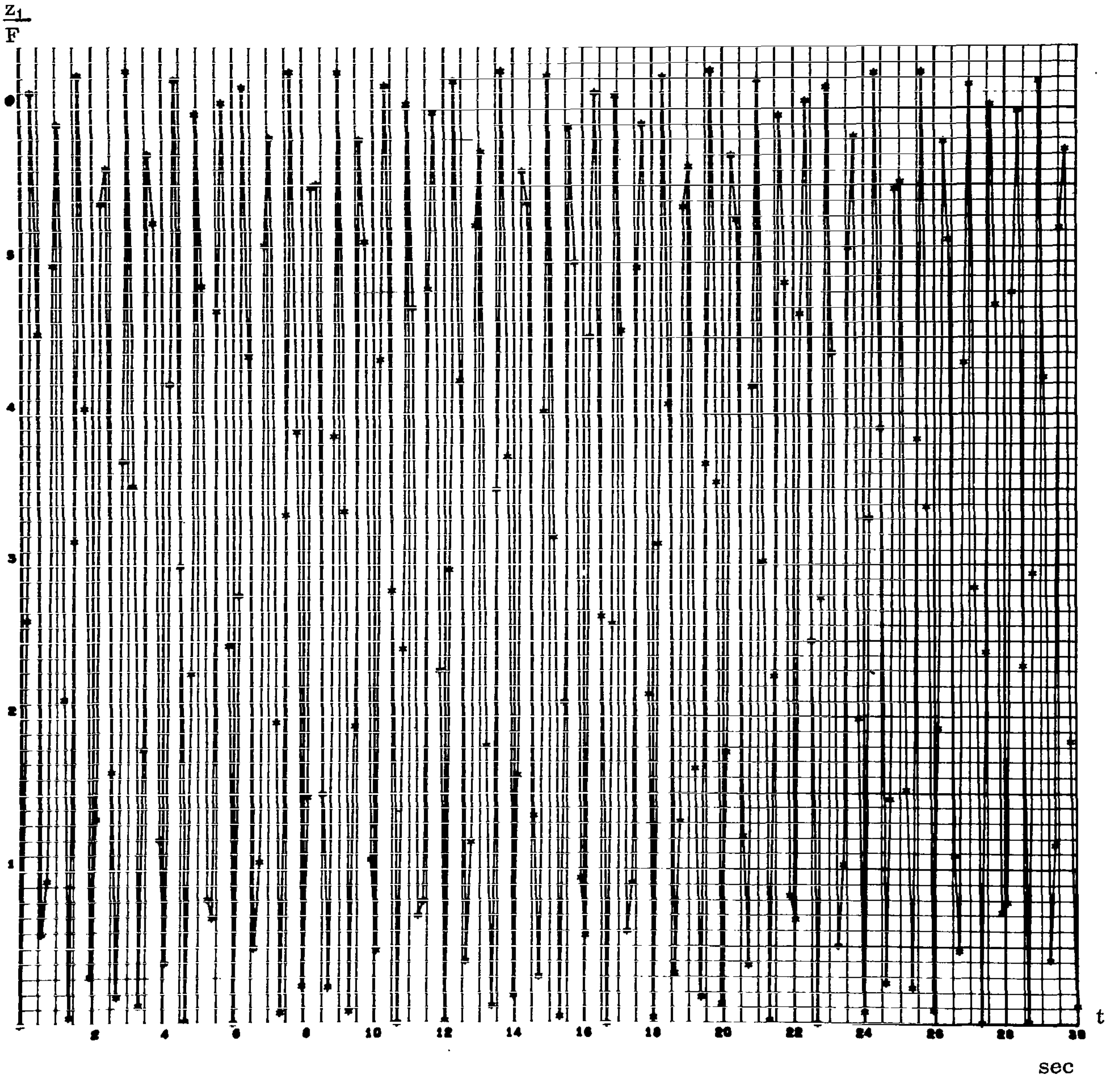

FIGURE 18. RESPONSE OF STRUCTURE TO STEP FUNCTION (α = 2)

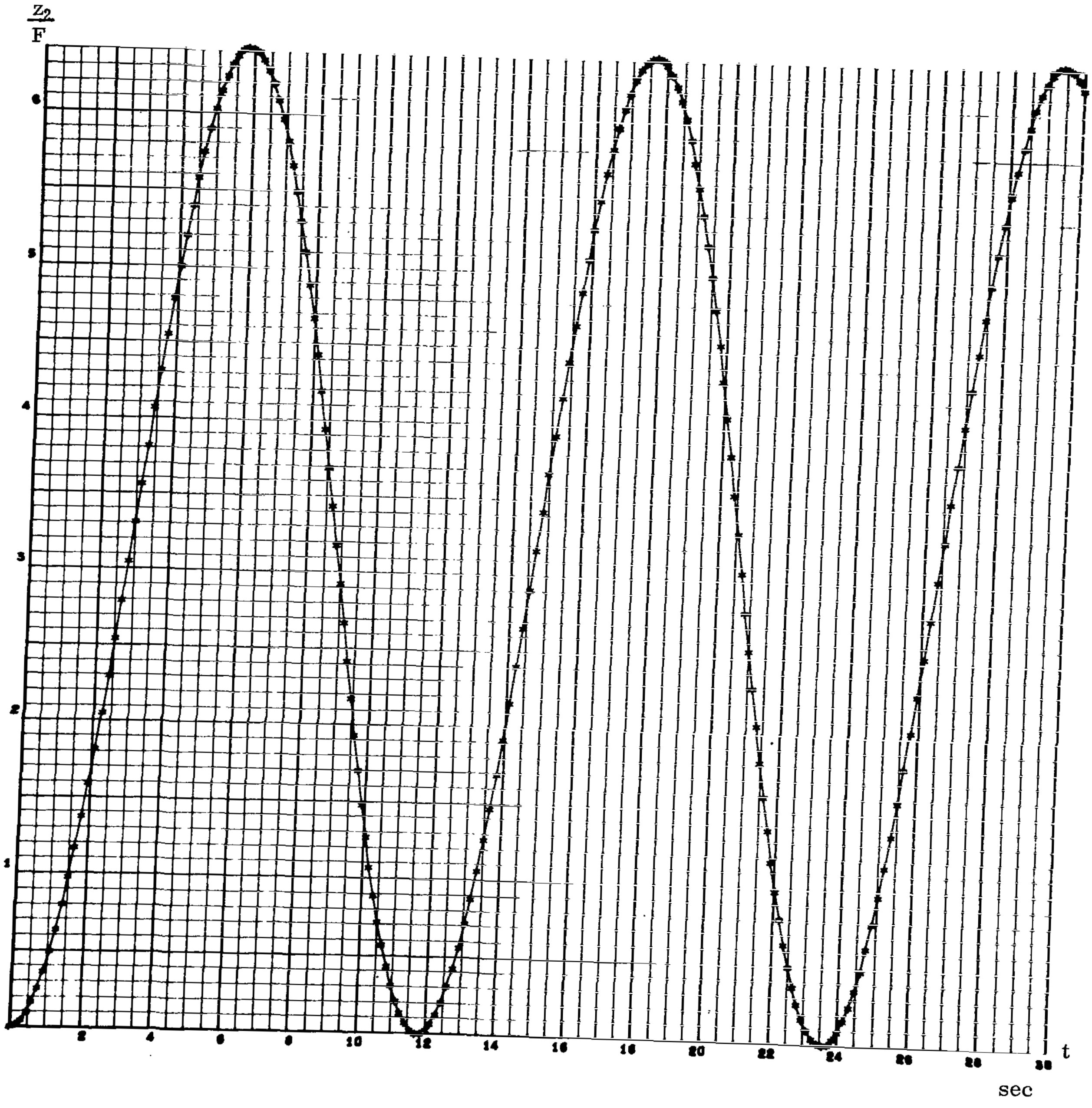

FIGURE 19. RESPONSE OF LIQUID TO STEP FUNCTION ($\alpha = 2$)

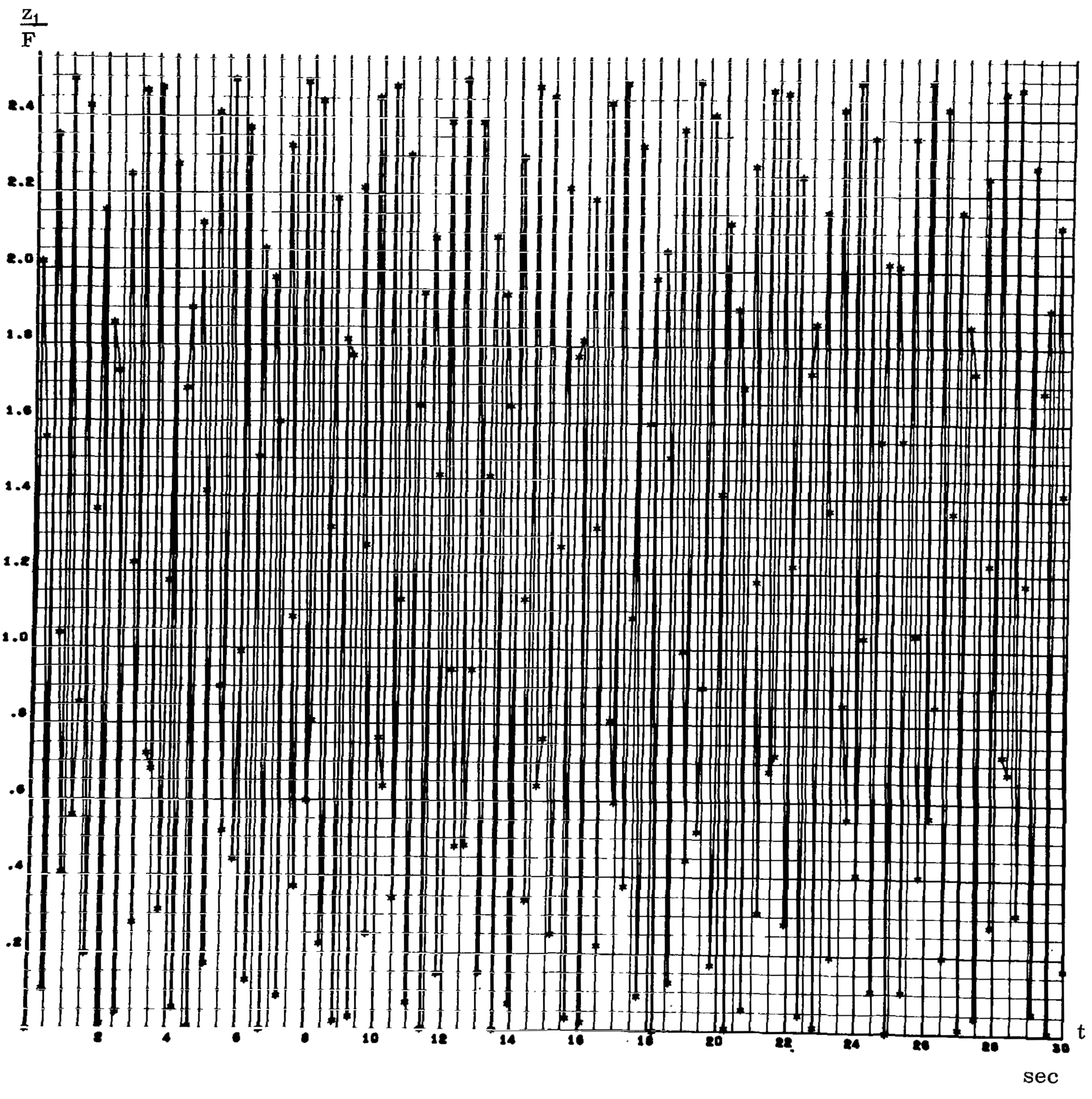

FIGURE 20. RESPONSE OF STRUCTURE TO STEP FUNCTION (α = 5)

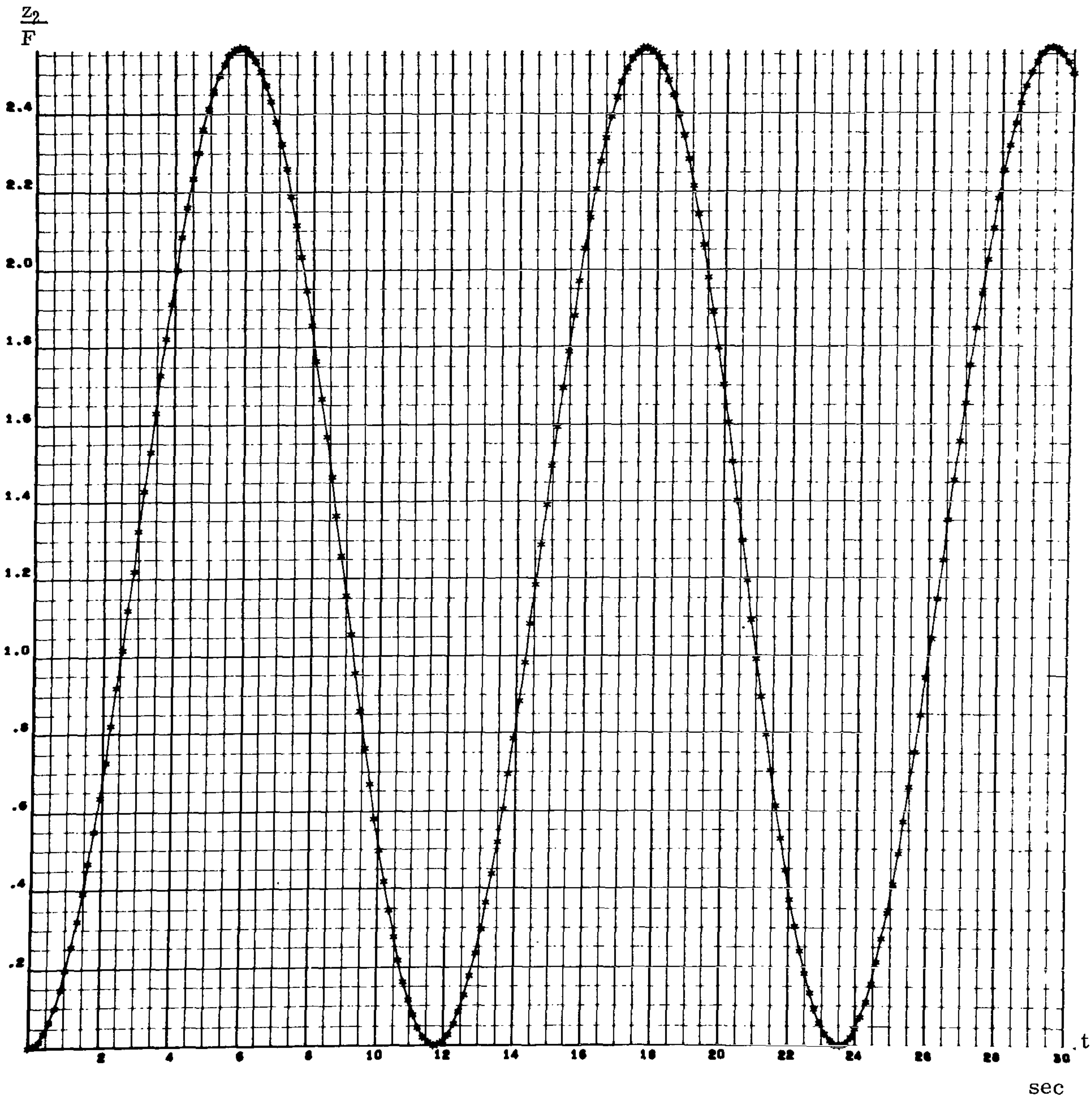

FIGURE 21. RESPONSE OF LIQUID TO STEP FUNCTION (α = 5)

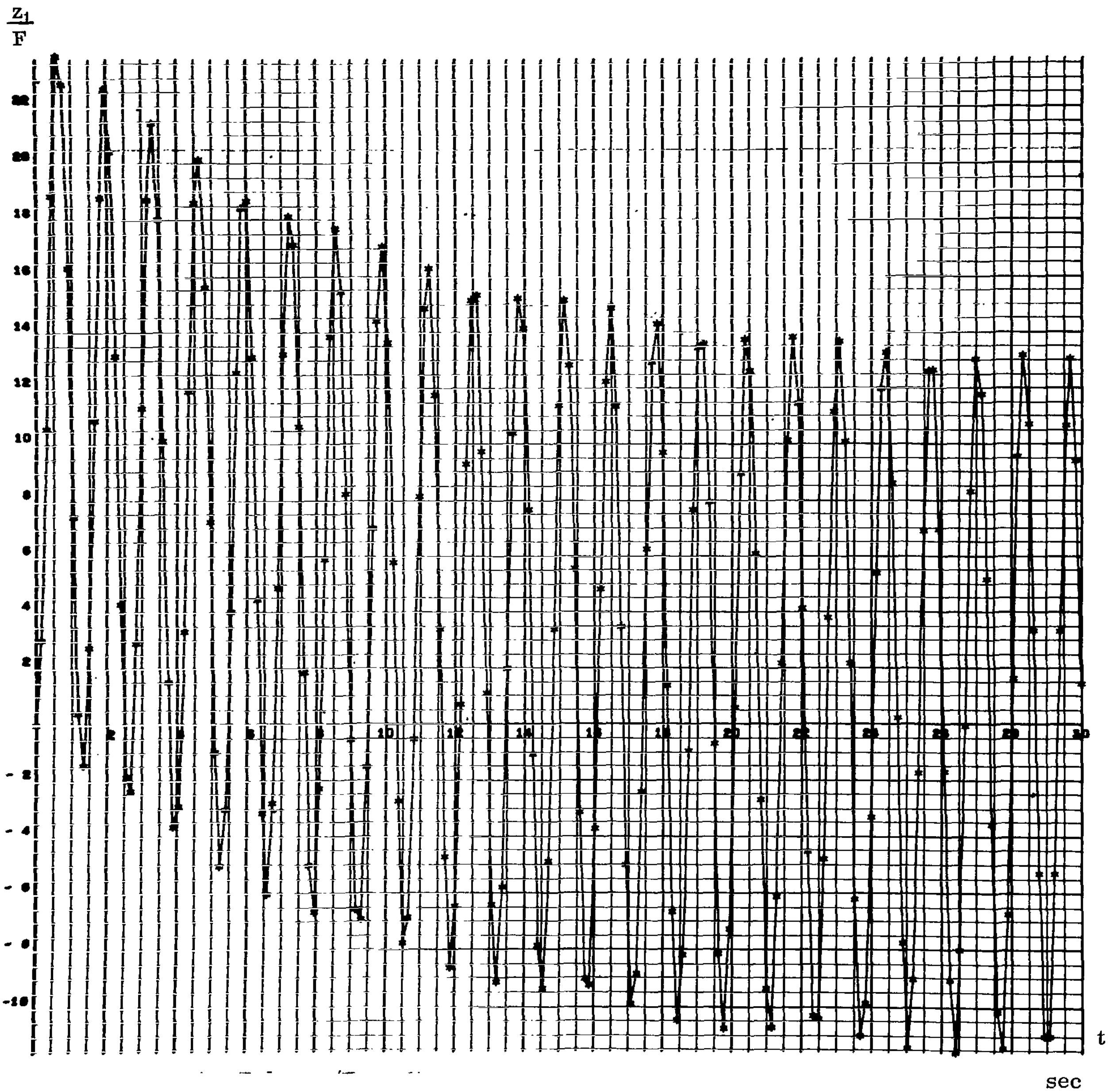

FIGURE 22. RESPONSE OF STRUCTURE TO EXPONENTIALLY DECAYING PULSE ($\beta = 0.1$) ($\alpha = \frac{1}{2}$)

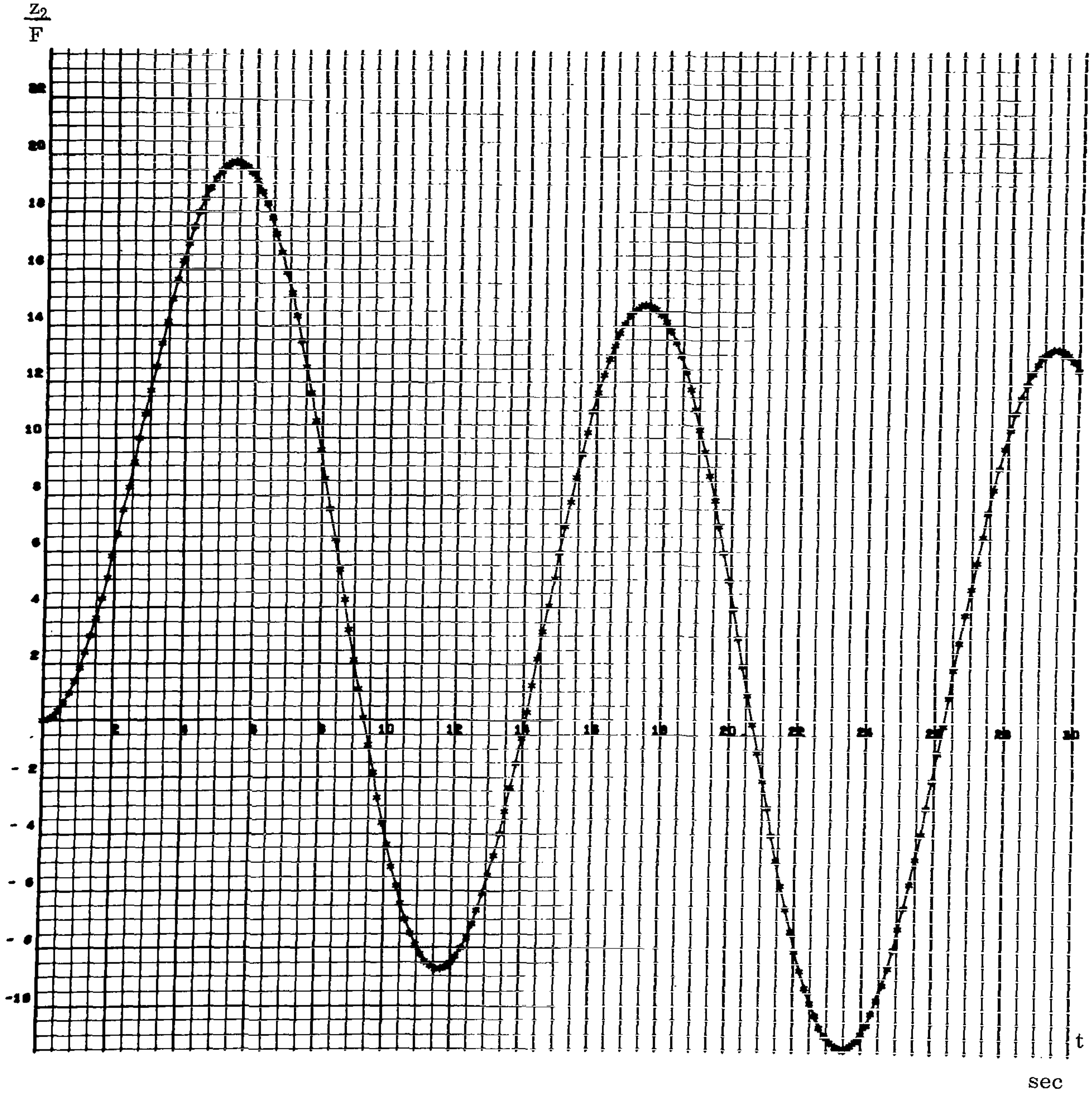

FIGURE 23. RESPONSE OF LIQUID TO EXPONENTIALLY DECAYING PULSE (β = 0.1) ($\alpha = \frac{1}{2}$)

