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METHODS OF QUADRATURE FOR EULER

TRANSFORM INTEGRALS

by

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ABSTRACT 15323

Several methods of treating Euler transform integrals exist. One such method follows from the expression of the Euler transform kernel as a bilinear series of independent solutions to the Jacoby equation valid for the integration variable in the real interval -1 to 1 and the transform variable outside. The transform function then is expressed as a series of solutions of the second kind to the Jacoby equation whose coefficients are the expansion coefficients of the function to be transformed in the complete set of Jacoby polynomials, provided the latter exist. Such a series is absolutely convergent for the transform variable not on the real interval cited above. Another method, due to MacRobert, permits quadrature of the Euler transform integral directly for certain integrands. Finally, the expansion of the Euler kernel in a bilinear series of Bessel functions and Neumann polynomials valid for the integration variable on the finite interval, 0 to a, is mentioned, and applied to several integrals. Examples of all three methods are given.

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INTRODUCTION

In certain applications of applied mathematics such as the study of potential problems in quantum mechanics or plasma physics, integrals of the Euler transform type occasionally arise;

$$g(z) = \int_{-1}^{1} (z-x)^{-\mu} f(x)\, dx \tag{1}$$

where f(x) is sufficiently well-behaved to be uniformly approximated on the interval $[-1,1]$ by a complete set of polynomials, and μ is any number such that Re $\mu > 0$. If the order of the transform, μ, is an integer, the restriction that z not lie on the real axis segment, $[-1,1]$ may be relaxed by taking the principal part of the integral. As stands, g(z) is an analytic function of z for all neighborhoods not overlapping the real axis cut as given above, and therefore the kernel may be expanded in a Taylor series such that the integration may be carried out term by term. There is, however, another expansion of the kernel in a bilinear series of functions which are solutions of the first and second kind of a hypergeometric equation. This representation of the kernel is also absolutely convergent for all z restricted as above, therefore term by term integration in equation 1 is also justified. However, solutions of the first kind with integer indices are polynomials that form closed sets on the interval, $[-1,1]$ with respect to specified weight functions. Thus g(z) may be expressed as a series in solutions of the second kind with coefficients that are the expansion coefficients of the arbitrary function in the polynomial set. An abvious advantage of this kernel representation follows of f(x) is orthogonal to all but one of the polynomials, in which case g(z) is proportional to a solution of the second kind. From a numerical point of view, expression of g(z) in a series of solutions permits the use of recursion relations to "build up" the series as might be done in a computer evaluation of g(z). Finally, the establishment of a bilinear expansion of the Euler kernel permits the extension of integral tables to cover integrals of the type shown in eq. 1 if the appropriate expansion coefficients are already evaluated. Examples of a few of these integrals are given in Appendix A.

The following three sections summarize the development of several bilinear expansions, and treat the Euler transform of a special integrand. The first of these reviews some basic properties of the Jacoby function system necessary to subsequent development.

The Jacoby Function System

The hypergeometric equation with three regular singular points located at ± 1 and ∞ is known as the Jacoby equation. Its two independent solutions are characterized by three parameters α, β and n; if the latter is integer, one of the two solutions is a polynomial of order n. The second solution is regular everywhere in the complex plane but has branch-points at ± 1, with a branch cut joining these two singularities to make it single valued. The Jacoby polynomials, or solutions of the first kind, form a complete set on the closed interval -1 to 1 with respect to the integer index n and a weight function given in the table below. At infinity, these polynomials have a simple pole of order n. Both solutions satisfy well-known recursion formulas given elsewhere.[1] For certain specified values of the parameters α , and β , the Jacoby polynomials are proportional to the Gegenbauer, Tchebycheff, and Legendre polynomials as shown in the table below:

<u>Table One</u>

(α,β)	Name	Symbol	Weight Function, $W^{(\alpha,\beta)}$	Normalization, $N_n^{(\alpha,\beta)}$
	Jacoby	$P_n^{(\alpha,\beta)}$	$(1+x)^{\beta}(1-x)^{\alpha}$	$\frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)}$
$\alpha=\beta=\lambda-\frac{1}{2}$	Gegenbauer	$C_n^{\lambda}=\frac{\Gamma(2\lambda+n)\Gamma(\lambda+\frac{1}{2})}{\Gamma(2\lambda)\Gamma(\lambda+\frac{1}{2}+n)}P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}$	$(1-x^2)^{\lambda-\frac{1}{2}}$	$\frac{\pi 2^{1-\lambda 2}\Gamma(2\lambda+n)}{n!(n+\lambda)[\Gamma(\lambda)]^2}$
$\alpha=\beta=-\frac{1}{2}$	Tchebycheff 1st kind	$T_n=\frac{n!\Gamma(\frac{1}{2})}{2\Gamma(n+\frac{1}{2})}P_n^{(-\frac{1}{2},-\frac{1}{2})}$	$(1-x^2)^{-\frac{1}{2}}$	$\frac{1}{2}\pi, n\neq 0$; $\pi, n=0$
$\alpha=\beta=\frac{1}{2}$	Tchebycheff 2nd kind	$U_n=\frac{(n+1)!\Gamma(\frac{3}{2})}{2\Gamma(n+\frac{3}{2})}P_n^{(\frac{1}{2},\frac{1}{2})}$	$(1-x^2)^{\frac{1}{2}}$	$\frac{1}{2}\pi, n\neq 0$; $\pi, n=0$
$\alpha=\beta=0$	Legendre	$P_n=P_n^{(0,0)}$	1	$\frac{2}{2n+1}$

The parameters α,β and λ have real parts greater than -1. Thus we see that any result that holds for the Jacoby polynomial system is also true for any of the systems listed in the table. For the purpose of conveniently expressing later results, we shall introduce a general polynomial/function system in the next section. These are the Jacoby solutions for arbitrary α and β but with an additional multiplicative factor to account for the various interrelations among the polynomial sets given by special values of α,β as listed above.

