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Translational and Rotational Motion of a Body Entering the Mars Atmosphere

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PREFACE

The following report was presented before the joint annual meeting of the WGLR-DGRR (Scientific Society for Air and Space Travel, E.V. —German Society for Rocketry and Space Travel, E.V.), Berlin, Germany, September 16, 1964, and was published in *Raumfahrtforschung*, Vol. 3, July–September 1965.

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SUMMARY

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The atmosphere model which is generally assumed for the planet Mars consists of two overlying layers, the troposphere and the stratosphere. In the troposphere, which encompasses altitudes between zero and h_T, one assumes a linear decrease of the atmosphere temperature with altitude. The stratosphere, which is adjacent to the troposphere, is at a constant temperature (T_S). Other quantities which determine the Martian atmosphere model are: r_0 (the radius of Mars), R (the gas constant), g_0 (the Mars acceleration), T_0 (the temperature), and ρ_0 (the density at the surface of Mars). The numerical values of the atmospheric parameters, which have been proposed by different scientists, are summarized in a table.

In the present paper, we first derive the density distribution within the Martian atmosphere as a function of altitude and the atmospheric parameters, assuming the perfect gas law holds. The equations of motion of simple bodies are established and solved. The solution consists of equations for the velocity, the acceleration, and time as explicit functions of altitude. The case of oblique entry into the atmosphere is also treated. It is shown how the altitudes at which acceleration and heating are maximum can be calculated.

In the second part, we consider a sphere entering the atmosphere described above. It is assumed that the center of gravity of the sphere does not coincide with its center. The diameter containing the sphere's center of gravity will be called its "axis." First, we consider the rotational motion of the sphere, assuming that the axis of the sphere is initially inclined with respect to the trajectory. The axis carries out an oscillation in a plane, the amplitude of which decreases and then increases after maximum acceleration is reached. If in addition the

(over)

sphere is spinning initially, it will carry out a complicated precessional motion around the velocity vector. As in the first case, the precession angle first converges, and then diverges.

In the third part, the results obtained above are compared with the exact solutions. For the exact solution, a numerical integration of the equations of motion is carried out taking into account all forces neglected above. It is seen that the new theory of motion is a considerable improvement over the old theories.

authors

I. DENSITY DISTRIBUTION ON MARS

A typical Mars model atmosphere consists of a troposphere extending from altitudes $h = 0$ to $h = h_T$, above which lies the stratosphere of infinite extension. Within the troposphere, the temperature is assumed to decrease linearly with altitude h; the stratosphere is assumed to be at constant temperature T_S (see Figure 1.1). The parameters which completely define a model Mars atmosphere are: the surface density ρ_0; the surface temperature T_0; the tropopause altitude h_T; the stratosphere temperature T_S; the gas constant R of the atmosphere, and the acceleration of gravity at the surface g_0. Table 1 shows the numerical values of these constants for Mars model atmospheres proposed by various scientists.

The density distribution $\rho(h)$ within the atmosphere will now be calculated, assuming that the perfect gas law holds and the atmosphere is in equilibrium with the local gravitational field and is nonrotating.

The equation of equilibrium for an elementary volume element located at a distance r from the planet center is

$$\frac{\rho r^2 \, d\Omega \, dr \, g_0 r_0^2}{r^2} = p r^2 d\Omega - (p + dp)(r + dr)^2 \, d\Omega \tag{1.1}$$

Here ρ is the density, p the pressure, and $d\Omega$ the solid angle of the volume element subtended at the planet center. The perfect gas law in differential form is:

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T} \tag{1.2}$$

The combination of Equations (1.1) and (1.2) results in the following differential equation for density ρ:

$$\frac{1}{\rho}\frac{d\rho}{dr} = -\left(\frac{2}{r} + \frac{g_0 r_0^2}{r^2 R T} + \frac{1}{T}\frac{dT}{dr}\right) \tag{1.3}$$

Within the troposphere, $T = T_0 + Gh$. The boundary conditions are $\rho(h = 0) = \rho_0$. Integration of Equation (1.3) then gives the following density distribution within the troposphere:

Table 1. Summary of Mars atmosphere constants

Atmosphere	Surface pressure p_0 (mb)	Surface density ρ_0 (g/cm^3)	Surface temperature T_0 (°K)	Tropopause altitude h_T (km)	Stratosphere temperature T_S (°K)	Gas constant $R = R'/M$ (m^2/sec^2 °K)	Ratio of specific heats $\gamma = C_p/C_v$ (nondim.)	Acceleration of gravity g_0 (m/sec^2)
"G"	11	$2.17 \cdot 10^{-5}$	260	25.09	130	195.17	1.37	3.75
"J"	30	$5.37 \cdot 10^{-5}$	210	19.75	130	265.6	1.4	3.75

$$\rho = \rho_0 \left(\frac{T_0}{T}\right)^{1+\omega} \left(\frac{r_0}{r}\right)^{2-\omega} \exp\left[-\alpha\left(\frac{1}{r_0} - \frac{1}{r}\right)\right] \tag{1.4}$$

where

$$\alpha = g_0\, r_0^2/R\,(T_0 - Gr_0) \qquad \omega = G\alpha/(T_0 - Gr_0)$$

The density ρ_T at the tropopause altitude h_T is then found by substituting $r = r_0 + h_T$ in Equation (1.4).

Within the stratosphere, $T = T_S$, and the boundary conditions are $\rho(h = h_T) = \rho_T$; integration of Equation (1.3) results in the following density distribution function within the stratosphere:

$$\rho = \rho_T \left(\frac{r_T}{r}\right)^2 \exp\left[-\beta r_0^2\left(\frac{1}{r_T} - \frac{1}{r}\right)\right] \tag{1.5}$$

$$\beta = g_0/RT_S$$

Note that for $h = \infty$, $\rho = 0$.

If the acceleration of gravity is assumed constant throughout the atmosphere or, equivalently, if a flat planet is assumed, the density distribution becomes the purely exponential one $[\rho = \rho_0 \exp(-\beta h)]$ often treated in the literature (Reference 1). Figures 1.2 and 1.3 show density as a function of altitude for model G. For comparison, the density distribution, assuming a flat planet, is also included. It is seen that at 800,000 feet, the densities vary by an order of magnitude.

II. VERTICAL ENTRY OF A BODY HAVING CONSTANT BALLISTIC COEFFICIENT

For sufficiently high approach speeds, such as will be encountered by nonretarded Mars entry vehicles approaching Mars from Earth, the aerodynamic drag will be several orders of magnitude larger than the acceleration of gravity during entry. Neglecting gravity, the equation of motion of a body descending vertically into the atmosphere is (see Figure 2.1)

$$\dot{V} = -B\rho V^2 \tag{2.1}$$

where $B = C_D A/2m$, i.e., one half of the reciprocal ballistic coefficient, and is assumed constant. We assume the boundary conditions

$$V = V_E \text{ for } h = \infty \tag{2.2}$$

where subscript E represents conditions at entry.

Within the stratosphere, the density is given by Equation (1.5), and integration of Equation (2.1) results in the following equation for velocity V as a function of distance r from the center of the planet:

$$V(h) = V_E \exp\left\{ -\frac{B\rho_T r_T^2 \exp\left(\dfrac{-\beta r_0^2}{r_T}\right)}{\beta r_0^2}\left[\exp\left(\frac{\beta r_0^2}{r}\right) - 1\right]\right\} \tag{2.3}$$

The velocity at the tropopause altitude h_T is given by

$$V_T = V_E \exp\left\{ -\frac{B}{\beta}\left(\frac{r_T}{r_0}\right)^2 \rho_T \left[1 - \exp\left(\frac{-\beta r_0^2}{r_T}\right)\right]\right\} \tag{2.4}$$

Within the troposphere, the density is given by Equation (1.4) and the boundary condition is $V(h = h_T) = V_T$. One obtains the following velocity function valid within the troposphere:

$$V(r) = V_T \exp\left\{ - \frac{\rho_0 B \exp\left(\frac{-\alpha}{r_0}\right)}{\beta} \left(\frac{T_0}{r_0}\right)^{\omega} \left[\left(\frac{r}{T}\right)^{\omega} \exp\left(\frac{\alpha}{r}\right) - \left(\frac{r_T}{T_S}\right)^{\omega} \exp\left(\frac{\alpha}{r_T}\right)\right]\right\} \tag{2.5}$$

Within either layer, the velocity is given by

$$V(h) = V_T \exp\left\{ - \frac{BR}{g_0 r_0^2} \left[\rho(r)\, T(r)\, r^2 - \rho_T\, T_S\, r_T^2\right]\right\} \tag{2.6}$$

where subscript T refers to conditions at the tropopause altitude h_T.