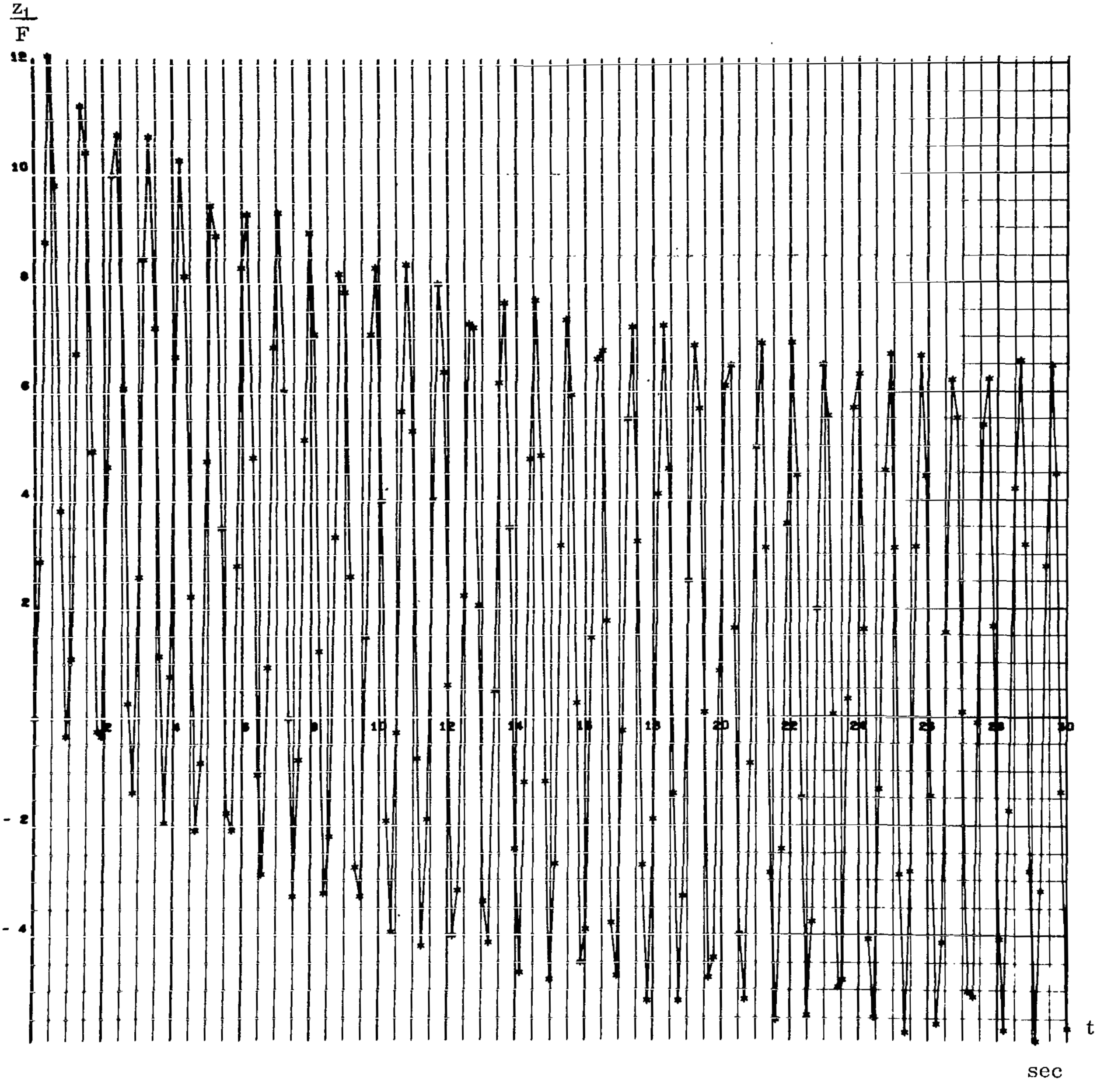

FIGURE 24. RESPONSE OF STRUCTURE TO EXPONENTIALLY DECAYING PULSE (α = 1) (β = 0.1)

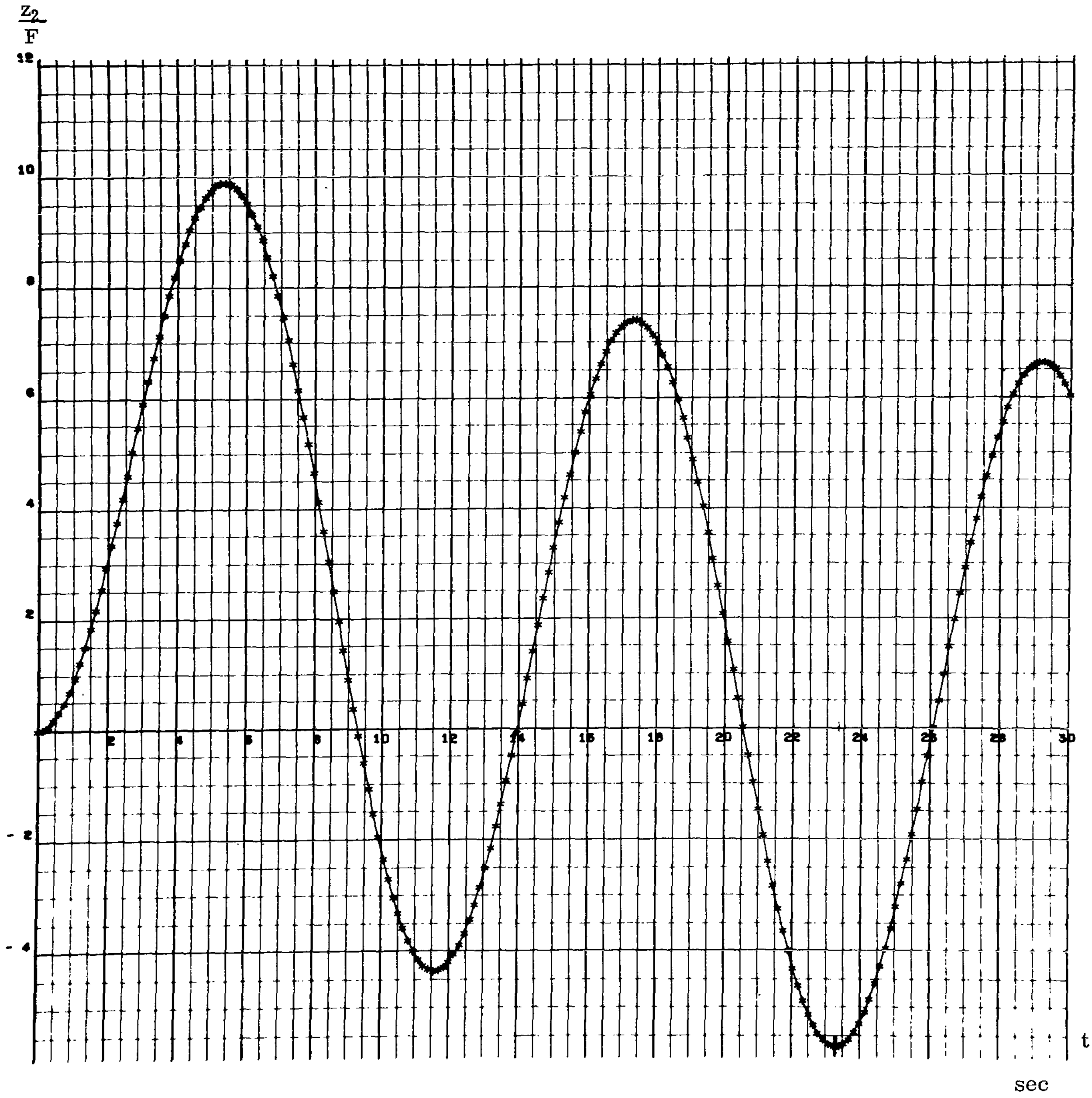

FIGURE 25. RESPONSE OF LIQUID TO EXPONENTIALLY DECAYING PULSE (α = 1) (β = 0.1)

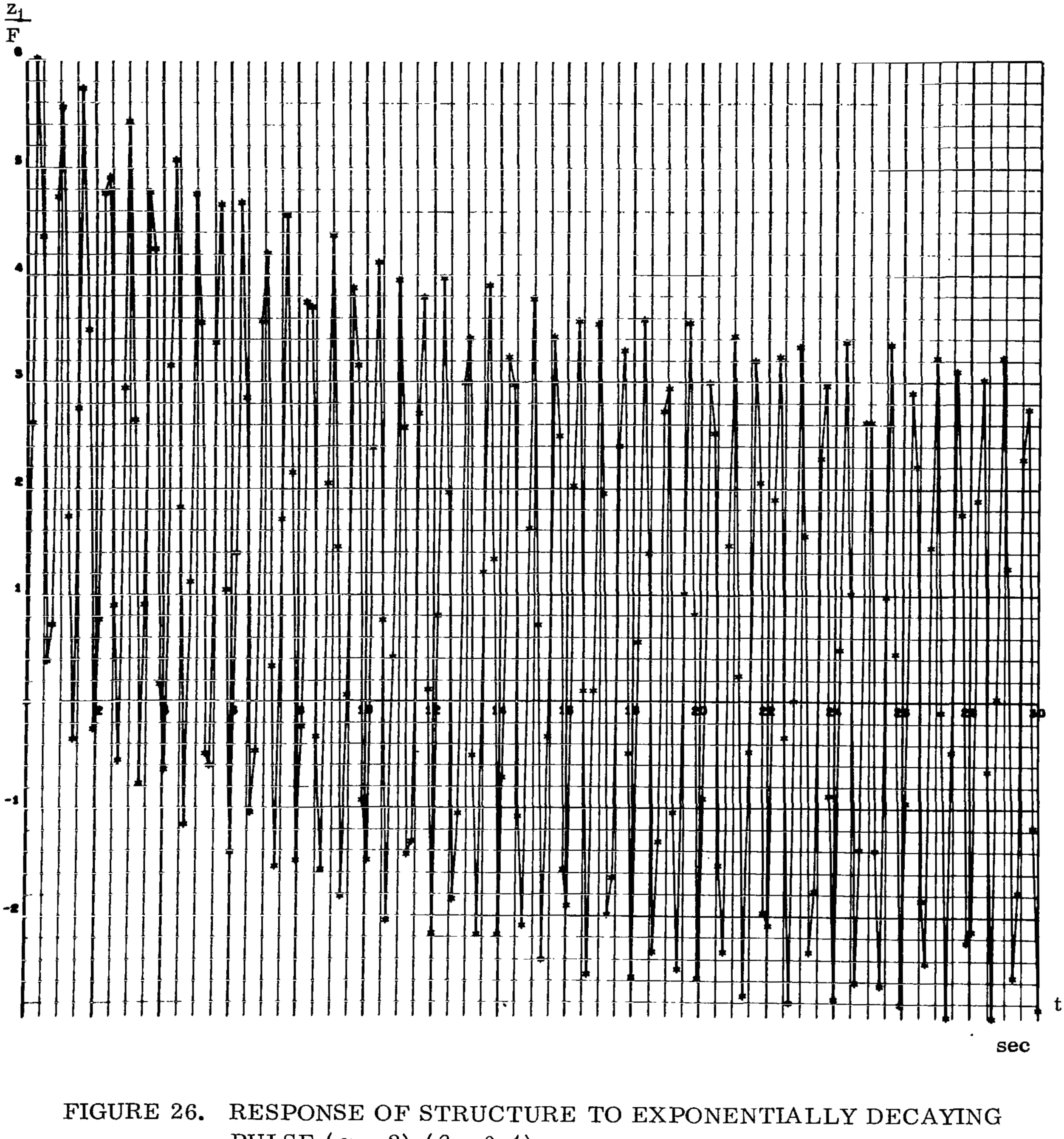

FIGURE 26. RESPONSE OF STRUCTURE TO EXPONENTIALLY DECAYING PULSE (α = 2) (β = 0. 1)

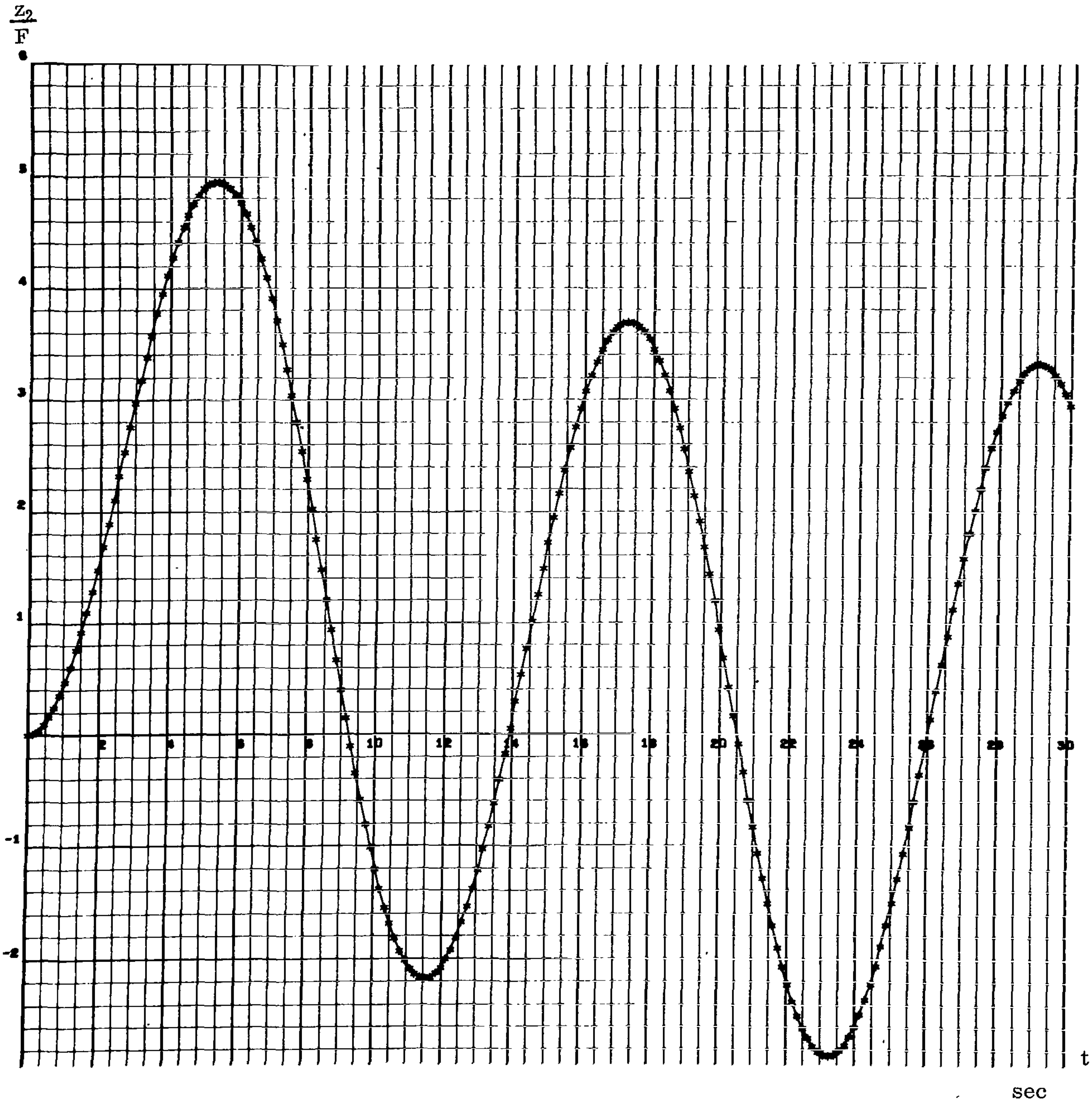

FIGURE 27. RESPONSE OF LIQUID TO EXPONENTIALLY DECAYING PULSE (α = 2) (β = 0.1)

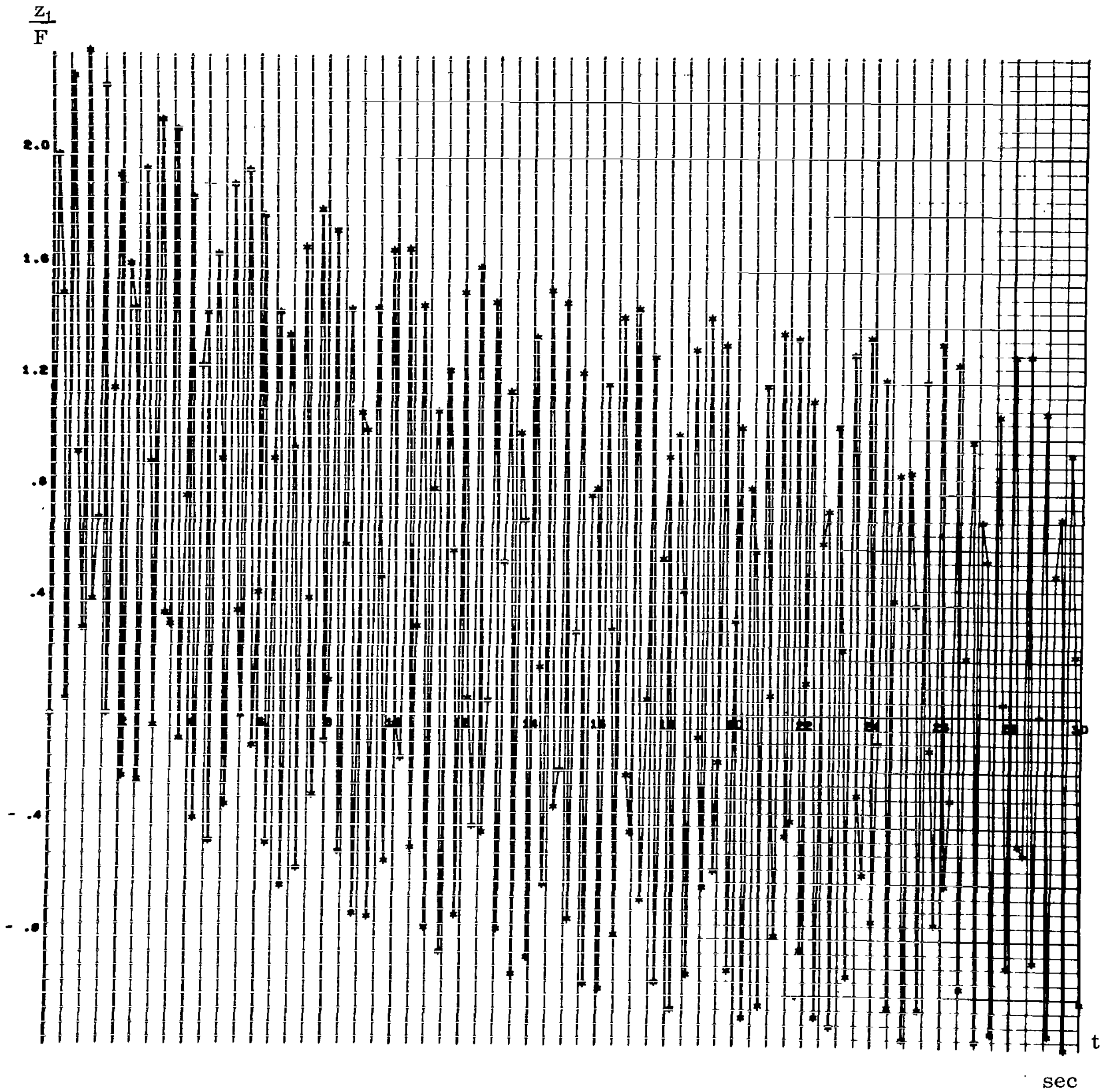

FIGURE 28. RESPONSE OF STRUCTURE TO EXPONENTIALLY DECAYING PULSE (α = 5) (β = 0.1)