As shown in Appendix B, the Euler transform method solution of the Jacoby equation leads to a convenient integral representation of its solutions. The choice of the contour determines which of the two independent solutions is represented; and, for the real interval $[-1,1]$, we obtain the integral form of the Jacoby function of the second kind:

$$Q_n^{(\alpha,\beta)}(z) = 2^{-n-1}\int_{-1}^{1}(z-t)^{-n-\alpha-\beta-1}(1+t)^{n+\alpha}(1-t)^{n+\beta}dt \qquad (2)$$

Here $z \notin [-1,1]$. This function satisfies the same recursion formulas as $P_n^{(\alpha,\beta)}$, except for n = 0, and if $\mathrm{Re}(\alpha,\beta)>-1$ and is analytic everywhere except for the branch cut between -1 and 1. Its value on the cut is defined to be:

$$Q_n^{(\alpha,\beta)}(x) \equiv \tfrac{1}{2}\left[Q_n^{(\alpha,\beta)}(x+i0) + Q_n^{(\alpha,\beta)}(x-i0)\right]$$

$$= \frac{-1}{2\pi\sin\alpha\pi}P_n^{(\alpha,\beta)}(x) + \frac{2^{\alpha+\beta-1}\cos\alpha\pi\,\Gamma(\alpha)\Gamma(n+\beta+1)(1-x)^{-\alpha}(1+x)^{-\beta}}{\Gamma(n+\alpha+\beta+1)} \qquad (3)$$

$$\times F(n+1, -n-\alpha-\beta; 1-\alpha; \tfrac{1}{2}-\tfrac{1}{2}x) \quad ; \quad -1<x<1$$

The latter equality follows from considering the contour integral about the branch cut and taking the limit Imz = 0; and F is the hypergeometric function, ${}_2F_1$. Additional properties of the solutions of the second kind associated with the Jacoby, Gegenbauer, Tchebycheff, and Legendre polynomials are found in standard texts.[1,2,3]

It is known from the theory of the hypergeometric function that these quantities map into themselves under the fractional linear substitution:

$$t' = (At+B)/(Ct+D) \; ; \; AD-BC \neq 0 \qquad (4)$$

Therefore it is possible to obtain several equivalent integral representations of the same solution. One very useful form is derived from eq. 2 by the substitution:

$$t = \left\{ \frac{z}{1+z}(1+u) - 1 \right\} \Big/ \left\{ \frac{1+u}{1+z} - 1 \right\} \tag{5}$$

Eq. 2 then reads:

$$Q_n^{(\alpha,\beta)}(z) = 2^{-n-1}(z-1)^{-\alpha}(z+1)^{-\beta}\int_{-1}^{1} du\,(z-u)^{-n-1}(1+u)^{n+\beta}(1-u)^{n+\alpha};\; z \notin [-1,1] \tag{6}$$

In a similar manner, the integral representations of the polynomial solution may be developed:

$$P_n^{(\alpha,\beta)}(z) = 2^{-n-1}(\pi i)^{-1}(1-z)^{-\alpha}(1+z)^{-\beta}\oint_C (z-u)^{-n-1}(1+u)^{n+\beta}(1-u)^{n+\alpha}du\,;\; -1 \le z \le 1 \tag{7}$$

Here the phase of the integrand has been chosen such that $P_n^{(\alpha,\beta)}$ is real for z on the positive real axis. The selected contour encloses the points z and 1 where the complex plane has been cut from -1 to $-\infty$. These two relations may be combined to give the key equation upon which this paper is based. First we see that the integrand of eq. 7 is analytic everywhere in C except at z, and hence its residue is $(n!)^{-1}(-1)^n$ times the n-th derivative of $(1+z)^{n+\beta}(1-z)^{n+\alpha}$. Therefore the polynomial $P_n^{(\alpha,\beta)}$ is expressed by Rodrigues' formula:

$$P_n^{(\alpha,\beta)}(x) = \frac{(1-x)^{-\alpha}(1+x)^{-\beta}(-1)^n}{2^n n!}\frac{d^n}{dx^n}(1+x)^{n+\beta}(1-x)^{n+\alpha} \tag{8}$$

Now, since the integrand of eq. 6 vanishes at $u=\pm 1$, the integrated terms vanish upon an integration by parts to give:

$$Q_n^{(\alpha,\beta)}(z) = \frac{(z-1)^{-\alpha}(z+1)^{-\beta}(-1)^n}{2^{n+1}\, n!}\int_{-1}^{1} \frac{1}{(z-t)} \frac{d^n}{dt^n}(1-t)^{n+\alpha}(1+t)^{n+\beta}\, dt \qquad (9)$$

Rodrigues' formula may be substituted for the integrand in the above equation to give the fundamental relationship between the polynomials and solutions of the second kind, often called the Neumann integral:

$$Q_n^{(\alpha,\beta)}(z) = 2^{-1}(z-1)^{-\alpha}(z+1)^{-\beta}\int_{-1}^{1}(z-t)^{-1}(1+t)^{\beta}(1-t)^{\alpha}P_n^{(\alpha,\beta)}(t)\, dt \qquad (10)$$

Here $z \notin [-1,1]$. The equation may be extended by definition to values of /Rez/<1, Imz=0, if the principal part of the above integral is taken. This Euler transform relationship holds for all of the specializations of the Jacoby polynomials and solutions of the second kind as listed in Table One.