By assigning specific values to the quantities C, j, and k in

$$F = C \rho^j V^k \tag{2.7}$$

several key trajectory parameters are obtained:

Linear acceleration: $|\dot{V}| = B \rho V^2$

Average heat-transfer rate: $\dot{H}_a = C_a C_F' \rho V^3$

Stagnation-point heat-transfer rate: $\dot{H}_S = C_S \rho^{1/2} V^3$

The altitude at which F reaches a maximum is found by solving the transcendental equation

$$j\left(\frac{2}{r} + \frac{g_0 r_0^2}{r^2 RT} + \frac{dT}{T}\right) = k B \rho \tag{2.8}$$

The time, measured from some convenient reference altitude, is given by the integral:

$$t = - \int \frac{dr}{V(r)} + t_0 \tag{2.9}$$

In the stratosphere, V is given by (2.3), and the time becomes:

$$t(r) = + \frac{\alpha}{C} \int \frac{e^{Y}\, dY}{Y\,(\ln Y/K)^2} \tag{2.10}$$

where

$$Y = \ln(V_E/V) \qquad K = \frac{B}{\beta}\left(\frac{r_T}{r_0}\right)\rho_T e^{-\gamma/r_T} \qquad C = V_E\, e^{K}$$

This integral must be evaluated numerically. A first approximation to (2.10) is obtained by setting $r_0 = \infty$, in which case the time can be expressed by the tabulated function Ei (x):

$$t = + \frac{1}{\beta V_E} \mathrm{Ei}(Y) + t_0 \tag{2.11}$$

III. NONVERTICAL ENTRY INTO A PLANETARY ATMOSPHERE

Let us now assume that the body approaches the planet along a rectilinear path which misses the planet's center by some nonzero distance $r_c \neq 0$. Let the coordinate y along the path be defined by Figure 2.1. Neglecting the acceleration of gravity as before, we obtain the equation of motion

$$\dot{V} = -B\rho V^2 \tag{3.1}$$

Transforming to the independent variable r, we obtain the equation of motion

$$\frac{dV}{V} = -B\rho\, dy = \frac{-B\rho(r)\, dr}{\left[1-\left(\frac{r_c}{r}\right)^2\right]^{1/2}} \tag{3.2}$$

Within the stratosphere, the density is given by Equation (1.3). If the boundary condition $V(h=\infty) = V_E$ is imposed, the final equation for the velocity within the stratosphere becomes

$$V = V_E \exp\left\{-\left[\frac{B\rho_T r_T^2 \exp\left(-\frac{\beta r_0^2}{r_T}\right)}{\beta r_0^2}\right]\left(J_0 + \frac{1}{2}J_2 + \cdots + C_n J_{2n} + \cdots\right)\right\} \tag{3.3}$$

where

$$J_0 = \exp(\delta Z) - 1 \qquad K = \frac{B\rho_T r_T^2 e^{-\beta r_0^2/r_T}}{r_c} \tag{3.4}$$

$$J_2 = \exp(\delta Z)\left(Z^2 - \frac{2Z}{\delta} + \frac{2}{\delta^2}\right) - \frac{2}{\delta^2} \qquad C_n = \frac{1\cdot3\cdot5\cdots(2n-1)}{2^n\, n!}$$

$$J_{2n} = I_{2n}\exp(\delta Z) - \frac{(2n)!}{\delta^{2n}} \qquad \delta = \frac{\beta r_0^2}{r_c}$$

$$Z = \frac{r_c}{r}$$

$$I_{2n} = Z^{2n} - 2n \left[\frac{Z^{2n-1}}{\delta} - \frac{(2n-1)}{\delta^2} I_{2n-2} \right]$$

$$I_0 = 1$$

Figure 3.1 shows velocity versus altitude for entry into the "G" model atmosphere. Curves are shown for two different ballistic coefficients ($B = 1$ and $10 \ ft^2/slug$) and two entry angles (−60 and −90°). Figure 3.2 shows the corresponding acceleration histories. The acceleration increases as the motion progresses; the altitude at which maximum acceleration is reached depends strongly on the ballistic coefficient, but not on entry velocity. The acceleration decreases again after reaching a maximum.

IV. THE ROTATIONAL MOTION OF A SPHERE ENTERING THE MARS ATMOSPHERE

We shall now investigate the six-degree-of-freedom motion of a sphere entering the Mars atmosphere. For this purpose, let (x, y, z) be the inertial cartesian coordinates of the center of gravity (S) of the sphere (see Figure 4.1). We also define two additional coordinate systems with origins at the c.g.: (x', y', z') with axes parallel to the inertial system, and (x_B, y_B, z_B) with axes fixed in the body. We assume that the sphere's c.g. (S) has coordinates $(x_B = 0, y_B = 0, z_B = 0)$, the coordinates of the sphere's center (M) are $(x_B = 0, y_B = 0, z_B = N)$, and the coordinates of the sphere's center of pressure (D) are $(x_B = 0, y_B = 0, z_B = -L)$. We assume N = constant and L = constant during entry. The z_B-axis of the sphere, which contains the c.g. (S), the center of the sphere (M), and its center of pressure (D), will be called its "axis."

Figure 4.1 shows the Euler angles (ψ, ϕ, θ), which relate the coordinate systems (x', y', z') and (x_B, y_B, z_B). The Euler angles and their time derivative have the following physical meaning:

> $\dot{\psi}$ is the inertial precession rate; ψ measures the angle between the line of nodes k and the fixed x'-direction. The line of nodes k is the intersection of the $x'y'$-plane and the x_By_B-plane.
>
> $\dot{\phi}$ measures the rate of self-rotation of the body about its "axis" z_B; ϕ measures the angle between the line of nodes and the x_B-axis.
>
> $\dot{\theta}$ is the angular rate of nutation of the body axis with respect to the inertial z-axis; θ is the inclination of the "axis" of the sphere with respect to the z-axis.

The six-degree-of-freedom equations of motion for the sphere will now be derived using the Lagrangian formulation. For this purpose, we first form the kinetic energy T of the vehicle as follows:

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2}(I_x P^2 + I_y Q^2 + I_z R^2) \tag{4.1}$$

It consists of a translational and a rotational part; $(\dot{x}, \dot{y}, \dot{z})$ are the cartesian velocity components of the c.g. and (P, Q, R) are the body-fixed components of angular rate. They are related to the Euler angles and their derivatives by

$$\begin{aligned} P &= \dot{\psi} \sin\theta \sin\phi + \dot{\theta} \cos\phi \\ Q &= \dot{\psi} \sin\theta \cos\phi - \dot{\theta} \sin\phi \\ R &= \dot{\psi} \cos\theta + \dot{\phi} \end{aligned} \tag{4.2}$$

If, in addition, we assume the mass distribution of the sphere to be axially symmetric, that is, $I_x = I_y$, then the kinetic energy becomes

$$T = \frac{I_x}{2} (\dot{\psi}^2 \sin^2 \theta + \dot{\theta}^2) + \frac{I_z}{2} (\dot{\psi} \cos \theta + \dot{\phi})^2 + \frac{m}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \tag{4.3}$$

The equations of motion for each degree of freedom (x, y, z, ψ, ϕ, θ) are found using Lagrange's equations. This has been carried out in Appendix 1. It is found that the motion of the c.g. is rectilinear, and the z-axis direction can be chosen as the direction of motion without loss of generality. The final rigorous equations of motion are

$$\ddot{x} = 0 \qquad x = x_0 \tag{4.41}$$

$$\ddot{y} = 0 \qquad y = y_0 \tag{4.42}$$

$$\ddot{z} = -\frac{C_D A \rho \dot{z}^2}{2m} \tag{4.43}$$

$$\frac{d}{dt} (I_x \dot{\psi} \sin^2 \theta + I_z (\dot{\psi} \cos \theta + \dot{\phi}) \cos \theta) = 0 \tag{4.44}$$

$$I_z \frac{d}{dt} (\dot{\psi} \cos \theta + \dot{\phi}) = 0 \tag{4.45}$$

$$I_x \ddot{\theta} - I_x \dot{\psi}^2 \sin \theta \cos \theta + I_z (\dot{\psi} \cos \theta + \dot{\phi}) \dot{\psi} \sin \theta = -\frac{C_D A \rho \dot{z}^2}{2} L \sin \theta \tag{4.46}$$

Equation (4.43) was solved in the former sections, if one sets $\dot{z} = V$, and will not be discussed here. Two integrals of the rotational motion follow immediately from (4.44) and (4.45):

$$(I_x \sin^2 \theta + I_z \cos^2 \theta) \dot{\psi} + I_z \dot{\phi} \cos \theta = I_x b \tag{4.5}$$

$$I_z (\dot{\psi} \cos \theta + \dot{\phi}) = I_z R = I_x a \tag{4.6}$$

Thus, we have

$$\dot{\phi} = \frac{I_x}{I_z} a - \dot{\psi} \cos \theta \tag{4.7}$$

$$\dot{\psi} = \frac{b - a \cos \theta}{\sin^2 \theta} \tag{4.8}$$

Equations (4.46), (4.7), and (4.8) constitute the rigorous rotational equations of motion for the sphere. The terms ρ and $\dot{z}^2 = V^2$ are known functions of altitude from the former analysis.

Two types of rotational motion will now be treated: fast precession and nutation.

V. FAST PRECESSIONAL MOTION OF A SPHERE ENTERING THE MARS ATMOSPHERE

We assume that the sphere is initially spinning at a high angular rate R about its "axis." (The axis of the sphere is the diameter which contains the c.g.) Assume also that the axis is inclined to the trajectory by some angle θ_E. For small angles of attack θ, we can set $\sin\theta = \theta$, $\cos\theta = 1$. The following differential equation is thus obtained for θ:

$$\frac{\ddot{\theta}\,\theta^3}{a^2} + \frac{\theta^4 U}{4} + c\,\theta^2 - c^2 = 0 \tag{5.1}$$

where $c = b - a$, $a^2 = (I_z R/I_x)^2$ and $U = (2 C_D A L I_x/I_z^2 R^2)\,\rho\,V^2$ are known functions.