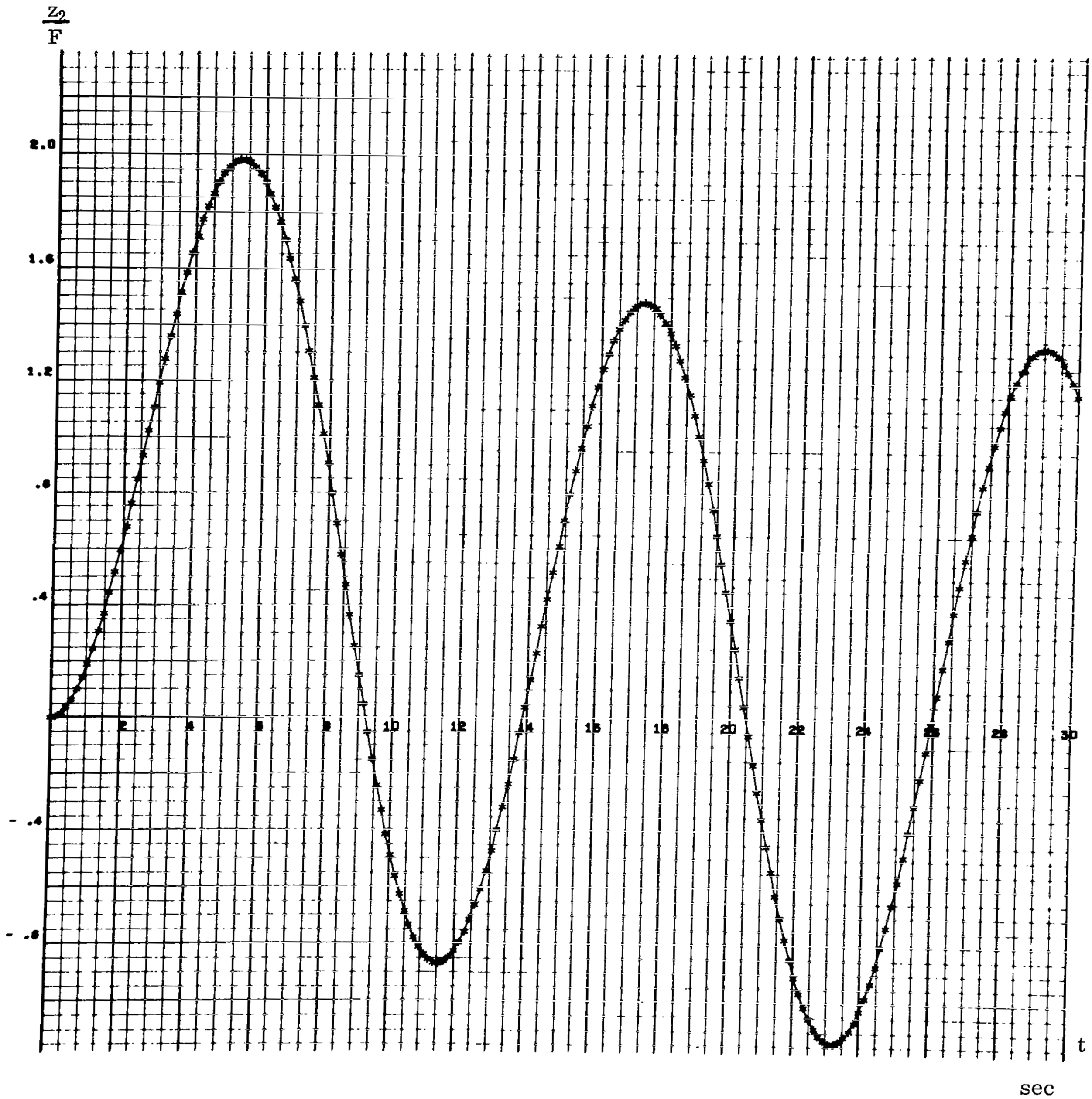

FIGURE 29. RESPONSE OF LIQUID TO EXPONENTIALLY DECAYING PULSE (α = 5) (β = 0.1)

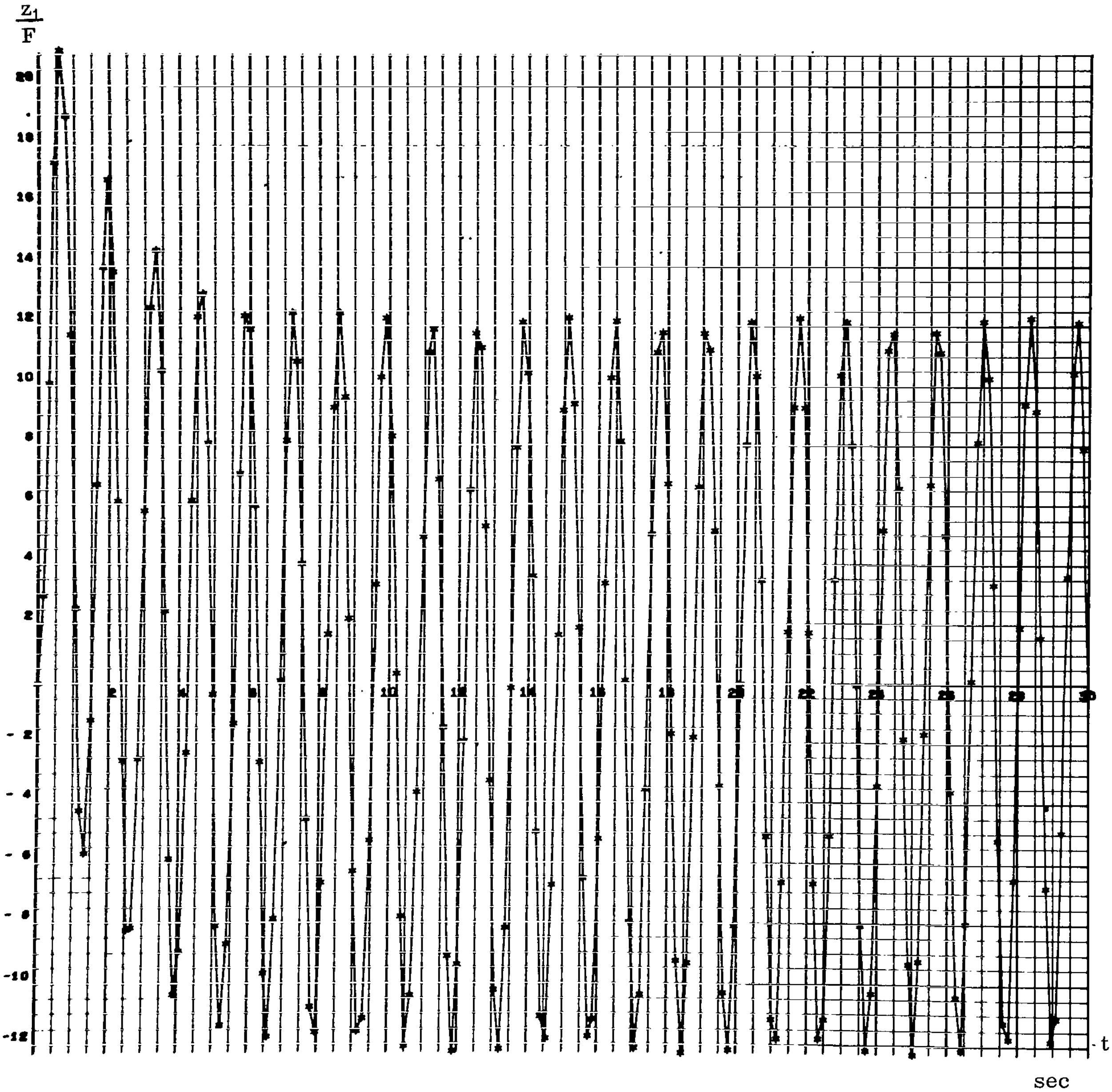

FIGURE 30. RESPONSE OF STRUCTURE TO EXPONENTIALLY DECAYING PULSE ($\alpha = \frac{1}{2}$) ($\beta = 0.5$)

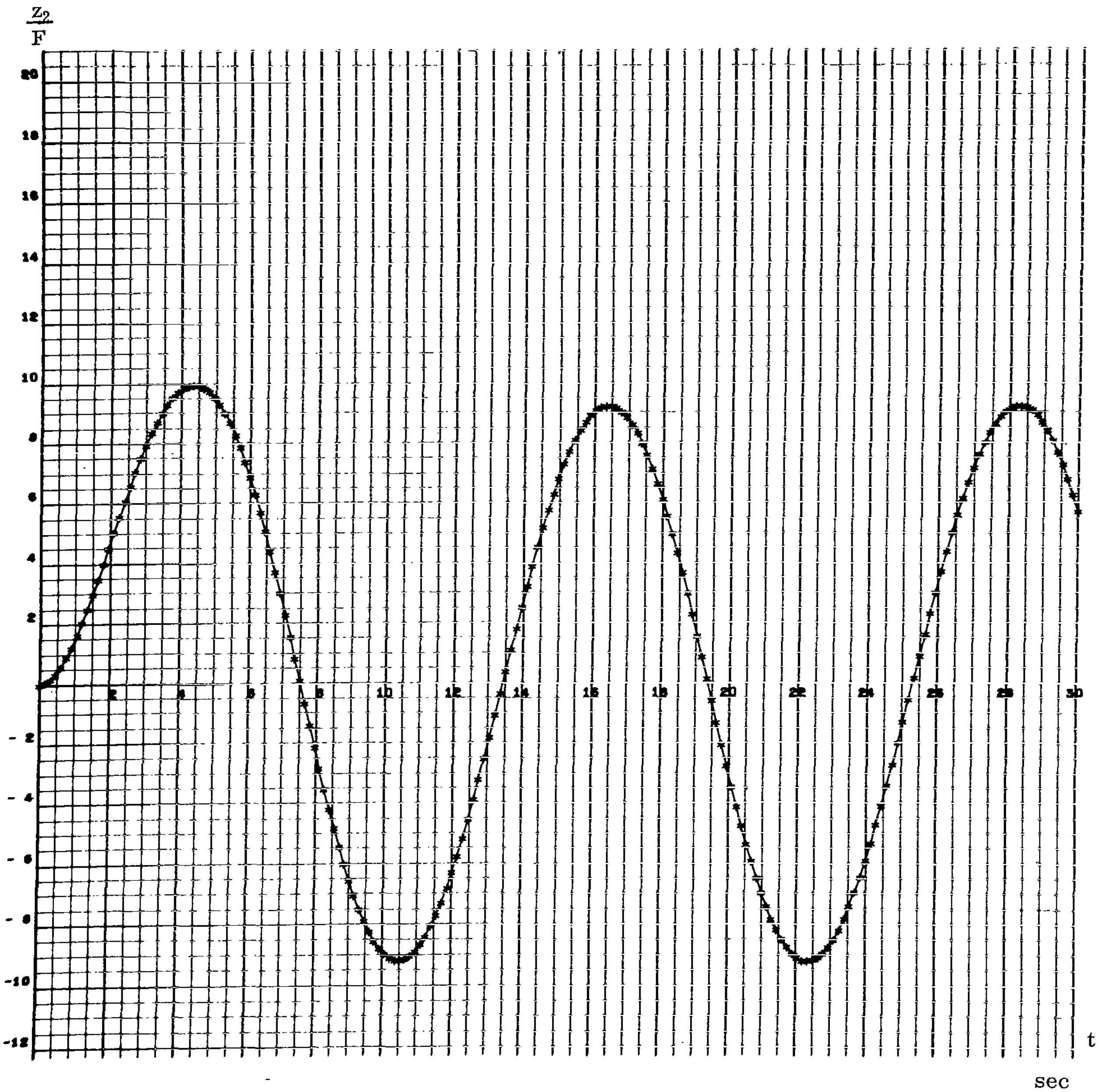

FIGURE 31. RESPONSE OF LIQUID TO EXPONENTIALLY DECAYING PULSE ($\alpha = \frac{1}{2}$) ($\beta = 0.5$)

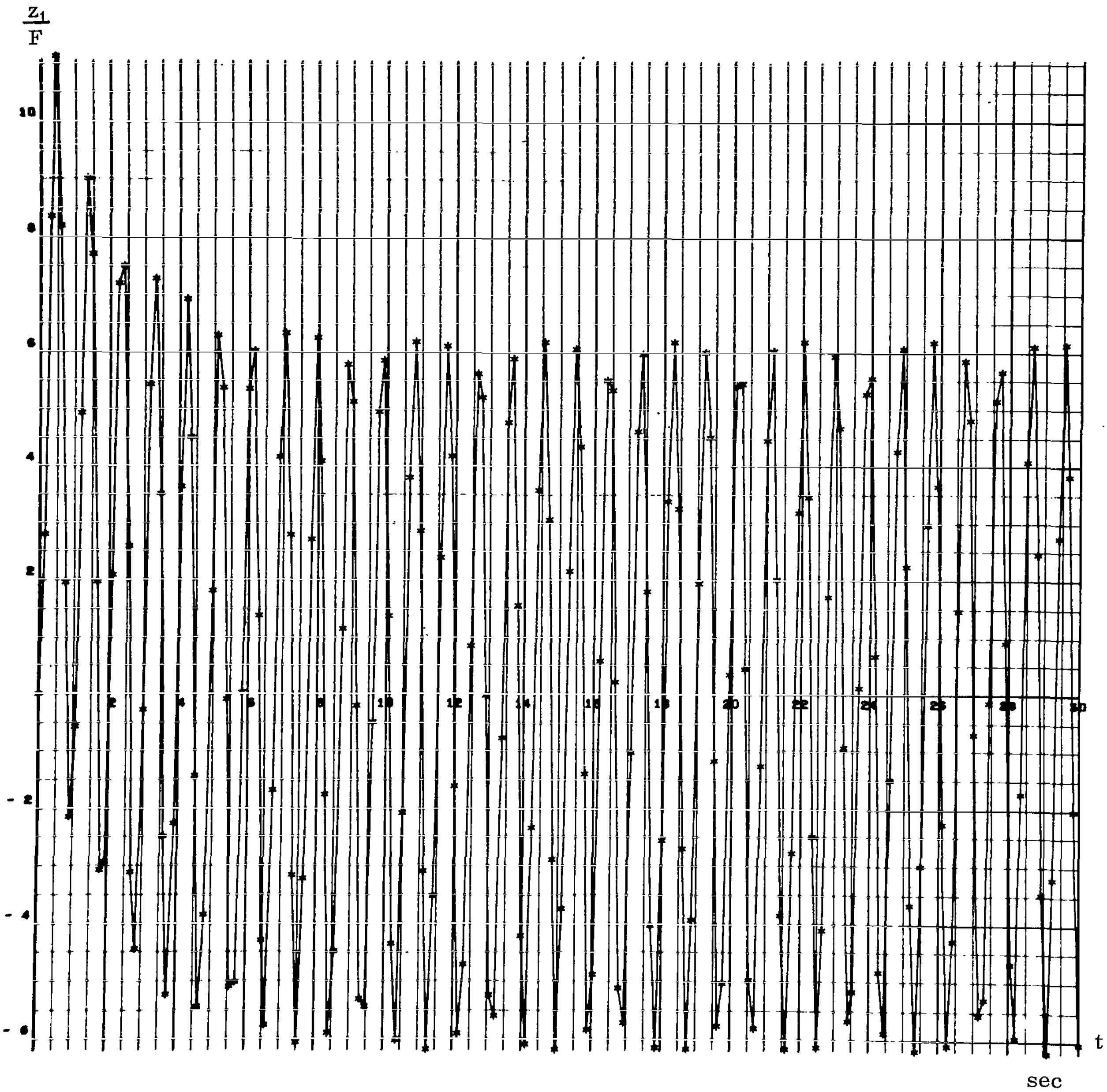

FIGURE 32. RESPONSE OF STRUCTURE TO EXPONENTIALLY DECAYING PULSE (α = 1) (β = 0. 5)

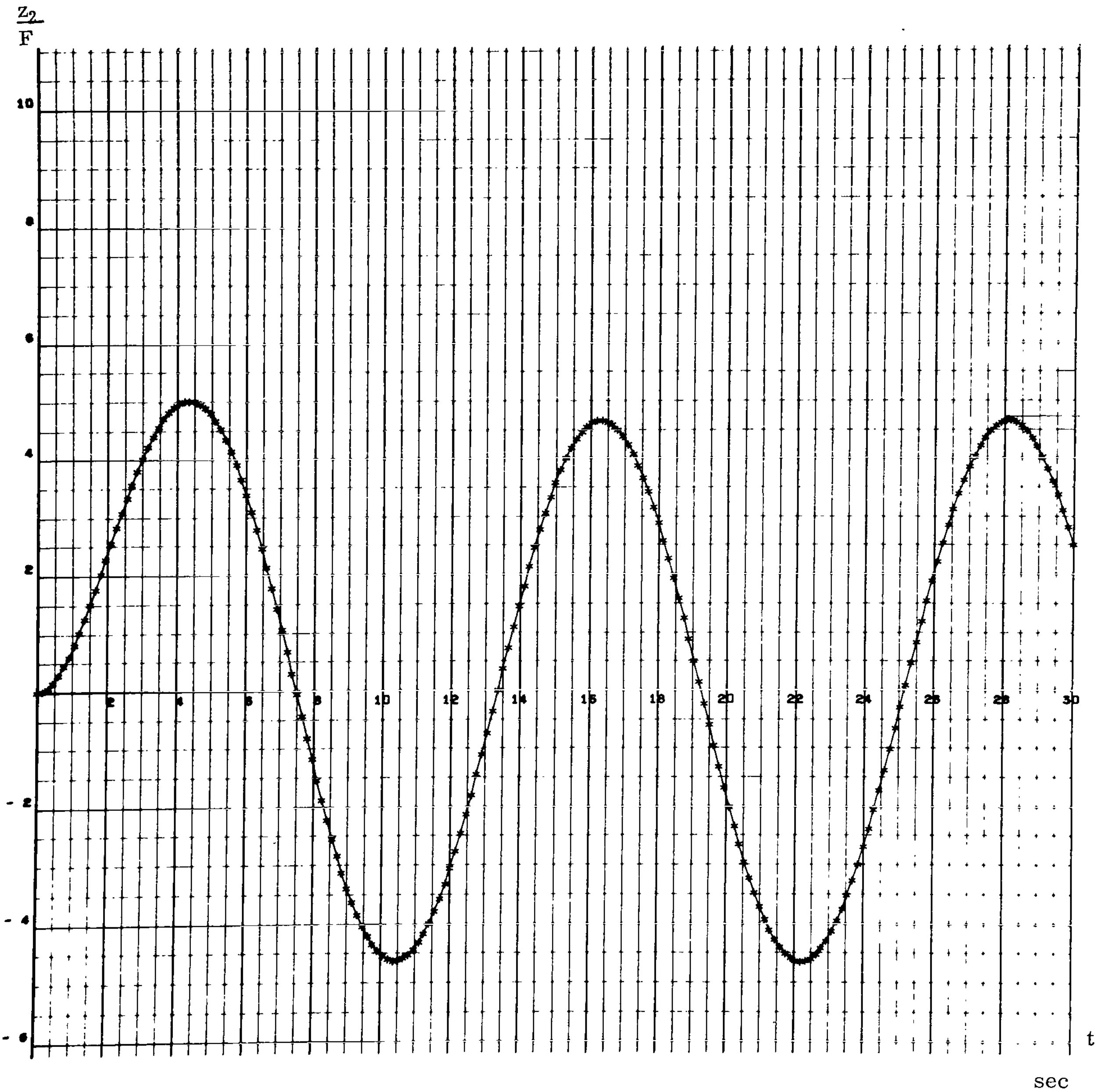

FIGURE 33. RESPONSE OF LIQUID TO EXPONENTIALLY DECAYING PULSE (α = 1) (β = 0.5)

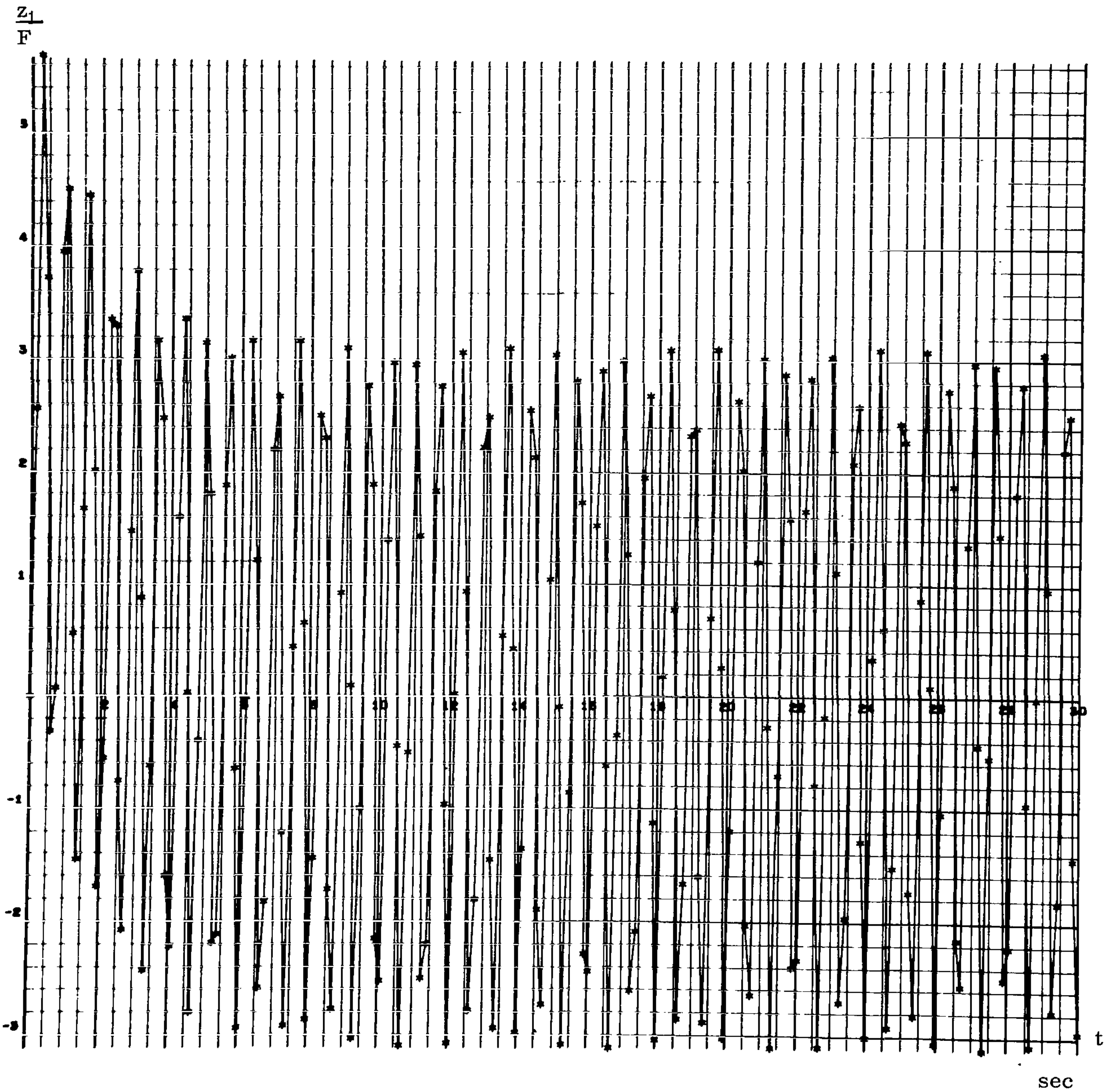

FIGURE 34. RESPONSE OF STRUCTURE TO EXPONENTIALLY DECAYING PULSE (α = 2) (β = 0.5)