Bilinear Expansions

The Jacoby polynomial system and its specializations form closed sets on the interval $[-1,1]$. In order to express this property as well as subsequent expansion formulas, it is convenient to introduce a general polynomial of the first kind $\psi_n^{(\alpha,\beta)}$, and a general function of the second kind $\varphi_n^{(\alpha,\beta)}$, which, for special values of their indices are proportional to various of the polynomial systems listed in Table One. Thus, for α and β as given in the table below, we have:

$$\left\{ \begin{matrix} \psi_n^{(\alpha,\beta)} \\ \varphi_n^{(\alpha,\beta)} \end{matrix} \right\} = B_n^{(\alpha,\beta)} \times \left\{ \begin{matrix} \text{any of the Jacoby Polynomial/function} \\ \text{systems given in Table 1} \end{matrix} \right\}$$

The constant of proportionality, $B_n^{(\alpha,\beta)}$ has the values:

Table Two

α, β	Name	Polynomial/Function		$B_n^{(\alpha,\beta)}$
α, β	Jacoby	$P_n^{(\alpha,\beta)}(x)$;$Q_n^{(\alpha,\beta)}(x)$		1
$\lambda-\frac{1}{2}, \lambda-\frac{1}{2}$	Gegenbauer	$C_n^{\lambda}(x)$	;$D_n^{\lambda}(x)$	$\Gamma(2\lambda)\Gamma(\lambda+\frac{1}{2}+n)/\{\Gamma(2\lambda+n)\Gamma(\lambda+\frac{1}{2})\}$
$-\frac{1}{2}, -\frac{1}{2}$	Tchebycheff 1st kind	$T_n(x)$	;$R_n(x)$	$\Gamma(n+\frac{1}{2})/\{\Gamma(\frac{1}{2})\Gamma(n+1)\}$
$\frac{1}{2}, \frac{1}{2}$	Tchebycheff 2nd kind	$U_n(x)$	;$S_n(x)$	$\Gamma(n+\frac{3}{2})/\{\Gamma(\frac{1}{2})\Gamma(n+2)\}$
$0, 0$	Legendre	$P_n(x)$	;$Q_n(x)$	1

The general polynomials are orthogonal on the closed interval from -1 to 1 with respect to the weight function w(x) for different integer indices:

$$\int_{-1}^{1} w(x)\, \psi_n^{(\alpha,\beta)}(x)\, \psi_m^{(\alpha,\beta)}(x)\, dx = \eta_n^{(\alpha,\beta)}\, \delta_{nm}$$

The normalization constant for the general polynomials is related to those given in Table One for the indicated choice of parameters by:

$$\eta_n^{(\alpha,\beta)} = \left[B_n^{(\alpha,\beta)} \right]^2 N_n^{(\alpha,\beta)}$$

The closure relation for the general case is:

$$w(x) \sum_{n=0}^{\infty} \left[\eta_n^{(\alpha,\beta)}\right]^{-1} \psi_n^{(\alpha,\beta)}(x)\, \psi_n^{(\alpha,\beta)}(x') = \delta(x-x') \tag{11}$$

It follows immediately from this result and the Euler transform relation between the polynomials of the first kind and functions of the second kind, eq. 10, that $(z-x)^{-1}$ has the expansion:

$$(z-x)^{-1} = 2(-1)^{\alpha} W(z) \sum_{n=0}^{\infty} \left\{\eta_n^{(\alpha,\beta)}\right\}^{-1} \psi_n^{(\alpha,\beta)}(x)\, \varphi_n^{(\alpha,\beta)}(z) \tag{12}$$

This equation is the familiar Christoffel-Darbroux identity as the upper limit on the index n passes to infinity. It may be derived from the recursion relations of three different indices satisfied by all of the Jacoby family for any upper limit, as shown in Bateman.[1] For fixed z, the region of absolute convergence for x is any point on the interior of an ellipse, passing through z in the complex plane with foci at ± 1. The quadrature formulas now become straight-forward in terms of this expansion.

A generalization of eq. 12 to integral powers of $(z-x)^{-1}$ follows from the definition of the associated polynomial and function of the second kind. Let these quantities be defined by:

$$\psi_n^{(\alpha,\beta;m)}(x) = \frac{(x^2-1)^{m/2}}{W(x)} \frac{d^m}{dx^m}\left\{W(x)\,\psi_n^{(\alpha,\beta)}(x)\right\} \tag{13}$$

and

$$\varphi_n^{(\alpha,\beta;m)}(z) = \frac{(z^2-1)^{m/2}}{W(z)} \frac{d^m}{dz^m}\left\{W(z)\,\varphi_n^{(\alpha,\beta)}(z)\right\} \tag{14}$$

where the index m is integer and positive. It may easily be shown that these functions satisfy the associated equation:

$$(1-x^2)y'' + \left(\beta-\alpha-(\alpha+\beta+2)x\right)y' + (n-m)(n+m+\alpha+\beta+1)\,y = 0 \tag{15}$$

where y represents either type of solution. Recursion relations for these functions follow from the above equation and the previous definitions. Integral representations follow from the definitions and the integral representations of the non-associated quantities previously discussed. Thus differentiation of eq. 12 with respect to z leads to the required generalization of that formula in terms of the associated functions of the second kind:

$$(z-x)^{-m-1} = (m!)^{-1}\,2\,(-1)^{\alpha+m}\,w(z)\,(z^2-1)^{-\frac{m}{2}} \sum_{n=0}^{\infty} \left\{\eta_n^{(\alpha,\beta)}\right\}^{-1} \psi_n^{(\alpha,\beta)}(x)\,\varphi_n^{(\alpha,\beta;m)}(z) \tag{16}$$