The precession rate is given by

$$\dot{\psi} = \frac{ca}{\theta^2} \tag{5.2}$$

Let the nondimensional time τ be introduced by the transformation $\tau = t/t_1$, where $t_1 = 1$ sec. Then, the term $\ddot{\theta}$ in Equation (5.1) can be reinterpreted to mean double differentiation of θ with respect to τ. The quantity a becomes dimensionless. R is the numerical value of the spin rate.

For large spin rates R, a solution of (5.1) can be found of the form

$$\theta(t,\epsilon) = \theta_0(t) + \epsilon\,\theta_1(t) + \frac{\epsilon^2}{2}\,\theta_2(t) + \cdots \tag{5.3}$$

where

$$\epsilon = \frac{1}{a^2} = \left(\frac{I_x}{I_z}\right)^2 \frac{1}{R^2} \tag{5.4}$$

Substitution of (5.3) in (5.1) results in the following equation for the zero-order term θ_0:

$$\theta_0 = \left(\frac{2c}{1 + \sqrt{1 + U}}\right)^{1/2} \tag{5.5}$$

The first- and second-order functions satisfy

$$\theta_1 = \frac{-\ddot{\theta}_0\,\theta_0^2}{\theta_0^2 U + 2c} \tag{5.6}$$

and

$$\theta_2 = -\frac{\theta_0^3\,\ddot{\theta}_1 + 3\,\ddot{\theta}_0\,\theta_0^2\,\theta_1 + 1.5\,U\,\theta_0^2\,\theta_1^2 + c\,\theta_1^2}{U\,\theta_0^2 + 2c\,\theta_0} \tag{5.7}$$

In (5.6) and (5.7), the functions $\ddot{\theta}_0$, $\ddot{\theta}_1$ are known by

$$\ddot{\theta}_i = V^2 \left(\frac{d^2\theta_i}{dr^2} + B\rho\,\frac{d\theta_i}{dr}\right) \quad i = 0,\ 1,\ \cdots\ n \tag{5.8}$$

For $t = 0$ ($r = \infty$), we assume that $\theta = \theta_E$, and the solutions become

$$\theta_0(0) = c^{1/2};\ \theta_1(0) = 0;\ \theta_2(0) = 0 \tag{5.9}$$

Thus, $c = \theta_E^2$.

The dependence of the zero-order solution $\theta_0(r)$ on U, which is proportional to the linear acceleration $B\rho V^2$, is shown in Figure 5.1, as is the zero-order inertial precession rate $\dot{\psi}_0 = ca/\theta_0^2$. It is seen that the angle of attack θ_0 decreases monotonically with linear acceleration until maximum acceleration is reached; thereafter, θ_0 increases. $\dot{\psi}_0$ manifests exactly the opposite behavior.

VI. NUTATION OF THE AXIS OF A SPHERE ENTERING A PLANETARY ATMOSPHERE

If no initial angular rates are present, and if we assume some small, nonzero initial angle of attack θ_E (i.e., $R(0) = 0$, $\dot{\psi}(0) = 0$, $\theta(0) = \theta_E \neq 0$), the rigorous equations of motion (4.7, 4.8, 4.46) become

$$I_z(\dot{\psi} \cos \theta + \dot{\phi}) = I_z R = I_z R_0 = 0 \tag{6.1}$$

$$I_x \dot{\psi} \sin^2 \theta = I_x \dot{\psi}_0 \sin^2 \theta_0 = 0 \tag{6.2}$$

$$\ddot{\theta} + \frac{C_D A \rho V^2 L}{2I_x} \sin \theta = 0 \tag{6.3}$$

There remains one second-order differential equation for θ. The coefficients of $\sin \theta$ include ρ and V^2, which are known functions of altitude from the analysis of Section 3.

We now change to the independent variable (y), assuming nonvertical entry into the planetary atmosphere. From the previous analysis, we have:

$$y^2 = r^2 - r_0^2 \qquad \frac{1}{V}\frac{dV}{dy} = B\rho \tag{6.4}$$

and we obtain the following second-order differential equation for θ, with r as the independent variable:

$$\theta_{rr}\left(\frac{y}{r}\right)^2 + \theta_r\left(\frac{B\rho y}{r} + \frac{r_c^2}{r^3}\right) + \frac{C_D A \rho L}{2I_x} \sin \theta = 0 \tag{6.5}$$

For vertical entry ($r_c = 0$, $y = r$), and changing to the independent variable $h = r - r_T$, this equation becomes

$$\theta_{hh} + B\rho\theta_h + \frac{C_D A L}{2I_x} \rho \sin \theta = 0 \tag{6.6}$$

Within the stratosphere, the density ρ is given by

$$\rho = \rho_T \left(\frac{r_T}{r_T + h} \right)^2 e^{-\left(\frac{\gamma h}{1 + h/r_T} \right)} \tag{6.7}$$

$$\gamma = \beta r_0^2 / r_T^2$$

The density distribution (6.7) is extremely close to the exponential one, which corresponds to $r_T = \infty$. We seek a solution to (6.6) of the form

$$\theta(h, \epsilon) = \theta_0(h) + \epsilon\, \theta_1(h) + \frac{\epsilon^2}{2} \theta_2(h) + \cdots \tag{6.8}$$

where $\epsilon = 1/r_T \gamma$ is a small parameter. The following expansion for the density distribution is then obtained:

$$\rho = \rho_T\, e^{-\gamma h} \left[1 + \epsilon \gamma h (\gamma h - 2) + \epsilon^2 (\ldots) \right] \tag{6.9}$$

Substituting Equations (6.8) and (6.9) into the differential equation (6.6), the following differential equations are obtained for $\theta_0(h)$ and $\theta_1(h)$:

$$\theta_0'' + B \rho_T\, e^{-\gamma h} \theta_0' + C \rho_T\, e^{-\gamma h} \theta_0 = 0 \tag{6.10}$$

$$\theta_1'' + B \rho_T\, e^{-\gamma h} \theta_1' + C \rho_T\, e^{-\gamma h} \theta_1 = \gamma h (\gamma h - 2)\, \theta_0'' \tag{6.11}$$

The differential equations for $\theta_0(h)$ and the homogeneous equation for $\theta_1(h)$ are both of the type

$$y_{HH} + 2 f e^{-H} y_H + g e^{-H} y = 0 \tag{6.12}$$

where

$$f = B \rho_T / 2\gamma \qquad\qquad g = C_D A L\, \rho_T / 2 I_x \gamma^2$$

and subscript H refers to differentiation with respect to $H = \gamma h$. The first-order term can be removed by the transformation

$$y = z\, e^{f e^{-H}} \tag{6.13}$$

which leads to the equation

$$z_{HH} + z\left(e^{-H}(f+g) - f^2 e^{-2H}\right) = 0 \tag{6.14}$$

It can be shown that the term containing f^2 can be ignored with respect to the others. The solution to (6.12) then becomes

$$y = e^{fe^{-\gamma h}}\left(C_1 J_0\left(2\sqrt{f+g}\, e^{-\gamma h/2}\right) + C_2 Y_0\left(2\sqrt{f+g}\, e^{-\gamma h/2}\right)\right) \tag{6.15}$$

where J_0 and Y_0 are the zero-order Bessel functions of the first and second kind, respectively. If we impose the boundary condition

$$\theta_0(h=\infty) = \theta_E, \quad \frac{d}{dh}\theta_0(h=\infty) = 0$$

the following zero-order solution is obtained:

$$\theta_0(h) = \theta_E\, e^{fe^{-\gamma h}} J_0\left(2\sqrt{f+g}\, e^{-\gamma h/2}\right) \tag{6.16}$$

For large values of the argument we can expand $J_0(x)$ and we have

$$\theta_0(h) = \theta_E\, e^{fe^{-\gamma h}}\left(\pi\sqrt{f+g}\, e^{-\gamma h/2}\right)^{-½} \cos\left(\frac{\pi}{4} - 2\sqrt{f+g}\, e^{-\gamma h/2}\right) \tag{6.17}$$

The general solution of (6.11) consists of the general solution of the corresponding homogeneous equation, (6.12), and a particular solution of (6.11):

$$\theta_1(h) = e^{fe^{-\gamma h}}\left\{ y_1(h)\left[C_3 + \pi\int R(h)\, e^{fe^{-\gamma h}} y_2(h)\, dh\right] + y_2(h)\left[C_4 + \pi\int R(h)\, e^{fe^{-\gamma h}} y_1(h)\, dh\right]\right\} \tag{6.18}$$

where

$$y_1(h) = J_0\left(2\sqrt{f+g}\ e^{-\gamma h/2}\right),\quad y_2(h) = Y_0\left(2\sqrt{f+g}\ e^{-\gamma h/2}\right)$$

$$R(h) = -\left(\gamma^2 h^2 - 2\gamma h\right) e^{-\gamma h}\left(2\sqrt{f+g}\ f\ e^{-\gamma h/2}\ J_1\left(2\sqrt{f+g}\ e^{-\gamma h/2}\right) + g\ y_1(h)\right)$$

and constants C_3 and C_4 are to be adjusted according to the boundary conditions

$$\theta_1(h=\infty) = 0,\quad \theta_1'(h=\infty) = 0$$

The solution is not obtainable in terms of elementary functions.