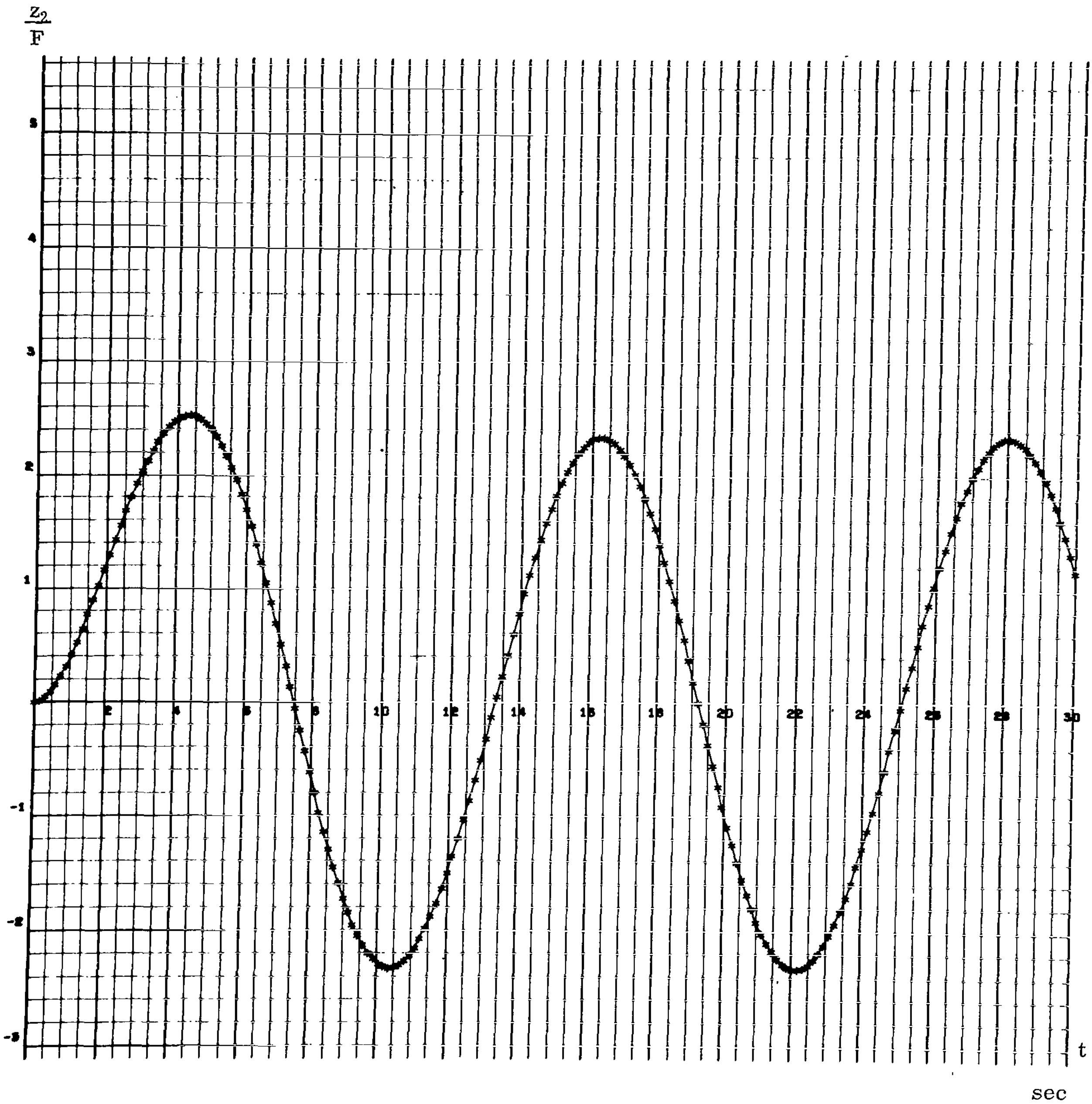

FIGURE 35. RESPONSE OF LIQUID TO EXPONENTIALLY DECAYING PULSE (α = 2) (β = 0.5)

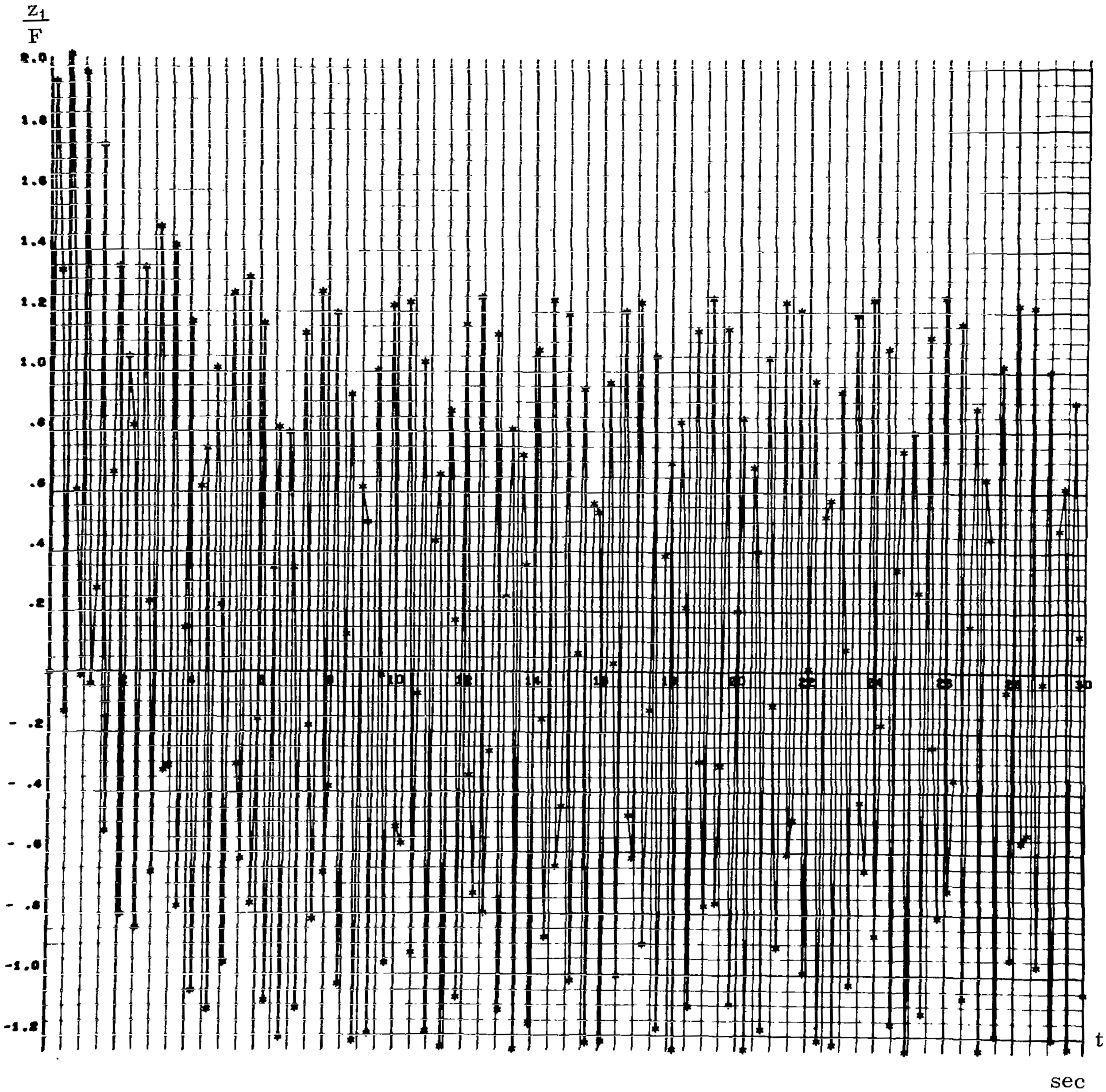

FIGURE 36. RESPONSE OF STRUCTURE TO EXPONENTIALLY DECAYING PULSE (α = 5) (β = 0.5)

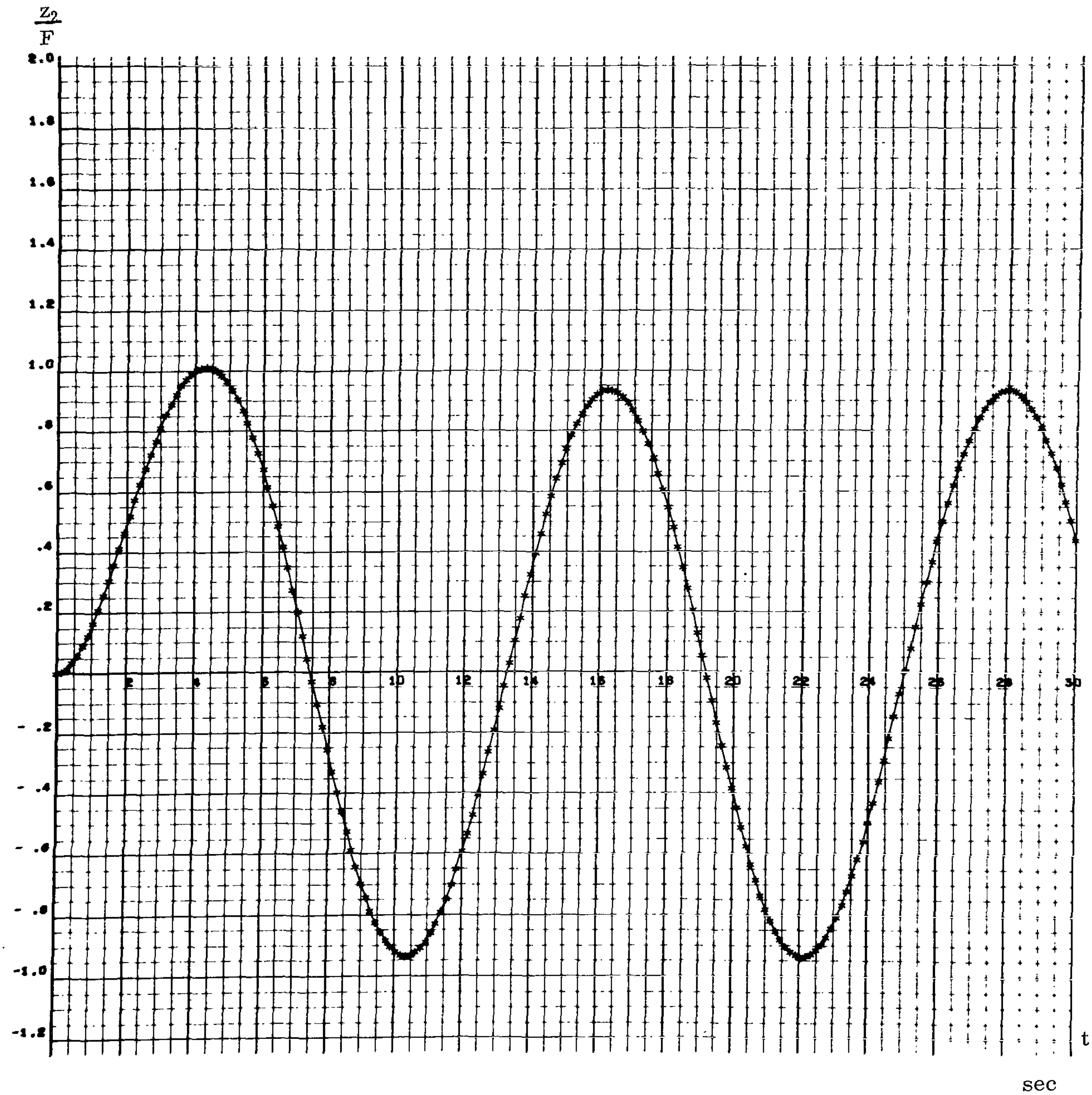

FIGURE 37. RESPONSE OF LIQUID TO EXPONENTIALLY DECAYING PULSE (α = 5) (β = 0.5)

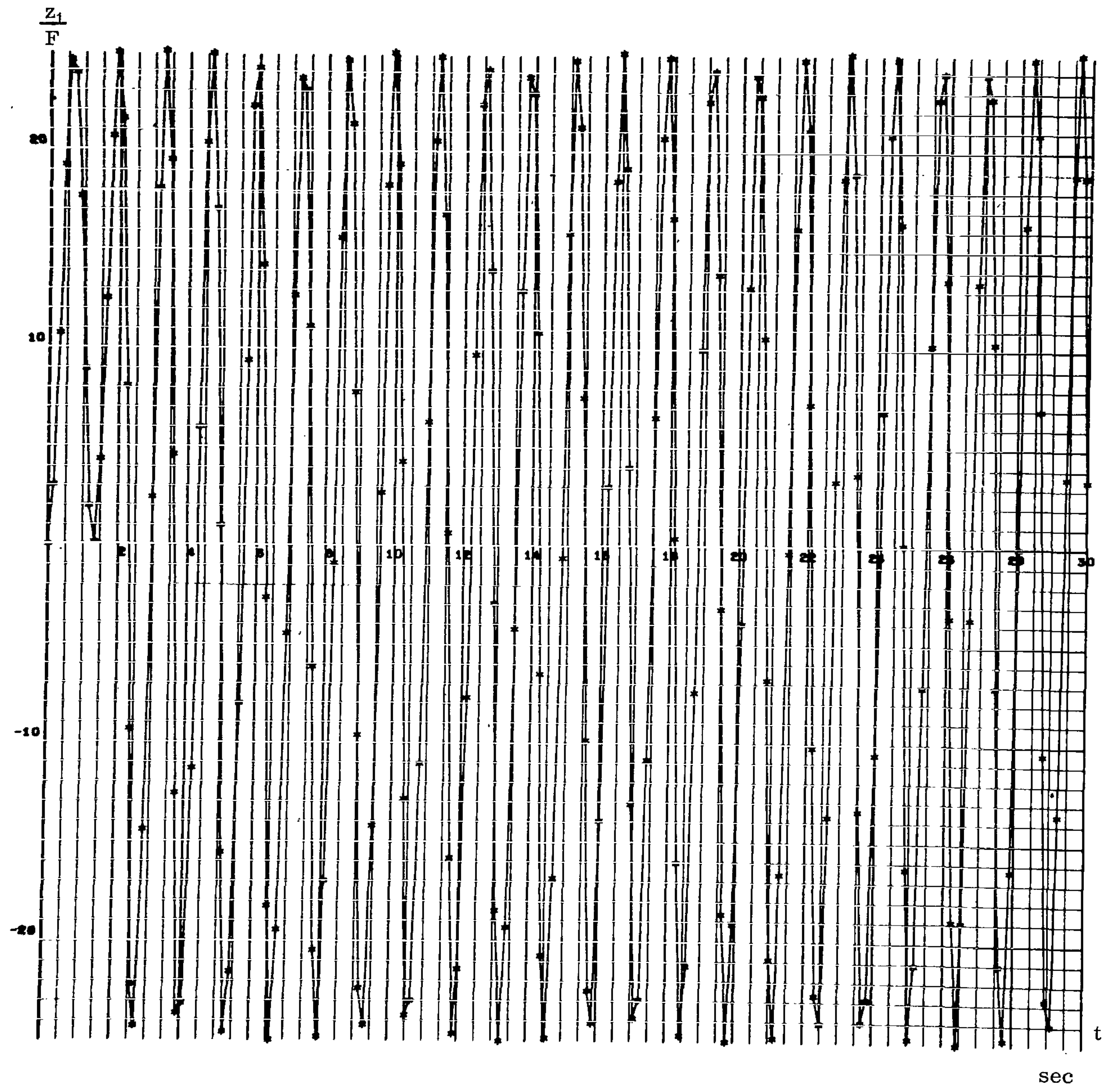

FIGURE 38. RESPONSE OF STRUCTURE TO RECTANGULAR PULSE OF DURATION $t_1 = 2$ SECONDS ($\alpha = \frac{1}{2}$)

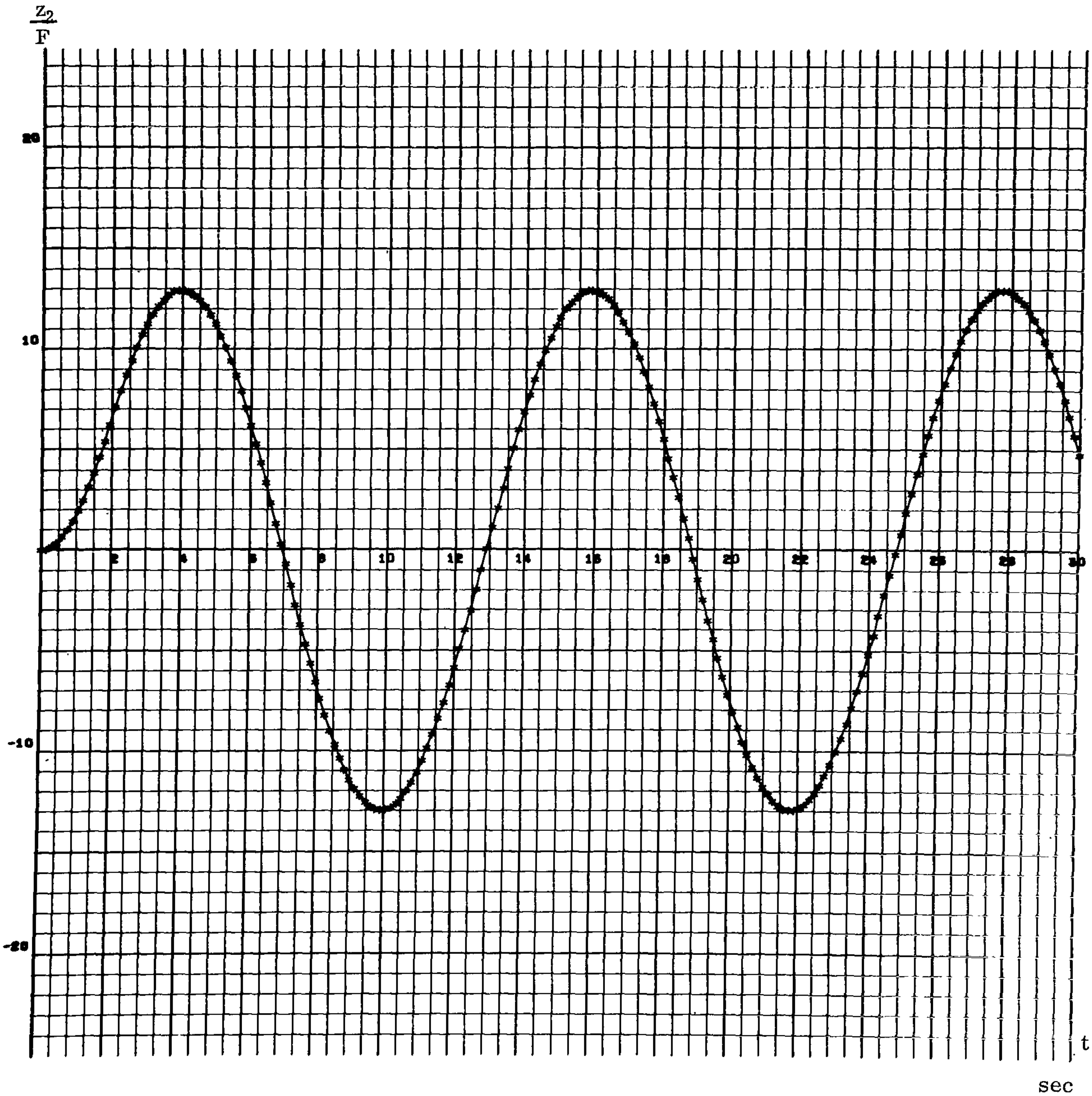

FIGURE 39. RESPONSE OF LIQUID TO RECTANGULAR PULSE OF DURATION $t_1 = 2$ SECONDS ($\alpha = \frac{1}{2}$)