If, in the integrand of eq. 1, h(x) is defined by $f(x) = w(x)h(x)$ and $\mu = m+1$ then that integral becomes, in view of the expansion developed above:

$$g(z) = \int_{-1}^{1} \frac{w(x)h(x)\,dx}{(z-x)^{m+1}} = \frac{(-1)^{\alpha+m}\,w(z)\,(z^2-1)^{m/2}}{m!} \sum_{n=0}^{\infty} A_n^{(\alpha,\beta)}\,\varphi_n^{(\alpha,\beta;m)}(z) \tag{17}$$

where the coefficients of $\varphi_n^{(\alpha,\beta;m)}$ are:

$$A_n^{(\alpha,\beta)} = 2\left\{\eta_n^{(\alpha,\beta)}\right\}^{-1} \int_{-1}^{1} w(x)\,\psi_n^{(\alpha,\beta)}(x)\,h(x)\,dx \tag{18}$$

By this means we have expressed g(z) in an absolutely convergent series for all $z \notin [-1,1]$, and have reduced the integral to the problem of determining the coefficients given in the above equation. In cases where these are listed in integral tables, or may be easily determined, the tables then can be expanded to include generalized Euler transform integrals of the type given in eq. 17. As an example, consider $h(x) = \cos\alpha x$; $w(x) = 1$; $\psi_n^{(\alpha,\beta)} = P_n(x)$; and $\varphi_n^{(\alpha,\beta)} = Q_n(z)$; then we have:

$$\int_{-1}^{1} \cos\alpha x (z-x)^{-1} dx = \sum_{n=0}^{\infty} A_n Q_n(z) = \sqrt{\frac{2\pi}{\alpha}} \sum_{\substack{n=0 \\ \text{even}}}^{\infty} (-1)^{n/2} (2n+1) J_{n+\frac{1}{2}}(\alpha) Q_n(z) \tag{19}$$

where

$$A_n = \begin{cases} \sqrt{\frac{2\pi}{\alpha}} (2n+1)(-1)^{n/2} J_{n+\frac{1}{2}}(\alpha) & ; n = \text{even} \\ 0 ; \; n = \text{odd}. \end{cases} \tag{20}$$

Other examples are listed in Appendix A.

A simple generalization of the integral shown in eq. 1 follows by replacing the denominator by a polynomial of finite order whose roots do not lie on the real axis segment, $[-1,1]$. By means of an improper fraction expansion, the integral may be reduced to a sum over distinct roots of integrals of the form shown in eq. 17. Here, the integer m+1 is the order of the degenerate root if $m > 0$. Integrals with polynomial denominators that do have a finite number of roots on the real axis segment may be handled in like manner, but with the real axis segment definition of functions of the second kind, eq. 3, and replacement of the integrals with their principal values.

A third method of expansion of a Euler kernel depends on the fact that the integral representation of the solutions of the second kind, eq. 6, is valid for any n whose real part is greater than minus one. Thus we shall show that a bilinear expansion exists for the quantity $(z-x)^{-\gamma-1}$, for Re $\gamma > -1$. For clarity, we shall explicitly indicate the parameter dependence of the weight function as superscripts. Now from the choice of the parameter, B, given in Table Two, it follows that eqs. 8 and 6 may be taken to be definitions of the general polynomial and function of the second kind respectively. If, in the latter equation, n is set equal to $l+\gamma$, where l is a positive integer, eq. 6 becomes:

$$\varphi_{l+\gamma}^{(\alpha,\beta)}(z) = \frac{(-1)^{\alpha}}{2^{l+\gamma+1}\, W^{(\alpha,\beta)}(z)} \int_{-1}^{1} \frac{dt\, W^{(\alpha,\beta)}(t)\,(1-t^2)^{l+\gamma}}{(z-t)^{l+\gamma+1}} \tag{21}$$

But

$$W^{(\alpha,\beta)}(t)(1-t^2)^{l+\gamma} = W^{(\alpha+\gamma,\,\beta+\gamma)}(t)(1-t^2)^{l}$$

and therefore, after integrating eq. 21 l times by parts, we have:

$$\varphi_{l+\gamma}^{(\alpha,\beta)}(z) = \frac{(-1)^{\alpha+l}\,\Gamma(\gamma+1)}{2^{l+\gamma+1}\, W^{(\alpha,\beta)}(z)\,\Gamma(l+\gamma+1)} \int_{-1}^{1} \frac{dt}{(z-t)^{\gamma+1}} \frac{d^{l}}{dt^{l}}\left\{ W^{(\alpha+\gamma,\beta+\gamma)}(t)(1-t^2)^{l} \right\} \tag{22}$$