Figure 6.1 shows the behavior of the first-order solution $\theta_0(h)$ for entry into the "G" atmosphere (Equations 6.16 and 6.17). The amplitude of (6.17) has a minimum at the altitude $h_m = 1/\gamma \ln(2B\rho_T/\gamma)$; below this altitude, it again diverges. This solution is discussed in detail in Reference 2.

VII. NUMERICAL COMPARISON WITH THE EXACT SOLUTION

In the following, a numerical comparison is drawn between the theory, which was derived under simplifying assumptions, and the exact solution of the problem. The six-degree-of-freedom equations of motion of a sphere with constant ballistic coefficient have been solved using an IBM 7094 computer. It is assumed that the atmosphere rotates rigidly with the planet and the gravitational acceleration acts on the body during entry. The following entry conditions were assumed:

Entry latitude	Equator plane
Entry altitude	$h_E = 400000$ ft
Inertial entry velocity	$V_E = 20000$ ft/sec
Inertial entry angle	$\theta_E = -90°$
Atmosphere	G
Ballistic coefficient	$B = 2$ ft^2/slug

Table 2 shows a comparison between the present theory, the exact theory, and the flat planet theory (Reference 1). In this Table, velocity, acceleration, and density are shown at various altitudes. Because no flat planet theory exists for an atmosphere with a temperature gradient, no values appear for altitudes less than 82300 ft (the tropopause altitude).

Table 3 shows a numerical comparison of the rotational motion theory (Equations 5.5, 5.7, 6.16) and the exact solution. In addition to those given above, the following initial conditions were assumed:

Initial angle of attack	$\theta_E = 20°$
Static moment L	$L = 5.17$ ft
Pitch moment of inertia	$I_x = 36.135$ slugs ft^2
Roll moment of inertia	$I_z = 48.18$ slugs ft^2
Spin rate	$R = 100$ RPM

Table 2. Numerical comparison with exact theory

h (ft)	Present theory Eqs. (2.3), (2.5), (2.7), (1.4), (1.5)			Exact theory (equatorial entry)		Flat planet theory		
	V (ft/sec)	a (ft/sec^2)	ρ (slugs/ft^3)	V_I (ft/sec)	a_I (ft/sec^2)	V (ft/sec)	a (ft/sec^2)	ρ (slugs/ft^3)
300000	19999.5	0.3802	0.475×10^{-9}	20000	11.28	19999.7	0.285	0.356×10^{-9}
200000	19966.9	28.71	0.36×10^{-7}	20023	19.73	19971.4	25.7	0.322×10^{-7}
150000	19707.9	250.6	0.32×10^{-7}	19778	254.0	19729.5	238.8	0.306×10^{-7}
100000	17505.8	1806.7	0.295×10^{-5}	17608	1818	17569.5	1802.2	0.292×10^{-5}
82300	14936.5	2891.4	0.648×10^{-5}	15038	2920	15000.4	2916.2	0.648×10^{-5}
50000	7491.	1767.	0.157×10^{-4}	7582	1792.5	- - - -	- - - - -	- - -
0	476.6	19.1	0.421×10^{-4}	628.5	20.96	- - - -	- - - - -	- - -

Table 3. Numerical comparison with exact theory: rotational motion

h (ft)	Precession			Nutation	
	New Theory Eqs. (5.5), (5.6)		Exact Theory	New Theory Eq. (6.16)	Exact Theory
	θ_0 (deg)	θ_1 (deg)	θ (deg)	Ampl. of θ (deg)	Ampl. of θ (deg)
300000	19.969831	19.969887	20.0	- -	20
200000	18.317042	18.317865	17.1	- -	7.3
150000	14.142225	14.142234	11.55	5.9	4.1
100000	9.6119784	9.6119360	7.27	3.41	2.57
80000	8.6408430	8.6408436	6.4713	2.7	2.3
75000	8.6358440	8.6358434	6.50065	- -	- -
70000	8.6937488	8.6937465	6.5486	- -	- -
50000	9.6580666	9.6580586	7.448155	- -	- -
0	18.777052	18.777048	17.8	- -	- -

REFERENCES

1. Allen, H. J., and Eggers, A. J., *A Study of the Motion and Aerodynamic Heating of Missiles Entering the Earth's Atmosphere*, NACA TN 4047, 1957.

2. Allen, H. J., *Motion of a Ballistic Missile Angularly Misaligned with the Flight Path Upon Entering the Atmosphere and Its Effect Upon Aerodynamic Heating, Aerodynamic Loads and Miss Distance*, NACA TN 4048, 1957.

NOTATION

Symbol	Dimensions	Explanation
a	(rad/sec)	See Eq. (5.1)
A	(ft^2)	Cross-sectional area of the reentry body
$B = C_D A/2m$	(ft^2/sl)	$(2 \times \text{Ballistic coefficient})^{-1}$
c		See Eq. (5.1)
C		See Eq. (2.10)
f, g		See Eq. (6.12)
F		See Eq. (2.7)
g_0	(ft/sec^2)	Acceleration of gravity at the surface of Mars
G	(°R/ft)	Temperature gradient in the troposphere
$h = r - r_0$	(ft)	Altitude
$h_T = r_T - r_0$	(ft)	Altitude of tropopause
j, k		See Eq. (2.7)
I_x, I_y, I_z	(sl ft^2)	Moments of inertia of sphere
J_0		Bessel function of the first kind
L	(ft)	Coordinate of sphere's center of pressure
m	(slugs)	Mass of entry body
N	(ft)	Coordinate of entry sphere's center
p	(mb)	Pressure
p_0	(mb)	Pressure at surface of Mars
P, Q, R	(rad/sec)	Body fixed angular rates
r	(ft)	Distance from planet center
r_0	(ft)	Radius of Mars
r_T	(ft)	Distance of tropopause from planet center

NOTATION (Cont'd)

Symbol	Dimensions	Explanation
R	($ft^2/sec^2\ ^\circ R$)	Gas constant
t	(sec)	Time (see Eq. 2.9)
T	($^\circ R$;sl ft/sec^2)	Temperature; kinetic energy
T_S	($^\circ R$)	Stratosphere temperature
T_0	($^\circ R$)	Surface temperature
U	(sec^{-2})	See Eq. (5.1)
V	(ft/sec)	Velocity
V_E	(ft/sec)	Velocity at entry
V_T	(ft/sec)	Velocity at $h = h_T$
$\left.\begin{array}{l} x, y, z \\ x', y', z' \\ x_B, y_B, z_B \end{array}\right\}$	(ft)	Defined in Section IV
y	(ft)	See Fig. 2.1
Y	(dimensionless)	$\ln (V_E/V)$
Y_0		Bessel function of the second kind
α		See Eq. (1.4)
β		See Eq. (1.5)
$\gamma = C_p/C_v$	(dimensionless)	Ratio of specific heats (see Eq. 6.7)
ρ	($slugs/ft^3$)	Density
ρ_0	($slugs/ft^3$)	Surface density
ρ_T	($slugs/ft^3$)	Tropopause density
(ψ, ϕ, θ)	(rad)	Euler angles (see Section IV)
ω		See Eq. (1.4)
Ω	(rad)	Solid angle (see Eq. 1.1)

NOTATION (Cont'd)

Symbol	Explanation
$(\dot{\ })$	Derivative with respect to time t
$(\ddot{\ })$	Double derivative with respect to time t
$(\ )_0$	Zero-order solution
$(\ )_1$	First-order solution
$(\ )_2$	Second-order solution
$(\ )_r$	Derivative with respect to r
$(\ )_{rr}$	Second derivative with respect to r
$(\ )_h$	Derivative with respect to h
$(\ )_{hh}$	Second derivative with respect to h
$(\ )_H$	Derivative with respect to $H = \gamma h$
$(\ )_{HH}$	Second derivative with respect to $H = \gamma h$

APPENDIX 1. DERIVATION OF THE ROTATIONAL EQUATIONS OF MOTION OF A SPHERE ENTERING THE MARS ATMOSPHERE

The kinetic energy of the sphere was given in Equation (4.3). The left-hand sides of Lagrange's equations for the system

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = F_q \qquad q = x, y, z, \psi, \phi, \theta \tag{1.1'}$$

for each of the degrees of freedom are:

$$\ddot{x} = F_x \quad \ddot{y} = F_y \quad \ddot{z} = F_z$$

$$\frac{d}{dt}\left(I_x \dot{\psi} \sin^2\theta + I_z(\dot{\psi}\cos\theta + \phi)\cos\theta\right) = F_\psi$$

$$I_z \frac{d}{dt}(\dot{\psi}\cos\theta + \dot{\phi}) = F_\phi \tag{1.2'}$$

$$I_x \ddot{\theta} - I_x \dot{\psi}^2 \sin\theta\cos\theta + I_z(\dot{\psi}\cos\theta + \dot{\phi})\dot{\psi}\sin\theta = F_\theta$$

The right sides of Equations (1.1′) are the generalized forces. They are obtained by subjecting the system to six virtual displacements dx, dy, dz, dP, dQ, dR; the generalized forces are the coefficients of dx, dy, dz, dψ, dϕ, dθ in the expression obtained for the virtual work dA. The only force acting on the system is $\vec{F}_A$, which is in the direction of the negative velocity vector:

$$\vec{F}_A = -\frac{C_D A \rho V}{2}\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} \tag{1.3'}$$

If we first consider only the translational virtual displacements dx, dy, dz, the following equations of motion are obtained:

$$m\ddot{x} = -C_D A \rho V \dot{x}/2 \qquad m\ddot{y} = -C_D A \rho V \dot{y}/2 \qquad m\ddot{z} = -C_D A \rho V \dot{z}/2 \tag{1.4'}$$

Thus, the motion of the c.g. is rectilinear, and we can assume, without loss of generality, that the motion is parallel to the z-axis (i.e., $\dot{x} = 0$, $\dot{y} = 0$, $|\dot{z}| = V$).