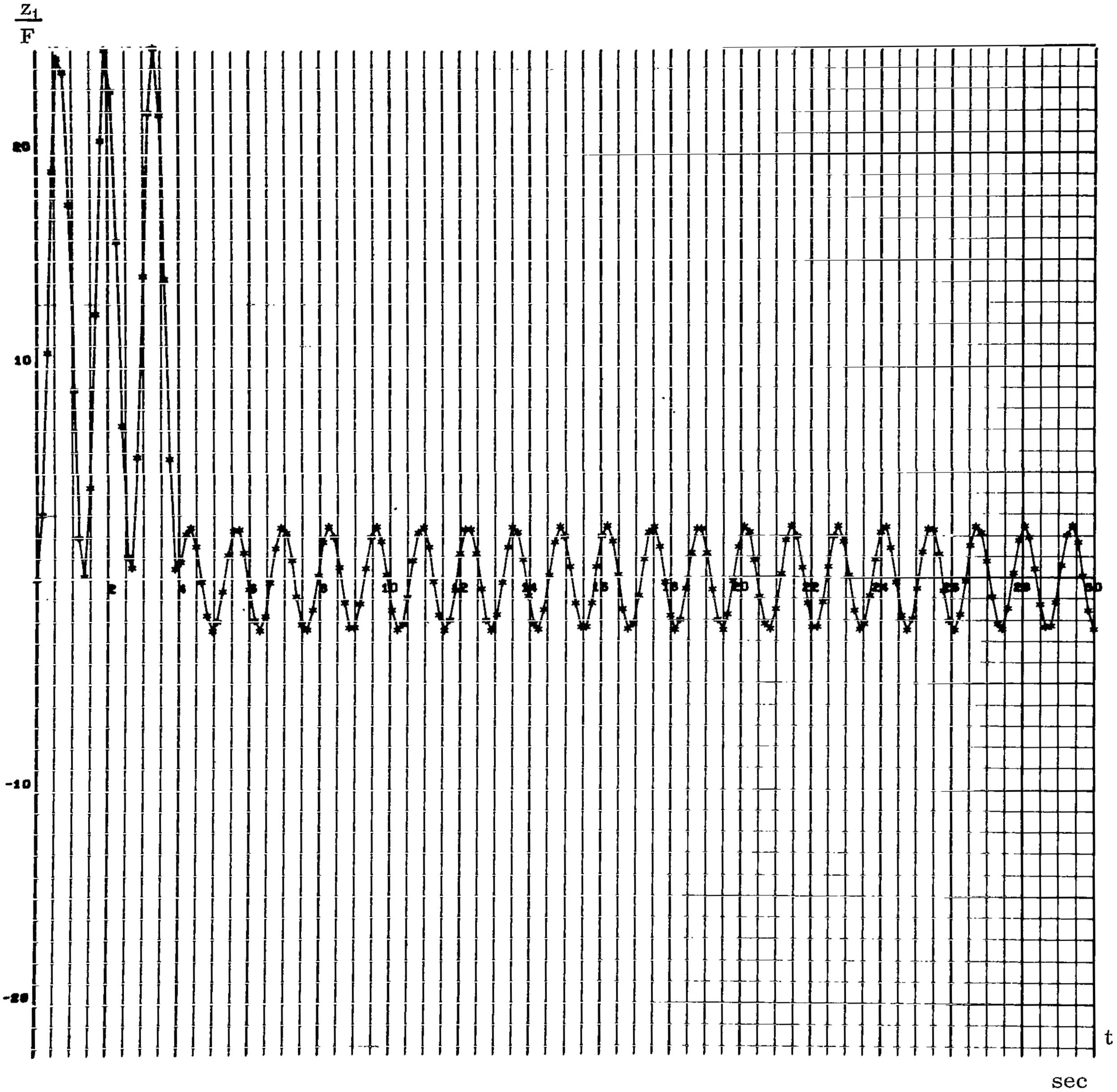

FIGURE 40. RESPONSE OF STRUCTURE TO RECTANGULAR PULSE OF DURATION $t_1 = 4$ SECONDS ($\alpha = 1$)

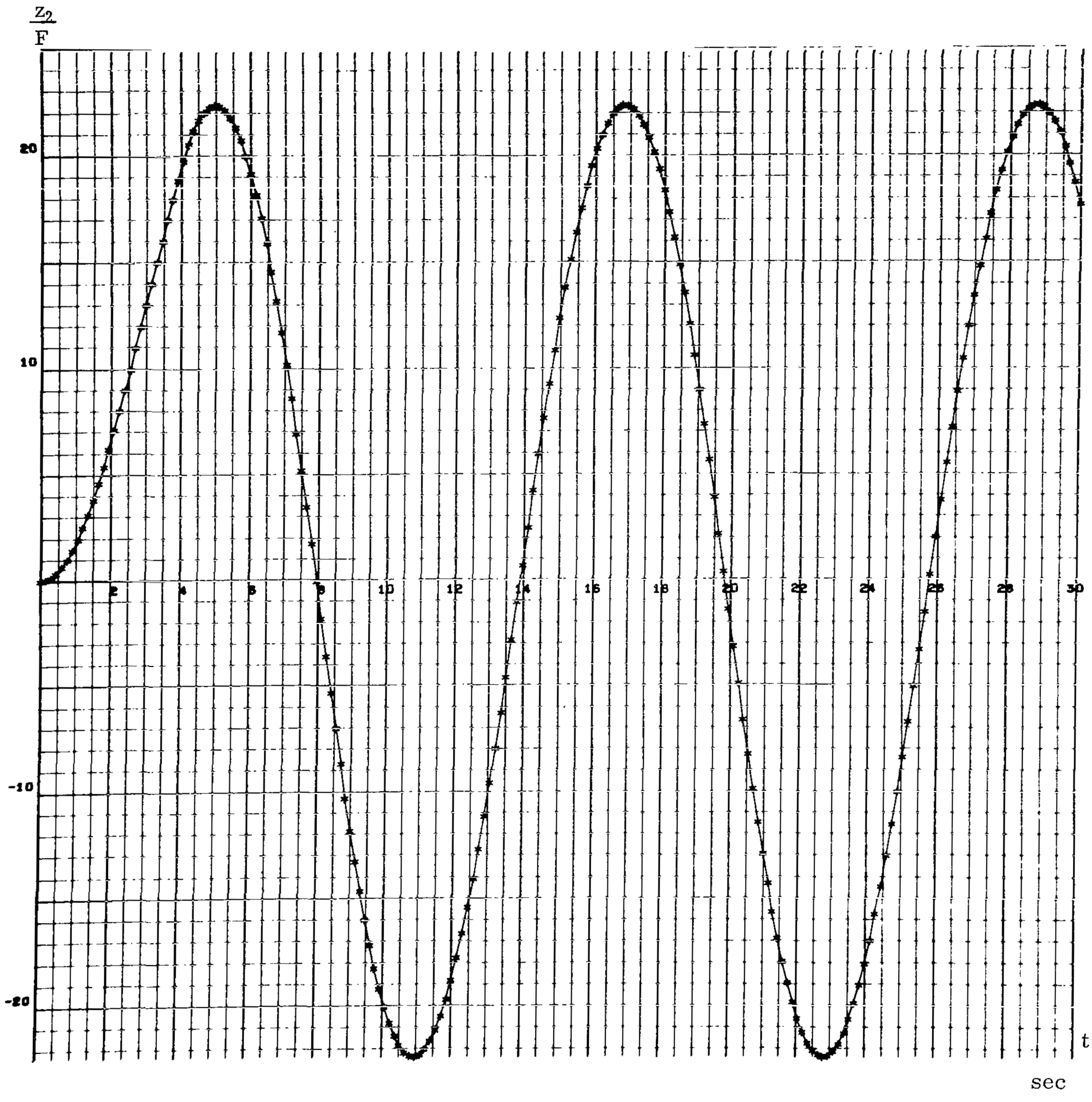

FIGURE 41. RESPONSE OF LIQUID TO RECTANGULAR PULSE OF DURATION t_1 = 4 SECONDS (α = 1)

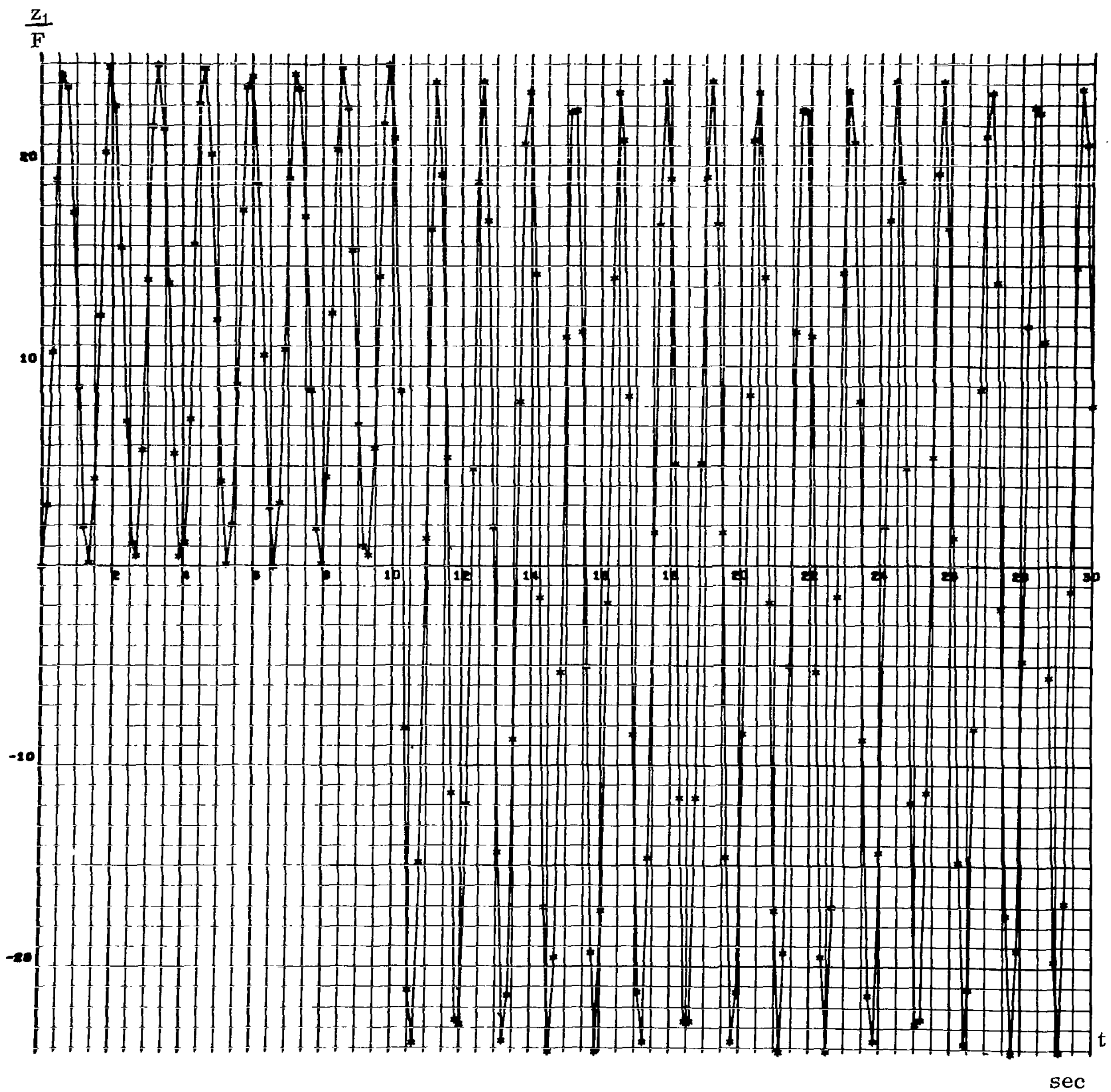

FIGURE 42. RESPONSE OF STRUCTURE TO RECTANGULAR PULSE OF DURATION t_1 = 10 SECONDS ($\alpha = \frac{1}{2}$)

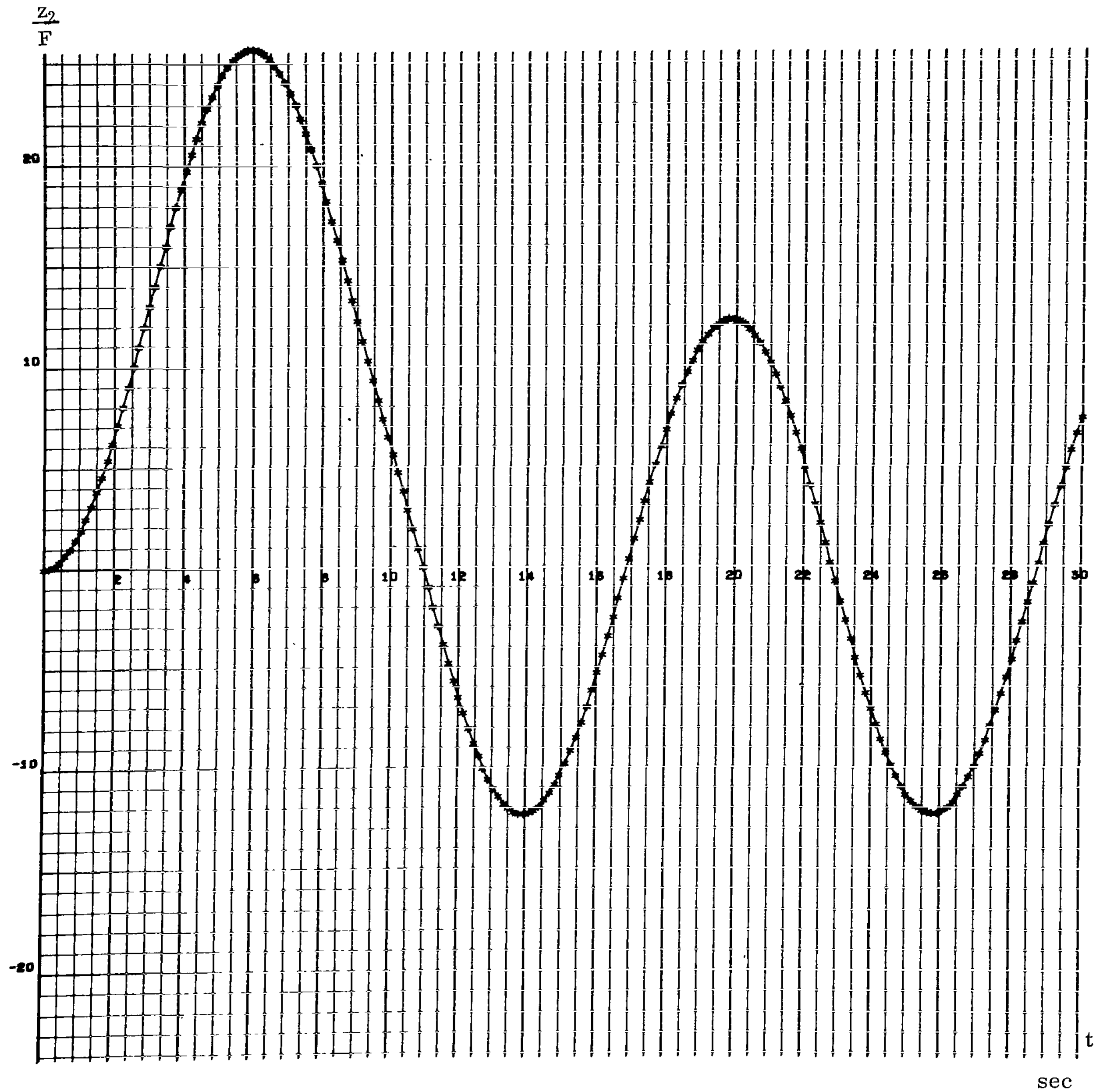

FIGURE 43. RESPONSE OF LIQUID TO RECTANGULAR PULSE OF DURATION $t_1 = 10$ SECONDS ($\alpha = \frac{1}{2}$)