As in the previous cases, the integrand vanishes at the end points, and thus the integrated terms are zero. Rodriques' formula may be used to replace the derivative appearing in the integrand with the appropriate general polynomial to give the following integral representation:

$$\varphi_{l+\gamma}^{(\alpha,\beta)}(z) = \frac{\Gamma(1+\gamma)\,\Gamma(l+1)\,(-1)^{\alpha}}{W^{(\alpha,\beta)}(z)\,\Gamma(l+\gamma+1)\, 2^{\gamma+1}} \int_{-1}^{1} \frac{dt}{(z-t)^{\gamma+1}} W^{(\alpha+\gamma,\beta+\gamma)}(t)\, \psi_{l}^{(\alpha+\gamma,\beta+\gamma)}(t) \tag{23}$$

The closure formula eq. 11, may now be applied to extract the denominator; thus;

$$(z-x)^{-\gamma-1} = (-1)^{\alpha} W^{(\alpha,\beta)}(z) \sum_{l=0}^{\infty} \left\{ \eta_l^{(\alpha+\gamma,\beta+\gamma)} \right\}^{-1} \psi_l^{(\alpha+\gamma,\beta+\gamma)}(x)\, \varphi_{l+\gamma}^{(\alpha,\beta)}(z)\, \frac{\Gamma(l+\gamma+1)\, 2^{\gamma+1}}{\Gamma(\gamma+1)\, l!} \tag{24}$$

In this formula, as in others, z is assumed not to lie on the cut. Several interesting cases arise from particular choices of the parameters α , β and γ . For example, if $\alpha = \beta = 0$ and $\gamma = -\frac{1}{2}$, the above equation reduces to:

$$(z-x)^{-\frac{1}{2}} = \frac{2^{\frac{1}{2}}}{\pi} Q_{-\frac{1}{2}}(z) + \frac{2\sqrt{2}}{\pi} \sum_{l=1}^{\infty} Q_{l-\frac{1}{2}}(z)\, T_n(x) \tag{25}$$

which, if the interval is changed to 0, π becomes the Fourier cosine expansion of the square root of z-cosθ. The functions, $Q_{l-\frac{1}{2}}$ are analytic everywhere in the complex plane with logarithmic singularities at ± 1, and discontinuous along the cut from -1 to 1. These functions arise in the theory of toroidal harmonics, and are discussed by Hobson.[3] This same quantity, $(z-x)^{-\frac{1}{2}}$, may be written as a bilinear series in Legendre polynomials by setting $\alpha = \beta = \frac{1}{2}$, $\gamma = -\frac{1}{2}$. The result is:

$$(z-x)^{-\frac{1}{2}} = (z^2-1)^{\frac{1}{2}}\, 2^{\frac{1}{2}} \sum_{l=1}^{\infty} l\, P_l(x)\, S_l(z) \tag{26}$$

where $S_l(z)$ is a Tchebycheff solution of the second kind defined by:

$$S_l(z) = 2^{-1}(z^2-1)^{-\frac{1}{2}} \int_{-1}^{1} (z-t)^{-1}(1-t^2)^{\frac{1}{2}} U_l(t)\, dt \tag{27}$$

Quadrature formulas applying these expansions are straight-forward, and examples are given in Appendix A.

As a concluding remark on this section on bilinear expansions, we shall briefly comment on one other bilinear form for the Euler kernel. As shown in Bateman,[1] for x in the finite interval, $[0, a]$, and for $|z| > |x|$, an expansion of the Euler kernel may be written:

$$(z-x)^{-1} = \sum_{n=0}^{\infty} \epsilon_n O_n(z) J_n(x) \; ; \qquad \epsilon_0 = 1,\ n=0\ ;\ \epsilon_n = 2,\ n \geq 1$$

The coefficients of the Bessel functions in the above equation are (Neumann) polynomials in z^{-1} of degree one greater than the order and are bounded for large z by an exponential form in z^2. Therefore the expansion shown above is absolutely convergent whenever $|x| < |z|$. These polynomials, however, do not satisfy Bessel's equation for arbitrary index n and therefore do not posses the same relationship to the Bessel functions as the Jacoby polynomials do to the Jacoby solutions of the second kind. Recursion relations, and integral properties are found in the above reference to Bateman. Several examples of this type of integral are included in Appendix A.

A Special Integrand

Special methods for carrying out the integration of eq. 1 for certain integrands exist. One, due to MacRobert,[4] is given below. Consider the quantity $\zeta_q^n(z)$ to be defined as:

$$\zeta_n^q(z) = z^q \varphi_n^{(\alpha,\beta)}(z) - \frac{(-1)^\alpha}{2W^{(\alpha,\beta)}(z)} \int_{-1}^{1} (z-t)^{-1} t^q \psi_n^{(\alpha,\beta)}(t) W^{(\alpha,\beta)}(t)\,dt \tag{28}$$

We now may show that for $q \leq n$, $\varrho_q^n(z) = 0$. From the Euler transform relation between solutions of the first and second kind as given in eq. 10, we may substitute for $\varphi_n^{(\alpha,\beta)}$ in the above integral to give:

$$\varrho_q^n(z) = \frac{(-1)^\alpha}{2W^{(\alpha,\beta)}(z)} \int_{-1}^{1} (z-t)^{-1}(z^q - t^q)\, \psi_n^{(\alpha,\beta)}(t)\, W^{(\alpha,\beta)}(t)\, dt \tag{29}$$

However for $q \leq n$, $(z^q - t^q)/(z-t)$ is a polynomial of $q-1$ order, and hence is orthogonal to the general polynomial $\psi_n^{(\alpha,\beta)}$, in eq. 29. Thus $\varrho_q^n = 0$, from which it follows that;

$$z^q \varphi_n^{(\alpha,\beta)}(z) = \frac{(-1)^\alpha}{2W^{(\alpha,\beta)}(z)} \int_{-1}^{1} (z-t)^{-1}\, t^q\, \psi_n^{(\alpha,\beta)}(t)\, W^{(\alpha,\beta)}(t)\, dt \tag{30}$$