The aerodynamic force $\vec{F}_A$ acting at the center of pressure D is equipollant to a force $\vec{F}_A$ and a moment $\vec{M}_A$ acting at the c.g. (see Figure 4.1). $\vec{M}_A$ is given by the vector product:

$$\vec{M}_A = \vec{S}_D \times \vec{F}_A = -L \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} X(A) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \left(-C_D A \frac{\rho \dot{z}^2}{2} \right) \tag{1.5'}$$

where (A) is the matrix of rotation from the (x', y', z') system to the (x_B, y_B, z_B) system. The moment becomes

$$\vec{M}_A = -\left(\frac{1}{2} L C_D A \rho \dot{z}^2 \right) [\vec{i}_B \cos\phi \sin\theta - \vec{j}_B \sin\phi \sin\theta] \tag{1.6'}$$

$\vec{i}_B$, $\vec{j}_B$ are unit vectors along the x_B and y_B body axes, respectively. The virtual displacement components corresponding to the rotational degrees of freedom along the (x_B, y_B, z_B) directions are

$$\begin{aligned} dP &= P\,dt = d\psi (\sin\theta \sin\phi) + d\theta \cos\phi \\ dQ &= Q\,dt = d\psi (\sin\theta \cos\phi) - d\theta \sin\phi \\ dR &= R\,dt = d\psi (\cos\theta) + d\phi \end{aligned} \tag{1.7'}$$

respectively. The virtual work performed by the moment $\vec{M}_A$ during the virtual displacement (dP, dQ, dR) is given by the scalar product

$$dA = \vec{M}_A \cdot \begin{pmatrix} dP \\ dQ \\ dR \end{pmatrix} = -L C_D A \rho \dot{z}^2 \sin\theta \, d\theta/2 \tag{1.8'}$$

Thus, the generalized forces corresponding to the rotational degrees of freedom are

$$F_\psi = 0 \qquad F_\phi = 0 \qquad F_\theta = -C_D A \rho \dot{z}^2 L \sin\theta/2$$

Substitution into Equations (1.2') results in the equations of motion (4.44-4.46) of Section IV.

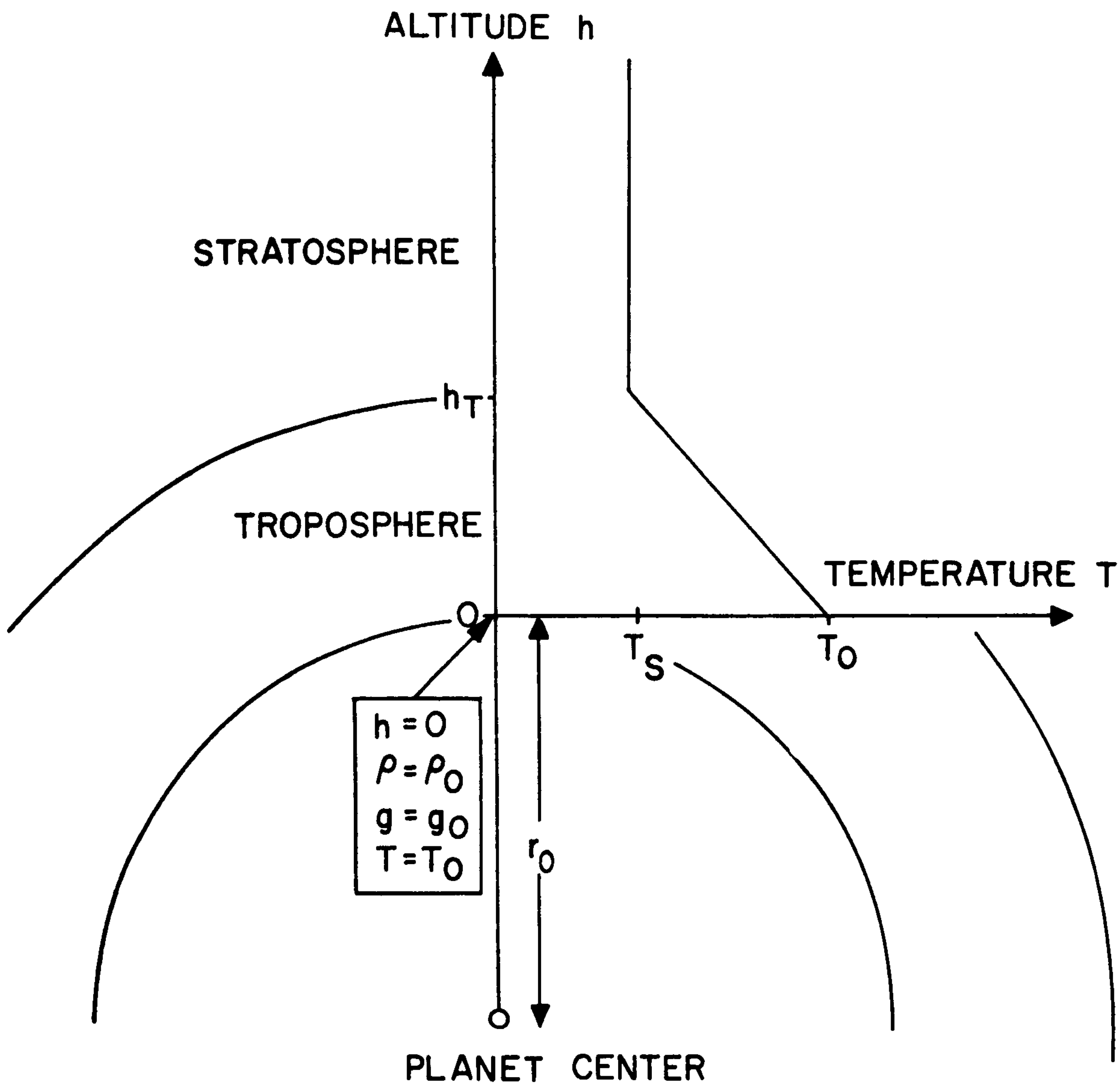

Fig. 1.1. Mars model atmosphere: temperature distribution

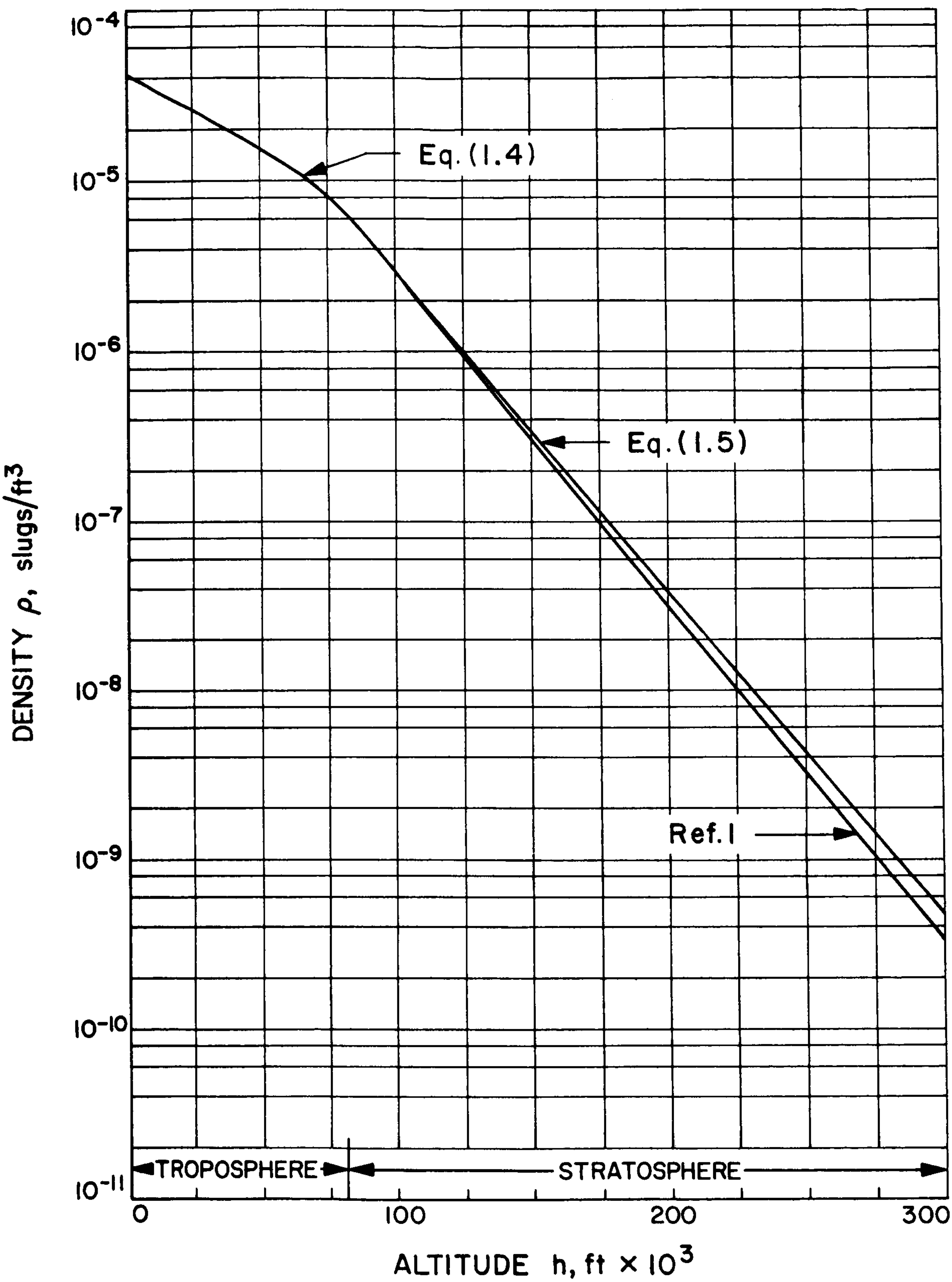

Fig. 1.2. Mars atmosphere density distribution; atmosphere "G"