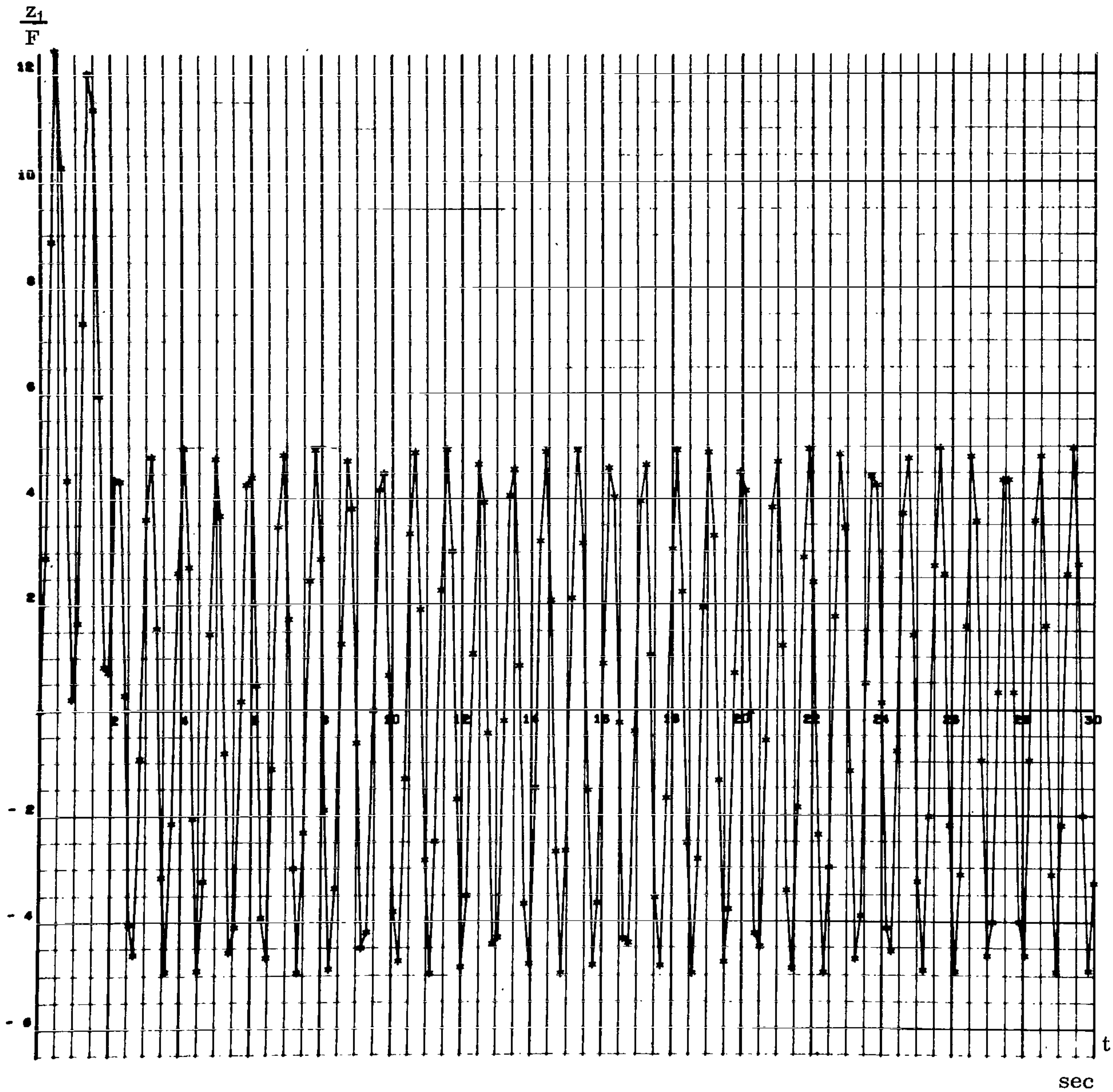

FIGURE 44. RESPONSE OF STRUCTURE TO RECTANGULAR PULSE OF DURATION (α = 1) t_1 = 2 SECONDS

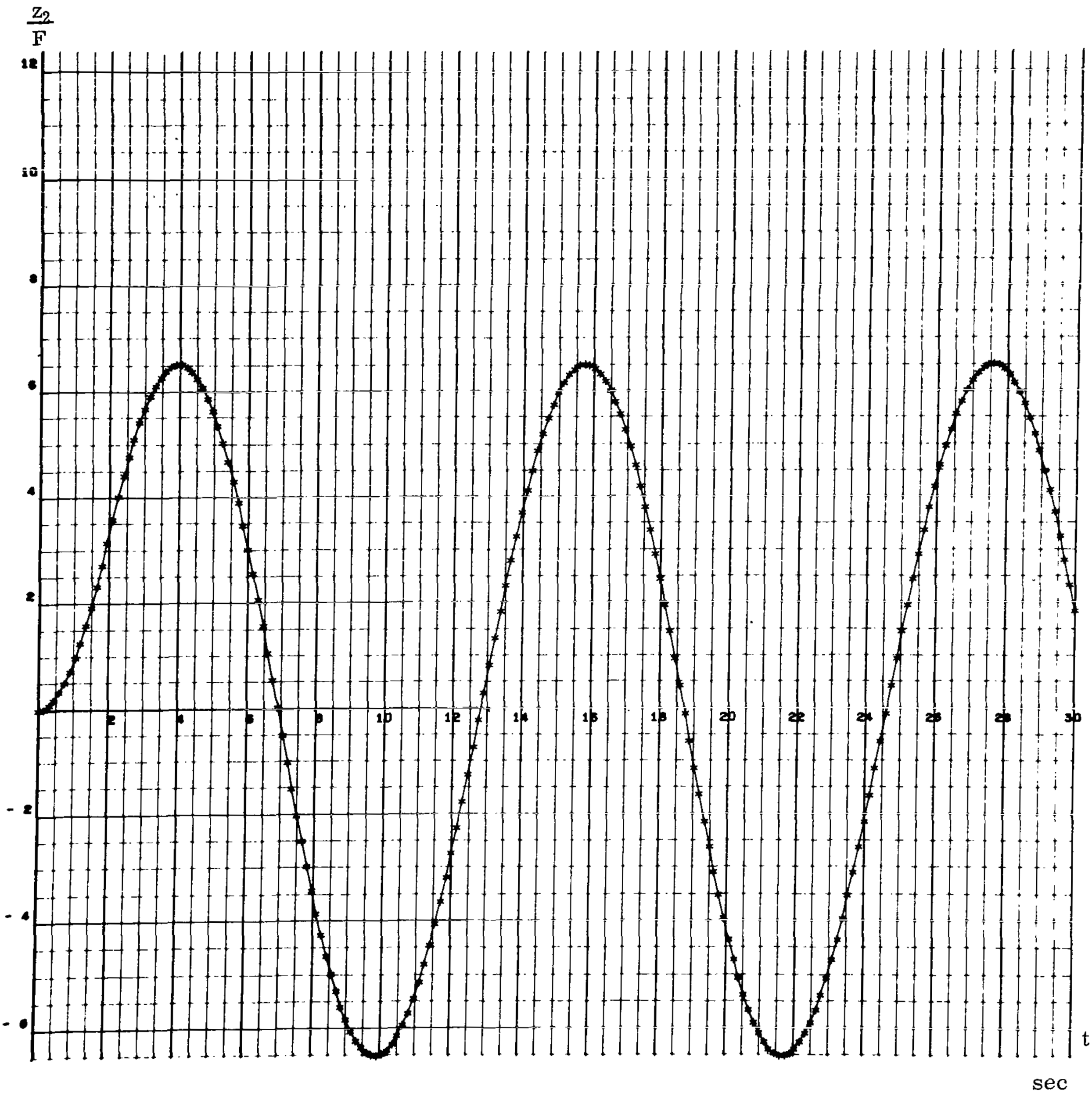

FIGURE 45. RESPONSE OF LIQUID TO RECTANGULAR PULSE OF DURATION (α = 1) t_1 = 2 SECONDS

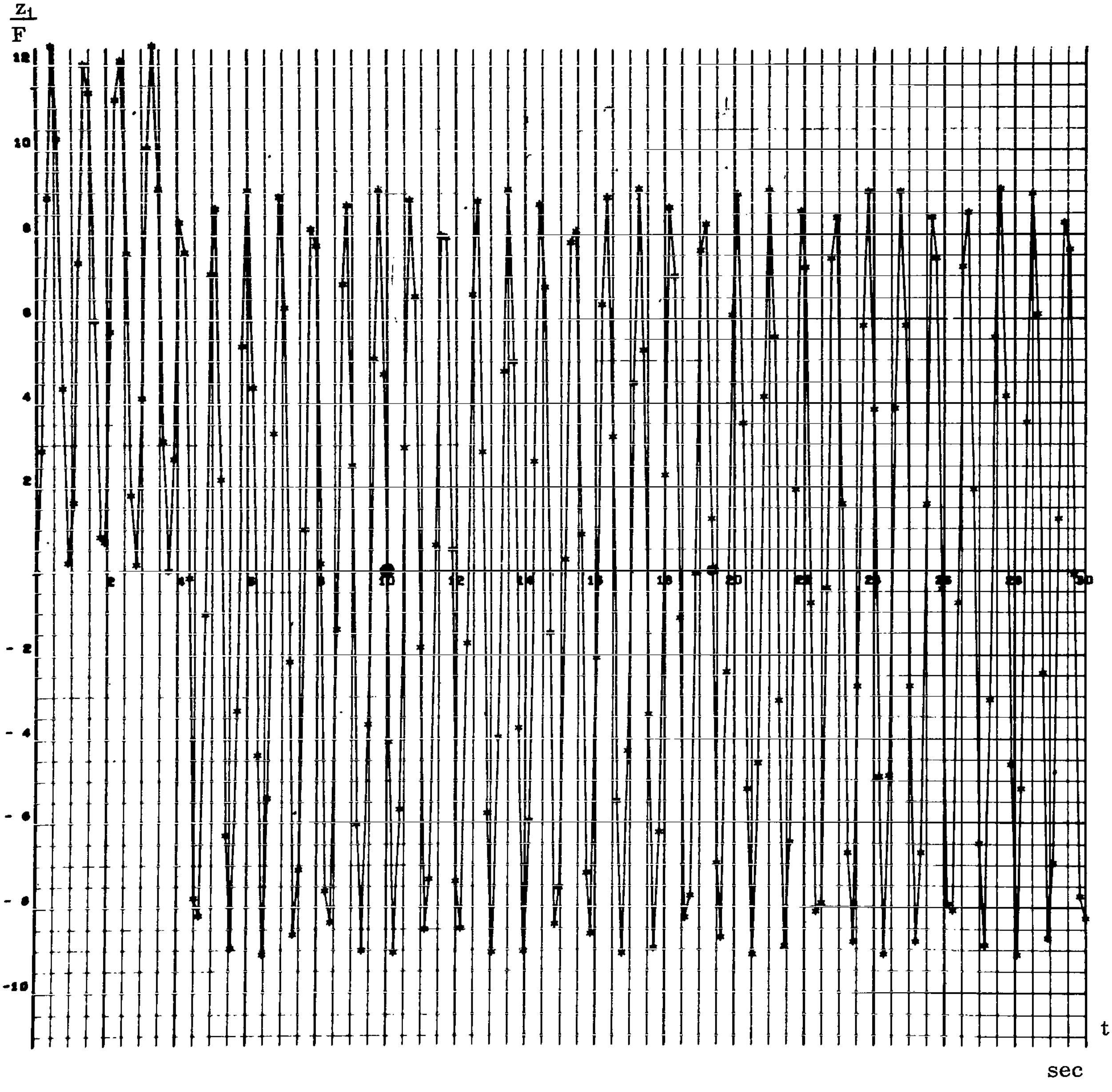

FIGURE 46. RESPONSE OF STRUCTURE TO RECTANGULAR PULSE OF DURATION ($\alpha = 1$) $t_1 = 4$ SECONDS

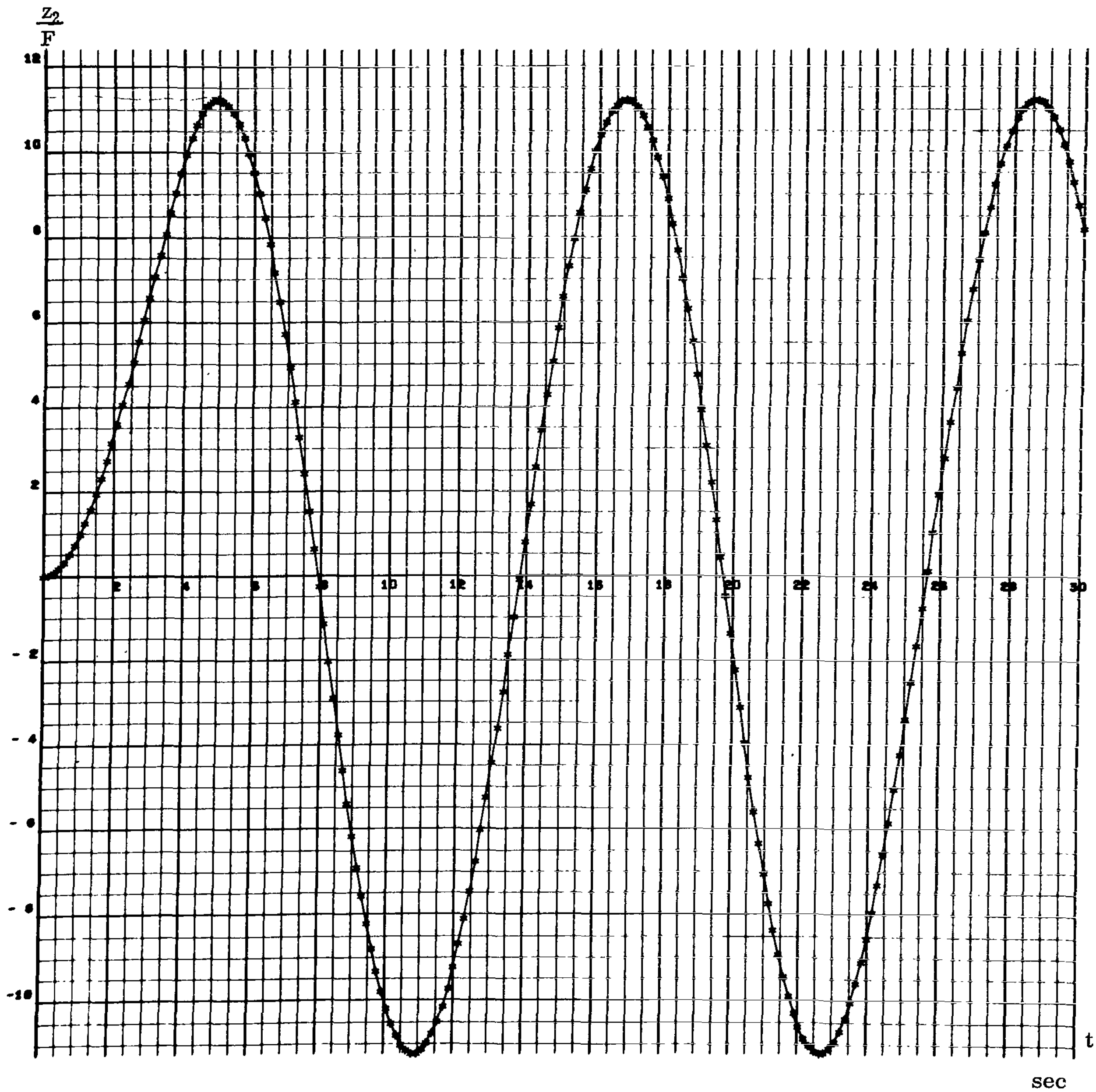

FIGURE 47. RESPONSE OF LIQUID TO RECTANGULAR PULSE OF DURATION (α = 1) t_1 = 4 SECONDS

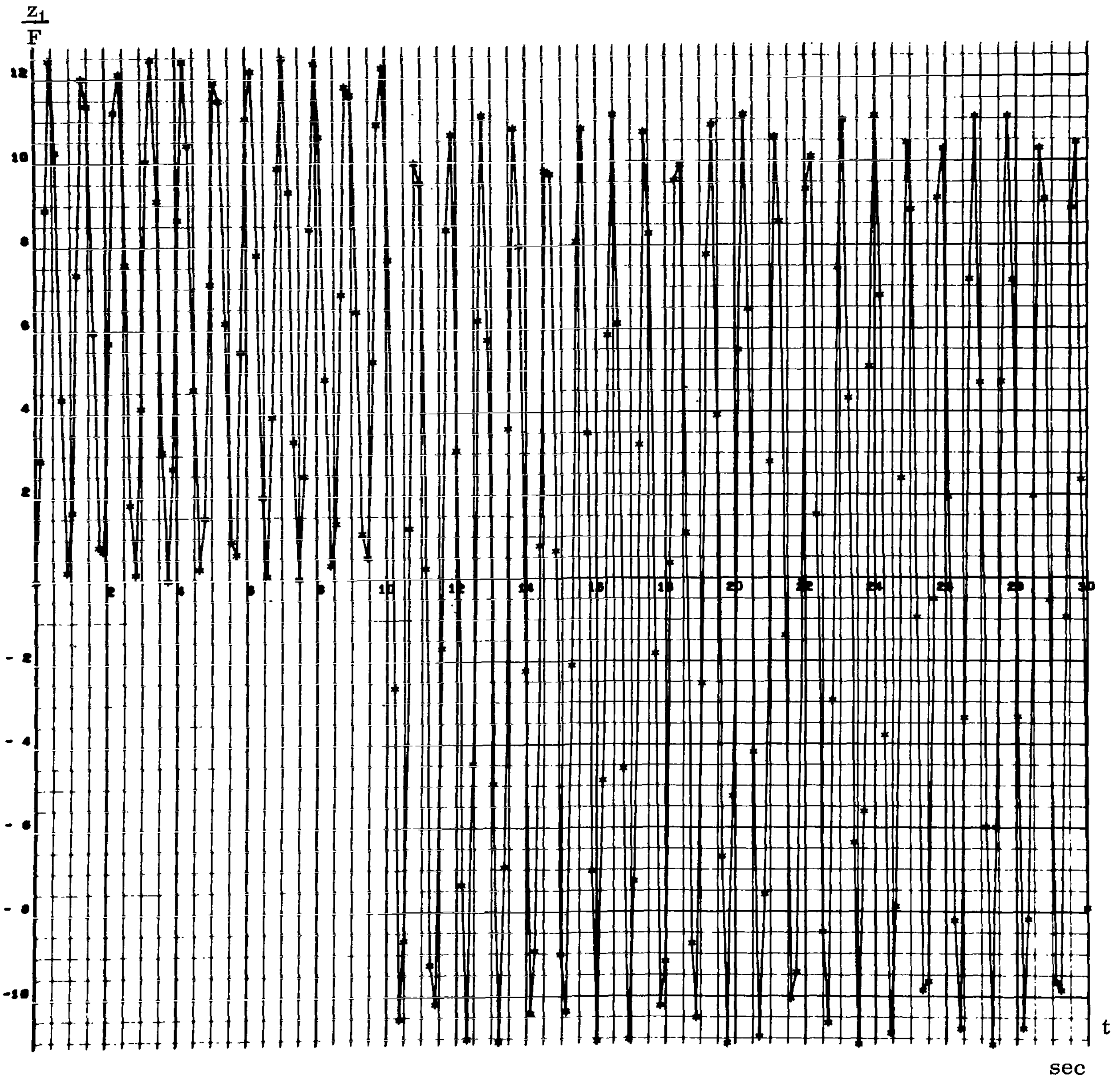

FIGURE 48. RESPONSE OF STRUCTURE TO RECTANGULAR PULSE OF DURATION (α = 1) t_1 = 10 SECONDS

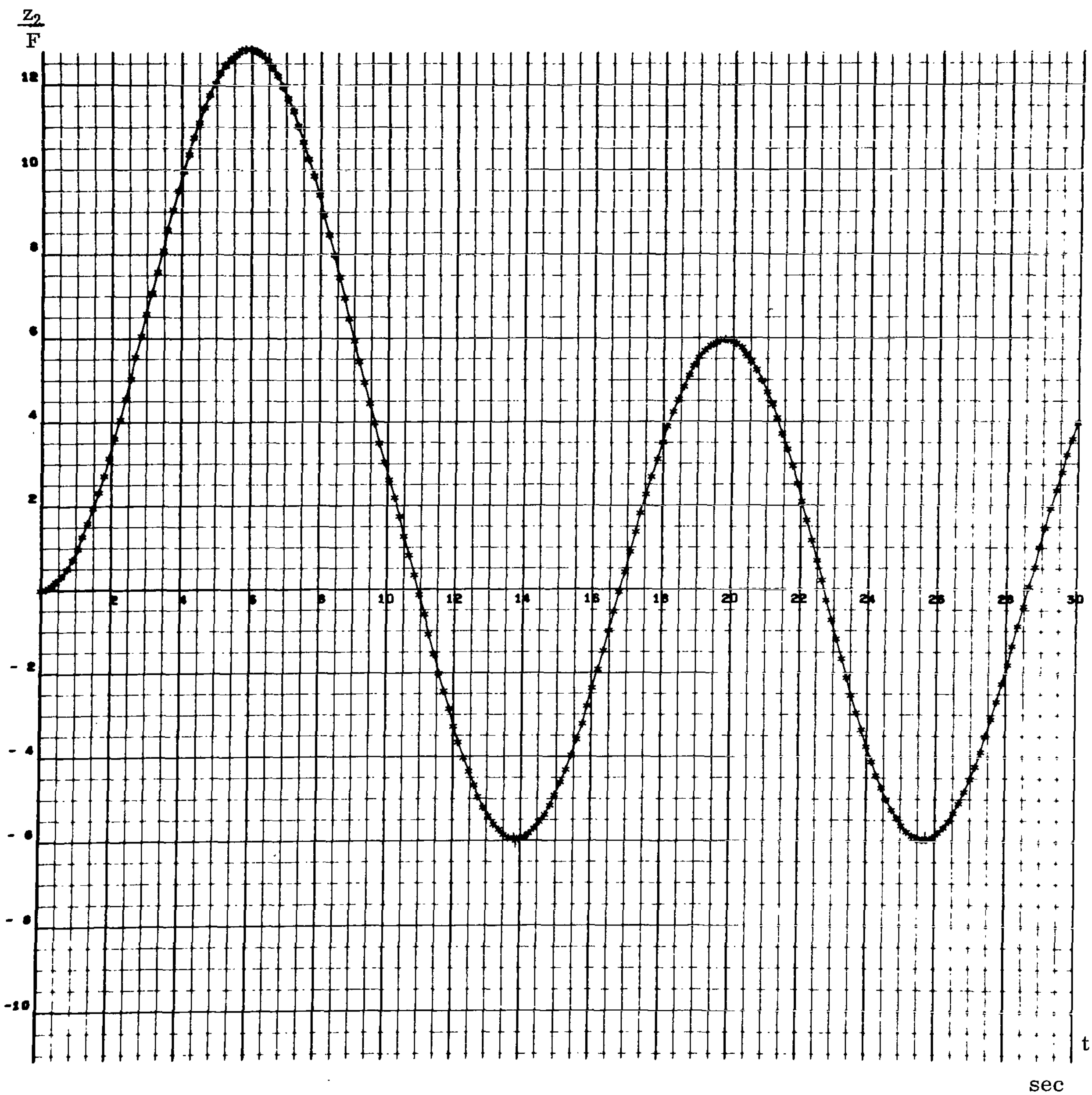

FIGURE 49. RESPONSE OF LIQUID TO RECTANGULAR PULSE OF DURATION (α = 1) t_1 = 10 SECONDS

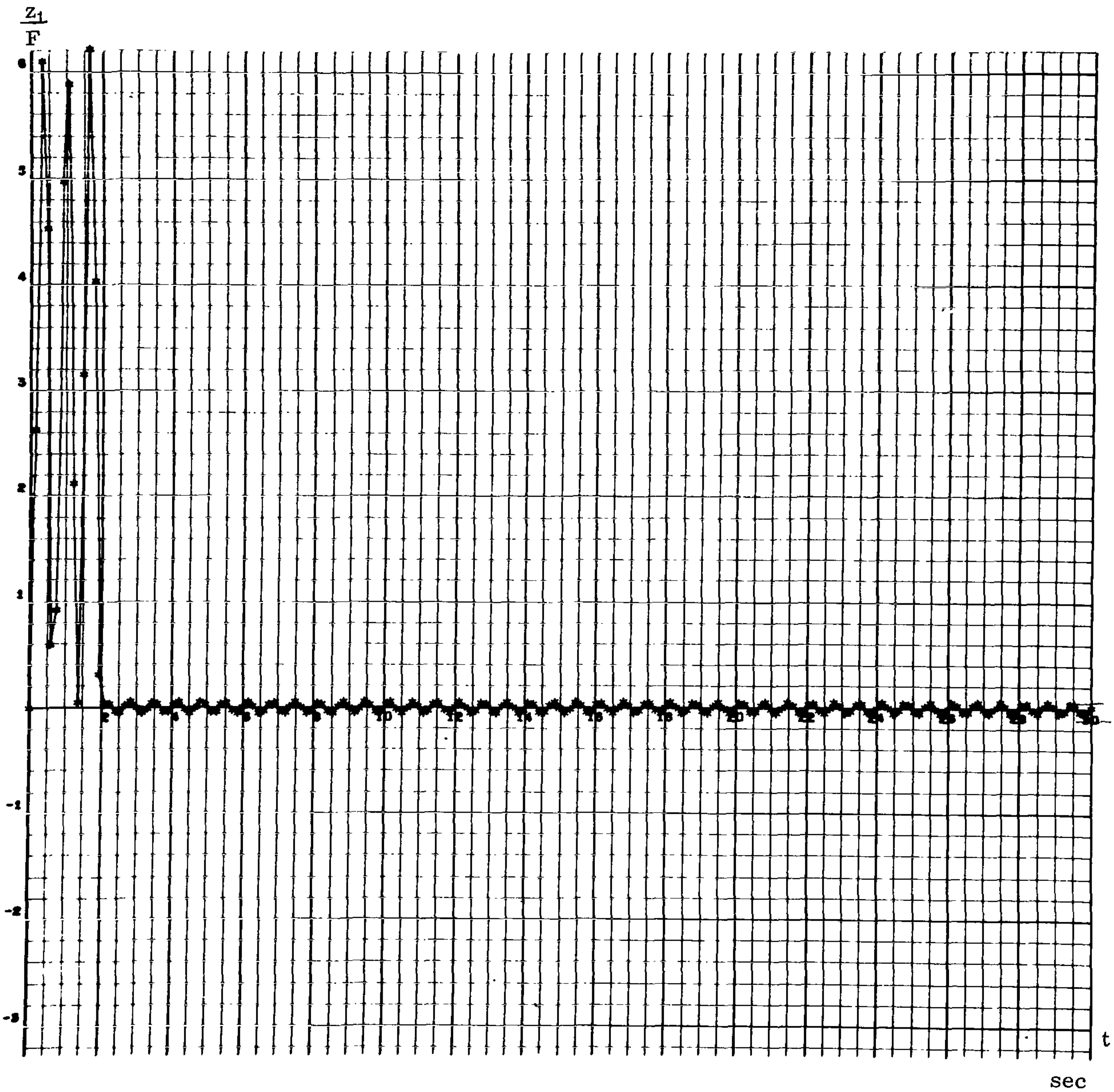