Or, by taking the appropriate linear combinations of this expression, it also follows that:

$$\psi_q^{(\alpha,\beta)}(z)\varphi_n^{(\alpha,\beta)}(z) = \frac{(-1)^\alpha}{2W^{(\alpha,\beta)}(z)} \int_{-1}^{1} (z-t)^{-1} \psi_q^{(\alpha,\beta)}(t)\, \psi_n^{(\alpha,\beta)}(t)\, W^{(\alpha,\beta)}(t)\, dt \tag{31}$$

This result may be generalized somewhat by setting q = n+1. Then, by the arguments above we have:

$$\varrho_{n+1}^n(z) = (-1)^\alpha 2^{-1} [W^{(\alpha,\beta)}(z)]^{-1} \int_{-1}^{1} t^n\, \psi_n^{(\alpha,\beta)}(t)\, W^{(\alpha,\beta)}(t)\, dt \tag{32}$$

To evaluate this integral, we replace the polynomial by Rodrigues' formula and integrate by parts n times. We then obtain:

$$\varrho_{n+1}^n(z) = (-1)^\alpha 2^{-n-1} [W^{(\alpha,\beta)}(z)]^{-1} \int_{-1}^{1} (1-t^2)^n\, W^{(\alpha,\beta)}(t)\, dt \tag{33}$$

This integral is just the normalization integral for the Jacoby polynomials with n set equal to zero, and α replaced by $n+\alpha$ and β replaced by $n+\beta$. Therefore:

$$\int_{n+1}^{n}(z) = 2^{n+\alpha+\beta}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Big/\left\{W^{(\alpha,\beta)}(z)\,\Gamma(2n+\alpha+\beta+2)\right\} \tag{34}$$

And finally giving the result:

$$z^{n+1}\varphi_n^{(\alpha,\beta)}(z) - \frac{(-1)^\alpha}{2W^{(\alpha,\beta)}(z)}\int_{-1}^{1}\frac{t^n\psi_n^{(\alpha,\beta)}(t)\,w^{(\alpha,\beta)}(t)\,dt}{(z-t)} = \frac{(-1)^\alpha 2^{n+\alpha+\beta}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{W^{(\alpha,\beta)}(z)\,\Gamma(2n+\alpha+\beta+2)} \tag{35}$$

Again, after taking the appropriate linear combinations of the above equation, it may be shown that:

$$\psi_{n+1}^{(\alpha,\beta)}(z)\,\varphi_n^{(\alpha,\beta)}(z) = (-1)^\alpha z^{-1}[W^{(\alpha,\beta)}(z)]^{-1}\int_{-1}^{1}(z-t)^{-1}\psi_n^{(\alpha,\beta)}\psi_{n+1}^{(\alpha,\beta)}w^{(\alpha,\beta)}\,dt$$
$$+\ \frac{2^{\alpha+\beta-1}(2n+\alpha+\beta+2)\Gamma(\alpha+n+1)\Gamma(\beta+n+1)(-1)^\alpha}{n!\,\Gamma(2n+\alpha+\beta+2)\,W^{(\alpha,\beta)}(z)} \tag{36}$$

The convergence of these integrals for large z is shown in the case of the Legendre functions for all real indices in the reference cited above.

Comments

The expansions of the Euler transform kernel and its generalizations given above represent a general technique for the reduction of the generalized Euler transform integrals as shown in eq. 1. In each case, the quadrature is expressed as a series, convergent for all values of z not on the branch cut, of functions that form the second solution to the variation of the Jacoby equation as listed in Table One. The well-known recursion relations and asymtotic behavior of these functions are valuable aids in the numerical computation and analytical study of integrals of the Euler transform type.

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Appendix A

A miscellaneous collection of integrals evaluated by the procedures discussed in the text is listed below. The appropriate expansion coefficients are taken from references 5 and 6. Integrals involving the general polynomial/function system defined in the text are valid for all of the special cases of the Jacoby polynomial/function set. Roman indices represent positive integers or zero; greek indices represent numbers restricted by the requirement that their real parts be greater than -1 unless otherwise noted. The variable z is an arbitrary complex number not lying on the real axis segment -1,1 unless otherwise noted. Integrals with a slash are principal parts.

$$\int_{-1}^{1} \frac{\psi_n^{(\alpha,\beta)}(x)\, W^{(\alpha,\beta)}(x)}{(a+bx^2)}\, dx = \frac{i\, W^{(\alpha,\beta)}\left(i\sqrt{\frac{a}{b}}\right)}{\sqrt{ab}} \left\{ \varphi_n^{(\alpha,\beta)}\left(i\sqrt{\frac{a}{b}}\right) - (-1)^{n+1} \varphi_n^{(\beta,\alpha)}\left(i\sqrt{\frac{a}{b}}\right) \right\} \tag{A1}$$

$$\frac{1}{2}\int_{-1}^{1} \frac{\psi_n^{(\alpha,\beta)}(x)\, \psi_{n+1}^{(\alpha,\beta)}(x)}{(z-x)}\, W^{(\alpha,\beta)}(x)\, dx = W^{(\alpha,\beta)}(z)(-1)^{\alpha}\, \psi_{n+1}^{(\alpha,\beta)}(z)\, \varphi_n^{(\alpha,\beta)}(z) - \frac{2^{\alpha+\beta-1}(2n+\alpha+\beta+2)\,\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)}{n!\,\Gamma(2n+\alpha+\beta+2)} \tag{A2}$$