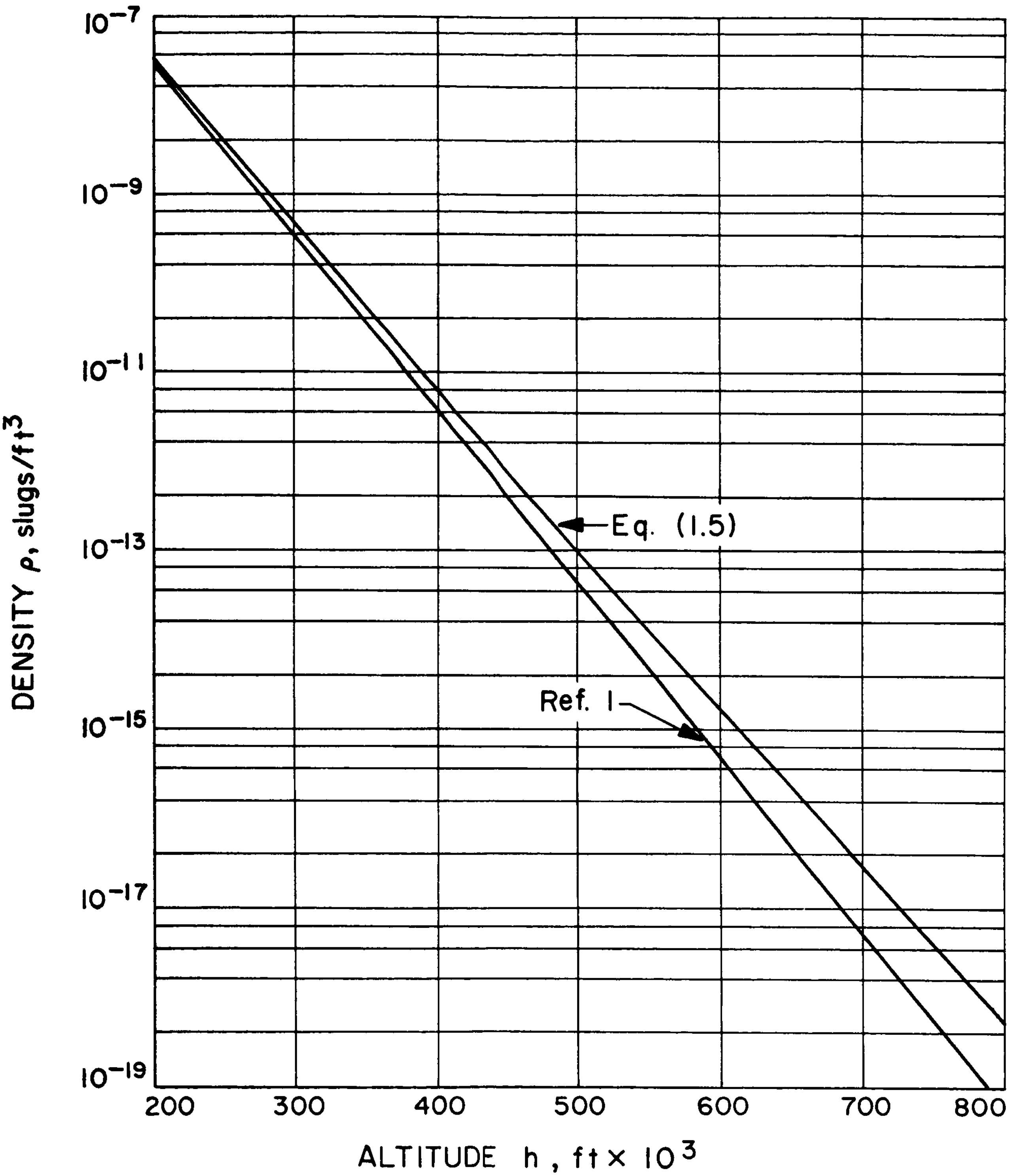

Fig. 1.3. Mars stratosphere density distribution; atmosphere "G"