FIGURE 50. RESPONSE OF STRUCTURE TO RECTANGULAR PULSE OF DURATION (α = 2) t_1 = 2 SECONDS

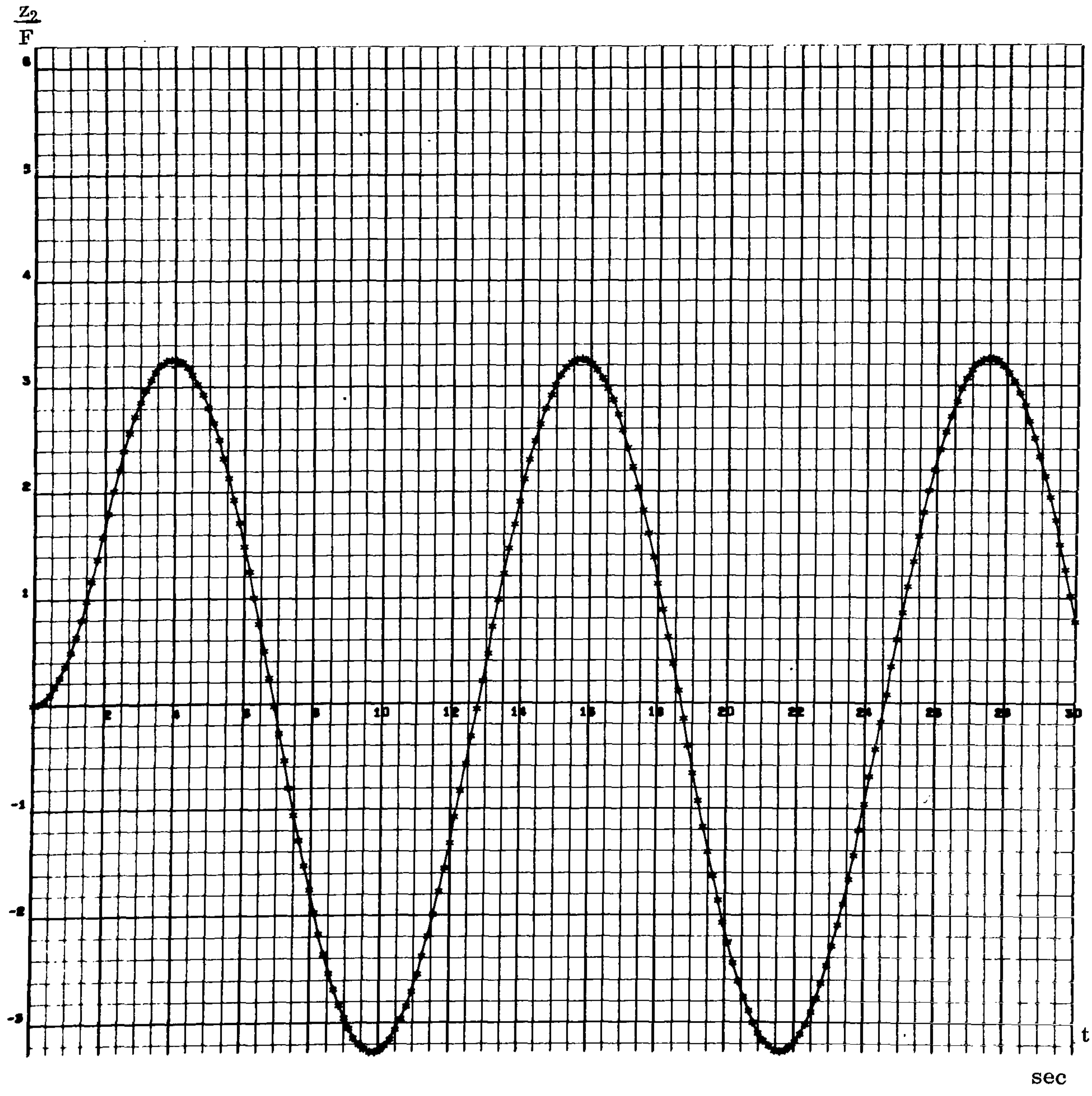

FIGURE 51. RESPONSE OF LIQUID TO RECTANGULAR PULSE OF DURATION (α = 2) t_1 = 2 SECONDS

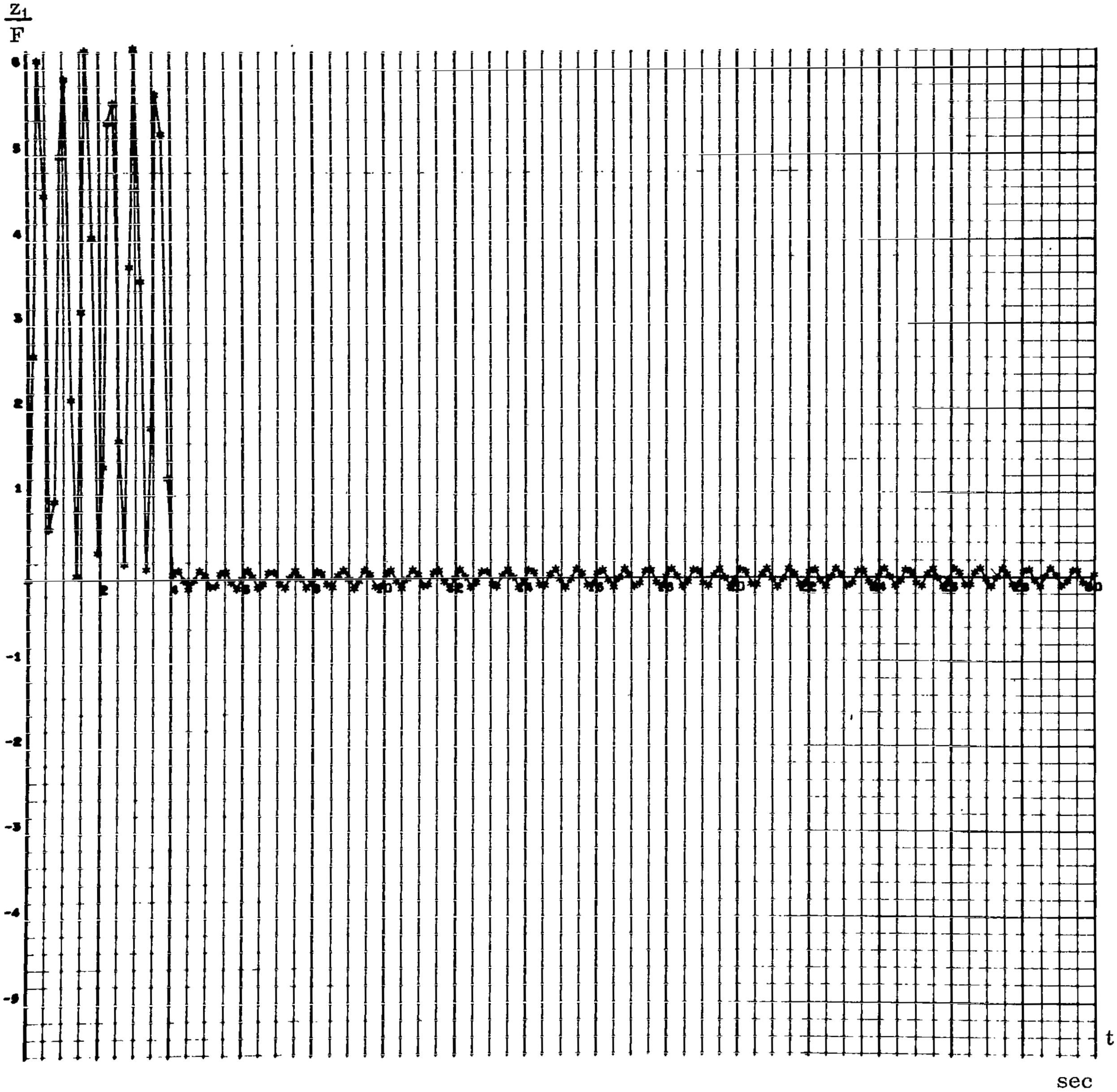

FIGURE 52. RESPONSE OF STRUCTURE TO RECTANGULAR PULSE OF DURATION ($\alpha = 2$) $t_1 = 4$ SECONDS

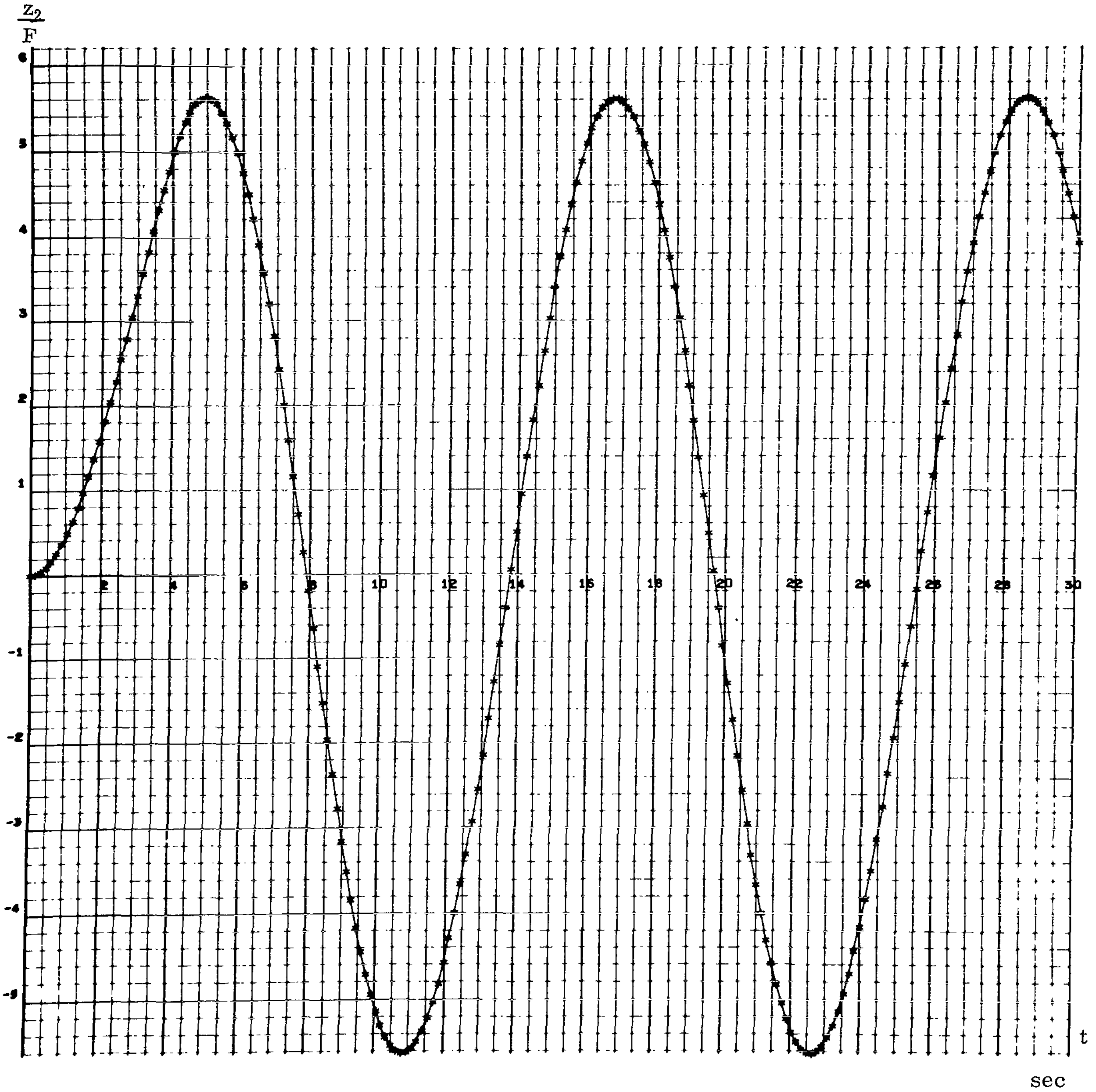

FIGURE 53. RESPONSE OF LIQUID TO RECTANGULAR PULSE OF DURATION (α = 2) t_1 = 4 SECONDS

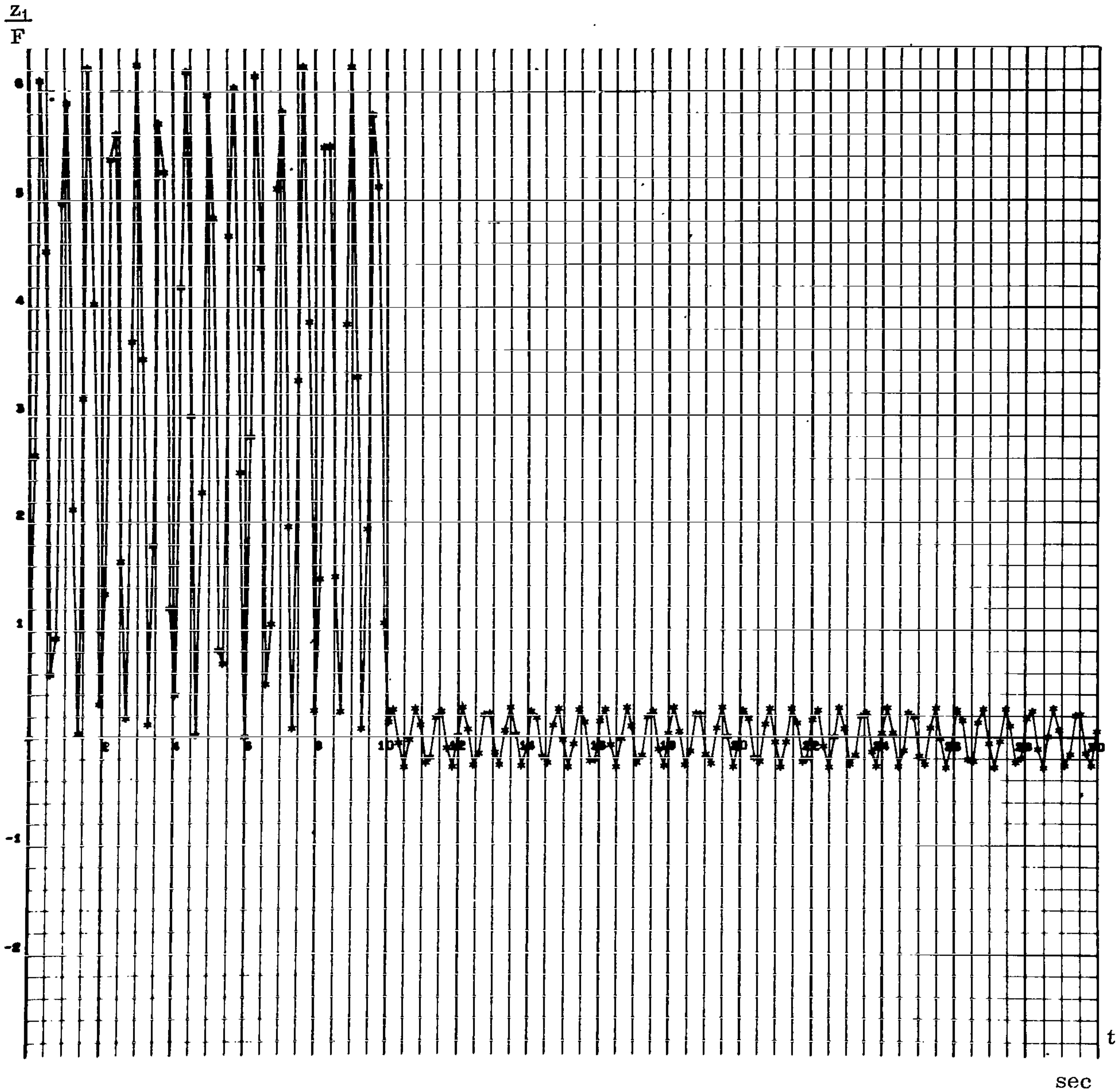

FIGURE 54. RESPONSE OF STRUCTURE TO RECTANGULAR PULSE OF DURATION (α = 2) t_1 = 10 SECONDS

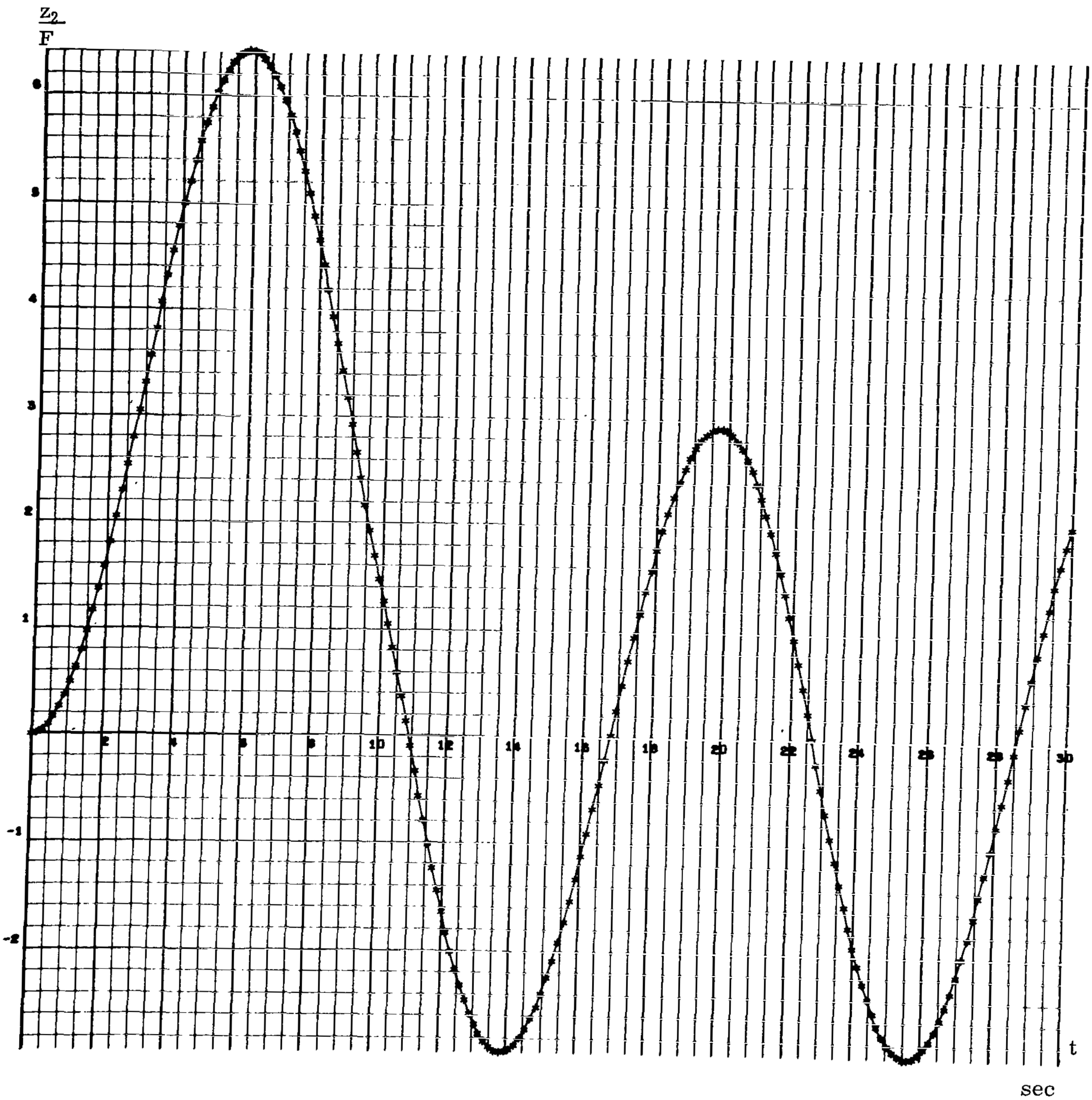

FIGURE 55. RESPONSE OF LIQUID TO RECTANGULAR PULSE OF DURATION (α = 2) t_1 = 10 SECONDS

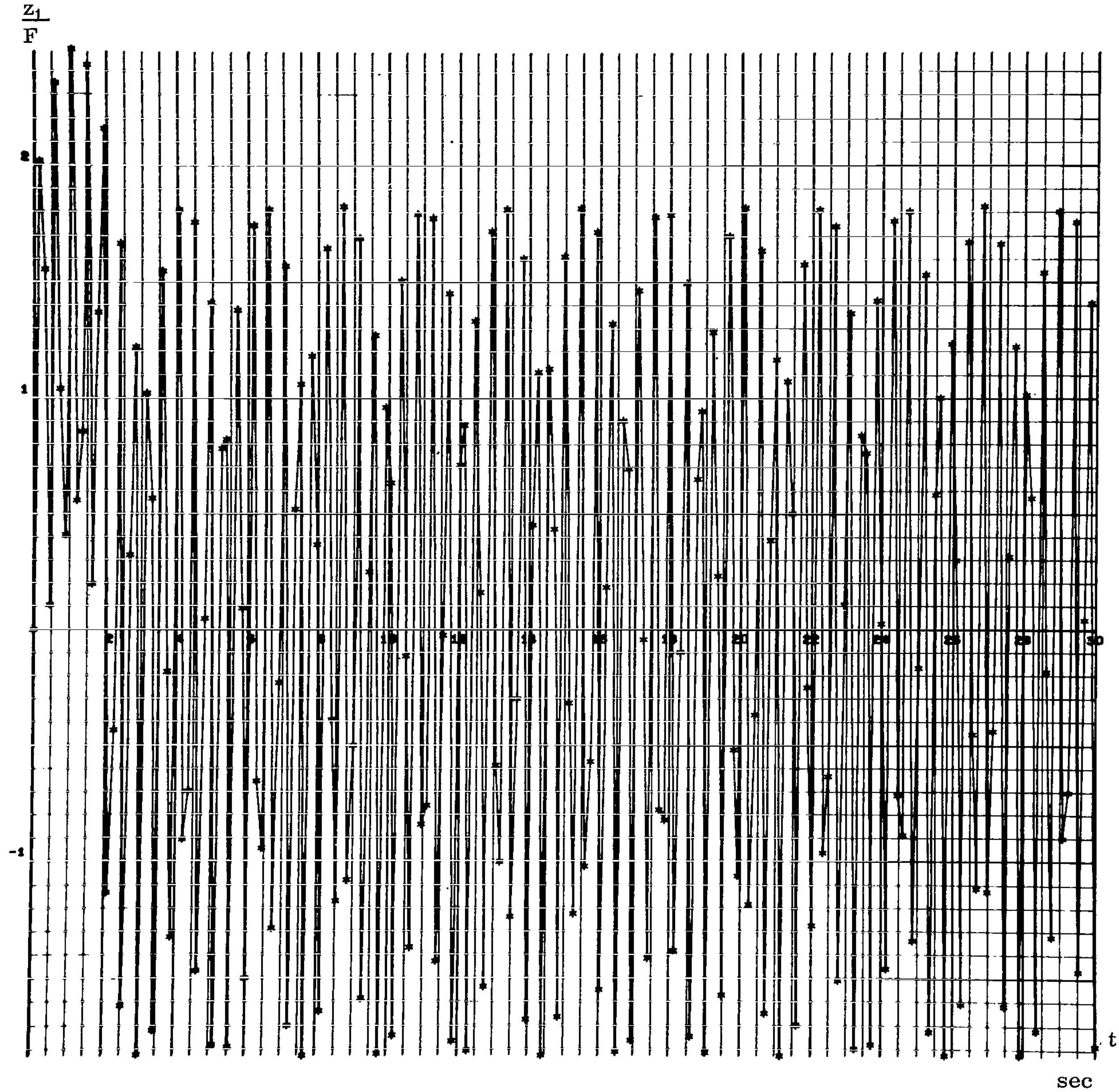

FIGURE 56. RESPONSE OF STRUCTURE TO RECTANGULAR PULSE OF DURATION (α = 5) t_1 = 2 SECONDS