$$\frac{1}{2}\int_{-1}^{1}\frac{x^n\psi_n^{(\alpha,\beta)}(x)\,W^{(\alpha,\beta)}(x)dx}{(z-x)} = z^{n+1}W^{(\alpha,\beta)}(z)(-1)^{\alpha}\varphi_n^{(\alpha,\beta)}(z) - \frac{2^{n+\alpha+\beta}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)} \tag{A3}$$

$$\frac{1}{2}\int_{-1}^{1}\frac{\psi_m^{(\alpha,\beta)}\psi_n^{(\alpha,\beta)}W^{(\alpha,\beta)}dx}{(z-x)} = (-1)^{\alpha}W^{(\alpha,\beta)}(z)\,\psi_m^{(\alpha,\beta)}(z)\,\varphi_n^{(\alpha,\beta)}(z);\; m\le n \tag{A4}$$

$$\int_{-1}^{1}\frac{\psi_n^{(\alpha+\gamma,\beta+\gamma)}(x)W^{(\alpha+\gamma,\beta+\gamma)}(x)\,dx}{(z-x)^{\gamma+1}} = \frac{(-1)^{\alpha}W^{(\alpha,\beta)}(z)\,2^{\gamma+1}\Gamma(n+\gamma+1)}{\Gamma(1+\gamma)\,n!}\varphi_{n+\gamma}^{(\alpha,\beta)}(z) \qquad \mathrm{Re}\,\gamma>-1 \tag{A5}$$

$$\int_{-1}^{1}\frac{\psi_n^{(\alpha,\beta)}(x)\,W^{(\alpha,\beta)}(x)\,dx}{(z-x)^{m+1}} = 2(-1)^{\alpha+m}[m!]^{-1}W^{(\alpha,\beta)}(z)(z^2-1)^{-m/2}\varphi_n^{(\alpha,\beta;m)}(z) \tag{A6}$$

$$\int_{-1}^{1}\frac{e^{iax}(1-x^2)^{\gamma+\lambda-\frac{1}{2}}dx}{(z-x)^{\gamma+1}} = (z^2-1)^{\gamma+\lambda-\frac{1}{2}}2^{1+\lambda+2\gamma}a^{-\lambda-\gamma}\Gamma(\lambda+\gamma)/\Gamma(\gamma+1) \times\sum_{\ell=0}^{\infty} i^{\ell}\frac{(\ell+\lambda+\gamma)}{\ell!}\Gamma(\ell+\gamma+1)D_{\ell+\gamma}^{\lambda}(z)\,J_{\lambda+\gamma+\ell}(a) \qquad \mathrm{Re}\,\gamma>-1;\ \mathrm{Re}(\gamma+\lambda)>-\tfrac{1}{2} \tag{A7}$$

$$\int_{-1}^{1}\frac{e^{iax}\,dx}{(z-x)} = \left(\frac{2\pi}{a}\right)^{1/2}\sum_{\ell=0}^{\infty} i^{\ell}(2\ell+1)\,Q_{\ell}(z)\,J_{\ell+\frac{1}{2}}(a) \tag{A8}$$

$$\int_{-1}^{1} \frac{\sin(\lambda R)}{(z-x)R}\, dx = \frac{\pi}{\sqrt{ab}} \sum_{n=0}^{\infty} (2n+1)\, Q_n(z)\, J_{n+\frac{1}{2}}(a\lambda)\, J_{n+\frac{1}{2}}(b\lambda) \tag{A9}$$

where $R = a^2+b^2-2abx$

$$\int_{0}^{a} \frac{\cos(\lambda S)}{S(z-x)}\, dx = \frac{1}{2}\pi \sum_{n=0}^{\infty} \epsilon_n O_n(z)\, J_{\frac{1}{2}n}\left\{\tfrac{1}{2}a\left(\lambda+\sqrt{1+\lambda^2}\right)\right\} J_{\frac{1}{2}n}\left\{\tfrac{1}{2}a\left(\lambda+\sqrt{1+\lambda^2}\right)^{-1}\right\}; \tag{A10}$$

$\epsilon_n = 1,\ n=0,\ \epsilon_n = 2,\ n \geq 1;\ |z| > |a|;\ S = (a^2-x^2)^{1/2}$

Properties of the Neumann polynomials are given in Ref. 1

$$\int_{0}^{a} (a-x)^{-1/2}(z-x)^{-1}\, dx = \pi\sqrt{\frac{a}{2}} \sum_{n=0}^{\infty} \epsilon_n O_n(z)\, J_{\frac{1}{2}n+\frac{1}{4}}\left(\tfrac{1}{2}a\right) J_{\frac{1}{2}n-\frac{1}{4}}\left(\tfrac{1}{2}a\right) \tag{A11}$$

$$= 2(z-a)^{-1/2} \tan^{-1}\left(\frac{\sqrt{a}}{\sqrt{z-a}}\right)$$

$$\int_{-1}^{1} (z-x)^{-1} Q_k(x)\, dx = \sum_{n=0}^{\infty} (2n+1)\, Q_n(z)\left[1-(-1)^{n+k}\right] / \left\{(n-k)(n+k+1)\right\}; \tag{A12}$$

$Q_k(x)$ is defined by Eq. 3.