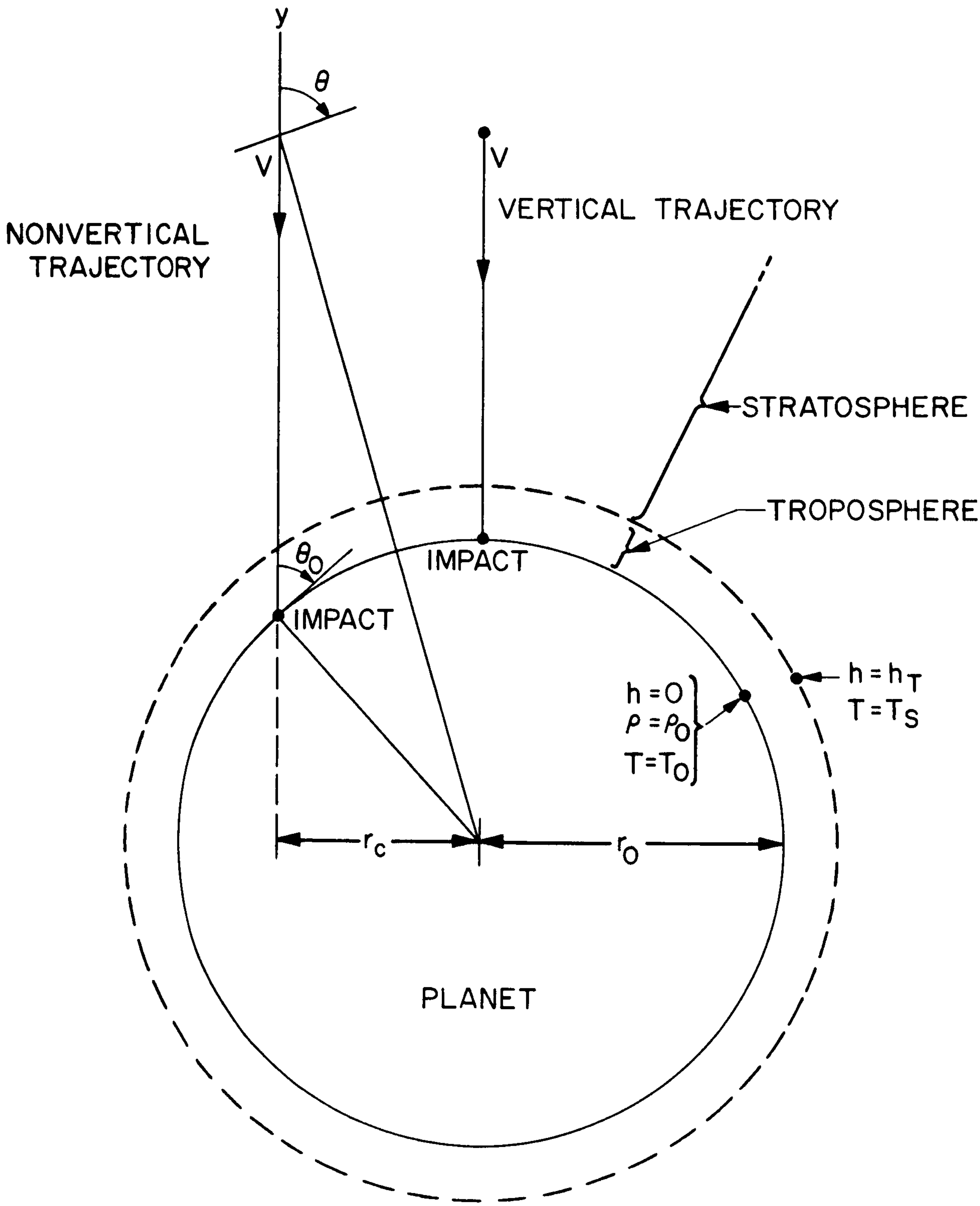

Fig. 2.1. Reentry trajectories: definition of terms

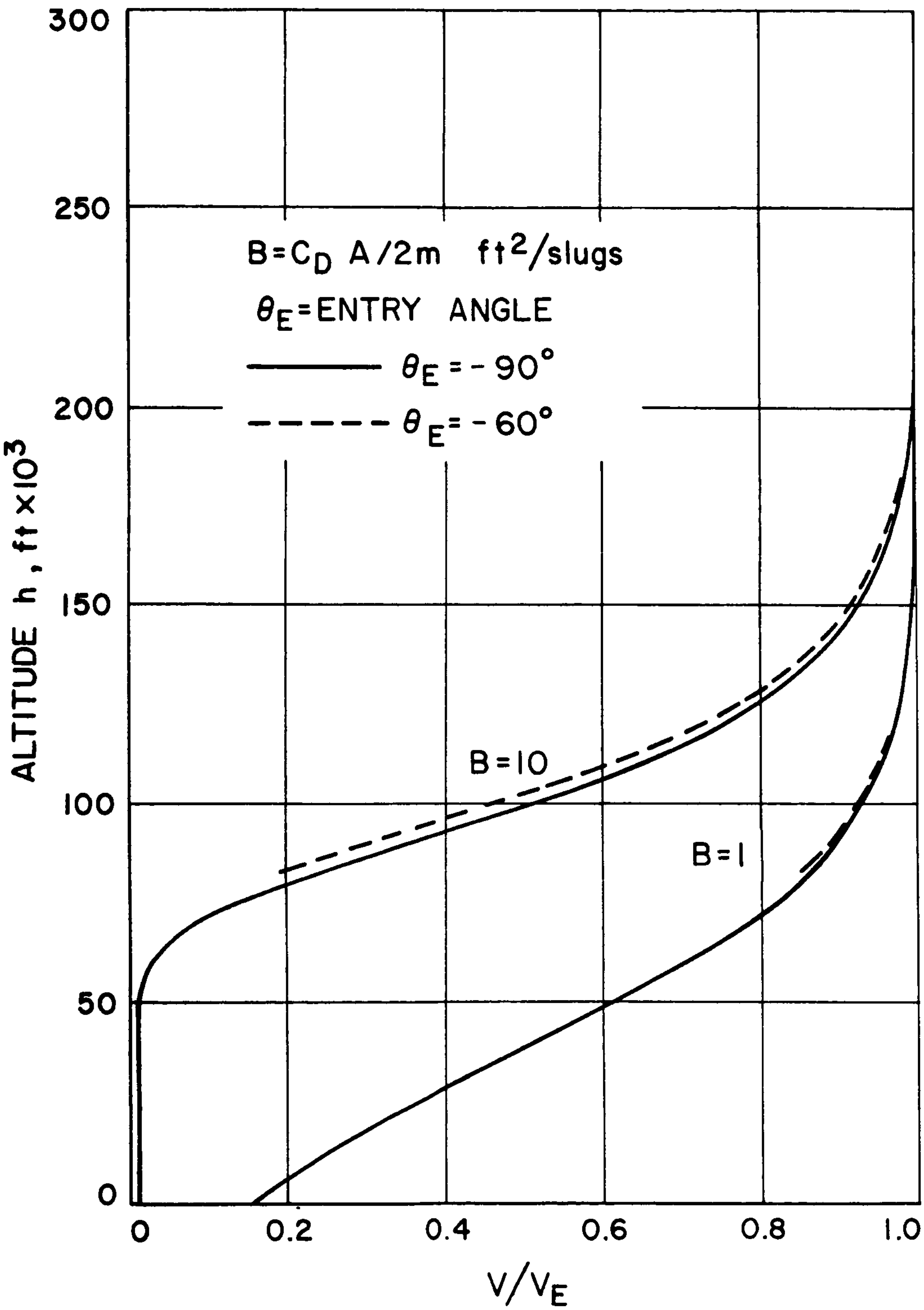

Fig. 3.1. Velocity V/V_E as a function of altitude h; atmosphere "G"

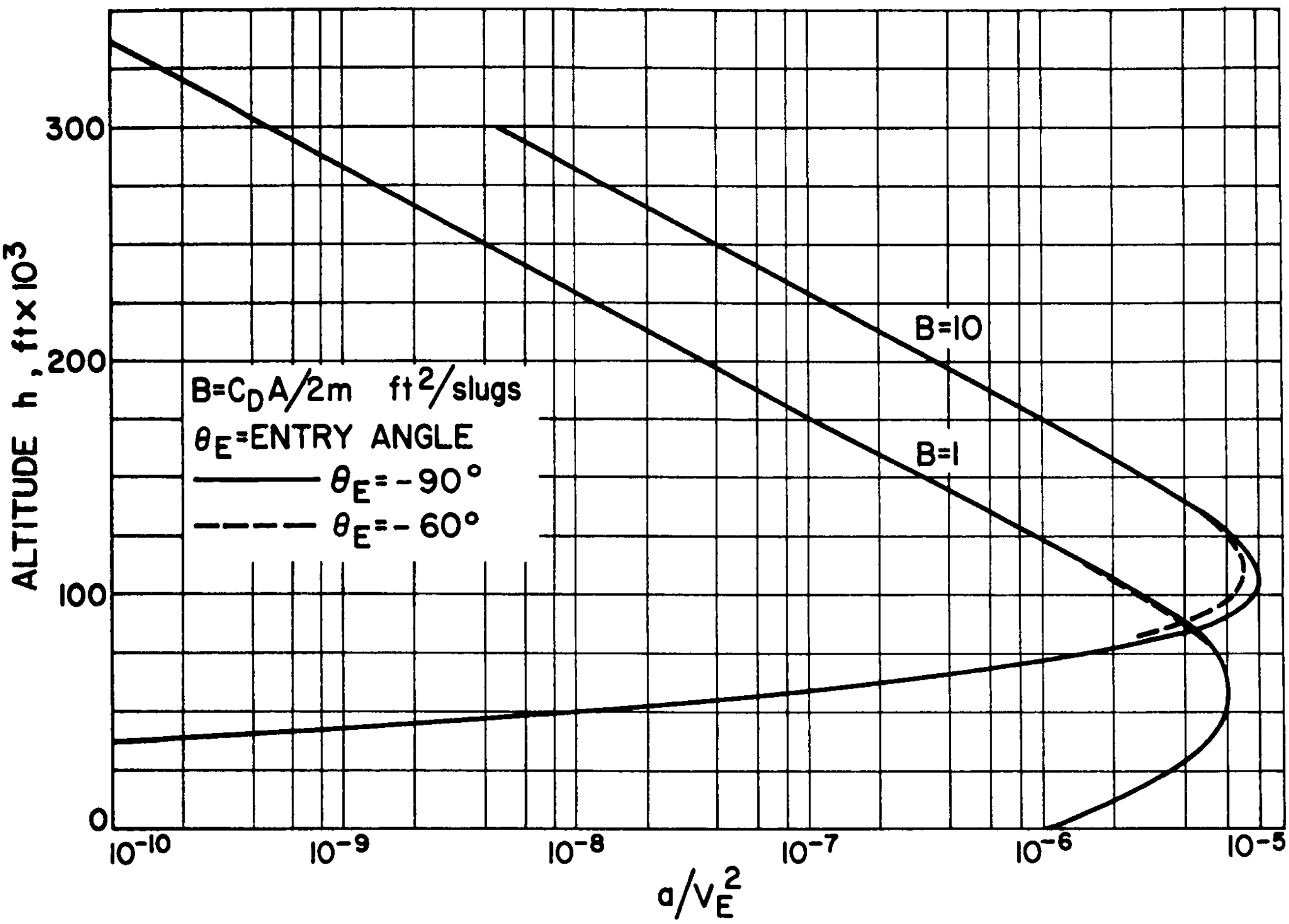

Fig. 3.2. Acceleration a/V_E^2 as a function of altitude h; atmosphere "G"