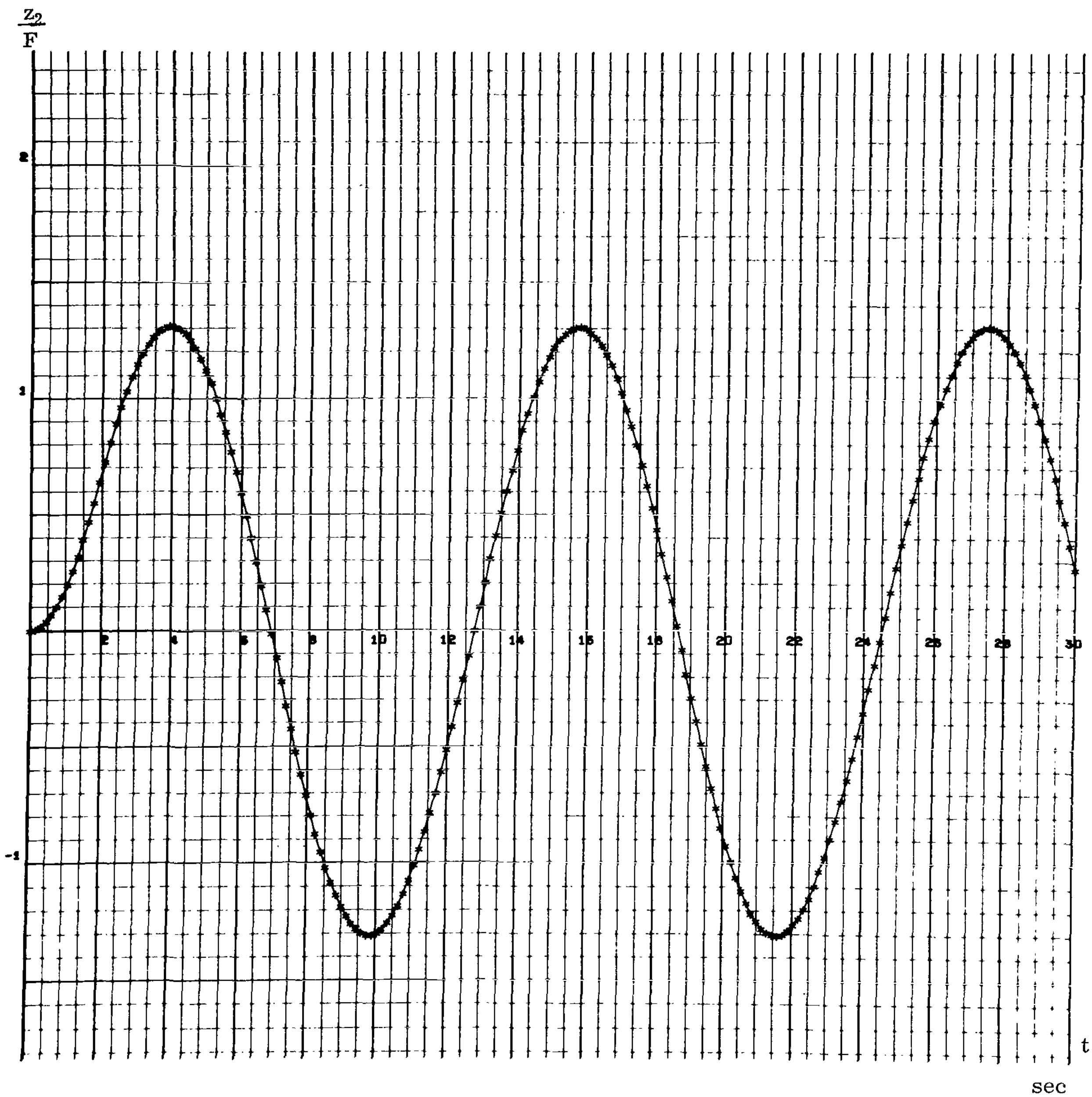

FIGURE 57. RESPONSE OF LIQUID TO RECTANGULAR PULSE OF DURATION (α =.5) t_1 = 2 SECONDS

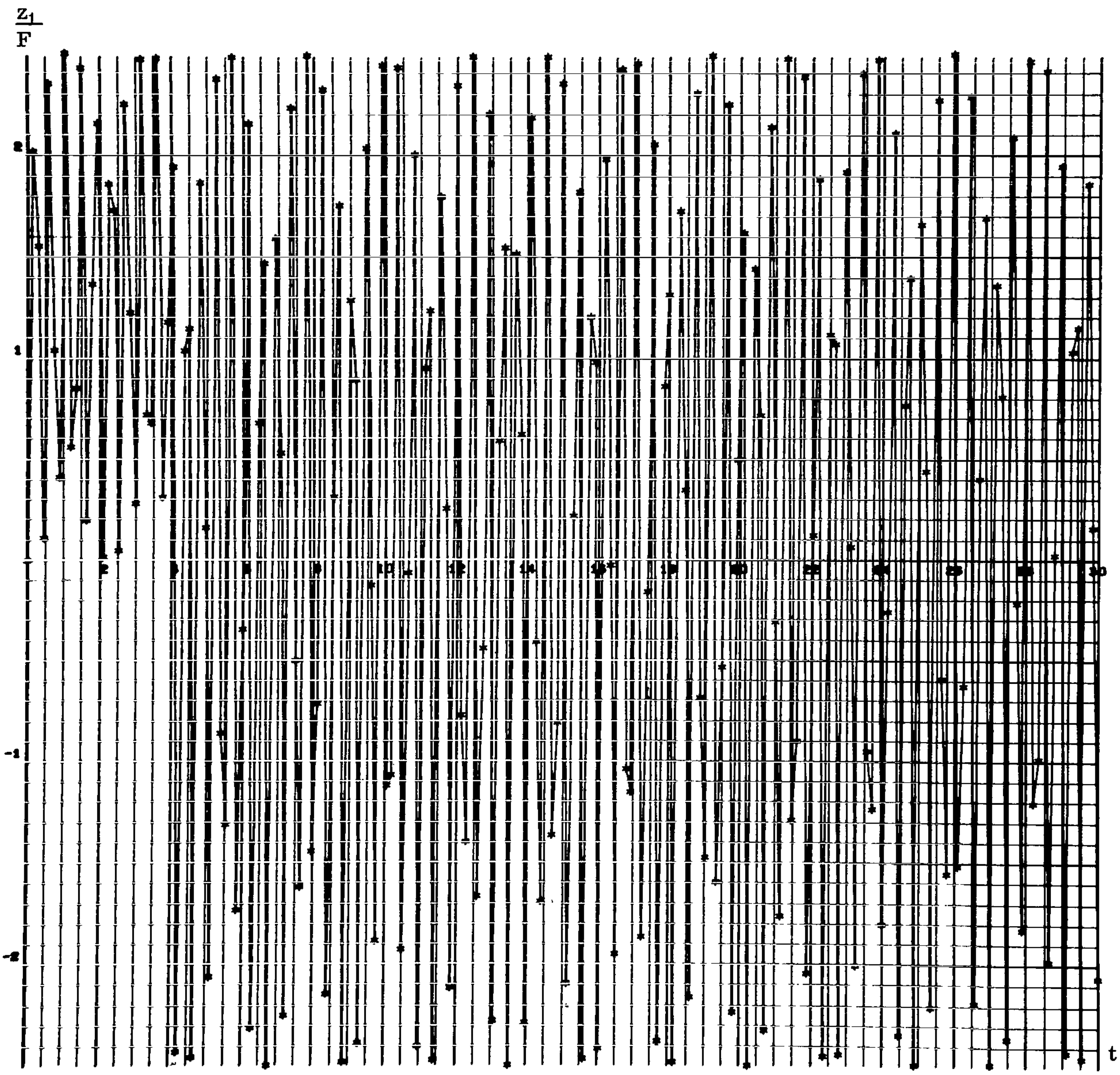

FIGURE 58. RESPONSE OF STRUCTURE TO RECTANGULAR PULSE OF DURATION (α = 5) t_1 = 4 SECONDS

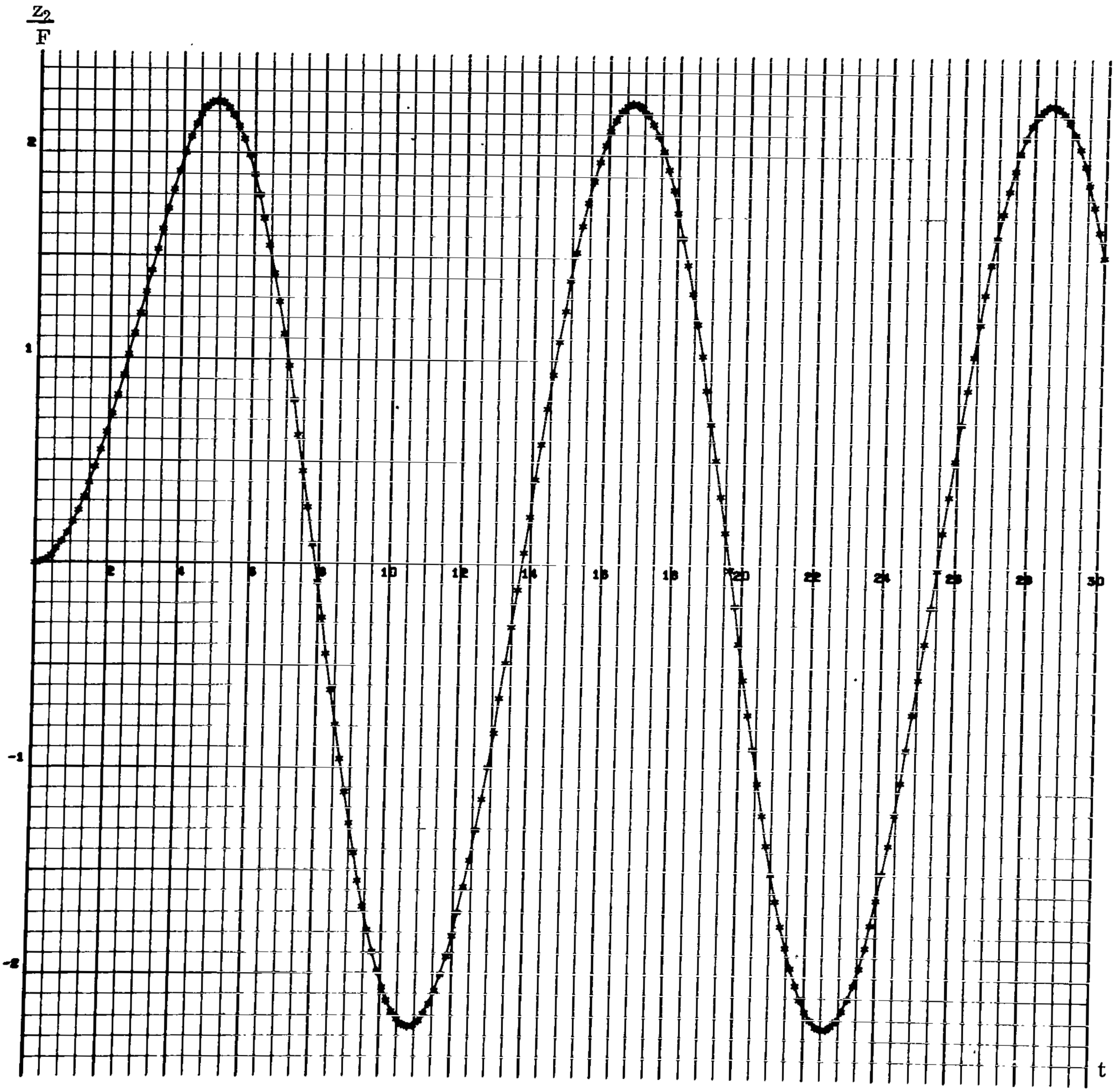

FIGURE 59. RESPONSE OF LIQUID TO RECTANGULAR PULSE OF DURATION ($\alpha = 5$) $t_1 = 4$ SECONDS

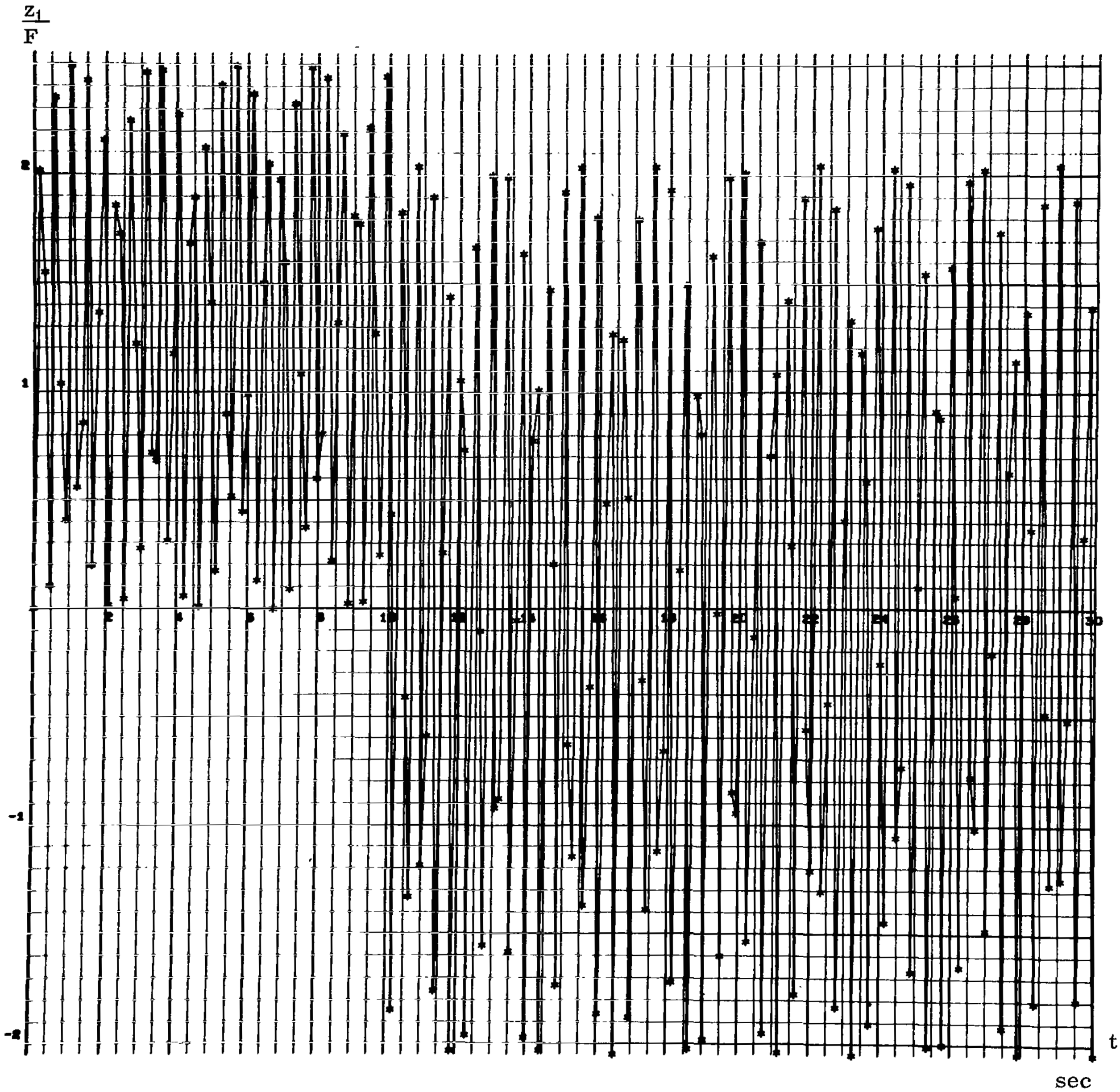

FIGURE 60. RESPONSE OF STRUCTURE TO RECTANGULAR PULSE OF DURATION (α = 5) t_1 = 10 SECONDS

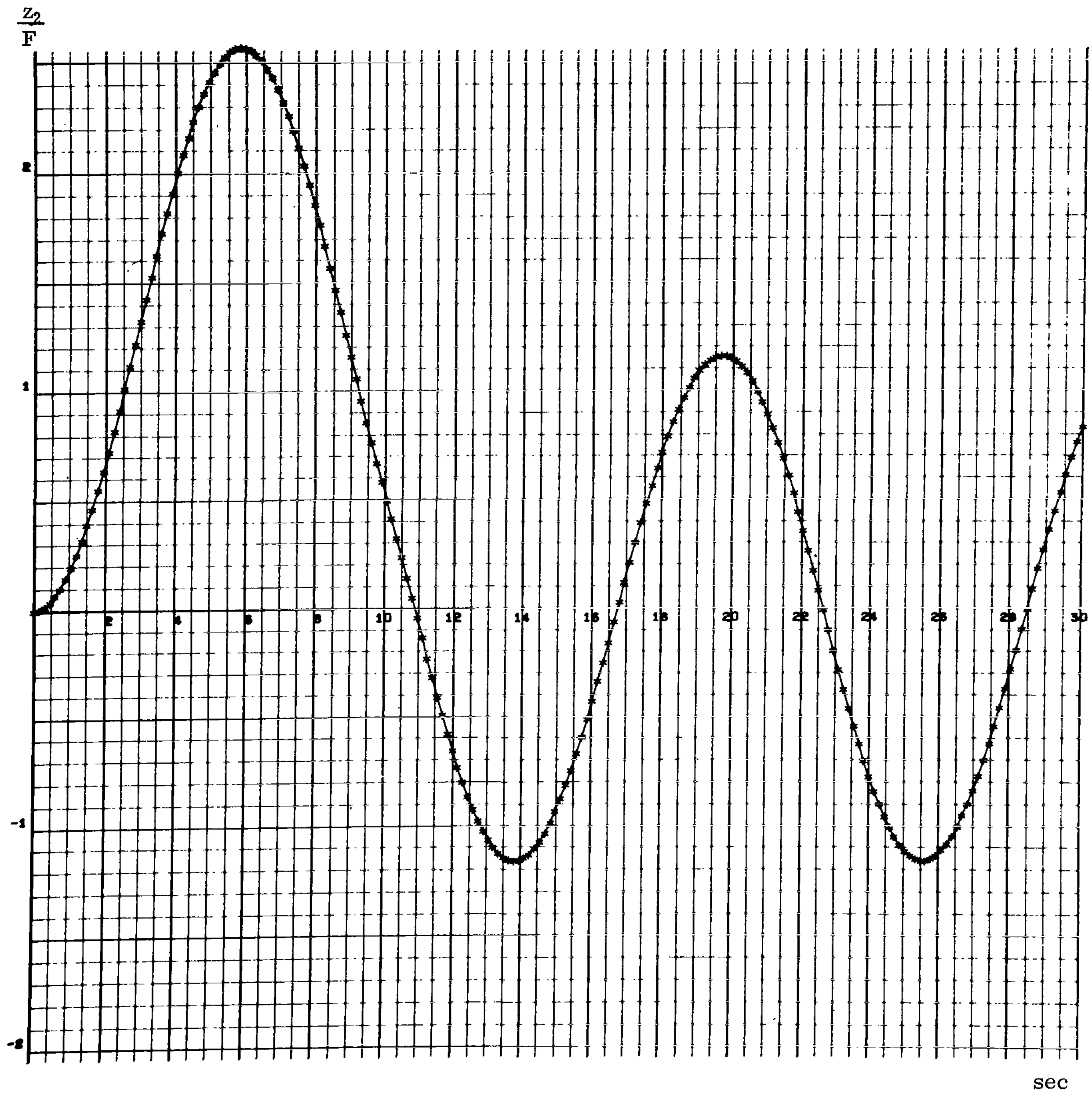

FIGURE 61. RESPONSE OF STRUCTURE TO RECTANGULAR PULSE OF DURATION (α = 5) t_1 = 10 SECONDS

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—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

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