$$\int_{-1}^{1} (1-x^2)^{-1/2}(z-x)^{-1}(1+a^2-2ax)^{-1}\, dx = -\pi \sum_{n=0}^{\infty} \epsilon_n U_{n-1}(x) \begin{cases} \dfrac{a^n}{1-a^2}; & |a|<1 \\ \dfrac{1}{a^n(a^2-1)}; & |a|>1 \end{cases} \tag{A13}$$

Appendix B

The Jacoby equation is derived from the hypergeometric equation with three regular singular points at 0, 1, and ∞ by an appropriate coordinate change and linear combination of parameters such that the new equation has its singular points at ± 1 and ∞ . Written in standard from, the Jacoby equation is:

$$\left\{ (1-x^2)\frac{d^2}{dx^2} + \left(\beta-\alpha-(\alpha+\beta+2)x\right)\frac{d}{dx} + n(n+\alpha+\beta+1) \right\} y(x) = 0 \tag{B1}$$

where the parameters α and β have real parts >-1, and the index n is taken as integer. In this case, one of the two independent solutions of the above equation is a polynomial, regular at ± 1 and having a simple pole of order n at infinity; the other solution is a function, regular at infinity and single-valued if a branch cut is made on the Riemann sheet between the branch points -1 and 1. An integral representation of the general solution of the above equation may be had by means of a generalized Euler transform:

$$y(x) = \oint_C (x-t)^{+\mu} v(t)\,dt \tag{B2}$$

where μ is a parameter to be fixed, and the contour C will determine what linear combination of the two independent solutions y(x) represents. Let L_x stand for the operator in eq. B1, then there exists an operator A such that:

$$L_x (x-t)^{+\mu} = A_t (x-t)^{+\mu} \tag{B3}$$

where A operates on the variable t and is a linear operator with the same type of coefficients of its second and first derivatives as appear in the operator L. These coefficients may readily be determined by a Taylor expansion about t of the correspondent coefficient of the operator L in the above equation. We find that:

$$A_t = (1-t^2)\frac{d^2}{dt^2} + (\alpha-\beta+(\alpha+\beta+2\mu)t)\frac{d}{dt} + n(n+\alpha+\beta+1) - \mu(\mu+\alpha+\beta+1) \tag{B4}$$

The adjoint operator, $\tilde{A}$, is determined from the above by Green's theorem and turns out to be:

$$\tilde{A}_t = \frac{d^2}{dt^2}(1-t^2) - \frac{d}{dt}(\alpha-\beta+(\alpha+\beta+2\mu)t) + n(n+\alpha+\beta+1) - \mu(\mu+\alpha+\beta+1) \tag{B5}$$

Now all of the above equations can be combined to give the following sequence of results:

$$\begin{aligned} 0 &= L_x y \\ &= \oint_c L_x (x-t)^{+\mu} V(t)\,dt \\ &= \oint_c A_t (x-t)^{+\mu} v(t)\,dt \\ &= \oint_c (x-t)^{+\mu}\tilde{A}_t V(t)\,dt + \oint_c \frac{d}{dt}\left\{(1-t^2)\left(V(t)\frac{d}{dt}(x-t)^{-\mu} - (x-t)^{+\mu}\frac{d}{dt}V(t)\right)\right\}dt \end{aligned} \tag{B6}$$

Now any contour that forms a complete circuit in the Riemann sheet or any open contour that begins and ends in a zero of the integrand of the second term (the bilinear concomitant) will cause that term to vanish. Since v(t) is as yet arbitrary, the satisfaction of the resulting equation demands that:

$$\tilde{A}_b V(t) = \frac{d^2}{dt^2}\left\{(1-t^2)v(t)\right\} - \frac{d}{dt}\left\{(\alpha-\beta-(\alpha+\beta+2\mu)t)\,v(t)\right\}$$

$$+n(n+\alpha+\beta+1) - \mu(\mu+\alpha+\beta+1) = 0 \qquad \text{(B7)}$$

We have, as yet, not chosen the parameter μ. Quadrature of the above equation follows immediately if the last term vanishes; therefore it follows that:

$$\mu = \begin{Bmatrix} n \\ -n-\alpha-\beta-1 \end{Bmatrix}$$

If v(t) is not to have an essential singularity at ∞, the constant of the first integration must be taken to be zero, therefore, the second integration can be readily performed to give:

$$V(t) = A\,(1+t)^{n+\alpha}(1-t)^{n+\beta} \qquad \text{(B8)}$$

Hence, the general solution to the Jacoby equation may be written:

$$y(x) = A\int_C (x-t)^{-n-\alpha-\beta-1}(1+t)^{n+\alpha}(1-t)^{n+\beta}\,dt \qquad \text{(B9)}$$

where we have made the choice $\mu = -n-\alpha-\beta-1$, and A is an integration constant. The combination of fundamental solutions that y(x) represents is determined by the choice of the contour in the above representation. It may be shown that if a simple loop enclosing the point x on the line segment -1,1 and 1 is selected, with the branch cut made from -1 to $-\infty$, y(x) is proportional to the Jacoby polynomial, $P_n^{(\alpha,\beta)}(x)$; and on the other hand, if the contour is chosen to be the real axis segment from -1 to 1, the second solution, $Q_n^{(\alpha,\beta)}(x)$ is represented if x does not lie on the real axis segment. The proportionality constant in both cases is the integration constant A, and turns out to be 2^{-n-1} as determined by comparison with the hypergeometric series solution of the same equation.

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