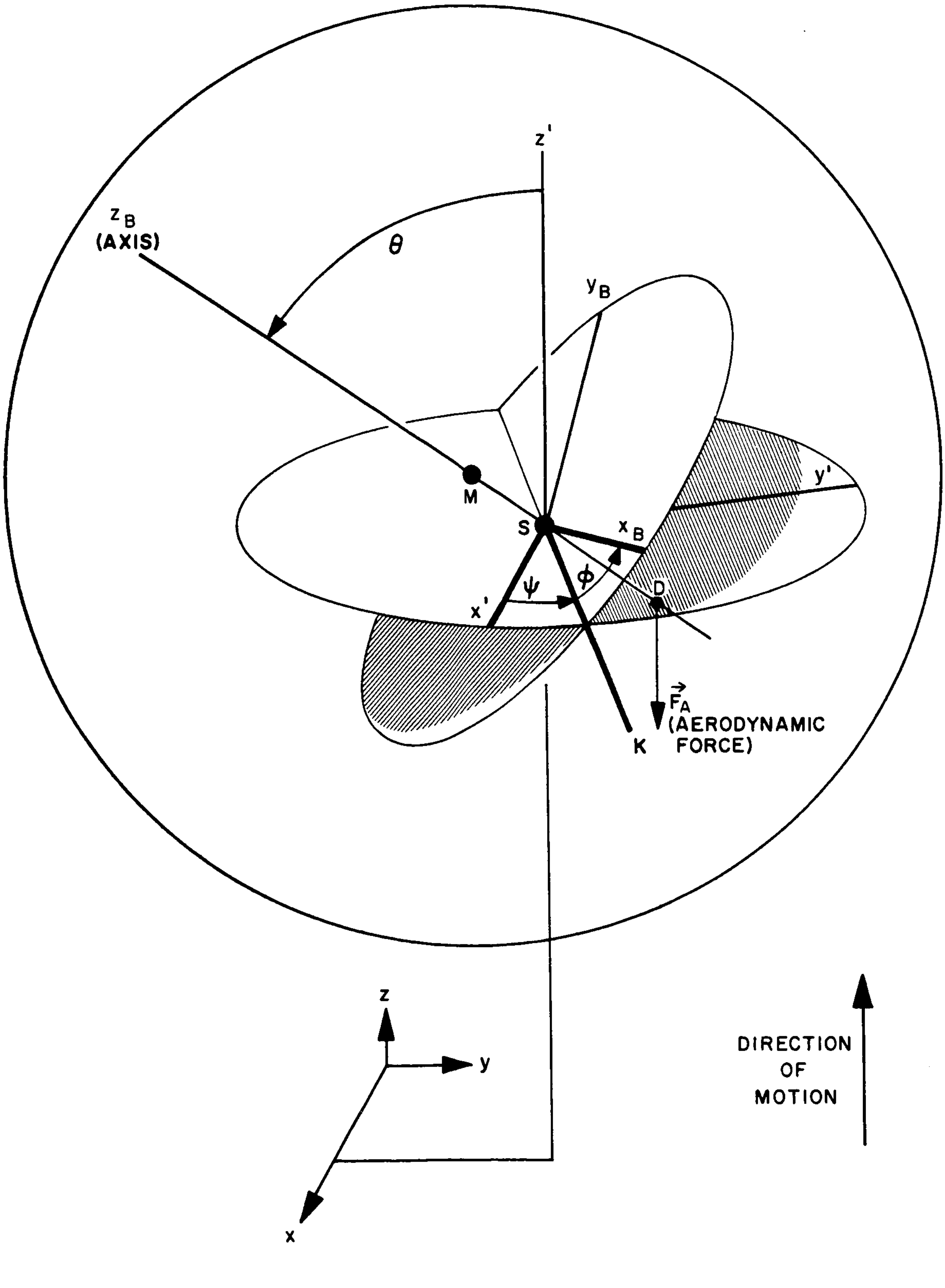

Fig. 4.1. Definition of Euler angles ψ, ϕ, θ

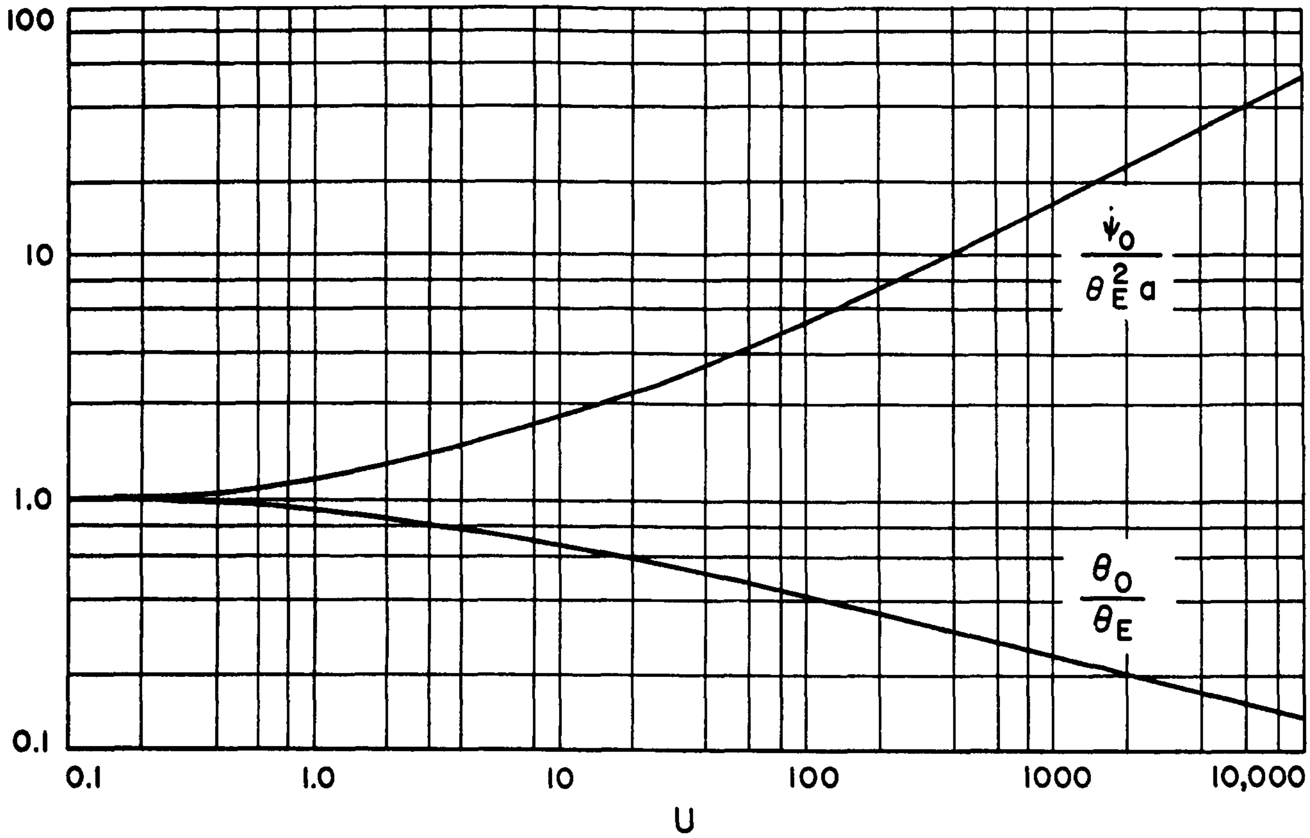

Fig. 5.1. Normalized zero-order precession angle θ_0/θ_E as a function of U = constant × linear acceleration; normalized zero-order precession rate as a function of U

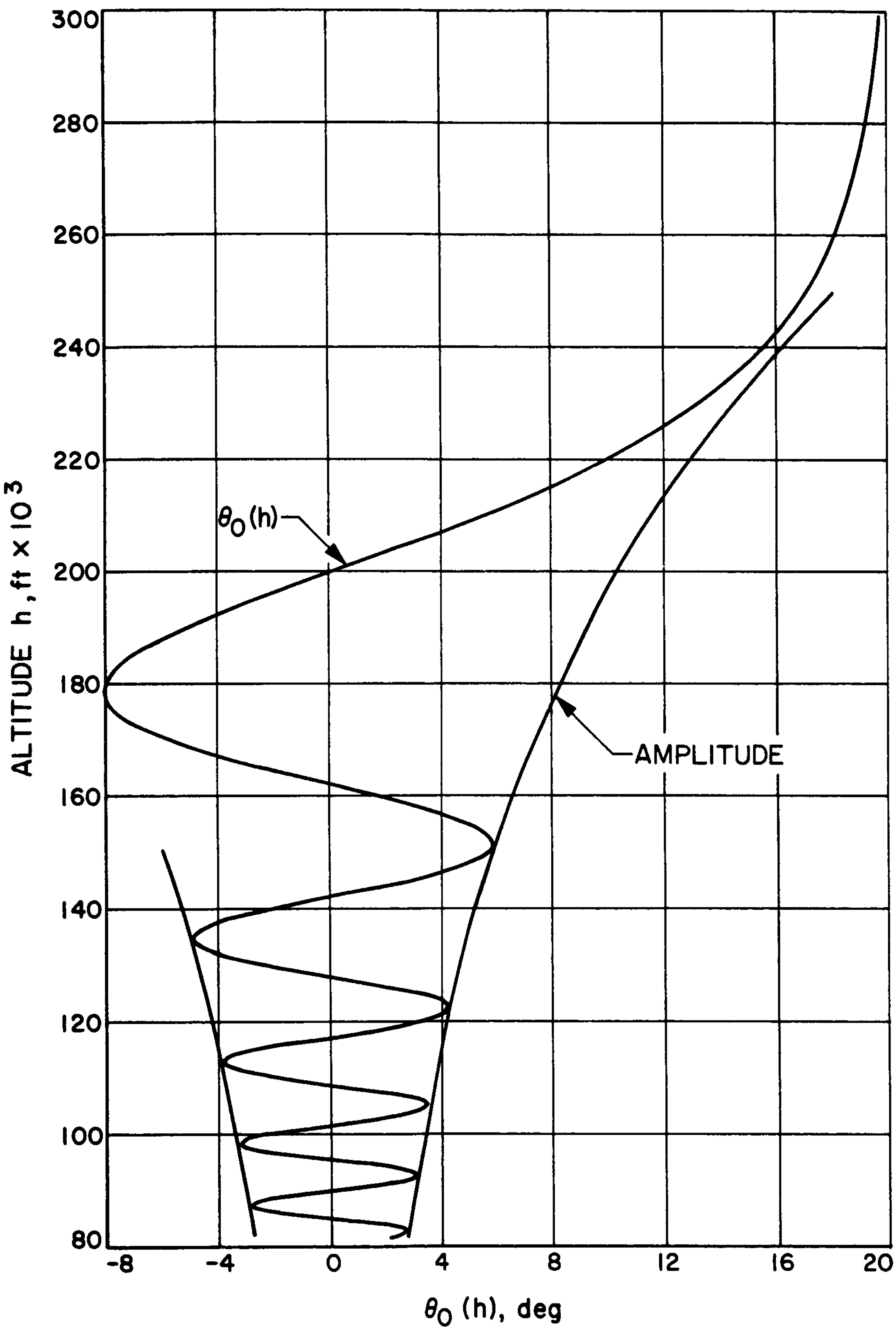

Fig. 6.1. Nutation: behavior of first-order solution; $\theta_0(h)$